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INNER GEOMETRY OF COMPLEX SURFACES: A VALUATIVE APPROACH

ANDRÉ BELOTTO DA SILVA, LORENZO FANTINI, AND ANNE PICHON

Abstract. Given a complex analytic germ $(X, 0)$ in $(\mathbb{C}^n, 0)$, the standard Hermitian metric of $\mathbb{C}^n$ induces a natural arc-length metric on $(X, 0)$, called the inner metric. We study the inner metric structure of the germ of an isolated complex surface singularity $(X, 0)$ by means of an infinite family of numerical analytic invariants, called inner rates. Our main result is a formula for the Laplacian of the inner rate function on a space of valuations, the non-archimedean link of $(X, 0)$. We deduce in particular that the global data consisting of the topology of $(X, 0)$, together with the configuration of a generic hyperplane section and of the polar curve of a generic plane projection of $(X, 0)$, completely determine all the inner rates on $(X, 0)$, and hence the local metric structure of the germ. Several other applications of our formula are discussed in the paper.

1. Introduction

Given a complex analytic germ $(X, 0) \subset (\mathbb{C}^n, 0)$ at the origin of $\mathbb{C}^n$, the standard Hermitian metric of $\mathbb{C}^n$ induces a natural metric on $(X, 0)$ by measuring the lengths of arcs on $(X, 0)$. This metric, called the *inner metric*, has been widely studied in several different contexts. For example, in [BL07] it is used to introduce a notion of metric tangent cone, while in [HP85] and [Nag89] it is used to study the the L^2-cohomology of singular algebraic surfaces, following original ideas of [Che80].

The study of the inner metric is motivated by the following fact: while it is well known that, for $\epsilon > 0$ sufficiently small, X is locally homeomorphic to the cone over its link $X^{(\epsilon)} = X \cap S_\epsilon$, where S_ϵ denotes the sphere centered at 0 with radius ϵ in $\mathbb{C}^n$, in general the metric germ $(X, 0)$ is not metrically conical. Indeed, there are parts of its link $X^{(\epsilon)}$ whose diameters with respect to the inner metric shrink faster than linearly when approaching the singular point. It is then natural to study how $X^{(\epsilon)}$ behaves metrically when approaching the origin.

In this paper, we study the metric structure of the germ of an isolated complex surface singularity $(X, 0)$ by means of an infinite family of numerical analytic invariants, called *inner rates*. A resolution of $(X, 0)$ is *good* if its exceptional locus is a normal crossing divisor. Given a good resolution $\pi\colon X_\pi \to X$ of $(X, 0)$ that factors through the blowup of the maximal ideal and through the Nash modification of $(X, 0)$, an irreducible component E of the exceptional divisor $\pi^{-1}(0)$ of π, and a small neighborhood $\mathcal{N}(E)$ of E in X_π with a neighborhood of each double point of $\pi^{-1}(0)$ removed, the inner rate q_E of E is a rational number that measures how fast the subset $\pi\big(\mathcal{N}(E)\big)$ of $(X, 0)$ shrinks when approaching the origin (see Figure 1 for a pictorial explanation and Definition 3.3 for a precise definition). The inner rate q_E is independent of the choice of an embedding of $(X, 0)$ into a smooth germ, only depending on the analytic type of $(X, 0)$ and on the divisor E.

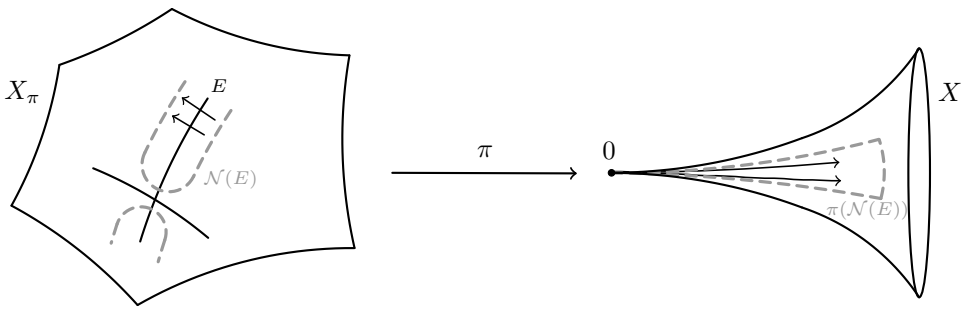

FIGURE 1. The inner rate q_E, which measures the size of the subset $\pi\big(\mathcal{N}(E)\big)$ of $(X,0)$ delimited by the dashed gray line on the right, can also bee defined as the inner contact of the pushforwards to X of two generic curvettes of E (see the discussion of Section 3).

As the sets of the form $\pi\big(\mathcal{N}(E)\big)$ cover the germ $(X,0)$ and can be taken to be arbitrarily small by refining the resolution π, the knowledge of all the inner rates of the exceptional divisors of all good resolutions of $(X,0)$ gives a very fine understanding of the metric structure of the germ. For example, one can use the inner rates to compute the contact order for the inner metric of any pair of complex curve germs in $(X,0)$, using a minimax procedure (see Remark 3.4).

The inner rates form an infinite family of analytic invariants of the germ $(X,0)$. Some specific inner rates have been studied in the context of Lipschitz geometry, but the bilipschitz class of $(X,0)$ can only determine finitely many of them. This is a consequence of the tameness of the bilipschitz classification of germs, which was proved by Mostovski in the complex analytic setting [Mos85] and by Parusiński in the real semi-analytic setting [Par88]. The finite set of inner rates which are bilipschitz invariants for the inner metric has been described explicitly in [BNP14].

While the inner rates have raised significant interest in the last decade, it was still an open question to determine how the global geometry of the singularity $(X,0)$ influences their behavior. Our first main result states that the global data of the topology of $(X,0)$, together with the configuration of a generic hyperplane section and of the polar curve of a generic plane projection $(X,0) \to (\mathbb{C}^2,0)$, not only influences the behavior of the inner rates, but in fact completely determines all of them.

Theorem A. *Let $(X,0)$ be an isolated complex surface singularity, let E be an irreducible component of the exceptional divisor of some resolution of $(X,0)$, and let $\pi\colon X_\pi \to X$ be the minimal good resolution of $(X,0)$ that factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$. Then the inner rate q_E of E is determined by (and can be computed in terms of) the following data:*

(i) the topological data consisting of the dual graph Γ_π of $\pi^{-1}(0)$ decorated with the Euler classes and the genera of its vertices;
(ii) on each vertex v of Γ_π, an arrow for each irreducible component of a generic hyperplane section of $(X,0)$ whose strict transform on X_π passes through the irreducible component E_v of $\pi^{-1}(0)$ corresponding to v;

(iii) on each vertex v of Γ_π, an arrow for each irreducible component of the polar curve of a generic projection $(X,0) \to (\mathbb{C}^2,0)$ whose strict transform on X_π passes through E_v.

In earlier papers on the subject, the inner rates were always computed by considering a generic projection $\ell\colon (X,0) \to (\mathbb{C}^2,0)$ and lifting the inner rates of the components of the exceptional divisor of a suitable resolution of the discriminant curve of ℓ. Outside of the simplest examples, this approach turned out to be very impractical and computationally expensive, since it is generally very hard to compute discriminant curves and to decide whether a projection is generic. On the other hand, given the data of the theorem, which is generally much simpler to obtain, our result also provides a very easy way to compute the inner rate q_E by means of an elementary linear algebra computation. To showcase this fact, in Example 5.8 we illustrate, for a fairly complicated singularity, how simple it is to obtain the inner rates, which were computed in [BNP14, Example 15.2] using Maple.

We obtain Theorem A as a consequence (see Corollary 5.4) of a stronger result about the Laplacian on the inner rate function on a space of valuations, a non-archimedean avatar of the link $X^{(\epsilon)}$ of $(X,0)$. Indeed, it is very natural to study the inner rates from the point of view of valuation theory, since q_E only depends on the *divisorial valuation* associated with an exceptional divisor E.

The *non-archimedean link* $\mathrm{NL}(X,0)$ of $(X,0)$ is defined as the set of (suitably normalized) semi-valuations on the completed local ring $\mathcal{O}_{X,0}$ of X at 0 that are trivial on $\mathbb{C}$ (see Definition 2.2). It is a Hausdorff topological space that contains the set of divisorial valuations of $(X,0)$ as a dense subset. For example, if X is smooth at 0 then the associated non-archimedean link is a well known object, the valuative tree of [FJ04]; the singular case was first studied in [Fav10].

If $\pi\colon X_\pi \to X$ is a good resolution of $(X,0)$, then the dual graph Γ_π of the exceptional divisor $\pi^{-1}(0)$ embeds naturally in $\mathrm{NL}(X,0)$, and the latter deforms continuously onto the former. Moreover, these retractions identify $\mathrm{NL}(X,0)$ with the projective limit of all the dual graphs of the good resolutions of $(X,0)$, allowing one to see $\mathrm{NL}(X,0)$ as a universal dual graph. This makes it a very convenient object in the study of the inner rates.

We endow each dual resolution graph Γ_π with a metric defined as follows. Let $e = [v,v']$ be an edge of Γ_π and let E_v be the component of the exceptional divisor $\pi^{-1}(0)$ corresponding to v. Then the *multiplicity* m_v of v is defined as the order of vanishing on E of the pullback on X_π of a generic hyperplane section of $(X,0)$ (that is, equivalently, m_v is the multiplicity of E_v in the exceptional divisor $\pi^{-1}(0)$). Similarly, denote by $m_{v'}$ the multiplicity of v'. Then we declare the length of the edge e to be $1/(m_v m_{v'})$. This length has a natural geometrical interpretation as the opposite of the screw number of the representative of the monodromy automorphism on the piece defined by the edge e of the Milnor fiber of a generic linear form on $(X,0)$ (see [MMA11, Theorem 7.3.(iv)] and the discussion of Remark 2.5). These lengths give rise to a metric on each Γ_π and, by refining the resolutions π, to a natural metric on $\mathrm{NL}(X,0)$.

The starting point of our exploration lies in the fact that the map sending a divisorial valuation v to the associated inner rate q_{E_v} extends canonically to a continuous function $\mathcal{I}_X\colon \mathrm{NL}(X,0) \to \mathbb{R}_+ \cup \{+\infty\}$ which has the remarkable property that its restriction $\mathcal{I}_X|_{\Gamma_\pi}$ to any dual resolution graph Γ_π is piecewise linear with integral slopes with respect to the metric defined above. This allows to

study the inner rate function $\mathcal{I}_X$ using classical tools of potential theory on metric graphs, as is done for example in an arithmetic setting in [BN16]. Namely, our main result is a formula computing the *Laplacian* $\Delta_{\Gamma_\pi}(\mathcal{I}_X)$ of the restriction of the inner rate function to a dual resolution graph Γ_π, that is the divisor on Γ_π whose coefficient $\Delta_{\Gamma_\pi}(\mathcal{I}_X)(v)$ at a vertex v of Γ_π is the sum of the slopes of $\mathcal{I}_X$ on the edges of Γ_π emanating from v (see Section 2.3 for a detailed explanation of all the relevant notions).

In order to measure quantitatively the geometric data appearing in the statement of Theorem A, for any vertex v of Γ_π denote by l_v (respectively by p_v) the number of arrows on v associated with a generic hyperplane section (respectively with a generic polar curve) of $(X,0)$, by $\check{E}_v$ the reduced curve obtained by removing from E_v the double points of $\pi^{-1}(0)$, and by $\chi(\check{E}_v)$ the Euler characteristic of $\check{E}_v$. We can now state a slightly weaker version of our main result, Theorem 4.2.

Theorem B (Laplacian of the inner rate function)**.** *Let $(X,0)$ be an isolated complex surface singularity and let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ that factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$. For every vertex v of Γ_π, we have:*

$$\Delta_{\Gamma_\pi}(\mathcal{I}_X)(v) = m_v\big(2l_v - p_v - \chi(\check{E}_v)\big).$$

Obtaining such a precise result is only possible because we are working with two-dimensional germs. Indeed, not only the statement of Theorem B itself is very specific to the dimension two, but our proof uses in an essential way the fact that the link of a complex surface is a 3-dimensional real manifold.

Besides the complete description of inner rates obtained in Theorem A, Theorem B has several interesting consequences. First, as an immediate by-product of our result we deduce that the inner rate function $\mathcal{I}_X$ is actually linear, and not just piecewise linear, along every string of the metric graph Γ_π (Corollary 4.5). Furthermore, the fact that the Laplacian is a divisor of degree zero of Γ_π (that is, the sum over all the vertices v of Γ_π of the integers $\Delta_{\Gamma_\pi}(\mathcal{I}_X)(v)$ vanishes) allows us to retrieve the Lê–Greuel–Teissier formula [LT81], which is a singular version of the classical Lê–Greuel formula, in the case of a generic linear form $h\colon (X,0) \to (\mathbb{C},0)$ on a complex surface singularity (Proposition 5.1).

The Laplacian formula of Theorem B also imposes strong restrictions on the possible configurations of the values of l_v and p_v on the vertices of the graph Γ_π. To perform a detailed study of this phenomenon, we prove an alternative, more numerical version of Theorem 4.2, Proposition 5.3, which not only permits us to easily derive an extended version of Theorem A (Corollary 5.4), but also has multiple other applications. For example, considering now the minimal resolution of $(X,0)$ which factors through the blowup of the maximal ideal (and not necessarily through the Nash transform) of $(X,0)$, then the data of the inner rates and the integers l_v on Γ_π determine the localization of the strict transform of the polar curve Π of a generic projection of $(X,0)$, and vice versa. More interestingly, prescribing the values of the inner rates is not required, as even without doing so Proposition 5.3 still gives us strong restrictions on the relative positions of the components of the polar curve and of the generic hyperplane section, and on the values of the inner rates as well, as illustrated in Example 5.9.

These restrictions on the relative behavior of the polar curve and of the generic hyperplane section are to be interpreted as a first step towards the famous question

of D. T. Lê (see [LÔ0, Section 4.3] or [BL02, Section 8]) inquiring about the existence of a duality between the two main algorithms of resolution of a complex surface, via normalized blowups of points ([Zar39]) or via normalized Nash transforms ([Spi90]). Further restrictions can be derived from the fact that not every degree zero divisor on a metric graph can be realized as the Laplacian of a piecewise linear function with integral slopes; this is part of an rapidly developing research area and we refer the interested reader to the recent book [CP18] for an overview of divisor theory on metric graphs that focuses on similar questions.

Finally, as explained at the end of the paper, Proposition 5.3 can also be considered from the point of [Mic08, Theorem 4.9], which gives information on the localization of the polar curve of the germ of a finite morphism $(f,g)\colon (X,0)\to(\mathbb{C}^2,0)$ in terms of the topology of the pair $\big(X,(fg)^{-1}(0)\big)$. In the case where f and g are generic linear forms on $(X,0)$, that is when the morphism $\ell=(f,g)$ is a generic projection of $(X,0)$, [Mic08, Theorem 4.9] does not give more information than the polar multiplicity described by Lê–Greuel–Teissier formula. However, our Proposition 5.3 refines this result, imposing strong restrictions on the possible localization of the polar curve.

◇

Let us conclude this introduction by discussing several future directions whose exploration is made possible by our valuative point of view on inner rates, and which will be the subjects of further study by the authors.

First of all, as observed in Remark 3.10, the inner rates can be used to define a ultrametric (that is, non-archimedean) inner distance on $\mathrm{NL}(X,0)$ via a standard minimax procedure which in this case has a very natural geometric interpretation. Ultrametric distances on valuations spaces of surface singularities have been recently studied in [GBGPPPR18]; however only a special class of surface singularities, those having contractible non-archimedean link (called *arborescent* in *loc.cit.*) were endowed there with an ultrametric distance carrying a geometric interpretation. Our non-archimedean inner distance on $\mathrm{NL}(X,0)$ will be used to define canonical metric decompositions of non-archimedean links, parallel to those of [BNP14] but much more intrinsic.

The inner rate of a divisorial valuation v of $(X,0)$ can be studied via birational techniques, namely using logarithmic Fitting ideals. More precisely, it can be shown that the inner rate $\mathcal{I}_X(v)$ of v equals the Mather log discrepancy of v (normalized by the multiplicity m_v) minus one (see Remark 3.11). In particular, in the smooth case the inner rate function $\mathcal{I}_{\mathbb{C}^2}$ equals the normalized log discrepancy function on $\mathrm{NL}(\mathbb{C}^2,0)$, as studied for example in [FJ04] and in [GBGPPP19], minus one. This approach would be well suited to be generalized to the higher dimensional case, for example by exploiting recent work on the resolution of singularities of cotangent sheaves of [BdSBGM17]. Indeed, the metric properties of singular varieties of higher dimension are poorly understood and the approach outlined above could shed some light on them, although we do not expect a result as sharp as our Theorem B.

It is also worth noticing that the Laplacian formula could be fairly easily extended to the case of a complex surface germ with non isolated singularity by adding a correcting term involving the multiplicities of generic hyperplane sections of X at points of $\mathrm{Sing}X\smallsetminus\{0\}$. This correcting term would appear in a generalized statement of Proposition 4.21, that is in the computation of the Euler Characteristic of the part

F_v of the Milnor fiber of a generic linear form on $(X, 0)$. Moreover, the fact that our formula enables us to recover a Lê–Greuel type formula for a particular function also suggests the possibility of extending the Laplacian formula to more general settings such as the metric behavior of a pair of holomorphic germs $(f, g)\colon (X, 0) \to (\mathbb{C}^2, 0)$, which as of now is still poorly understood.

Finally, let us remark that the inner rates were used in [BNP14] to study the Lipschitz classification of the inner metric surface germs $(X, 0)$. While on this paper we chose to focus on a metric germ $(X, 0)$, and not on its Lipschitz class, our methods seem very well adapted to study questions of bilipschitz geometry. For example, the valuation-theoretic point of view permits to give an elegant valuative version of the complete invariant on the inner Lipschitz geometry of $(X, 0)$ constructed in [BNP14].

◇

Let us give a short outline of the structure of the paper. All the material about non-archimedean links, dual graphs, and potential theory on metric graphs that we need in the paper is recalled in section 2. Section 3 is devoted to the construction of the inner rate function $\mathcal{I}_X$ and to the proof of its basic properties, such as its piecewise linearity. In Section 4 we state and prove our main result, Theorem 4.2, which is a stronger version of Theorem B. Section 5 is devoted to proving a slightly more precise version of Theorem A, Corollary 5.4, and discussing other applications of Theorem 4.2.

We aimed to make the paper entirely self-contained. The main definitions and results are illustrated with the help of a recurring example, that of the E_8 surface singularity, which is treated in detail throughout the paper (see Examples 2.4, 2.6, 2.8, 3.7, 4.1, and 4.4).

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2. Preliminaries

2.1. **Non-archimedean links.** Throughout the paper, $(X, 0)$ will always be an isolated complex surface singularity. We will begin by introducing the valuation space we will work with, the non-archimedean link of $(X, 0)$.

Denote by $\mathcal{O} = \widehat{\mathcal{O}_{X,0}}$ the completion of the local ring of X at 0 with respect to its maximal ideal. A (rank 1) *semivaluation* on $\mathcal{O}$ is a map $v\colon \mathcal{O} \to \mathbb{R} \cup \{+\infty\}$ such that, for every f and g in $\mathcal{O}$ and every λ in $\mathbb{C}^\times$, we have

(i) $v(fg) = v(f) + v(g)$;
(ii) $v(f + g) \geq \min\{v(f), v(g)\}$;
(iii) $v(\lambda) = 0$.

In particular, it follows from the first condition that $v(0) = +\infty$. However, we do not require 0 to be the only element sent to $+\infty$, which is why our maps are only semivaluations rather than valuations. If v is a semivaluation on $\mathcal{O}$, the valuation of v on the maximal ideal $\mathfrak{M}$ of $\mathcal{O}$ is defined as $v(\mathfrak{M}) = \inf\{v(f) \mid f \in \mathfrak{M}\}$. Observe that by definition this integer can be computed as the valuation of the equation of a generic hyperplane section of $(X, 0)$.

Example 2.1. The main example of (semi)valuation that we will consider in this paper is the following. Let $\pi\colon X_\pi \to X$ be a good resolution of $(X, 0)$ and let E be an irreducible component of the exceptional divisor $\pi^{-1}(0)$. Then the map

$$\begin{aligned} v_E\colon \mathcal{O} &\longrightarrow \mathbb{R}_+ \cup \{+\infty\} \\ f &\longmapsto \frac{\mathrm{ord}_E(f)}{\mathrm{ord}_E(\mathfrak{M})} \end{aligned}$$

is a valuation on $\mathcal{O}$. We call it the *divisorial valuation* associated with E. Note that this valuation does not depend on the choice of π, in the following sense: if $\pi' = \pi \circ \rho$ is another good resolution of $(X, 0)$ that dominates π, then $v_E = v_{E'}$, where E' denotes the strict transform of E in $X_{\pi'}$. If v is the divisorial valuation associated with a component of the exceptional divisor $\pi^{-1}(0)$ of a good resolution $\pi\colon X_\pi \to X$ of $(X, 0)$, we will generally denote by E_v this prime divisor, and by m_v the positive integer $\mathrm{ord}_{E_v}(\mathfrak{M})$, which we call the *multiplicity* of E_v. This terminology is justified by the fact that m_v is also the multiplicity of E_v in $\pi^{-1}(0)$, when the latter is considered with its natural scheme structure. In practice, m_v is usually computed as the order of vanishing along E_v of the total transform by π of a generic hyperplane section of $(X, 0)$. For a plethora of examples of semivaluations we refer the reader to [FJ04, Chapter 1].

We are interested in a projectivisation of the space semivaluations on $\mathcal{O}$. This is be achieved by only considering those semivaluations that are normalized by requiring that the valuation of the maximal ideal is 1.

Definition 2.2. The *non-archimedean link* $\mathrm{NL}(X, 0)$ of $(X, 0)$ is the topological space whose underlying set is

$$\mathrm{NL}(X, 0) = \left\{ v\colon \mathcal{O} \to \mathbb{R}_+ \cup \{+\infty\} \text{ semivaluation} \;\middle|\; v(\mathfrak{M}) = 1 \text{ and } v|_{\mathbb{C}} = 0 \right\}$$

and whose topology is induced from the product topology of $\left(\mathbb{R}_+ \cup \{+\infty\}\right)^{\mathfrak{M}}$ (that is, it is the topology of the point-wise convergence).

Remark 2.3. The non-archimedean link can be endowed with an additional analytic structure, by considering the space of semi-valuations on $\mathcal{O}$ as a non-archimedean analytic space, in the sense of Berkovich [Ber90], over the field $\mathbb{C}$ endowed with the trivial absolute value. This point of view was developed in [Fan18], where it was used to obtain a non-archimedean characterization of the essential valuations of a surface singularities in arbitrary characteristic, and later in [FFR18] to give a characterization of sandwiched surface singularities. Moreover, this can be done independently from classical resolution of singularities, as it is possible to study the analytic structure of non-archimedean links to deduce the existence of resolutions of singularities of complex surfaces (see [FT18, Theorem 8.6]). However, as the analytic structure of $\mathrm{NL}(X, 0)$ plays no role in this paper, we will not discuss it further.

If $(X,0) = (\mathbb{C}^2,0)$ is a smooth surface germ, it's non-archimedean link $\mathrm{NL}(\mathbb{C}^2,0)$ is the so-called *valuative tree*, the main object of study of the monograph [FJ04]. It is an infinite real tree which has several applications in commutative algebra and to the study of the dynamics of complex polynomials in two variables.

The non-archimedean link $\mathrm{NL}(X,0)$ is a useful object if one wants to study the resolutions of $(X,0)$, as we will now explain. We call *good resolution* of $(X,0)$ a proper morphism $\pi\colon X_\pi \to X$ such that X_π is regular, π is an isomorphism outside of its exceptional locus $\pi^{-1}(0)$, and the latter is a strict normal crossing divisor on X_π. The fact that the exceptional divisor has normal crossing allows us to associate with it its *dual graph* Γ_π, which is the graph whose vertices are in bijection with the irreducible components of $\pi^{-1}(0)$, and where two vertices of Γ_π are joined by an edge for each point of the intersection of the corresponding components. We will generally write v for a vertex of Γ_π and E_v for the associated component of $\pi^{-1}(0)$ (note that this is consistent the notation introduced in Example 2.1). We also write $g(v)$ for the genus of the component E_v; it will sometimes be useful to think of $g(v)$ as a function with finite support on the topological space underlying Γ_π.

While in general not all good resolutions of $(X,0)$ factor through the blowup of its maximal ideal (see Example 5.9), throughout the paper this will be the case most of the times. We call the subset of $\mathrm{NL}(X,0)$ consisting of the divisorial valuations associated with the exceptional components of the blowup of the maximal ideal of $\mathrm{NL}(X,0)$ the set of $\mathcal{L}$-*nodes* of $(X,0)$. In other words, this is the set of the Rees valuations of the maximal ideal of $(X,0)$.

Example 2.4. Consider the standard singularity $X = E_8$, which is the hypersurface in $\mathbb{C}^3$ defined by the equation $x^2 + y^3 + z^5 = 0$. A good resolution $\pi\colon X_\pi \to X$ of $(X,0)$ can be obtained by the method described in [Lau71, Chapter II]; this method is particularly useful for us because it is based on projections to $\mathbb{C}^2$ and thus fits well with our point of view. It considers the projection $\ell = (y,z)\colon (X,0) \to (\mathbb{C}^2,0)$ and, given a suitable embedded resolution $\sigma\colon Y \to \mathbb{C}^2$ of the associated discriminant curve $\Delta\colon y^3 + z^5 = 0$, gives a simple algorithm to compute a resolution of $(X,0)$ as a cover of Y. In this example, the dual graph Γ_σ of the minimal embedded resolution σ of Δ in $\mathbb{C}^2$ is depicted on the left of Figure 2. Its vertices are labeled as ν_0, ν_1, ν_2, and ν_3 in their order of appearance as exceptional divisors of blowups in the resolution process. The negative number attached to each vertex denotes the self-intersection of the corresponding exceptional curve, while the arrow denotes the strict transform of Δ. In this case Laufer's algorithm requires us to refine the resolution σ to another resolution σ' obtained by blowing up once each double point of the exceptional divisor $\sigma^{-1}(0)$. This yields the dual graph $\Gamma_{\sigma'}$ depicted on the right of Figure 2. Again, each vertex is decorated by the self-intersection of the corresponding exceptional curve and the arrow denotes the strict transform of Δ.

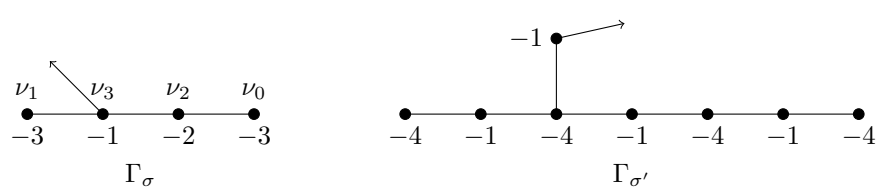

FIGURE 2. Dual resolution graphs for the plane curve $\{y^3 + z^5 = 0\}$.

Then it follows from Laufer's algorithm that the exceptional divisor $\pi^{-1}(0)$ of π is a tree of eight rational curves whose dual graph Γ_π is depicted in Figure 3. We label the vertices of Γ_π as $v_0, \dots, v_7$, since this will make it simple to refer to this example in the rest of the paper. Each vertex of Γ_π is mapped by ℓ to the corresponding vertex of Γ_σ. As before, the negative numbers attached to the v_i are the self-intersections of the corresponding exceptional components; as no self-intersection is equal to -1 this resolution is minimal. Observe that π factors through the blowup of the maximal ideal of $(X, 0)$ and the only $\mathcal{L}$-node of $(X, 0)$ is v_0. This example will be detailed throughout the whole paper, see Examples 2.6, 2.8, 3.7, 4.1, and 4.4.

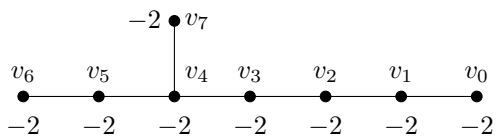

FIGURE 3. The dual resolution graph Γ_π for the singularity E_8.

We will now explain how to describe the structure of a non-archimedean link $\mathrm{NL}(X, 0)$ in terms of the dual graphs of the good resolutions of $(X, 0)$. The results below can be deduced fairly easily from the analogous result for the valuative tree $\mathrm{NL}(\mathbb{C}^2, 0)$, originally proven in [FJ04, Chapter 6] and elegantly summarized in [Jon15, Section 7], as was discussed for example in [GR17, Section 2.5].

Given any good resolution $\pi\colon X_\pi \to X$ with dual graph Γ_π, there exist a natural embedding

$$i_\pi\colon \Gamma_\pi \hookrightarrow \mathrm{NL}(X, 0)$$

and a canonical continuous retraction

$$r_\pi\colon\ \mathrm{NL}(X, 0) \longrightarrow \Gamma_\pi$$

such that $r_\pi \circ i_\pi = \mathrm{Id}_{\Gamma_\pi}$. The embedding i_π maps each vertex v of Γ_π to the divisorial valuation associated with the component E_v, and each edge $e = [v, v']$ that corresponds to a point p of the intersection $E_v \cap E'_v$ to the set of monomial valuations on X_π at p. More precisely, given $0 \leq \omega \leq 1$ and $f \in \mathcal{O}$, we choose an isomorphism $\widehat{\mathcal{O}_{X_\pi,P}} \cong \mathbb{C}[[x, y]]$ such that E_v and $E_{v'}$ are defined locally ay p by $x = 0$ and $y = 0$, and define a valuation $v_{\pi,p,\omega}$ on $\mathcal{O}$ by setting

$$v_{\pi,p,\omega}(f) = \inf\big\{(i, j) \cdot ((1 - \omega)/m_{v'}, \omega/m_v) \,\big|\, a_{i,j} \neq 0\big\},$$

where f is an element $\mathcal{O}$ and we denote by $\pi^* f = \sum a_{i,j} x^i y^j$ the image in $\widehat{\mathcal{O}_{X_\pi,P}}$ of the pullback of f through π. Observe that $v_{\pi,p,0} = v$ and $v_{\pi,p,1} = v'$, while the points $v_{\pi,p,\omega}$ for $0 < \omega < 1$ form a segment joining the two divisorial valuations v and v' in $\mathrm{NL}(X, 0)$.

The retraction r_π is defined as follows. Given a point w of $\mathrm{NL}(X, 0)$, it follows from the valuative criterion of properness that w has a non-empty center $\mathrm{c}_\pi(w)$ on X_π. The center $\mathrm{c}_\pi(w)$ is the biggest irreducible subvariety Y of X_π such that for every regular function f on X we have $w(f) \geq 0$ and $w(f) > 0$ if and only if f vanishes on Y. If $\mathrm{c}_\pi(w)$ is a whole component E_v of $\pi^{-1}(0)$ or a point of E_v that is smooth in $\pi^{-1}(0)$, set $r_\pi(w) = v$. If $\mathrm{c}_\pi(w) = p$ lies on the intersection of two

components E_v and $E_{v'}$ of $\pi^{-1}(0)$, set $r_\pi(w) = v_{\pi,p,\omega}$, with $\omega = w(y)/\big(w(x)+w(y)\big)$, where as above x and y are local equations for E_v and $E_{v'}$ at p.

The continuous retractions r_π induce then a natural homeomorphism

$$\mathrm{NL}(X,0) \xrightarrow{\cong} \varprojlim_{\pi} \Gamma_\pi$$

from $\mathrm{NL}(X,0)$ to the inverse limit of the dual graphs Γ_π, where π ranges over the filtered family of good resolutions of $(X,0)$ (see [GR17, Theorem 2.25] and [Jon15, Theorem 7.9]). This means that the non-archimedean link $\mathrm{NL}(X,0)$ can be thought of as a universal dual graph, making it a very convenient object for studying the combinatorics of the resolutions of $(X,0)$.

Dual resolution graphs can be endowed with a natural metric as follows. Let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ which factors through the blowup of the maximal ideal of $(X,0)$. We define a metric on Γ_π by declaring the length of an edge $e = [v,v']$ to be

$$\mathrm{length}(e) = \frac{1}{m_v m_{v'}}.$$

Remark 2.5. This metric has a natural geometric interpretation following [MMA11, Theorem 7.3.(iv)]. Indeed, if the edge $e = [v,v']$ corresponds to a point P of $E_v \cap E_{v'}$, which in turn corresponds to a union of annuli in the Milnor fiber of a generic linear form on $(X,0)$, then the opposite $-\mathrm{length}(e) = -\frac{1}{m_v m_{v'}}$ of the length of e coincides with the screw number of the representative of the monodromy automorphism (which is a product of Dehn twists) on these annuli. This observation, which will be explained in detail in Section 4.4, will play an important role in the proof of our main result.

Example 2.6. Consider again the singularity $(X,0) = (E_8,0)$ of Example 2.4 and its minimal good resolution $\pi\colon X_\pi \to X$. In Figure 4, the vertices of Γ_π are decorated with the multiplicities of the corresponding exceptional components (in parenthesis), which can be computed by choosing a generic linear form $h\colon (X,0) \to (\mathbb{C},0)$ on $(X,0)$, for example $h = z$. The edges of Γ_π are decorated with the corresponding lenghts. For example, we can observe that the length of the path from v_0 to v_7 is $1/6+1/12+1/20+1/30+1/18 = 7/18$. The $\mathcal{L}$-node v_0 is decorated with one arrow, representing the fact that the strict transform h^* of h on X_π is an irreducible curve passing through the divisor E_{v_0}.

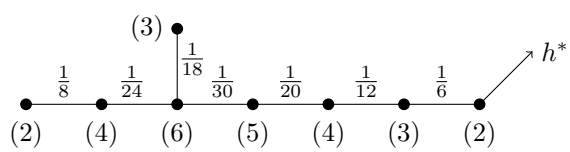

FIGURE 4.

Let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ that factors through the blowup of the maximal ideal of $(X,0)$, let p be a point of the exceptional divisor $\pi^{-1}(0)$ of π that lies on the intersection of two irreducible components E_v and $E_{v'}$ of $\pi^{-1}(0)$, and let $\pi'\colon X_{\pi'} \to X$ be the good resolution of $(X,0)$ obtained from π by blowing up X_π at the point p. Then the exceptional component E_w of

$(\pi')^{-1}(0)$ arising from the blowup of p has multiplicity $m_w = m_v + m_{v'}$. Since $1/m_v(m_v + m_{v'}) + 1/m_{v'}(m_v + m_{v'}) = 1/m_v m_{v'}$, this means that the canonical inclusion $\Gamma_\pi \hookrightarrow \Gamma_{\pi'}$, which is bijective at the level of the underlying topological spaces, is also an isometry. Therefore, by passing to the limit, the metrics on the dual graphs Γ_π define a metric on the non-archimedean link $\mathrm{NL}(X,0)$. We call this metric the *skeletal metric* on $\mathrm{NL}(X,0)$.

2.2. **Generic projections.** We will study the surface germ $(X,0)$ using suitable projections to $(\mathbb{C}^2,0)$, making use of a classical notion of generic projection due to Teissier.

Fix an embedding of $(X,0)$ in some smooth germ $(\mathbb{C}^n,0)$, and consider the morphism $\ell_{\mathcal{D}}\colon (X,0) \to (\mathbb{C}^2,0)$ obtained by restricting to X the projection along a $(n-2)$-dimensional linear subspace $\mathcal{D}$ of $\mathbb{C}^n$. Whenever $\ell_{\mathcal{D}}$ is finite, the associated *polar curve* $\Pi_{\mathcal{D}}$ is the closure in $(X,0)$ of the ramification locus of $\ell_{\mathcal{D}}$ in $X \smallsetminus \{0\}$. The Grassmanian variety of $(n-2)$-planes in $\mathbb{C}^n$ contains a dense open set Ω such that $\ell_{\mathcal{D}}$ is finite and the family $\{\Pi_{\mathcal{D}}\}_{\mathcal{D}\in\Omega}$ is well behaved (for example, it is equisingular in a strong sense). We refer the reader to [Tei82, Lemme-clé V 1.2.2] and to [BNP14, Section 3] for a precise definition. We say that a morphisms $\ell\colon (X,0) \to (\mathbb{C}^2,0)$ is a *generic projection* of $(X,0)$ if $\ell = \ell_{\mathcal{D}}$ for some $\mathcal{D}$ in Ω.

The smallest modification of $(X,0)$ that resolves the basepoints of the family of polar curves $\{\Pi_{\mathcal{D}}\}_{\mathcal{D}\in\Omega}$ is the *Nash transform* of $(X,0)$ (see [Spi90, Part III, Theorem 1.2] and [GS82, Section 2]), which is the blowup of $(X,0)$ along its Jacobian ideal. We call the subset of $\mathrm{NL}(X,0)$ consisting of the divisorial valuations associated with the exceptional components of the Nash transform of $(X,0)$ the set of $\mathcal{P}$*-nodes* of $(X,0)$. In another terminology, this is the set of the Rees valuations of the Jacobian ideal of X.

If $\pi\colon X_\pi \to X$ is a good resolution of $(X,0)$ which factors through its Nash transform, we say that a projection $\ell\colon (X,0) \to (\mathbb{C}^2,0)$ is *generic with respect to* π if it is a generic projection in the sense above and the strict transform of its polar curve via π only intersects the components of $\pi^{-1}(0)$ corresponding to the $\mathcal{P}$-nodes.

A generic projection $\ell\colon (X,0) \to (\mathbb{C}^2,0)$ induces a natural morphism

$$\widetilde{\ell}\colon \mathrm{NL}(X,0) \to \mathrm{NL}(\mathbb{C}^2,0).$$

Indeed, ℓ induces a map $\ell^{\#}\colon \widehat{\mathcal{O}_{X,0}} \to \widehat{\mathcal{O}_{\mathbb{C}^2,0}}$, hence a point of $\mathrm{NL}(X,0)$ (which is a semivaluation on $\widehat{\mathcal{O}_{X,0}}$) gives rise to a point of $\mathrm{NL}(\mathbb{C}^2,0)$ simply by pre-composing the semivaluation with $\ell^{\#}$. A concrete way to compute $\widetilde{\ell}(v)$ for a divisorial valuation $v \in \mathrm{NL}(X,0)$ goes as follows. Take a good resolution π of $(X,0)$ such that v is associated with an irreducible component E_v of $\pi^{-1}(0)$ and consider a generic pair of curvettes γ and γ' of E_v. Let σ be the minimal sequence of blowups of $(\mathbb{C}^2,0)$ such that the strict transforms of $\ell(\gamma)$ and $\ell(\gamma')$ by σ meet the exceptional divisor $\sigma^{-1}(0)$ at distinct points. Then, being generic, they meet $\sigma^{-1}(0)$ at smooth points along the exceptional curve C_ν created at the last blowup. It can be seen via a standard Hirzebruch–Jung argument that neither the morphism σ nor the divisorial valuation $\nu \in \mathrm{NL}(\mathbb{C}^2,0)$ depend on the choice of the generic pair γ,γ', and indeed we have $\widetilde{\ell}(v) = \nu$.

Since ℓ is finite, it is not hard to show that $\widetilde{\ell}$ is a branched covering, that is a finite map that is a finite topological covering out of a nowhere dense *ramification locus*. On the level of dual graphs, if $\pi\colon X_\pi \to X$ is a good resolution of $(X,0)$ that

factors through its Nash transform and $\sigma\colon Y_\sigma \to \mathbb{C}^2$ is any sequence of blowups of $\mathbb{C}^2$ above the origin such that $\widehat{\ell}\big(V(\Gamma_\pi)\big) \subset V(\Gamma_\sigma)$, then $\widehat{\ell}(\Gamma_\pi)$ is the subgraph of Γ_σ spanned by the vertices in $\widehat{\ell}\big(V(\Gamma_\pi)\big)$, and the restriction of $\widetilde{\ell}$ to Γ_π is a (ramified) covering of graphs onto its image.

2.3. **Laplacians on metric graphs.** We will briefly recall some basic notions of divisor theory on metric graphs.

For us a *graph* is a finite and connected metric graph

$$\Gamma = \big(V(\Gamma), E(\Gamma), l\colon E(\Gamma) \to \mathbb{Q}_{>0}\big),$$

where $V(G)$ is the set of vertices of Γ, $E(G)$ is the set of its edges, and l attaches a length to each edge of Γ. We allow Γ to have loops and multiple edges. We will freely identify Γ with its geometric realization, which is the metric space whose metric is induced by the lengths of its edges.

A *divisor* $D = \sum a_v[v]$ of Γ is a finite sum of points of Γ with integer coefficients $a_v \in \mathbb{Z}$. We also denote by $D(v)$ the coefficient a_v of a divisor D at a point v of Γ, and by $\mathrm{Div}(\Gamma) = \bigoplus_{v\in\Gamma} \mathbb{Z}[v]$ the abelian group of divisors of Γ. The *degree* $\deg(D)$ of D is the integer $\deg(D) = \sum_{v\in\Gamma} D(v)$.

A function $F\colon \Gamma \to \mathbb{R}$ is said to be *piecewise linear* if F is a continuous piecewise affine map with integral slopes (with respect to the metric induced by l on Γ) and F has only finitely many points of non-linearity on each edge of Γ.

Definition 2.7. If $F\colon \Gamma \to \mathbb{R}$ is a piecewise linear map, its *Laplacian* $\Delta_\Gamma(F)$ is the divisor of Γ whose coefficient $\Delta_\Gamma(F)(v)$ at a point v of Γ is the sum of the outgoing slopes of F at v.

Example 2.8. Consider the metric graph Γ_π associated with the dual graph of the minimal resolution of the singularity E_8, as described in Examples 2.4 and 2.6. Let F be the function on Γ_π which is linear on its edges and such that $\big(F(v_i)\big)_{i=0}^7 = \big(1, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{7}{4}, 2, 2\big)$; we will see in Example 3.7 that this function F is the inner rate function of E_8. Then it is easy to see that F grows linearly with slope 2 on the path from v_0 to v_6, while it grows linearly with slope 6 on the edge $[v_4, v_7]$. Therefore, its Laplacian is the divisor $\Delta_{\Gamma_\pi}(F) = 2[v_0] + 6[v_4] - 2[v_6] - 6[v_7]$.

Observe that, as we do not require F to be linear inside the edges of Γ, its Laplacian $\Delta_\Gamma(F)$ is not necessarily supported on $V(\Gamma)$. Moreover, since every segment $[v, v']$ in Γ such that $F|_{[v,v']}$ is linear contributes with the same slope but opposite signs to the Laplacian $\Delta_\Gamma(F)(v)$ and $\Delta_\Gamma(F)(v')$ at v and v' respectively, the Laplacian $\Delta_\Gamma(F)$ of F is a divisor of degree 0.

A function $f\colon \Gamma' \to \Gamma$ between two metric graphs induces a natural map $f_*\colon \mathrm{Div}(\Gamma') \to \mathrm{Div}(\Gamma)$ defined by sending a divisor $D = \sum_{v'\in\Gamma'} a_{v'}[v']$ to the divisor $f_*D = \sum_{v\in\Gamma} b_v[v]$, where $b_v = \sum_{v'\in f^{-1}(v)} a_v$.

Similarly, we call *divisor* on $\mathrm{NL}(X,0)$ a finite sum of points of $\mathrm{NL}(X,0)$ with integer coefficients, and we denote by $\mathrm{Div}\big(\mathrm{NL}(X,0)\big) = \bigoplus_{v\in\mathrm{NL}(X,0)} \mathbb{Z}[v]$ the abelian group of divisors on $\mathrm{NL}(X,0)$. Then, if $\pi\colon X_\pi \to X$ is a good resolution of $(X,0)$, the retraction map $r_\pi\colon \mathrm{NL}(X,0) \to \Gamma_\pi$ induces a map of divisors $(r_\pi)_*\colon \mathrm{Div}\big(\mathrm{NL}(X,0)\big) \to \mathrm{Div}(\Gamma_\pi)$ by the same formula as above.

3. The inner rate function

Let $(X,0) \subset (\mathbb{C}^n,0)$ be a surface germ with an isolated singularity. In this section we define the inner rate function on a non-archimedean link $\mathrm{NL}(X,0)$ and prove its basic properties.

We will use the big-Theta asymptotic notations of Bachmann–Landau in the following form: given two function germs $f,g\colon \big([0,\infty),0\big) \to \big([0,\infty),0\big)$ we say f is *big-Theta* of g and we write $f(t) = \Theta\big(g(t)\big)$ if there exist real numbers $\eta > 0$ and $K > 0$ such that for all t, if $f(t) \leq \eta$ then $K^{-1}g(t) \leq f(t) \leq Kg(t)$.

Let $(\gamma,0)$ and $(\gamma',0)$ be two distinct germs of complex curves on the surface germ $(X,0) \subset (\mathbb{C}^n,0)$, and denote by S_ϵ the sphere in $\mathbb{C}^n$ having center 0 and radius $\epsilon > 0$. Denote by d_i the inner distance on $(X,0)$. The *inner contact* between γ and γ' is the rational number $q_i = q_i^X(\gamma,\gamma')$ defined by

$$d_i\big(\gamma \cap S_\epsilon, \gamma' \cap S_\epsilon\big) = \Theta(\epsilon^{q_i}).$$

Remark 3.1. While the existence of the inner contact $q_i^X(\gamma,\gamma')$ and its rationality can be deduced from the work of [KO97], in the case that interests us they can also be seen as a consequence of the next lemma.

The following lemma is fundamental, as it will allow us to define the inner rate of a divisorial valuation of $\mathrm{NL}(X,0)$.

Recall that if $\pi\colon X_\pi \to X$ is a resolution of X and if E is an irreducible component of $\pi^{-1}(0)$, a *curvette* of E is a smooth curve germ (γ,p) in X_π, where p is a point of E which is a smooth point of $\pi^{-1}(0)$ and such that γ and E intersect transversely.

Lemma 3.2. *Let $v \in \mathrm{NL}(X,0)$ be a divisorial valuation on $(X,0)$ and let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ which factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$ and such that v is the divisorial valuation associated with an irreducible component E_v of $\pi^{-1}(0)$. Consider two curvettes γ^* and $\widetilde{\gamma}^*$ of E_v meeting it at distinct points, and write $\gamma = \pi(\gamma^*)$ and $\widetilde{\gamma} = \pi(\widetilde{\gamma}^*)$. Then the inner contact $q_i^X(\gamma,\widetilde{\gamma})$ between γ and $\widetilde{\gamma}$ only depends on v and not on the choice of π, γ^*, and $\widetilde{\gamma}^*$. Moreover, $m_v q_i^X(\gamma,\widetilde{\gamma})$ is an integer, $q_i^X(\gamma,\widetilde{\gamma}) \geq 1$, and if $\ell\colon (X,0) \to (\mathbb{C}^2,0)$ is a generic projection with respect to π we have $q_i^X(\gamma,\widetilde{\gamma}) = q_i^{\mathbb{C}^2}\big(\ell(\gamma),\ell(\widetilde{\gamma})\big)$.*

Definition 3.3. We denote by q_v the rational number $q_i^X(\gamma,\widetilde{\gamma})$, and call it the *inner rate* of v.

Remark 3.4. It is worth noticing that the knowledge of all the inner rates q_v allows one to compute the inner contact between any two complex curve germs $(\gamma,0)$ and $(\gamma',0)$ on $(X,0)$. Indeed, assume that the good resolution $\pi\colon X_\pi \to X$ of $(X,0)$ also resolves the complex curve $\gamma \cup \gamma'$ and let v and v' be the vertices of Γ_π such that $\gamma^* \cap E_v \neq \emptyset$ and $\gamma'^* \cap E_{v'} \neq \emptyset$ respectively. Then $q_i^X(\gamma,\gamma') = q_{v,v'}$, where $q_{v,v'}$ is the maximum, taken over all injective paths γ in Γ_π between v and v', of the minimum of the inner rates of the vertices of Γ_π contained in γ. We refer to [NPP18, Proposition 15.3] for details.

Proof of Lemma 3.2. In the course of the proof we will use the outer contact between γ and γ', which is the rational number $q_o = q_o(\gamma,\gamma')$ defined by $d_o\big(\gamma \cap S_\epsilon, \gamma' \cap S_\epsilon\big) = \Theta(\epsilon^{q_o})$, where d_o is the outer distance $d_o(x,y) = ||x-y||_{\mathbb{C}^n}$. It is simple to see that q_o can also be defined by $d_o\big(\gamma \cap \{z=\epsilon\}, \gamma' \cap \{z=\epsilon\}\big) = \Theta(\epsilon^{q_o})$, whenever $z\colon (X,0) \to (\mathbb{C},0)$ is a generic linear form on $(X,0)$.

First observe that in the smooth case the result comes from classical theory of plane curves singularities. Indeed, in $\mathbb{C}^2$ the inner and outer metrics coincide, and if C_i is an exceptional component of a composition of blowups of points $\sigma\colon Y \to \mathbb{C}^2$ starting with the blowup of the origin, then for every pair of distinct curves $(\delta, 0)$ and $(\delta', 0)$ whose strict transforms by σ meet C_i at distinct smooth points of $\sigma^{-1}(0)$, the contact $q_i^{\mathbb{C}^2}(\delta, \delta') = q_o(\delta, \delta')$ coincides with the contact exponent between their Puiseux series (see for example [GBT99, page 401]) and does not depend on the choice of the pair δ, δ'.

Let us now focus on the general case. Denote by p the point of E_v where $\gamma*$ passes through and consider coordinates $(z_1, \dots, z_n)$ of $\mathbb{C}^n$ such that $\ell|_X\colon (z_1, \dots, z_n) \to (z_1, z_2)$ is a generic projection with respect to π and such that the strict transform Π^* of the polar curve Π of ℓ by π does not pass through p, nor do the strict transforms of the curves $X \cap \{z_i = 0\}$ for all $i = 1, \dots, n$. Choose local coordinates (u_1, u_2) centered at p such that E_v has local equation $u_1 = 0$, $\gamma*$ has local equation $u_2 = 0$, and such that $(z_1 \circ \pi)(u_1, u_2) = u_1^{m_v}$. One can then express the other coordinates in terms of (u_1, u_2) as follows:

$$(z_2 \circ \pi)(u_1, u_2) = u_1^{m_v} f_{2,0}(u_1) + u_1^{m_v q_2} \sum_{j \geq 1} u_2^j f_{2,j}(u_1), \tag{1}$$

and for $i = 3, \dots, n$,

$$(z_i \circ \pi)(u_1, u_2) = u_1^{m_v} f_{i,0}(u_1) + u_1^{m_v q_i} \sum_{j \geq 1} u_2^j f_{i,j}(u_1),$$

for suitable choices of $f_{i,j}(u_1) \in \mathbb{C}\{u_1\}$ and $1 \leq q_i \in \mathbb{Q}$. Replacing z_2 by a generic combination of the functions $z_1, \dots, z_n$, we can assume that $q_i \geq q_2 \geq 1$ for all $i \geq 3$. We will prove the following claim:

Claim 1. For any curvette $\widetilde{\gamma}^*$ of E_v meeting E_v at a point distinct from p, we have $q_i^X(\gamma, \widetilde{\gamma}) = q_2$.

Denote by γ_t^* the curvette of E_v defined by the equation $u_2 = t$ (so that in particular we have $\gamma_0 = \gamma$), set $\gamma_t = \pi(\gamma_t^*)$, and let D be a disc neighborhood of p in E_v which is contained in a neighborhood on which the local coordinates (u_1, u_2) are defined. Then for every $t \in D \smallsetminus \{0\}$ we have

$$d_o\big(\gamma \cap \{z_1 = \epsilon\}, \gamma_t \cap \{z_1 = \epsilon\}\big) = \Theta(\epsilon^{q_2}).$$

Now, for every $t \in D \smallsetminus \{0\}$, replacing γ_t by any other curvette $\widetilde{\gamma}^*$ passing through $(u_1, u_2) = (0, t)$ still gives

$$d_o\big(\gamma \cap \{z_1 = \epsilon\}, \widetilde{\gamma} \cap \{z_1 = \epsilon\}\big) = \Theta(\epsilon^{q_2}),$$

since $\widetilde{\gamma}^*$ is defined by a parametrization of the form $u_2 = t + h.o.$, where $h.o.$ denotes a sum of higher order terms. In particular, we have $q_o(\gamma, \widetilde{\gamma}) = q_2$.

On the other hand, let $\ell\colon (X, 0) \to (\mathbb{C}^2, 0)$ be a generic projection for $(X, 0)$. By [BNP14, Proposition 3.3], the local bilipschitz constant of the cover ℓ is bounded outside $\pi(W)$, where W is any analytic neighborhood of Π^* in X_π. Since Π^* does not pass through p, we can take D and W small enough that $D \cap W = \emptyset$. Therefore, the local bilipschitz constant of ℓ being bounded on $\bigcup_{t \in D} \gamma_t$, we have $q_i^X(\gamma, \widetilde{\gamma}) = q_i\big(\ell(\gamma), \ell(\widetilde{\gamma})\big)$ as long as $\widetilde{\gamma}^*$ passes through a point of $E_v \cap D$.

Since the coincidence exponent between the curves $\ell(\gamma)$ and $\ell(\widetilde{\gamma})$ in $(\mathbb{C}^2, 0)$ equals q_2, we deduce that $q_i^X(\gamma, \widetilde{\gamma}) = q_o\big(\ell(\gamma), \ell(\widetilde{\gamma})\big) = q_2$. This proves that $q_i^X(\gamma, \widetilde{\gamma}) = q_2$

does not depend on the choice of the curvette $\widetilde{\gamma}^*$ providing $\widetilde{\gamma}^* \cap E_v$ is in the neighborhood D of γ^*.

We now have to prove that q_2 does not depend on the point p, and that $q_i^X(\gamma, \widetilde{\gamma}) = q_2$ for any pair of curvettes γ^* and $\widetilde{\gamma}^*$ meeting E_v at distinct points.

Claim 2. The contact order q_2 does not depend on the point p.

Let p' be another smooth point of $\pi^{-1}(0)$ on E_v, let δ^* be a curvette of E_v through p' and let $\widetilde{\delta}^*$ be a neighbor curvette. Let δ and $\widetilde{\delta}$ be the images through π of δ^* and $\widetilde{\delta}^*$ respectively. Let $q_2(p')$ be the rate q_2 obtained as above by taking local coordinates centered at p' instead of p. We can assume without loss of generality that the projection ℓ is also generic for these coordinates in the sense above. Then we have $q_o\big(\ell(\delta), \ell(\widetilde{\delta})\big) = q_2(p')$. Consider the minimal sequence of blowups $\sigma\colon Y \to \mathbb{C}^2$ such that one of the irreducible components C of $\sigma^{-1}(0)$ corresponds to the valuation $\widetilde{\ell}(v)$ (where $\widetilde{\ell}$ is defined in Section 2.2). Then the strict transforms of the four curves γ, $\widetilde{\gamma}$, δ and $\widetilde{\delta}$ intersect C at four distinct points of C which are smooth points of $\sigma^{-1}(0)$. Therefore, we have $q_2 = q_o(\gamma, \widetilde{\gamma}) = q_o(\delta, \widetilde{\delta}) = q_2(p')$. This proves Claim 2.

Let us now take any pair of curvettes γ^* and $\widetilde{\gamma}^*$ meeting E_v at distinct points p and $\widetilde{p}$. Since the contact of γ with the π-image of any neighbor curvette equals q_2, we then have $q_i^X(\gamma, \widetilde{\gamma}) \leq q_2$. Moreover, by compactness of E, we can choose a finite sequence of smooth points $p_1, p_2, \ldots, p_s$ of $\pi^{-1}(0)$ on E_v, such that $p_1 = p$, $p_s = \widetilde{p}$ and for all $i = 1, \ldots, s-1$, $q_i^X(\gamma_i, \gamma_{i+1}) = q_2$ where γ_i^* and γ_{i+1}^* pass through p_i and p_{i+1} respectively. We then have

$$q_i^X(\gamma, \widetilde{\gamma}) \geq \min_{i=1,\ldots,s-1} \big(q_i^X(\gamma_i, \gamma_{i+1})\big) = q_2.$$

Putting this all together, we deduce that $q_i^X(\gamma, \widetilde{\gamma}) = q_2$.

Finally, observe that $q_i^X(\gamma, \widetilde{\gamma})$ only depends on v and not on E_v, since all the computations performed above are unchanged if we first blowup a closed point of E_v. □

Remark 3.5. The second claim in the proof of Lemma 3.2 can also be proved by a computation using local coordinates in the resolution as follows. Let p' be another smooth point of $\pi^{-1}(0)$ on E_v. By the genericity of z_1 and z_2, we can assume that the normal form of equation (1) is also valid at p', that is, there exists a coordinate system (u_1', u_2') centered at p' such that:

$$\begin{aligned}
&(z_1 \circ \pi)(u_1', u_2') = u_1'^{m_v}, \\
&(z_2 \circ \pi)(u_1', u_2') = u_1'^{m_v} g_{2,0}(u_1') + u_1'^{m_v q_v(p')} \sum_{j \geq 1} u_2'^j g_{2,j}(u_1').
\end{aligned}$$

Observe that:

$$(dz_1 \circ \pi) \wedge (dz_2 \circ \pi) = u_1'^{m_v(1+q_v(p'))-1} \Big(\sum\nolimits_{j \geq 1} j u_2'^{j-1} g_{2,j}(u_1') \Big) du_1' \wedge du_2'.$$

We can interpret the exponent $m_v\big(1 + q_v(p')\big) - 1$ as the maximal order of the exceptional divisor that factors through $(dz_1 \circ \pi) \wedge (dz_2 \circ \pi)$. Since this order is independent of the point $p' \in E_v$, we conclude that $q_v(p) = q_v(p')$ for every point $p' \in E_v$.

The following result is what allows us to compute in a simple way the inner rate of any divisorial valuation of $(\mathbb{C}^2, 0)$, and more generally that of any divisorial

valuation of a singular germ $(X,0)$ if we know the inner rates of the vertices of the dual graph Γ_π of a suitable good resolution π of $(X,0)$.

Lemma 3.6. *Let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ that factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$. Let p be a point of the exceptional divisor $\pi^{-1}(0)$ and let E_w be the exceptional component created by the blowup of X_π at p. Then:*

(i) If p is a smooth point of $\pi^{-1}(0)$ and E_v is the irreducible component of $\pi^{-1}(0)$ on which p lies, then

$$m_w = m_v \quad \text{and} \quad q_w = q_v + \frac{1}{m_v}.$$

(ii) If p lies on the intersection of two irreducible components E_v and $E_{v'}$ of $\pi^{-1}(0)$, then

$$m_w = m_v + m_{v'} \quad \text{and} \quad q_w = \frac{q_v m_v + q_{v'} m_{v'}}{m_v + m_{v'}}.$$

Proof. Assume first that p is a smooth point of $\pi^{-1}(0)$. We use again the notations of the proof of Lemma 3.2: in local coordinates (u_1,u_2) centered at p, we have $(z_1\circ\pi)(u_1,u_2) = u_1^{m_v}$ and

$$(z_2\circ\pi)(u_1,u_2) = u_1^{m_v} f_{2,0}(u_1) + u_1^{m_v q_2}\sum_{j\geq 1} u_2^j f_{2,j}(u_1).$$

Let us prove that $f_{2,1}(0)\neq 0$. Let Π be the polar curve of ℓ. Its total transform $\pi^{-1}(\Pi)$ by π is the critical locus of $\ell\circ\pi$. The Jacobian matrix of $\ell\circ\pi$ is:

$$\begin{pmatrix} m_v u_1^{m_v-1} & 0 \\ * & u_1^{m_v q_2}\big(f_{2,1}(u_1) + 2u_2 f_{2,2}(u_1) + \cdots\big) \end{pmatrix}$$

thus $\pi^{-1}(\Pi)$ has equation $m_v u_1^{m_v+m_v q_2-1}\big(f_{2,1}(u_1)+2u_2 f_{2,2}(u_1)+\cdots\big)=0$ and the strict transform Π^* has equation $f_{2,1}(u_1)+2u_2 f_{2,2}(u_1)+\cdots = 0$. Since Π^* does not pass through p, this implies that $f_{2,1}(0)\neq 0$.

Let e_p be the blowup of X_π at p. In the coordinates chart $(u_1',u_2')\mapsto(u_1',u_2'u_1')$, we have $(z_1\circ\pi\circ e_p)(u_1',u_2') = {u_1'}^{m_v}$ and

$$(z_2\circ\pi\circ e_p)(u_1',u_2') = {u_1'}^{m_v} f_{2,0}(u_1') + {u_1'}^{m_v q_2+1}\sum_{j\geq 1} {u_2'}^j {u_1'}^{j-1} f_{2,j}(u_1').$$

Therefore $m_w = m_v$. Fixing $u_2'\neq 0$ and comparing with the equation (1) of the proof of Lemma 3.2, we obtain $m_w q_w = m_v q_2 + 1$ by using Claim 1 of the proof of Lemma 3.2. This proves (i).

Assume now that p is an intersecting point between two irreducible components E_v and $E_{v'}$. In local coordinates (u_1,u_2) centered at p, we can assume without loss of generality that $(z_1\circ\pi)(u_1,u_2) = u_1^{m_v}u_2^{m_{v'}}$ and $(z_i\circ\pi)(u_1,u_2)\in\mathbb{C}\{u_1,u_2\}$; in particular, we consider their Taylor expansion

$$(z_i\circ\pi)(u_1,u_2) = \sum_{\alpha\in\mathbb{N}^2} T_{i\alpha} u_1^{\alpha_1} u_2^{\alpha_2}.$$

Now, let $\Lambda = \big\{\alpha\in\mathbb{N}^2 \,\big|\, \exists q\in\mathbb{Q} \text{ such that } q\cdot\alpha = (m_v,m_{v'})\big\}$. Since $(z_i\circ\pi)(u_1,u_2)$ are convergent power series, a sub-series is also convergent, and therefore

$$\widetilde{g}_i(u_1,u_2) = \sum_{\alpha\in\Lambda} T_{i\alpha} u_1^{\alpha_1} u_2^{\alpha_2}$$

is an analytic function. Furthermore, we know that $u_1^{m_v}u_2^{m_{v'}}$ divides $(z_i\circ\pi)(u_1,u_2)$, so $\widetilde{g}_i(u_1,u_2)=u_1^{m_v}u_2^{m_{v'}}g_i(u_1,u_2)$ and:

$$(z_i\circ\pi)(u_1,u_2)=u_1^{m_v}u_2^{m_{v'}}g_i(u_1,u_2)+u_1^{b_v}u_2^{b_{v'}}h_i(u_1,u_2)$$

where $b_v\geq m_v$, $b_{v'}\geq m_{v'}$ and, without loss of generality, $h_2(u_1,u_2)$ is not identically zero over E_v and $E_{v'}$. Now consider a point $(0,a_2)$ (the same computation works for $(a_1,0)$) and let us compare this Taylor expansion with the normal form given in equation (1). Consider the analytic change of coordinates:

$$u=u_1(u_2-a_2)^{\frac{m_{v'}}{m_v}},\quad v=u_2$$

which is centered at $(0,a_2)$ and note that $(z_1\circ\pi)(u_1,u_2)=u^{m_v}$. Furthermore, it follows from direct computation and the definition of Λ that:

$$u_1^{m_v}u_2^{m_{v'}}g_i(u_1,u_2)=u^{m_v}g_i(u)$$

so, these terms contribute only to the terms f_{i0} of the normal form. Furthermore, if $\alpha\in\mathbb{N}\smallsetminus\Lambda$, then:

$$u_1^{\alpha_1}u_2^{\alpha_2}=u^{\alpha_1}\,(v-a_2)^{\frac{\alpha_2 m_{v'}-\alpha_1 m_v}{m_{v'}}}=u^{\alpha_1}U_\alpha(u,v)$$

where $U_\alpha(u,v)$ is a non-constant unit whose derivative in respect to v is non-zero. By comparing the normal forms, it follows, that $b_v=m_vq_v$ (and by the analogous argument, that $b_{v'}=m_{v'}q_{v'}$), which yields to:

$$(z_i\circ\pi)(u_1,u_2)=u_1^{m_v}u_2^{m_{v'}}g_i(u_1,u_2)+u_1^{m_vq_v}u_2^{m_{v'}q_{v'}}h_i(u_1,u_2)$$

Now, since π factors through the Nash transform of $(X,0)$, we can suppose without loss of generality that the polar curve in respect to (z_1,z_2) does no pass through p. This implies that the following 2-form has support in the exceptional divisor:

$$d(z_1\circ\pi)\wedge d(z_2\circ\pi)=d(u_1^{m_v}u_2^{m_{v'}})\wedge d\big(u_1^{q_vm_v}u_2^{q_{v'}m_{v'}}h_2(u_1,u_2)\big)$$

Denoting by $\partial=m_{v'}u_1\partial_{u_1}-m_vu_2\partial_{u_2}$, we get that:

$$d(z_1\circ\pi)\wedge d(z_2\circ\pi)=u_1^{m_v-1}u_2^{m_{v'}-1}\partial\big(u_1^{q_vm_v}u_2^{q_{v'}m_{v'}}h_2(u_1,u_2)\big)du_1\wedge du_2$$

this implies that h_2 must be a unit at p. Then part (ii) of the Lemma follows from a simple direct computation. □

Example 3.7. Let us consider again the singularity $X=E_8$ of example 2.4. The inner rates on the dual embedded resolution graphs Γ_σ and $\Gamma_{\sigma'}$ of the discriminant curve Δ can be easily computed as contact exponent between Puiseux series of neighbor curvettes, or simply using Lemma 3.6. They are then lifted to the graph Γ_π via $\widetilde{\ell}$ thanks to Lemma 3.2, since ℓ is generic with respect to π. This proves that the inner rate function on the vertices of Γ_π coincides with the function introduced in Example 2.8 and depicted in Figure 5.

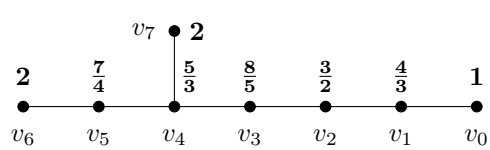

FIGURE 5. The inner rates of the vertices of the graph Γ_π

The starting point of our study of inner rates via potential theory on dual graphs is the following result, which states that the inner rates extend to a continuous, and that this function is piecewise-linear with respect to the metric defined in Section 2.3.

Lemma 3.8. *There exists a unique continuous function*

$$\mathcal{I}_X\colon \mathrm{NL}(X,0) \to \mathbb{R}_{\geq 1} \cup \{\infty\}$$

such that $\mathcal{I}_X(v) = q_v$ for every divisorial point v of $\mathrm{NL}(X,0)$. If π is a good resolution of $(X,0)$ that factors through the blowup of the maximal ideal and the Nash transform of $(X,0)$, then $\mathcal{I}_X$ is linear on the edges of Γ_π with integer slopes.

Proof. Let π be as in the statement. We only need to show that the inner rates extend uniquely to a continuous map on Γ_π which is linear on its edges with integer slopes, as the first part of the statement will then follow immediately from the description of $\mathrm{NL}(X,0)$ as inverse limit of dual graphs. The fact that the slopes are integer can be verified directly, as on an edge $e = [v, v']$ the slope is $(q_{v'} - q_v)/\operatorname{length}(e) = (q_{v'} - q_v)m_v m_{v'}$, which is an integer by Lemma 3.2. To prove the linearity on the edges, since the subset of Γ_π consisting of the divisorial points is dense in Γ_π, it is enough to show that the inner rates are linear on this set. Let $e = [v, v']$ be an edge of Γ_π corresponding to an intersection point p of two components E_v and $E_{v'}$ of $\pi^{-1}(0)$. Since any divisorial point of e is associated with a divisor appearing after a finite composition of point blowups centered over p, it is sufficient to prove that $\mathcal{I}_X$ is linear on the set $\{v, v'', v'\}$, where v'' is the divisorial valuation associated with the exceptional divisor $E_{v''}$ of the blowup of X_π at p. Therefore, all we have to show is that

$$\frac{\mathcal{I}_X(v'') - \mathcal{I}_X(v)}{\operatorname{length}([v, v''])} = \frac{\mathcal{I}_X(v') - \mathcal{I}_X(v)}{\operatorname{length}([v, v'])},$$

which follows from the definition of the lengths and from Lemma 3.6.(ii). □

Let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ that factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$ and let $\ell\colon (X,0) \to (\mathbb{C}^2, 0)$ be projection that is generic with respect to π. Denote by $p_1, \ldots, p_k$ the points where the strict transform of the polar curve of ℓ intersects the exceptional divisor $\pi^{-1}(0)$ of π. For each $j = 1, \ldots, k$, let E_{v_j} be the exceptional curve created by the blowup of p_j. As an immediate consequence of the description of $\widetilde{\ell}$ given in Section 2.2 and of the computation of the inner rates of Lemma 3.2, we deduce that for every $v \in \mathrm{NL}(X,0)$ we have we have $\mathcal{I}_X(v) = \mathcal{I}_{\mathbb{C}^2}\big(\widetilde{\ell}(v)\big)$ if v does not belong to one of the k connected components of $\mathrm{NL}(X,0) \smallsetminus \Gamma_\pi$ containing one of the v_j, and $\mathcal{I}_X(v) < \mathcal{I}_{\mathbb{C}^2}\big(\widetilde{\ell}(v)\big)$ otherwise.

In particular, this implies that $\mathcal{I}_X$ takes the value 1 precisely on the set of $\mathcal{L}$ nodes of $\mathrm{NL}(X,0)$, and is greater than one elsewhere. Indeed, the set of $\mathcal{L}$ nodes of $\mathrm{NL}(X,0)$ coincides with $\widetilde{\ell}^{-1}(\mathrm{ord}_0)$, where ord_0 is the divisorial valuation associated with the blowup of $\mathbb{C}^2$ at 0 (that is, ord_0 is the unique $\mathcal{L}$-node of $\mathrm{NL}(\mathbb{C}^2,0)$), we have $\mathcal{I}_{\mathbb{C}^2}(\mathrm{ord}_0)$ immediately from the definition and the fact that the inner rate function $\mathcal{I}_{\mathbb{C}^2}$ on $\mathrm{NL}(\mathbb{C}^2,0)$ is greater than 1 elsewhere follows for example from Lemma 3.6. Moreover, the fact that $\mathcal{I}_X$ can be computed on $\mathbb{C}^2$ and lifted via the finite cover $\widetilde{\ell}$ imposes strong conditions on the way it grows along paths in $\mathrm{NL}(X,0)$. The following result, which illustrates this phenomenon, is not needed in the rest of

this paper but can be helpful to the reader in order to acquire a better intuition about the behavior of inner rates. It can be compared to the main result of [MM17].

Proposition 3.9. *Let v be a point of* $\mathrm{NL}(X,0)$*. Then there exist an $\mathcal{L}$-node w of* $\mathrm{NL}(X,0)$ *and a path from v to w in* $\mathrm{NL}(X,0)$ *along which $\mathcal{I}_X$ is strictly increasing.*

Proof. It suffices to prove this when v is a divisorial point of $\mathrm{NL}(X,0)$. Let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ that factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$ and such that v is the divisorial point associated with an irreducible component E_v of $\pi^{-1}(0)$. Consider a generic projection $\ell\colon (X,0) \to (\mathbb{C}^2)$ with respect to π. Let $\sigma\colon Y \to \mathbb{C}^2$ be a sequence of blowups of points which resolves the discriminant curve Δ of ℓ and such that $\tilde{\ell}(w)$ corresponds to an exceptional curve of $\sigma^{-1}(0)$ for each vertex w of Γ_π. Let us perform the usual Hirzebruch–Jung construction for σ: taking the pull-back of σ by ℓ and its normalization, we obtain morphisms $\hat{\ell}\colon Z \to Y$ and $\rho\colon Z \to X$ such that $\ell \circ \rho = \sigma \circ \hat{\ell}$. Let $\pi'\colon X_{\pi'} \to X$ be the good resolution of $(X,0)$ obtained by composing ρ with the minimal good resolution of Z. Since the singularities of Z are quasi-ordinary singularities and the branching locus of the projection $\hat{\ell}$ is contained in the normal crossing divisor $\sigma^{-1}(\Delta)$, it follows that the inverse image by $\tilde{\ell}$ of an edge of Γ_σ joining two vertices ν and ν' is a union of segments in $\mathrm{NL}(X,0)$ each joining a vertex of $\tilde{\ell}^{-1}(\nu)$ to a vertex of $\tilde{\ell}^{-1}(\nu')$. This implies that the unique injective path $\widetilde{\gamma}$ from ord_0 to $\tilde{\ell}(v)$ in $NL(\mathbb{C}^2,0)$ lifts via the ramified cover $\widetilde{\ell}\colon \mathrm{NL}(X,0) \to \mathrm{NL}(\mathbb{C}^2,0)$ to a union of injective paths joining the vertices of $\tilde{\ell}^{-1}\big(\tilde{\ell}(v)\big)$ to $\mathcal{L}$-nodes. We can choose one such injective lifting γ of $\tilde{\gamma}$ from a vertex w to v, and as $\widetilde{\ell}(w) = \mathrm{ord}_0$ it follows that w is a $\mathcal{L}$-node of $\mathrm{NL}(X,0)$. Since ℓ is generic with respect to π, we deduce that $\mathcal{I}_X$ and $\mathcal{I}_{\mathbb{C}^2} \circ \tilde{\ell}$ coincide on γ, and since $\mathcal{I}_{\mathbb{C}^2}$ is strictly increasing along $\tilde{\gamma}$ from ord_0 to $\tilde{\ell}(v)$, it follows that $\mathcal{I}_X$ is strictly increasing along γ from w to v. □

Remark 3.10. With any continuous map $F\colon \mathrm{NL}(X,0) \to \mathbb{R}_+ \cup \{+\infty\}$ one can naturally associate an ultrametric (that is, non-archimedean) distance on $\mathrm{NL}(X,0)$ via a standard minimax procedure: the distance between two points v and v' is set to be $e^{-F_{v,v'}}$, where $F_{v,v'}$ is the maximum, taken over all injective paths γ in $\mathrm{NL}(X,0)$ between v and v', of the minimum of F on γ. The observation contained in Remark 3.4 allows us to give a natural geometric interpretation to the ultrametric distance associated with the inner rate function on $\mathrm{NL}(X,0)$.

Remark 3.11. As mentioned in the introduction, the inner rate of a divisorial valuation v of $(X,0)$ can be approached using a tool commonly used in birational geometry, namely logarithmic Fitting ideals. More precisely, it can be shown via a careful local computation as the one performed in the proof of Lemma 3.2, employing suitable local coordinates (such as the Hsiang–Pati coordinates studied in [HP85] and [BdSBGM17]), that if $\widehat{k}_v + 1$ denotes the Mather log discrepancy of v, then we have

$$\mathcal{I}_X(v) = \frac{\widehat{k}_v + 1}{m_v} - 1.$$

In particular, in the smooth case the inner rate function $\mathcal{I}_{\mathbb{C}^2}$ equals the normalized log discrepancy function on $\mathrm{NL}(\mathbb{C}^2,0)$, as studied for example in [GBGPPP19], minus one. This point of view, combined with the fact that the inner rate take its minimal value 1 on the $\mathcal{L}$-nodes of $\mathrm{NL}(X,0)$, can be used to deduce the main

result of [Ish13], which discusses minimal (non-normalized) Mather log discrepancies. Indeed, restricting ourselves to the case of surfaces, since $\widehat{k}_v + 1 = m_v \mathcal{I}_X(v) + 1$, with both $\mathcal{I}_X(v)$ and m_v at least one, we deduce that $\widehat{k}_v + 1 \geq 2$ and that the equality holds if and only if v is a $\mathcal{L}$-node of $\mathrm{NL}(X,0)$ of multiplicity 1, which is the two-dimensional case of [Ish13, Theorem 1.1] (see conditions (i) and (v) there). Observe that what is written above will generalizes to arbitrary dimension, hence the general case of [Ish13, Theorem 1.1] could be proven in the same way. This birational point of view to the study of inner rates will be adopted in a future work by the authors.

4. The Laplacian of the inner rate

In this section we prove our main result, Theorem 4.2, which computes the Laplacian of the restriction of the inner rate function to the dual graph of any good resolution of $(X,0)$ that factors through the blowup of the maximal ideal. In order to state it precisely we need to collect a few more definitions.

Let $\pi\colon X_\pi \to X$ be a good resolution and consider the map $g\colon \Gamma_\pi \to \mathbb{Z}_{\geq 0}$ sending a vertex of Γ_π to the genus of the associated irreducible component of the exceptional divisor and everything else to zero. We then define the *canonical divisor* K_{Γ_π} of the graph Γ_π as the divisor $K_{\Gamma_\pi} = \sum_{v\in\Gamma_\pi} m_v\big(\mathrm{val}_{\Gamma_\pi}(v) + 2g(v) - 2\big)[v]$ of Γ_π, where $\mathrm{val}_{\Gamma_\pi}(v)$ denotes the valency of v in Γ_π (that is the number of edges of Γ_π adjacent to e). Observe that this is indeed an element of $\mathrm{Div}(\Gamma_\pi)$ because for every point v of the metric graph Γ_π that is not a vertex we have $\mathrm{val}_{\Gamma_\pi}(v) = 2$ and $g(v) = 0$.

In particular, the canonical divisor K_{Γ_π} will account for the fact that, since the inner rate grows linearly after blowing up a smooth point on an exceptional component of a good resolution, the Laplacian of $\mathcal{I}_X|_{\Gamma_\pi}$ at a point v does depend on the choice of a resolution π such that $v \in \Gamma_\pi$, and more precisely on the valency $\mathrm{val}_{\Gamma_\pi}(v)$. For this reason, the Laplacians $\Delta_{\Gamma_\pi}(\mathcal{I}_X|_{\Gamma_\pi})$ will not define a divisor on $\mathrm{NL}(X,0)$ by a limit procedure.

Finally, we need to introduce two divisors on $\mathrm{NL}(X,0)$. Define $L = \sum m_v l_v[v]$, where v ranges over the set of divisorial valuations of $(X,0)$ associated with the irreducible components of the exceptional divisor the the blowup of the maximal ideal of $(X,0)$, and l_v is the number of irreducible components of a generic hyperplane section of $(X,0)$ whose strict transforms by the blowup intersect the divisor E_v associated with v. Similarly, set $P = \sum m_v p_v[v]$, for v ranging over the set of divisorial valuations of $(X,0)$ associated with the Nash transform of $(X,0)$, where p_v is the number of components of the polar curve of a generic projection of $(X,0)$ whose strict transforms by the Nash transform intersect E_v. Observe that by definition the divisor L is supported on the set of $\mathcal{L}$-nodes of $(X,0)$, while P is supported on the set of its $\mathcal{P}$-nodes.

Example 4.1. Consider again the singularity E_8, whose resolution data has been described in Examples 2.4, 2.6, 2.8, and 3.7. Then the canonical divisor of Γ_π is $K_{\Gamma_\pi} = -2[v_0] + 6[v_4] - 2[v_6] - 3[v_7]$. Moreover, as can be seen from the discussion of Example 2.6, we have $L = 2[v_0]$. Observe that the divisor P is not supported on Γ_π, since the resolution π, while a resolution of the polar curve of a generic projection, does not factor through the Nash transform. In fact, one can verify that π factors through the Nash transform after blowing up a suitable smooth point of E_{v_7} and then a suitable smooth point of the resulting exceptional divisor (see [BNP14, Example 3.5] for details). In particular, we deduce that $(r_\pi)_* P = 3[v_7]$.

4.1. **Statement of the main theorem.** We have now collected all the ingredients needed to state our main theorem in full generality. Since no risk of confusion will arise, we will make a small abuse of notation and simply write $\Delta_{\Gamma_\pi}(\mathcal{I}_X)$ for the Laplacian $\Delta_{\Gamma_\pi}(\mathcal{I}_X|_{\Gamma_\pi})$ of the restriction $\mathcal{I}_X|_{\Gamma_\pi}$ of the inner rate function on a dual graph Γ_π.

Theorem 4.2 (Laplacian of the inner rate function)**.** *Let $(X,0)$ be the germ of a complex surface with an isolated singularity. Let $\pi\colon X_\pi \to X$ be a good resolution that factors through the blowup of the maximal ideal of $(X,0)$. Then the following equality*

$$\Delta_{\Gamma_\pi}(\mathcal{I}_X) = K_{\Gamma_\pi} + 2L - (r_\pi)_* P$$

holds in $\mathrm{Div}(\Gamma_\pi)$.

Remark 4.3. In particular, if π is a good resolution of $(X,0)$ which factors through the blowup of the maximal ideal of X at 0 and through the Nash transform of $(X,0)$, then we obtain the statement of Theorem B as given in the introduction. Namely, the coefficient of the Laplacian of the inner rate function on the dual graph Γ_π at a vertex v is precisely $m_v\big(2l_v - p_v - \chi(\check{E}_v)\big)$, where l_v and p_v are defined as above, since $K_{\Gamma_\pi}(v) = -m_v\chi(\check{E}_v)$.

Example 4.4. The formula of Theorem 4.2 can be readily verified in the case of the singularity E_8 by combining Example 2.8 and Example 4.1. For example, on the vertex v_4, we indeed get $\Delta_{\Gamma_\pi}(\mathcal{I}_X)(v_4) = 6 = K_{\Gamma_\pi}(v_4) = K_{\Gamma_\pi}(v_4) + 2L(v_4) - P(v_4)$.

We introduce another simple combinatorial definition. Let Γ_π be the dual graph of a good resolution of $(X,0)$ that factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$. We call *string* of Γ_π a segment $\mathcal{S}$ in Γ_π that starts and ends at vertices v and v' of Γ_π respectively and such that $\mathcal{S} \smallsetminus \{v,v'\}$ contains no $\mathcal{L}$-nodes, no $\mathcal{P}$-nodes, and no points whose genus is strictly positive or whose valency in Γ_π is at least 3. Observe that every edge of Γ_π is a string. The following result, while a straightforward consequence of Theorem 4.2, is of independent interest.

Corollary 4.5. *Let $(X,0)$ be the germ of a complex surface with an isolated singularity. Let $\pi\colon X_\pi \to X$ be a good resolution that factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$. Then, if $\mathcal{S}$ is a string of Γ_π, the inner rate function $\mathcal{I}_X$ is linear on $\mathcal{S}$.*

The remaining part of the section is devoted to the proof of Theorem 4.2. An outline of the proof's method can be found in the introduction.

4.2. **Reduction to dominant resolution and the smooth case.** We start by proving the formula of our main theorem in the case of a smooth surface germ. This is the content of Proposition 4.7.

In order to do so, we will first establish a simple combinatorial lemma that will be used several times throughout the proof of Theorem 4.2.

Lemma 4.6. *Let $\pi\colon X_\pi \to X$ and $\pi'\colon X_{\pi'} \to X$ be two good resolutions of $(X,0)$ and assume that π' factors through π which in turn factors through the blowup of the maximal ideal of $(X,0)$. If*

$$\Delta_{\Gamma_{\pi'}}(\mathcal{I}_X) = K_{\Gamma_{\pi'}} + 2L - (r_{\pi'})_* P$$

holds, then

$$\Delta_{\Gamma_\pi}(\mathcal{I}_X) = K_{\Gamma_\pi} + 2L - (r_\pi)_* P$$

holds as well. Moreover, the converse implication is also true if π factors through the the Nash transform of $(X, 0)$.

Proof. To unburden the notation, we will write Γ and Γ' for Γ_π and $\Gamma_{\pi'}$ respectively. Since π' dominates π, the metric graph Γ' contains Γ. Denote by $\tilde{r}\colon \Gamma' \to \Gamma$ the restriction of r_π to Γ', so that $r_\pi = \tilde{r} \circ r_{\pi'}$. By applying $\tilde{r}_*$ to the first equation of the statement we obtain

$$\tilde{r}_* \Delta_{\Gamma'}(\mathcal{I}_X) = \tilde{r}_* K_{\Gamma'} + \tilde{r}_* 2L - \tilde{r}_* \circ (r_{\pi'})_* P.$$

We will first prove that the left-hand side is equal to $\Delta_\Gamma(\mathcal{I}_X)$ and that $r_*(K_{\Gamma'}) = K_\Gamma$. As π' is obtained by composing the resolution π with a finite sequence of point blowups, by induction we can assume without loss of generality that π' is obtained by blowing up a single point q of $\pi^{-1}(0)$. If q lies on the intersection of two components of $\pi^{-1}(0)$ then topologically $\Gamma' = \Gamma$, the map $\tilde{r}$ is the identity, and $K_{\pi'} = K_\pi$, so in this case there is nothing to prove. We can therefore assume that Γ' is obtained from Γ by adding a single vertex v' (corresponding to the exceptional component of the blowup of q) and precisely one edge e connecting v' to a vertex v of Γ. Then we have

$$\Delta_{\Gamma'}(\mathcal{I}_X)(v) = \Delta_\Gamma(\mathcal{I}_X)(v) + s,$$

where s is the slope of $\mathcal{I}_X$ on the edge e going from v to v'. Since the Laplacian of the restriction of $\mathcal{I}_X$ to the edge $e = \tilde{r}^{-1}(v)$ has degree 0, we have

$$s = -\sum_{v'' \in \tilde{r}^{-1}(v) \smallsetminus \{v\}} \Delta_{\Gamma'}(\mathcal{I}_X)(v'').$$

This implies that

$$\Delta_\Gamma(\mathcal{I}_X)(v) = \Delta_{\Gamma'}(\mathcal{I}_X)(v) + \Delta_{\Gamma'}(\mathcal{I}_X)(v') = \sum_{v'' \in \tilde{r}^{-1}(v)} \Delta_{\Gamma'}(\mathcal{I}_X)(v''),$$

which is what we wanted to prove. Moreover, observe that $\tilde{r}_* K_{\Gamma'}(v) = K_{\Gamma'}(v) + K_{\Gamma'}(v') = K_{\Gamma'}(v) - m_{v'} = K_{\Gamma'}(v) - m_v$, which is precisely $K_\Gamma(v)$ since the valency of Γ in v is equal to the valency of v in Γ' minus one. This shows that $\tilde{r}_*$ respects the canonical divisors. Since $\tilde{r}_* \circ (r_\pi)_* = (\tilde{r} \circ r_{\pi'})_* = (r_\pi)_*$, this proves the first part of the lemma. To establish the second part of the lemma, observe that, under the hypothesis that π factors through the blowup of the maximal ideal and though the Nash transform of $(X, 0)$, both L and P are divisors on Γ_π and therefore they are stable under $(r_\pi)_*$ and $(r_{\pi'})_*$, and moreover the slope s on the edge $[v, v']$ is equal to m_v (and therefore to $m_{v'}$) thanks to Lemma 3.6.(i). This means that

$$\begin{aligned} \Delta_{\Gamma'}(\mathcal{I}_X)(v) - \Delta_\Gamma(\mathcal{I}_X)(v) &= m_v = K_{\pi'} - K_\pi \\ &= \big(K_{\pi'} + 2L - (r_{\pi'})_* P\big) - \big(K_\pi + 2L - (r_\pi)_* P\big), \end{aligned}$$

which concludes the proof. □

In the smooth case the non-archimedean link $\mathrm{NL}(\mathbb{C}^2, 0)$ is a tree, it has divisors $L = [\mathrm{ord}_0]$, the only $\mathcal{L}$-node being the divisorial valuation ord_0 associated with the blowup of $\mathbb{C}^2$ at 0, and $P = 0$, as a generic projection to $\mathbb{C}^2$ is unramified and thus there are no $\mathcal{P}$-nodes. The Laplacian of the inner rate function in this very special case is simple to compute thanks to the previous lemma.

Proposition 4.7. *Let $\pi\colon X_\pi \to \mathbb{C}^2$ be a sequence of point blowups of $(C^2, 0)$ that is not the identity. Then the Laplacian of the inner rate function $\mathcal{I}_{\mathbb{C}^2}$ on the dual graph Γ_π associated with π is*

$$\Delta_{\Gamma_\pi}(\mathcal{I}_{\mathbb{C}^2}) = K_{\Gamma_\pi} + 2[\mathrm{ord}_o].$$

Proof. We will argue by induction on the number of blowups in π. If π is obtained by blowing up the origin of $\mathbb{C}^2$ once, then the associated dual graph consists of the divisorial valuation ord_0 and the formula is immediate, as $\Delta_{\{\mathrm{ord}_0\}}(\mathcal{I}_{\mathbb{C}^2}) = 0$ and $K_{\{\mathrm{ord}_0\}}(\mathrm{ord}_0) = -2$. The inductive step is a direct consequence of the second part of Lemma 4.6. □

We state as a separate result the smooth case of Corollary 4.5, which we obtain as an immediate consequence of Proposition 4.7, since we will need to refer to it in the course of our proof of Theorem 4.2.

Corollary 4.8. *Let $\pi\colon X_\pi \to \mathbb{C}^2$ be a sequence of point blowups starting with the blowup of the origin of $\mathbb{C}^2$. Then, if $\mathcal{S}$ is a string of Γ_π, the inner rate function $\mathcal{I}_{\mathbb{C}^2}$ is linear on $\mathcal{S}$.*

4.3. **Dehn twists and screw numbers.** Set $S^1 = \{z \in \mathbb{C}\colon |z| = 1\}$. Let $f\colon A \to A$ be an orientation preserving diffeomorphism of a disjoint union $A = \coprod_{i=1}^r A_i$ of annuli $A_i \cong S^1 \times [0,1]$ which cyclically exchanges the annuli A_i and which is periodic on the union $\partial A = \coprod_{i=1}^r \partial A_i$ of their boundaries. Let N be an integer such that f^N is the identity on ∂A. Then, up to isotopy fixed on ∂A, the map f is characterized by a rational number τ defined as follows. Let us choose an annulus A_{i_0} among the A_i and fix an isomorphism $A_{i_0} \cong S^1 \times [0,1]$. Observe that the restriction of f^N to A_{i_0} is, up to isotopy fixed on ∂A_{i_0}, a product of Dehn twists. Consider the transverse oriented path $\delta = \{1\} \times [0,1]$ inside $S^1 \times [0,1]$, and let us orient the circle $c = S^1 \times \{1\}$ in such a way that the intersection number $\delta.c$ of δ with c on the oriented surface $S^1 \times [0,1]$ is equal to $+1$.

Definition 4.9. The *screw number* τ of f is the rational number defined by the following equality in $H_1(S^1 \times [0,1], \mathbb{Z})$:

$$N\tau c = f^N(\delta) - \delta.$$

Observe that τ does not depend on the choice of the integer N such that f^N is the identity on ∂A, nor on the choice of the annulus A_{i_0}.

In the sequel, we will use the two following simple lemmas.

Lemma 4.10. *Set $A = S^1 \times [0,1]$ and decompose the annulus A as the union of the two concentric annuli $A_1 = S^1 \times [0, \frac{1}{2}]$ and $A_2 = S^1 \times [\frac{1}{2}, 1]$. Let $f\colon A \to A$ be an orientation preserving diffeomorphism such that $f(A_1) = A_1$, $f(A_2) = A_2$, and there exists $N > 0$ such that f^N is the identity on $S^1 \times \{0, \frac{1}{2}, 1\}$. Then, if $f|_{A_1}$ and $f|_{A_2}$ have screw numbers τ_1 and τ_2 respectively, f has screw number $\tau_1 + \tau_2$.*

Proof. Set $\delta_1 = \{1\} \times [0, \frac{1}{2}]$, $\delta_2 = \{1\} \times [\frac{1}{2}, 1]$ and $\delta = \delta_1 \cup \delta_2 = \{1\} \times [0,1]$. Set also $c_1 = S^1 \times \{\frac{1}{2}\}$ and $c_2 = S^1 \times \{1\}$, oriented so that $\delta.c_i = +1$ in A, so c_1 and c_2 have the same homology class on A. Then $N\tau_1 c_1 = f^N(\delta_1) - \delta_1$ and $N\tau_2 c_2 = f^N(\delta_2) - \delta_2$ hold as equalities of cycles in A_1 and A_2 respectively. Since c_1 is homolog to c_2 in A and $\delta = \delta_1 \cup \delta_2$, we deduce that $N(\tau_1 + \tau_2)c_1 = f^N(\delta) - \delta$ in $H_1(A, \mathbb{Z})$. This proves that the screw number of f equals $\tau_1 + \tau_2$. □

Lemma 4.11. *Let $A = S^1 \times [0,1]$ be an annulus and let $\ell\colon A \to A$ be a cyclic cover of degree $\deg(\ell)$. Let $\phi\colon A \to A$ and $\phi'\colon A \to A$ be two orientation preserving diffeomorphisms such that $\ell \circ \phi = \phi' \circ \ell$ and such that both ϕ and ϕ' have a power which is the identity on the boundary of A, and let τ and τ' denote the screw numbers of ϕ and ϕ' respectively. Then $\tau' = \deg(\ell)\tau$.*

Proof. Let N be a positive integer such that both ϕ^N and $(\phi')^N$ are the identity on ∂A. Then, using the same notations as before, the screw number τ of ϕ is defined by $\phi^N\delta - \delta = N\tau c$. Therefore $\ell(\phi^N\delta) - \ell(\delta) = N\tau\ell(c)$. Observe that we have $\ell(\delta) - \delta = 0$ and $\ell(c) - \deg(\ell)c = 0$ in $H_1(A,\mathbb{Z})$. Since $\ell \circ \phi^N = (\phi')^N \circ \delta$, this implies that $(\phi')^N\delta - \delta = N\tau\deg(\ell)c$, proving that the screw number τ' of ϕ' is equal to $\deg(\ell)\tau$. □

4.4. **Skeletal metric and the monodromy of the Milnor fibration.** Let $(X,0)$ be the germ of a complex surface with an isolated singularity, let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ which factors through the blowup of the maximal ideal, and let $h\colon (X,0) \to (\mathbb{C},0)$ be a generic linear form on $(X,0)$ (that is, $h^{-1}(0)$ is a generic hyperplane section of $(X,0)$). In this section we describe the lengths of the edges of Γ_π as screw numbers of the monodromy of some pieces of the Milnor fibration of h.

Given $,\epsilon,\eta > 0$, consider balls $B_\epsilon = \{z \in \mathbb{C}^n; \|z\| \leq \epsilon\}$ in $\mathbb{C}^n$ and $D_\eta = \{z \in \mathbb{C} \,|\, |z| \leq \eta\}$ and $D_\eta^* = D_\eta \smallsetminus \{0\}$ in $\mathbb{C}$. If $\eta << \epsilon$ are small enough, then B_ϵ is a Milnor ball for X and the restriction $h|_{T_{\epsilon,\eta}}\colon T_{\epsilon,\eta} \to D_\eta^*$ of h to $T_{\epsilon,\eta} = B_\epsilon \cap h^{-1}(D_\eta^*)$, is a topologically trivial fibration called the *Milnor–Lê fibration* of h. We also call $T_{\epsilon,\eta}$ a *Milnor tube* for L, and given t in D_η^* denote by $F_t = (h|_{T_{\epsilon,\eta}})^{-1}(t)$ a fiber of this fibration.

The monodromy of $h|_{T_{\epsilon,\eta}}$ admits a quasi-periodic representative ϕ which can be described using a suitable resolution of $(X,0)$. More precisely, let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ which factors through the blowup of the maximal ideal. In what follows, we identify the fibers F_t with their inverse image through π (recall that π is a diffeomorphism outside 0) without changing the notation.

For each component E_v of $E = \pi^{-1}(0)$, let $N(E_v)$ be a tubular neighborhood of E_v in X_π, which is the total space of a normal disc-bundle to E_v in X_π. Observe that $\pi^{-1}(B_\epsilon)$ can be identified with a tubular neighborhood of E obtained by plumbing the disc-bundles $N(E_v)$. For each vertex v of Γ_π, we set

$$\mathcal{N}(E_v) = \overline{N(E_v) \smallsetminus \bigcup_{v' \neq v} N(E_{v'})}.$$

For each irreducible component γ of $h^{-1}(0)$ in $(X,0)$, denote by $\gamma*$ the strict transform of γ in X_π, let A_{γ^*} be a small disc in $E_v \smallsetminus \bigcup_{v' \neq v} E_{v'}$ centered at $\gamma^* \cap E_v$, and let $N(\gamma^*)$ be the restriction of $N(E_v)$ over A_{γ^*}, so that $N(\gamma^*)$ is a small polydisc neighborhood of γ^* in X_π. Set

$$\check{\mathcal{N}}(E_v) = \overline{\mathcal{N}(E_v) \smallsetminus \bigcup_{\gamma \subset \{h=0\}} N(\gamma^*)} \quad \text{and} \quad F_{v,t} = F_t \cap \check{\mathcal{N}}(E_v).$$

We will often simply denote $F_{v,t}$ by F_v; since all fibers are diffeomorphic no risk of confusion will arise.

The intersection $T_{\epsilon,\eta} \cap \check{\mathcal{N}}(E_v)$ is fibered by the oriented circles obtained by intersecting the disc-fibers of $N(E_v)$ with $T_{\epsilon,\eta}$. Then the monodromy $\phi\colon F_t \to F_t$ is

defined on each $F_{v,t}$ as the diffeomorphism of first return of these circles, so it is a periodic diffeomorphism.

Now consider an edge $e = [v, v']$ in Γ_π and let $q \in E_v \cap E_{v'}$ be the corresponding intersection point. Let $\mathcal{N}(e)$ be the component of $N(E_v) \cap N(E_{v'})$ containing q. The intersection $F_{e,t} = F_t \cap \mathcal{N}(e)$ (or simply F_e) is the disjoint union of $\gcd(m_v, m_{v'})$ annuli which are cyclically exchanged by ϕ. Its monodromy is related to the length of the edge e by the following proposition.

Proposition 4.12. *([MMA11, Theorem 7.3.(iv)]) Denote by ϕ_e the restriction $\phi|_{F_e} \colon F_e \to F_e$ of the monodromy ϕ to F_e. Then $\phi_e^{m_v m_{v'}} = id_{F_e}$ and ϕ_e has screw number $-\frac{1}{m_v m_{v'}}$. In other words, this screw number is the opposite of* $\mathrm{length}(e)$.

4.5. **The resolution adapted to a projection.** Let $\ell \colon (X, 0) \to (\mathbb{C}^2, 0)$ be a generic projection as in Section 2.2 and let $\pi \colon X_\pi \to X$ be a good resolution of $(X, 0)$. Consider the minimal sequence $\sigma_\pi \colon Y_{\sigma_\pi} \to \mathbb{C}^2$ of blowups of points starting with the blowup of the origin of $\mathbb{C}^2$ such that $\widetilde{\ell}\big(V(\Gamma_\pi)\big) \subset V(\Gamma_\sigma)$. Now let $\tilde{\pi} \colon X_{\tilde{\pi}} \to X$ be the minimal good resolution of $(X, 0)$ obtained by pulling back σ_π, through ℓ, normalizing the resulting surface, and then resolving the remaining singularities. Denote by $\ell_\pi \colon X_{\tilde{\pi}} \to Y_{\sigma_\pi}$ the projection morphism, so that $\sigma_\pi \circ \ell_\pi = \ell \circ \tilde{\pi}$. The resolution $\tilde{\pi}$ factors through π, and if we denote by $\alpha_\pi \colon X_{\tilde{\pi}} \to X_\pi$ the resulting morphism we obtain the following commutative diagram:

$$\begin{tikzcd}
X_{\tilde{\pi}} \arrow[r, "\alpha_\pi"] \arrow[rr, bend left, "\tilde{\pi}"] \arrow[dr, "\ell_\pi"] & X_\pi \arrow[r, "\pi"] & X \arrow[d, "\ell"] \\
& Y_{\sigma_\pi} \arrow[r, "\sigma_\pi"] & \mathbb{C}^2
\end{tikzcd}$$

By construction, the following properties are satisfied:

(i) For every vertex v of Γ_π, we have $\tilde{\ell}(v) \in V(\Gamma_{\sigma_\pi})$;
(ii) For every vertex ν of Γ_{σ_π}, we have $\tilde{\ell}^{-1}(\nu) \subset V(\Gamma_{\tilde{\pi}})$.

Definition 4.13. We call the map $\tilde{\pi} \colon X_{\tilde{\pi}} \to X$ defined as above the resolution of $(X, 0)$ *adapted to π and ℓ.*

As part of the data of the adapted resolution we also keep the morphisms $\sigma_\pi \colon Y_{\sigma_\pi} \to \mathbb{C}^2$ and $\ell_\pi \colon X_{\tilde{\pi}} \to Y_{\sigma_\pi}$.

Observe that if the good resolution π factors through the Nash transform of $(X, 0)$ and if ℓ is generic with respect to π, then ℓ is also generic with respect to the adapted resolution $\tilde{\pi}$.

Example 4.14. Consider the hypersurface $(X, 0) \subset (\mathbb{C}^3, 0)$ defined by the equation $x^2 + y^2 + z^3 = 0$. This is the standard singularity A_2. The exceptional divisor of its minimal resolution $\pi \colon X_\pi \to X$ consists of two $\mathbb{P}^1$-curves E_{v_0} and $E_{v_0'}$ that intersect transversally at a point; its dual graph Γ_π is depicted on the right of Figure 6. To see this, we consider the generic projection $\ell = (y, z) \colon (X, 0) \to (\mathbb{C}^2, 0)$ and we perform Laufer's algorithm from [Lau71] (already used in Example 2.4. Denote by $\sigma \colon Y \to \mathbb{C}^2$ the minimal resolution of the discriminant curve Δ of ℓ, which is the cusp $y^2 + z^3 = 0$; its dual graph Γ_σ is the tree depicted on the left of Figure 6, with vertices labeled ν_0, ν_1, ν_2 in their order of appearance in the sequence of blowups. Then, performing Laufer's algorithm, we obtain the resolution $\pi' \colon X_{\pi'} \to X$ whose

dual graph Γ_π is at the middle of Figure 6. Its vertices are labeled v_0, v_0', v_1 and v_2, and we have $\tilde{\ell}(v_0) = \tilde{\ell}(v_0') = \nu_0$, $\tilde{\ell}(v_1) = \nu_1$ and $\tilde{\ell}(v_2) = \nu_2$. Denote by π'' the good resolution of $(X, 0)$ obtained by blowing down the exceptional curve E_{v_1}. Then π'' is the minimal good resolution of $(X, 0)$ that factors through its Nash transform of $(X, 0)$. If we further blow down the exceptional curve E_{v_2}, which has self-intersection -1 in $X_{\pi''}$, we obtain the minimal resolution π of $(X, 0)$ that we described before.

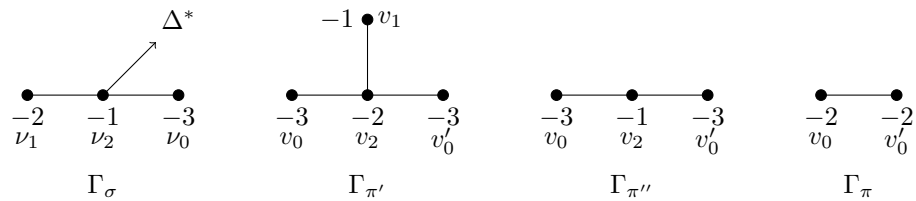

FIGURE 6.

Observe that, since $\tilde{\ell}(v_0) = \tilde{\ell}(v_0') = \nu_0$, the resolution π is already adapted to itself and to ℓ, and in this case σ_π is the blowup of $\mathbb{C}^2$ at the origin, so that $\Gamma_{\sigma_\pi} = \{\nu_0\}$; observe that $\widetilde{\ell}(\Gamma_\pi)$ is not mapped into Γ_{σ_π}. However, $\sigma_{\pi''} = \sigma$, hence the resolution adapted to π'' and ℓ is π'. This time we have $\widetilde{\ell}(\Gamma_{\pi''}) \subset \Gamma_{\sigma_{\pi''}}$, which was to be expected since π'' factors through the Nash transform of $(X, 0)$.

4.6. **The local degree formula.** Let $(X, 0)$ be a complex surface germ. For each divisorial valuation v in $\mathrm{NL}(X, 0)$, we are going to define the local degree $\deg(v) \in \mathbb{N}^*$ of v, which is an integer that measures the local topological degree at v of a generic projection of $(X, 0)$ onto $(\mathbb{C}^2, 0)$.

Let $\pi\colon X_\pi \to X$ be a good resolution of $(X, 0)$ that factors through its Nash transform and such that v is associated with a component E_v of its exceptional divisor $\pi^{-1}(0)$. Pick a projection $\ell\colon (X, 0) \to (\mathbb{C}^2, 0)$ which is generic with respect to π', let $\tilde{\pi}\colon X_{\tilde{\pi}} \to X$ be the resolution of $(X, 0)$ adapted to π and ℓ, and let $\ell_\pi\colon X_{\tilde{\pi}} \to Y_{\sigma_\pi}$ be the natural morphism (see Definition 4.13). Set $\nu = \widetilde{\ell}(v)$ and let us denote by C_ν the component of $\sigma_\pi^{-1}(0)$ associated with ν. Note that $\ell_\pi(E_v) = C_\nu$. For each component E_i of $\tilde{\pi}^{-1}(0)$ (respectively C_j of $\sigma_\pi^{-1}(0)$), let us choose a tubular neighborhood disc bundle $N(E_i)$ (resp. $N(C_j)$), and consider the sets $\mathcal{N}(E_v)$ and $\mathcal{N}(C_\nu)$ introduced in Section 4.4. We can then adjust the disc bundles $N(E_i)$ and $N(C_j)$ in such a way that the cover ℓ restricts to a cover

$$\ell_v\colon \tilde{\pi}\big(\mathcal{N}(E_v)\big) \to \sigma_\pi\big(\mathcal{N}(C_\nu)\big)$$

branched precisely on the polar curve of ℓ (if v is not a $\mathcal{P}$-node, the branching locus is just the origin). Its degree only depends on v and not on the choice of a generic projection ℓ.

Definition 4.15. The *local degree* $\deg(v)$ of v is the degree $\deg(v) = \deg(\ell_v)$ of the cover ℓ_v.

We now prove two simple lemmas about the local degrees. If $\pi\colon X_\pi \to X$ is a good resolution of $(X, 0)$ that factors through the blowup of the maximal ideal and through the Nash transform of $(X, 0)$, we denote by $V_N(\Gamma_\pi)$ the set of *nodes* of Γ_π, that is the subset of $V(\Gamma_\pi)$ consisting of the $\mathcal{P}$-nodes, the $\mathcal{L}$-nodes, and of all the vertices of genus strictly grater than zero or whose valency in Γ_π is at least three.

Lemma 4.16. *Let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ that factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$. Then* deg *is constant on each connected component of $\Gamma_\pi \smallsetminus V_N(\Gamma_\pi)$.*

Proof. Given a string $\mathcal{S}$ of Γ_π, we set $\mathcal{N}(\mathcal{S}) = N(e)$ if $\mathcal{S}$ is an edge e of Γ_π, and $\mathcal{N}(\mathcal{S}) = \bigcup_v N(E_v)$, where the union is taken over the set of vertices of Γ_π contained in $\mathcal{S}$, otherwise. By definition, a connected component $\mathcal{S}$ of $\Gamma_\pi \smallsetminus V_N(\Gamma_\pi)$ is a string of Γ_π. In particular, as $\mathcal{S}$ contains no $\mathcal{P}$-nodes, the polar curve of ℓ does not intersect $\mathcal{N}(\mathcal{S}) \smallsetminus \{0\}$, and so the restriction $\ell_{\mathcal{S}} = \ell|_{\mathcal{N}(\mathcal{S})}$ of $\ell_{\mathcal{S}}$ to $\mathcal{N}(\mathcal{S})$ is a regular cover outside 0. Denote by d the degree of this cover. If v is any divisorial point in the interior of $\mathcal{S}$, by further blowing up double points of the exceptional divisor we can assume without loss of generality that v is a vertex of Γ_π. Then ℓ_v is the restriction of $\ell_{\mathcal{S}}$ to $\mathcal{N}(E_v)$, and therefore $\deg(v) = d$. □

Remark 4.17. Although this is not needed in the paper, it is worth noticing that Lemma 4.16 allows us to extend the local degree to a map $\deg\colon \mathrm{NL}(X,0) \to \mathbb{Z}$ on the whole non-archimedean link $\mathrm{NL}(X,0)$. This map can be characterized as the unique upper semi-continuous map on $\mathrm{NL}(X,0)$ that takes the value $\deg(v)$ on any divisorial valuation v.

As a consequence of Lemma 4.16, if π is a resolution of $(X,0)$ that factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$ then we can define $\deg(e)$ for any edge e of the dual graph Γ_π by setting $\deg(e) = \deg(v)$, where v is any divisorial point in the interior of e. The following result gives an alternative way to compute local degrees.

Lemma 4.18. *Let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ which factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$, let $\ell\colon (X,0) \to (\mathbb{C}^2,0)$ be a generic projection with respect to π, let $\tilde{\pi}\colon X_{\tilde{\pi}} \to X$ be the resolution of $(X,0)$ adapted to π and ℓ, let v be a vertex of Γ_π, and let e be an edge in Γ_π adjacent to v. Denote by W_e the set of the edges e' of $\Gamma_{\tilde{\pi}}$ that are adjacent to v and such that $\widetilde{\ell}(e') \subset \widetilde{\ell}(e)$. Then we have*

$$\deg(v) = \sum_{e' \in W_e} \deg(e').$$

Proof. Consider an embedding $(X,0) \subset (\mathbb{C}^n,0)$ and choose coordinates $(z_1 \dots, z_n)$ on $\mathbb{C}^n$ in such a way that that z_1 and z_2 are generic linear forms and $\ell = (z_1, z_2)\colon X \to \mathbb{C}^2$ is our generic linear projection. We consider the generic linear form $h = z_1|_X\colon (X,0) \to (\mathbb{C},0)$ on $(X,0)$. As in [BNP14, Section 4], instead of considering a standard Milnor ball defined via the standard round ball in $\mathbb{C}^n$, we will consider a Milnor tube $T_{R\epsilon,\epsilon}$ of h. More precisely, given some sufficiently small ϵ_0 and some $R > 0$, for every $\epsilon \le \epsilon_0$ we define

$$B_\epsilon = \left\{(z_1,\dots,z_n) \,\middle|\, |z_1| \le \epsilon, |(z_1,\dots,z_n)| \le R\epsilon\right\} \quad \text{and} \quad S_\epsilon = \partial B_\epsilon\,.$$

By [BNP14, Proposition 4.1] we can choose ϵ_0 and R so that, as soon as $\epsilon \le \epsilon_0$, the fiber $h^{-1}(t)$ intersects the standard sphere $\left\{(z_1,\dots,z_n) \in \mathbb{C}^n \,\middle|\, ||(z_1,\dots,z_n)|| = \epsilon\right\}$ transversely for $|t| \le \epsilon$, and the polar curve of ℓ meets S_ϵ in the part where $|z_1| = \epsilon$. We set $B'_\epsilon = \ell(B_\epsilon)$. In what follows, the Milnor fibers of $z\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ are denoted by F', while the Milnor fibers of h are denoted by F. Note that by the choice of B_ϵ and B'_ϵ, we have $\ell(F_t) = F'_t$ for every t in D^*_η. Write $e = [v,w]$ and set $\tilde{v} = \tilde{\ell}(v)$ and $\tilde{w} = \tilde{\ell}(w)$. Then $\tilde{\ell}(e)$ is a string joining $\tilde{v}$ to $\tilde{w}$ in Γ_{σ_π}, and the

intersection $F'_{\tilde{\ell}(e)} = F'_t \cap \sigma_\pi\big(\mathcal{N}\big(\tilde{\ell}(e)\big)\big)$, where $\mathcal{N}\big(\tilde{\ell}(e)\big)$ for the string $\tilde{\ell}(e)$ is defined as at the beginning of the proof of Lemma 4.16, is a disjoint union of annuli. For each edge $e' = [v, w']$ of W_e, the intersection $F_{e'} = F_t \cap \tilde{\pi}\big(\mathcal{N}(e')\big)$ is a disjoint union of annuli. After adjusting the bundles $N(E_i)$ and $N(C_j)$ if necessary, we can assume that the cover ℓ restricts to a (possibly disconnected) regular cover

$$\ell' \colon \bigcup_{e' \in W_e} F_{e'} \to F'_{\tilde{\ell}(e)}.$$

On the one hand, the maps ℓ' and ℓ_v coincide on the boundary of $F'_{\tilde{\ell}(e)}$ that is inside $\mathcal{N}(C_{\tilde{v}})$, and by computing the degree of ℓ' over a point on this boundary we see that $\deg(\ell') = \deg(v)$. On the other hand, computing the degree of ℓ' over a point on the boundary of $F'_{\tilde{\ell}(e)}$ that is inside $\mathcal{N}(C_{\tilde{w}})$, we obtain

$$\deg(\ell') = \sum_{e' \in W_e} \deg(\ell|_{F_{e'}}) = \sum_{e' \in W_e} \deg(w').$$

Finally, by blowing-up all the intersections $E_v \cap E_{v'}$ where v, v' belong to the set of nodes $V_N(\Gamma_\pi)$, we can assume that there are no adjacent nodes in Γ_π. Therefore, for each edge $e' = [v, w']$ of W_e, the vertex w' has valency 2, $\mathcal{N}(E_{w'}) \cap F_t$ is a disjoint union of annuli each of whom has common boundary with one of the annuli of $F_{e'}$, and so we have $\deg(e') = \deg(w')$. This proves the Lemma. □

The following proposition, which we call the local degree formula, will play a key role in our proof of Theorem 4.2.

Proposition 4.19. *Let $\pi \colon X_\pi \to X$ be a good resolution of $(X, 0)$ which factors through the blowup of the maximal ideal and through the Nash transform of $(X, 0)$, let $\ell \colon (X, 0) \to (\mathbb{C}^2, 0)$ be a generic projection with respect to π, and let e be an edge of Γ_π. Then*

$$\operatorname{length}\big(\widetilde{\ell}(e)\big) = \deg(e)\operatorname{length}(e).$$

Observe that $\widetilde{\ell}(e)$ in general is not an edge of a dual graph of a sequence of blowups above the origin of $\mathbb{C}^2$, but only a string of edges. Here we consider its length for the skeletal metric of $\mathrm{NL}(\mathbb{C}^2, 0)$.

Proof. We use again the linear form $h = z_1|_X$ and Milnor balls B_ϵ and $B'_\epsilon = \ell(B_\epsilon)$ as at the beginning of the proof of Lemma 4.18. Consider the fibers $F_t = h^{-1}(t) \cap B_\epsilon$ in $(X, 0)$ and $F'_t = \{z_1 = t\} \cap B'_\epsilon$ in $(\mathbb{C}^2, 0)$. Denote by v and v' the two vertices of Γ_π adjacent to e. Let $\tilde{\pi} \colon X_{\tilde{\pi}} \to X$ be the resolution of $(X, 0)$ adapted to π and ℓ, and consider the commutative diagram $\ell \circ \tilde{\pi} = \ell_\pi \circ \sigma_\pi$ introduced in Section 4.5. Consider the string $\mathcal{S} = \widetilde{\ell}(e)$ of the dual tree Γ_{σ_π} of σ_π, and name its vertices $\nu_1, \dots, \nu_n$ following the order of the string from $\nu_1 = \widetilde{\ell}(v)$ to $\nu_n = \widetilde{\ell}(v')$. Let us re-define $\sigma_\pi\big(\mathcal{N}(\mathcal{S})\big) = B'_\epsilon \cap \sigma_\pi\big(\mathcal{N}(\mathcal{S})\big)$ and $\pi\big(\mathcal{N}(e)\big) = B_\epsilon \cap \pi\big(\mathcal{N}(e)\big)$ and note that that $\deg(e)$ is by definition the degree of the restriction $\ell|_{\pi(\mathcal{N}(e))} \colon \pi\big(\mathcal{N}(e)\big) \to \sigma_\pi\big(\mathcal{N}(\mathcal{S})\big)$. Next, set $F_{e,t} = F_t \cap \pi\big(\mathcal{N}(e)\big)$ and $F_{\mathcal{S},t} = F'_t \cap \sigma_\pi\big(\mathcal{N}(\mathcal{S})\big)$. Observe that the cover ℓ restricts to a regular cover $\ell_e \colon F_e \to F_{\mathcal{S}}$, because there are no polar curves passing by e since π factor through the Nash modification On the one hand, F_e is a disjoint union of $K = \gcd(m_v, m_{v'})$ annuli, and by Proposition 4.12, the monodromy $\phi \colon F_t \cap B_\epsilon \to F_t \cap B_\epsilon$ of the Milnor–Lê fiber of $h \colon (X, 0) \to (\mathbb{C}, 0)$ restricts to a map

$\phi_e\colon F_e \to F_e$ with screw number

$$\tau = -\frac{1}{m_v m_{v'}} = -\mathrm{length}(e).$$

On the other hand, for each $i = 1, \dots, n-1$, denote by $e_i = [\nu_i, \nu_{i+1}]$ the edge of Γ_{σ_π} connecting ν_i to ν_{i+1}. The number $K' = \gcd(m_{\nu_i}, m_{\nu_{i+1}})$ does not depend on the choice of i, as can be verified by computing the intersection number on Y_{σ_π} of the divisor $\sum_{j=1}^n m_{\nu_j} E_{\nu_j}$ with each of the E_{ν_i}, for $i = 2, \dots, n-1$. By Proposition 4.12, $F_{e_i} = F'_t \cap \sigma_\pi\big(\mathcal{N}(e_i)\big)$ is a disjoint union of K' annuli which are cyclically exchanged by the monodromy $\phi'\colon F'_t \cap B'_\epsilon \to F'_t \cap B'_\epsilon$ of the Milnor–Lê fiber of $z_1\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ and the restriction of $\phi'|_{F_{e_i}}\colon F_{e_i} \to F_{e_i}$ has screw number $-\mathrm{length}(e_i)$. Now, $F_{\mathcal{S}}$ is a disjoint union of K' annuli and the intersections $F_{e_i} = F'_t \cap \sigma_\pi\big(\mathcal{N}(e_i)\big)$, for $i = 1, \dots, n-1$, are concentric unions of K' annuli inside $F_{\mathcal{S}}$. Therefore, applying Lemma 4.10 we obtain that the restriction $\phi'|_{F_{\mathcal{S}}}\colon F_{\mathcal{S}} \to F_{\mathcal{S}}$ has screw number

$$\tau' = -\sum_{i=1}^{n-1} \mathrm{length}(e_i) = -\mathrm{length}(\mathcal{S}).$$

A representative of the monodromy $\phi'\colon F'_t \to F'_t$ is the first return diffeomorphism on F'_t of a 1-dimensional flow transverse to all the Milnor fibers, lifted from the standard unit tangent vector field to the circle $S^1_{|t|}$. Since $h = z_1 \circ \ell$, a representative of the monodromy $\phi\colon F_t \to F_t$ of the Milnor–Lê fibration of h is obtained by lifting this flow by ℓ and by taking its first return diffeomorphism on F_t. Therefore, we have $\ell \circ \phi = \phi' \circ \ell$. Moreover, the restriction $\ell|_{F_e}\colon F_e \to F_{\mathcal{S}}$ is a cover with degree $\deg(e)$. Applying Lemma 4.11 to $\phi^{K'}$ and $(\phi')^{K'}$ we then obtain $\tau' = \deg(e)\tau$. This concludes the proof. □

4.7. **Lifting Δ under $\widetilde{\ell}$.** We now explain how to use adapted resolutions and the local degree formula to relate the Laplacian of $\mathcal{I}_X$ to that of $\mathcal{I}_{\mathbb{C}^2}$ via the generic projection $\widetilde{\ell}$.

Proposition 4.20. *Let $\pi\colon X_\pi \to X$ be a good resolution of $(X, 0)$ which factors through the blowup of the maximal ideal and through the Nash transform of $(X, 0)$ and let $\ell\colon (X, 0) \to (\mathbb{C}^2, 0)$ be a generic projection with respect to π. Let $\tilde{\pi}\colon X_{\tilde{\pi}} \to X$ be the good resolution of $(X, 0)$ adapted to π and ℓ. For each vertex v of $V(\Gamma_\pi)$, we have*

$$\Delta_{\Gamma_{\tilde{\pi}}}\big(\mathcal{I}_X\big)(v) = \deg(v)\Delta_{\Gamma_{\sigma_\pi}}\big(\mathcal{I}_{\mathbb{C}^2}\big)\big(\widetilde{\ell}(v)\big).$$

Proof. Let E_v denote the set of those edges of $\Gamma_{\tilde{\pi}}$ that are adjacent to v, and for every edge e in E_v, denote by v_e the second extremity of e. On the other hand, set $\nu = \widetilde{\ell}(v)$ and let $\nu_1, \dots, \nu_r$ be the vertices of Γ_{σ_π} which are adjacent to the vertex ν. Since π factors through the Nash transform of $(X, 0)$, the strict transform of the polar curve of ℓ on X_π is a union of curvettes of the irreducible components E_v, for v ranging among the vertices of Γ_π. Therefore, the strict transform of the discriminant curve $\Delta = \ell(\Pi)$ by σ_π intersects the exceptional divisor $\sigma_\pi^{-1}(0)$ of σ_π at smooth points, not passing through the intersecting points $C_{\nu_i} \cap C_\nu$. This means that the remaining singularities over the points $p_j = C_{\nu_j} \cap C_\nu$ after the pull-back of σ_π by ℓ in the construction of Subsection 4.5 are quasi-ordinary singularities branched over the curve germ $(C_{\nu_j} \cup C_\nu, p_j)$. Therefore, for each edge e in $\mathrm{E}(v)$ the image $\tilde{\ell}(e)$ is contained in the edge $[\nu, \nu_j]$ for some j and conversely, for each $j = 1, \dots, r$, the

inverse image $\tilde{\ell}^{-1}\big([\nu,\nu_j]\big)$ of the edge $[\nu,\nu_j]$ through $\tilde{\ell}$ is a union of segments in $\Gamma_{\tilde{\pi}}$ that depart from v, and each such segment contains exactly one of the edges of E_v. For each j, let $e_{j,1},\dots,e_{j,k_j}$ be these vertices. We have $\mathrm{St}_v=\coprod_{j=1}^r\{e_{j,1},\dots,e_{j,k_j}\}$, and so the Laplacian of $\mathcal{I}_X$ on $\Gamma_{\tilde{\pi}}$ at v is

$$\Delta_{\Gamma_{\tilde{\pi}}}\big(\mathcal{I}_X\big)(v)=\sum_{e\in\mathrm{St}_v}\frac{\mathcal{I}_X(v_e)-\mathcal{I}_X(v)}{\mathrm{length}(e)}=\sum_{j=1}^r\sum_{i=1}^{k_j}\frac{\mathcal{I}_X(v_{e_{j,i}})-\mathcal{I}_X(v)}{\mathrm{length}(e_{j,i})}.$$

By Lemma 3.2, we have $\mathcal{I}_X(w)=\mathcal{I}_{\mathbb{C}^2}\big(\widetilde{\ell}(w)\big)$ for each vertex of $\Gamma_{\tilde{\pi}}$. On the other hand, by Proposition 4.19 for every $j=1,\dots,r$ and $i=1,\dots,k_j$ we have

$$\deg(e_{j,i})\mathrm{length}(e_{j,i})=\mathrm{length}([\nu,\tilde{\ell}(v_{e_{j,i}})]).$$

Moreover, since $\mathcal{I}_{\mathbb{C}^2}$ is linear on the edge $[\nu,\nu_j]$ and $\tilde{\ell}(v_{e_{j,i}})\in[\nu,\nu_j]$, we also have

$$\frac{\mathcal{I}_{\mathbb{C}^2}\big(\tilde{\ell}(v_{e_{j,i}})\big)-\mathcal{I}_{\mathbb{C}^2}(\nu)}{\mathrm{length}([\nu,\tilde{\ell}(v_{e_{j,i}})])}=\frac{\mathcal{I}_{\mathbb{C}^2}(\nu_j)-\mathcal{I}_{\mathbb{C}^2}(\nu)}{\mathrm{length}([\nu,\nu_j])}\,.$$

By putting this all together, we obtain

$$\Delta_{\Gamma_{\tilde{\pi}}}\big(\mathcal{I}_X\big)(v)=\sum_{j=1}^r\Big(\sum\nolimits_{i=1}^{k_j}\deg(e_{j,i})\Big)\frac{\mathcal{I}_{\mathbb{C}^2}(\nu_j)-\mathcal{I}_{\mathbb{C}^2}(\nu)}{\mathrm{length}([\nu,\nu_j])}\,.$$

Finally, by Lemma 4.18, for every $j\in1,\dots,r$ we have $\sum_{i=1}^{k_j}\deg(e_{j,i})=\deg(v)$. We deduce that

$$\Delta_{\Gamma_{\tilde{\pi}}}\big(\mathcal{I}_X\big)(v)=\deg(v)\Delta_{\Gamma_{\sigma_\pi}}\big(\mathcal{I}_{\mathbb{C}^2}\big)(\nu),$$

which is what we wanted to prove. □

4.8. **Lifting K under $\widetilde{\ell}$.** The last ingredient we need for the proof of Theorem 4.2 is a tool to compute also the canonical divisor of a dual resolution graph of our singular surface via a generic projection.

Proposition 4.21. *Let $\pi\colon X_\pi\to X$ be a good resolution of $(X,0)$ which factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$ and let $\ell\colon(X,0)\to(\mathbb{C}^2,0)$ be a generic projection with respect to π. Let $\tilde{\pi}\colon X_{\tilde{\pi}}\to X$ be the good resolution of $(X,0)$ adapted to π and ℓ, and let v be a vertex of Γ_π. Then*

$$K_{\Gamma_{\tilde{\pi}}}(v)-m_vp_v=\deg(v)K_{\Gamma_{\sigma_\pi}}\big(\widetilde{\ell}(v)\big).$$

Proof. We use again the notations of Sections 4.4 and 4.6: we write E_v for the divisor on $X_{\tilde{\pi}}$ associated with v, and C_ν for the divisor on Y_{σ_π} associated with $\nu=\widetilde{\ell}(v)$, and again we consider the cover $\ell_v\colon\tilde{\pi}\big(\mathcal{N}(E_v)\big)\to\sigma_\pi\big(N(C_\nu)\big)$ introduced in Section 4.6. We choose again a linear form $z_1\colon(\mathbb{C}^2,0)\to(\mathbb{C},0)$ on $(\mathbb{C}^2,0)$ such that $h=\ell\circ z_1\colon(X,0)\to(\mathbb{C},0)$ is a generic linear form on $(X,0)$, as in the proof of the degree formula (Proposition 4.19). For $t>0$ small enough, let $F_t=h^{-1}(t)\cap B_\epsilon$ be the Milnor–Lê fiber of L and let $F'_t=\{z_1=t\}\cap B'_\epsilon$ be that of h, so that we have $\ell(F_t)=F'_t$. Set $F_v=F_t\cap\tilde{\pi}\big(\mathcal{N}(E_v)\big)$ and $F'_\nu=F'_t\cap\sigma_\pi\big(\mathcal{N}(E_\nu)\big)$. Then ℓ_v restricts to a map $\ell_v|_{F_v}\colon F_v\to F'_\nu$.

Let l_v (respectively p_v) be the number of irreducible components of $h=0$ (resp. of the polar curve of ℓ) whose strict transforms by π intersects E_v. In particular, $l_v>0$ (resp. $p_v>0$) if and only if v is an $\mathcal{L}$-node (resp. a $\mathcal{P}$-node).

The cover $\ell_v\colon F_v\to F'_\nu$ is branched on p_vm_v points of F_v. Since ℓ is generic, then the images of these points are p_vm_v distinct points of F'_ν and if $p\in F'_\nu$ is one of

them, $\ell_v^{-1}(p)$ consists of $\deg(v)-1$ points. Therefore, applying Hurwitz formula we obtain $\chi(F_v)-\big(\deg(v)-1\big)m_vp_v=\deg(v)\big(\chi(F'_\nu)-m_vp_v\big)$, that is

$$\chi(F_v)+m_vp_v=\deg(v)\chi(F'_\nu). \tag{2}$$

Let us identify F_v with its pull-back by $\tilde\pi$ and let us consider again the generic linear form $h\colon(X,0)\to(\mathbb{C}^2,0)$ on $(X,0)$. If p is a smooth point of E_v which does not intersect the strict transform h^* of h on $X_{\tilde\pi}$, the total transform of h is of the form $(h\circ\tilde\pi)(u,v)=u^{m_v}$ in suitable local coordinates (u,v) centered at p. Therefore, the map $\mathcal{N}(E_v)\to E_v\cap\mathcal{N}(E_v)$ which sends each fiber-disc of $\mathcal{N}(E_v)$ to its intersecting point with E_v restricts to a regular cover $\rho_v\colon F_v\to E_v\cap\mathcal{N}(E_v)$ of degree m_v. Then, applying Hurwitz formula again, we obtain $\chi(F_v)=m_v\chi\big(E_v\cap\mathcal{N}(E_v)\big)$. By the same argument, we also have $\chi(F'_\nu)=m_\nu\chi\big(C_\nu\cap\mathcal{N}(C_\nu)\big)$.

Assume first that v is not an $\mathcal{L}$-node, that is $l_v=0$. Then $\chi\big(E_v\cap\mathcal{N}(E_v)\big)=2-2g(v)-\mathrm{val}_{\Gamma_{\tilde\pi}}(v)$. This implies that $\chi(F_v)=-K_{\Gamma_\pi}(v)$. By the same argument, we have $\chi(F'_\nu)=-K_{\Gamma_{\sigma_\pi}}(\nu)$. Using equation (2), we deduce that $K_{\Gamma_{\tilde\pi}}(v)-m_vp_v=\deg(v)K_{\Gamma_{\sigma_\pi}}(\nu)$, which is what we wanted.

Assume now that v is an $\mathcal{L}$-node. Then $\chi\big(E_v\cap\mathcal{N}(E_v)\big)=2-2g(v)-\mathrm{val}_{\Gamma_{\tilde\pi}}(v)-l_v$, which implies $\chi(F_v)=-K_{\Gamma_\pi}(v)-m_vl_v$, and since ν is the root vertex of Γ_{σ_π}, we have $m_\nu=1$, and we get $\chi(F'_\nu)=\chi(C_\nu)-1=-K_{\Gamma_{\sigma_\pi}}(\nu)-1$. Using equation (2), we then obtain $K_{\Gamma_{\tilde\pi}}(v)-m_vp_v+m_vl_v=\deg(v)\big(K_{\Gamma_{\sigma_\pi}}(\nu)+1\big)$, that is

$$K_{\Gamma_{\tilde\pi}}(v)-m_vp_v=\deg(v)K_{\Gamma_{\sigma_\pi}}(\nu)+\big(\deg(v)-m_vl_v\big). \tag{3}$$

Since v is an $\mathcal{L}$-node, then $\nu=\widetilde{\ell}(v)$ is the root vertex of the tree Γ_{σ_π}. Let $(\gamma,0)\subset(\mathbb{C}^2,0)$ be a generic complex line. Its strict transform γ^* by σ_π is a curvette of E_ν and since $m_\nu=1$, the intersection $\gamma\cap F'_v$ consists of a single point p. The cardinal of $\ell_v^{-1}(p)$ is equal to the degree $\deg(v)$ of the cover ℓ_v and also to the number of points in the intersection $\ell^{-1}(\gamma)\cap F_v$, which is exactly m_vl_v. This proves that $\deg(v)-m_vl_v=0$. Replacing this in equation (3), we obtain $K_{\Gamma_{\tilde\pi}}(v)-m_vp_v=\deg(v)K_{\Gamma_{\sigma_\pi}}(\nu)$, which is what we wanted to prove. □

4.9. **Proof of Theorem 4.2.** We are finally ready to put together all the pieces we prepared so far and finish the proof of our main theorem.

Thanks to Lemma 4.6, it suffices to prove the theorem for a good resolution π of $(X,0)$ that factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$. Again by the same lemma, it's enough to prove if for the resolution $\tilde\pi$ of $(X,0)$ adapted to π and to a given generic projection ℓ generic with respect to π. As the formula can be easily verified at all vertices of $\Gamma_{\tilde\pi}$ that are not vertices of Γ_π thanks to Lemma 3.6 by the same argument as the one in the proof of Proposition 4.7, we just have to prove that the formula holds for $\Delta_{\Gamma_{\tilde\pi}}(\mathcal{I}_X|_{\Gamma_{\tilde\pi}})$ at a vertex v of Γ_π. We want to prove that

$$\Delta_{\Gamma_{\tilde\pi}}(\mathcal{I}_X)(v)-K_{\Gamma_{\tilde\pi}}(v)-2m_vl_v+m_vp_v=0.$$

Set $\nu=\widetilde{\ell}(v)$. As we have already proved the theorem for the modification σ_π of $(\mathbb{C}^2,0)$ in Proposition 4.7, we have $\Delta_{\Gamma_{\sigma_\pi}}(\mathcal{I}_{\mathbb{C}^2})(\nu)-K_{\Gamma_{\sigma_\pi}}(\nu)-2=0$ if $\nu=\mathrm{ord}_0$ is the root of $\mathrm{NL}(\mathbb{C}^2,0)$, while $\Delta_{\Gamma_{\sigma_\pi}}(\mathcal{I}_{\mathbb{C}^2})(\nu)-K_{\Gamma_{\sigma_\pi}}(\nu)=0$ otherwise. By Proposition 4.20, we have $\Delta_{\Gamma_{\tilde\pi}}(\mathcal{I}_X)(v)=\deg(v)\Delta_{\Gamma_{\sigma_\pi}}(\mathcal{I}_{\mathbb{C}^2})(\nu)$, while by Proposition 4.21 we have $K_{\Gamma_{\tilde\pi}}(v)-m_vp_v=\deg(v)K_{\Gamma_{\sigma_\pi}}(\nu)$. Therefore, we have

$$\Delta_{\Gamma_{\tilde\pi}}(\mathcal{I}_X)(v)-K_{\Gamma_{\tilde\pi}}(v)-2m_vl_v+m_vp_v=\deg(v)\big(\Delta_{\Gamma_{\sigma_\pi}}(\mathcal{I}_{\mathbb{C}^2})(\nu)-K_{\Gamma_{\sigma_\pi}}(\nu)\big)-2m_vl_v.$$

If v is not an $\mathcal{L}$-node, then $l_v = 0$ and we get

$$\Delta_{\Gamma_{\bar{\pi}}}(\mathcal{I}_X)(v) - K_{\Gamma_{\bar{\pi}}}(v) - 2m_v l_v + m_v p_v = \deg(v)\big(\Delta_{\Gamma_{\sigma_\pi}}(\mathcal{I}_{\mathbb{C}^2})(\nu) - K_{\Gamma_{\sigma_\pi}}(\nu)\big) = 0.$$

If v is an $\mathcal{L}$-node, then $\nu = \mathrm{ord}_0$ and we obtain:

$$\Delta_{\Gamma_{\bar{\pi}}}(\mathcal{I}_X)(v) - K_{\Gamma_{\bar{\pi}}}(v) - 2m_v l_v + m_v p_v = \deg(v)\big(\Delta_{\Gamma_{\sigma_\pi}}(\mathcal{I}_{\mathbb{C}^2})(\nu) - K_{\Gamma_{\sigma_\pi}}(\nu) - 2\big) = 0.$$

This completes the proof of Theorem 4.2. □

5. Applications

In this section we will derive several consequences of Theorem 4.2.

5.1. **Lê–Greuel–Teissier formula.** As a first application of Theorem 4.2, we recover the Lê–Teissier formula of [LT81] in the particular case of a generic linear form $h\colon (X,0) \to (\mathbb{C},0)$. It is a singular version of the Lê–Greuel formula (see [Gre75, L$\hat{\text{T}}$74] for the classical statement and [Mas96, Theorem 4.2(A)] for a more general version).

Proposition 5.1. *Let $(X,0)$ be a surface germ with isolated singularity and let $h\colon (X,0) \to (\mathbb{C},0)$ be a generic linear form. Consider the Milnor–Lê fiber $F_t = \{h = t\} \cap B_\epsilon$ of h, for given real numbers $\epsilon > 0$ and $\eta > 0$ such that $\eta << \epsilon < 1$. Then we have*

$$m_\Pi(X,0) = m(X,0) - \chi(F_t),$$

where $m(X,0)$ denotes the multiplicity of $(X,0)$ and $m_\Pi(X,0)$ denotes the multiplicity of the polar curve Π of a generic linear projection $\ell\colon (X,0) \to (\mathbb{C}^2,0)$.

Proof. Let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ that factors through the blowup of the maximal ideal and through the Nash transform of $(X,0)$ and let Γ_π be the dual graph of π. As already observed in Subsection 2.3, the degree of the Laplacian of a piecewise linear function on a metric graph is equal to zero. Applying this on the inner rate function $\mathcal{I}_X$, we get

$$\sum_{v \in V(\Gamma_\pi)} \Delta_{\Gamma_\pi}\big(\mathcal{I}_X\big)(v) = 0.$$

If we replace the coefficients of this Laplacian by their expression given by Theorem B (which as observed in Remark 4.3 is an immediate consequence of Theorem 4.2), we obtain

$$\sum_{v \in V(\Gamma_\pi)} m_v l_v - \sum_{v \in V(\Gamma_\pi)} m_v p_v - \sum_{v \in V(\Gamma_\pi)} m_v\big(\chi(\check{E}_v) - l_v\big) = 0.$$

The first sum equals $m(X,0)$, while the second one equals $m_\Pi(X,0)$. Moreover, using again the notations of the proof of Proposition 4.21, for all $v \in V(\Gamma_\pi)$ we have

$$\chi(F_v) = m_v \chi\big(E_v \cap \mathcal{N}(E_v)\big) = m_v\big(\chi(\check{E}_v) - l_v\big),$$

as $E_v \cap \mathcal{N}(E_v)$ is obtained from $\check{E}_v$ by removing l_v small discs, one for each point of E_v through which a component of the pullback of a generic linear form passes. Since the connected components of $F_t \smallsetminus \coprod_{v \in V(\Gamma_\pi)} F_v$ are annuli and have therefore trivial Euler characteristic, the additivity of χ gives

$$\chi(F_t) = \sum_{v \in V(\Gamma_\pi)} \chi(F_v) = \sum_{v \in V(\Gamma_\pi)} m_v\big(\chi(\check{E}_v) - l_v\big),$$

which completes the proof. □

5.2. **A numeric version of Theorem 4.2.** We will now prove an alternative, more numeric version of Theorem 4.2 which will be later used to show our Theorem A from the introduction.

First, we need to introduce some notation. Let $(X,0)$ be a complex surface germ with isolated singularities and let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ that factors through the blowup of the maximal ideal of $(X,0)$. Let $\ell\colon (X,0) \to (\mathbb{C}^2,0)$ be a generic projection with respect to π. For each vertex v of the dual graph Γ_π of π, denote by Π_v the union of components of Π whose strict transform by π intersect E_v and set

$$p_\pi(v) = \frac{1}{m_v} \sum_{\delta \subset \Pi_v} \mathrm{mult}_0(\delta),$$

where δ runs over the components of Π_v and $\mathrm{mult}_0(\delta)$ denotes the multiplicity of δ at 0.

Remark 5.2. Let $h\colon (X,0) \to (\mathbb{C},0)$ be a generic linear form and let $(\delta,0)$ be an irreducible curve germ on $(X,0)$ which is not a component of $h^{-1}(0)$ and whose strict transform δ^* by π meets an exceptional component E_v at a point p. Then, by writing $h \circ \pi$ in local coordinates centered at p, we easily see that $\mathrm{mult}_0(\delta) = (E_v, \delta^*)_p m_v$, where $(E_v, \delta^*)_p$ denotes the intersection multiplicity of the curve germs (E_v, p) and (δ^*, p) at p. Therefore, we have

$$p_\pi(v) = \sum_{\delta \subset \Pi_v, p \in E_v} (E_v, \delta^*)_p.$$

In particular, $p_\pi(v)$ is a positive integer. Moreover, observe that if the good resolution π factors through the Nash transform of $(X,0)$, then for every vertex v of Γ_π we have $p_\pi(v) = p_v$, where, as in the introduction, p_v denotes the number of connected components of the strict transform of the polar curve of Π which meet E_v.

Recall that for every vertex v of Γ_π, the integer l_v, as defined in the introduction, denotes the number of irreducible components of $h^{-1}(0)$ which meet E_v. To keep the notation short and consistent with the statements given in the introduction, we denote by $q_v = \mathcal{I}_X(v)$ the inner rate of v. We can now state the main result of the subsection.

Proposition 5.3. *Let $(X,0)$ be the germ of a complex surface with isolated singularities, let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$ that factors through the blowup of the maximal ideal of $(X,0)$, and let Π be the polar curve of a projection $\ell\colon (X,0) \to (\mathbb{C}^2,0)$ that is generic with respect to π. Let $v_1,\dots,v_n$ be the vertices of the dual graph Γ_π and let $E_{v_1},\dots,E_{v_n}$ be the corresponding irreducible components of $\pi^{-1}(0)$. Consider the vectors $\underline{a} = {}^t(m_{v_1}q_{v_1},\dots,m_{v_n}q_{v_n})$, $\underline{L} = {}^t\big(l_{v_1},\dots,l_{v_n}\big)$, $\underline{P} = {}^t\big(p_\pi(v_1),\dots,p_\pi(v_n)\big)$, and $\underline{K} = {}^t\big(k_\pi(v_1),\dots,k_\pi(v_n)\big)$, where $k_\pi(v_i) = \mathrm{val}_{\Gamma_\pi}(v_i) + 2g(v_i) - 2 = m_{v_i}^{-1} K_{\Gamma_\pi}(v_i)$, and denote by $M = (E_{v_i} \cdot E_{v_j})_{1 \le i,j \le n}$ the intersection matrix of the exceptional divisor $\pi^{-1}(0)$ of π. Then we have*

$$M\underline{a} = \underline{K} + \underline{L} - \underline{P}.$$

Proof. Let $h\colon (X,0) \to (\mathbb{C},0)$ be a generic linear form on $(X,0)$ such that the strict transform $h^*$of $h^{-1}(0)$ by π does not intersect the strict transform of the polar curve Π. Assume for the time being that $\pi\colon X_\pi \to X$ factors through Nash transform of $(X,0)$. Since the total transform (h) of h by π, which is $(h) = \sum_{i=1}^n m_{v_i} E_{v_i} + h^*$,

is a principal divisor (see [Lau71, Theorem 2.6]), then for every vertex v of Γ_π we have $(h).E_v = 0$, that is

$$\sum_{[v,v']} m_{v'} + m_v E_v^2 + l_v = 0, \tag{4}$$

where the sum runs over the edges of Γ_π adjacent to v. Consider the coefficient of the Laplacian at a vertex v, as given by the formula of Theorem 4.2, and divide both sides by m_v. This yields the equality

$$\sum_{[v,v']} m_{v'}(q_{v'} - q_v) = \mathrm{val}_{\Gamma_\pi}(v) + 2g(E_v) - 2 + 2l_v - p_v,$$

which combined with equation (4) gives

$$q_v m_v E_v^2 + \sum_{[v,v']} m_{v'} q_{v'} = \mathrm{val}_{\Gamma_\pi}(v) + 2g(E_v) - 2 + (2 - q_v)l_v - p_v.$$

Now, observe that l_v vanishes unless v is an $\mathcal{L}$-node of $(X, 0)$, in which case $q_v = 1$, so that for every v we have $(2 - q_v)l_v = l_v$. This proves that $M\underline{a} = \underline{K} + \underline{L} - \underline{P}$.

Let us now prove the formula without the assumption that $\pi\colon X_\pi \to X$ factors through Nash transform of $(X, 0)$. Since there exists a good resolution $\hat{\pi}$ of $(X, 0)$ that dominates π and factors through the Nash transform of $(X, 0)$, it suffices to prove that if $\rho\colon X_{\pi'} \to X_\pi$ is the blowup of a point $q \in \pi^{-1}(0)$ and if the proposition holds for $\pi' = \pi \circ \rho$, then it also holds for π. We then conclude by induction.

Claim Let $E_{v''}$ be the rational curve on $X_{\pi'}$ that is created by the blowup ρ. Then we have

$$p_\pi(v) = p_{\pi'}(v) + p_{\pi'}(v'').$$

Let us now prove this claim. We will use the expression of $p_\pi(v)$ given in Remark 5.2. If $(\gamma, 0) \subset (X, 0)$ is a curve germ, let us denote by γ^* (respectively by γ^{**}) the strict transform of γ by π (resp. by $\pi' = \pi \circ \rho$). Let us decompose the germ (Π^*, q) of Π^* at p as the union $(\Pi^*, q) = (\Pi_1^*, q) \cup (\Pi_2^*, q)$, where Π_1^* (resp. Π_2^*) is the union of components of (Π^*, q) that are transverse (resp. tangent) to E_v (any component of (Π^*, q) belong to either one of those two sets by genericity of ℓ). Since Π_1^* is transverse to the curve E_v at q, then $(E_v, \Pi_1^*)_q$ equals the multiplicity $\mathrm{mult}_q(\Pi_1^*)$ of Π_1^* at q. Therefore, on the one hand we have

$$p_\pi(v) = \mathrm{mult}_q(\Pi_1^*) + (E_v, \Pi_2^*)_q + \sum_{p \in E_v \smallsetminus \{q\}} (E_v, \Pi^*)_p. \tag{5}$$

One the other hand, if we set $\tilde{q} = E_v \cap E_{v''} \subset X_{\pi'}$, since $\Pi_1^{**} \cap E_v = \emptyset$ and $\Pi_2^{**} \cap E_v = \tilde{q}$, we have

$$p_{\pi'}(v) = (E_v, \Pi_2^{**})_{\tilde{q}} + \sum_{p \in E_v \smallsetminus \{\tilde{q}\}} (E_v, \Pi^*)_p.$$

By computing the blowup ρ using local coordinates on X_π centered at q, we obtain

$$(E_v, \Pi_2^{**})_{\tilde{q}} = (E_v, \Pi_2^*)_q - \mathrm{mult}_q(\Pi_2^*).$$

Finally, again by writing ρ in local coordinates centered at q, we observe that $\sum_{p'' \in E_{v''}} (E_{v''}, \Pi_1^{**})_{p''} = \mathrm{mult}_q(\Pi_1^*)$ and $(E_{v''}, \Pi_2^{**})_{\tilde{q}} = \mathrm{mult}_q(\Pi_2^*)$. Therefore,

$$p_{\pi'}(v'') = \mathrm{mult}_q(\Pi_1^*) + \mathrm{mult}_q(\Pi_2^*). \tag{6}$$

Combining the equalities (5) and (6) we obtain $p_\pi(v) - p_{\pi'}(v) - p_{\pi'}(v'') = 0$, which proves the Claim.

Now assume that $q \in E_v \cap E_{v'}$ where E_v and $E_{v'}$ are two irreducible components of $\pi^{-1}(0)$. Since the formula that we need to prove is true for the vertex v'' of $\Gamma_{\pi'}$ by assumption, we have

$$-(m_v + m_{v'})q_{v''} + m_v q_v + m_{v'} q_{v'} = -p_{\pi'}(v''). \tag{7}$$

On the other hand, the formula is also true for v in $\Gamma_{\pi'}$, so we have:

$$(E_v^2 - 1)m_v q_v + m_{v''} q_{v''} + \sum_{[v,w], w \neq v''} m_w q_w = k_{\pi'}(v) - l_v - p_{\pi'}(v), \tag{8}$$

where E_v^2 denotes the self-intersection of E_v in X_π. Observe that the integer $\sum_{[v,w], w\neq v''} m_w q_w$, where the sum runs over edges of $\Gamma_{\pi'}$, is equal to the sum $\sum_{[v,w], w\neq v'} m_w q_w$ running over edges of Γ_π, and that moreover we have $k_{\pi'}(v) = k_\pi(v)$. Therefore, by summing the equalities (7) and (8) and using the Claim, we obtain

$$E_v^2 m_v q_v + m_{v'} q_{v'} + \sum_{[v,w], w \neq v'} m_w q_w = k_\pi(v) - l_v - p_\pi(v),$$

which is exactly what we wanted to prove for the vertex v. By symmetry, the formula also holds for v' in Γ_π. Moreover, the formula remains unchanged for all the vertices different from v and v'.

Assume now that q is a smooth point of $\pi^{-1}(0)$ on E_v. The formula for v'' in $\Gamma_{\pi'}$ is:

$$-m_v q_{v''} + m_v q_v = 1 - p_{\pi'}(v''), \tag{9}$$

while the formula for v in $\Gamma_{\pi'}$ is exactly the same one as in the previous case. By the same arguments as before, we obtain the formula for v in Γ_π by summing the two equalities (8) and (9). □

5.3. **From global geometric data to inner rates.** The following result, which is a generalization of Theorem A given in the Introduction, is a simple consequence of Proposition 5.3. It explains in which sense the metric structure of the germ $(X, 0)$, which is a very local datum, is determined by more global geometric data, namely by the topology of a good resolution of $(X, 0)$ and the position of the components of a generic hyperplane section of $(X, 0)$ and of the components of the polar curve of a generic projection of $(X, 0)$ onto $(\mathbb{C}^2, 0)$.

Corollary 5.4. *Let $(X, 0)$ be an isolated complex surface singularity and let $\pi\colon X_\pi \to X$ be a good resolution of $(X, 0)$ that factors through the blowup of the maximal ideal of $(X, 0)$. Then the inner rates of the vertices of Γ_π are determined by the following data:*

- *(i) the topological data consisting of the dual graph Γ_π decorated with the Euler classes and the genera of its vertices;*
- *(ii) the number l_v, for every vertex v of Γ_π;*
- *(iii) the number $p_\pi(v)$, for every vertex v of Γ_π.*

Moreover, if π also factors through the Nash transform of $(X, 0)$, then the data above determines completely the inner rate function $\mathcal{I}_X$ on the whole of $\mathrm{NL}(X, 0)$, and hence the local inner metric structure of the germ $(X, 0)$.

Observe that we do not need to require the metric on Γ_π to be part of the initial data. Indeed, the multiplicities m_v which we use to define it can be easily deduced from the data given in (i) and (ii), as will be explained in the proof.

Proof. Number the vertices of Γ_π as $v_1, \dots, v_n$, and denote by $M = (E_{v_i} \cdot E_{v_j})_{1 \le i,j \le n}$ the intersection matrix of the exceptional divisor $\pi^{-1}(0)$ of π. Than, in the notation of Proposition 5.3, the data described in the statement determine the three vectors $\underline{K}$, $\underline{L}$ and $\underline{P}$. By combining the equations (4) for $v = v_1, \dots, v_n$ we obtain the linear system $M\underline{m} = -L$, where $\underline{m} = {}^t(m_{v_1}, \dots, m_{v_n})$ is the vector of the multiplicities of h along the exceptional components. The matrix M, being negative definite, is invertible, and therefore we can retrieve the vector of multiplicities as $\underline{m} = -M^{-1}L$. Set $\underline{b} = \underline{K} + \underline{L} - \underline{P}$. By Proposition 5.3, we have $M\underline{a} = \underline{b}$. We then have $\underline{a} = M^{-1}\underline{b}$, and therefore we obtain the inner rates q_v for every vertex v of Γ_π by dividing each entry of $\underline{a}$ by the corresponding multiplicity m_v. Now, if π factors through the Nash transform of $(X, 0)$, as $\mathcal{I}_X$ is linear on the edges of Γ_π (Corollary 4.5) this determines the inner rate function $\mathcal{I}_X$ on the dual graph Γ_π. It remains to show that $\mathcal{I}_X$ is completely determined by its restriction to Γ_π. This is the content of the next lemma (Lemma 5.5). □

The next result will require us to consider a different metric d on $\mathrm{NL}(X, 0)$. This is defined on a dual graph Γ_π in a similar way as the one introduced in Section 2.1, by declaring the length of an edge connecting two divisorial valuations v and v' to be equal to $\mathrm{lcm}\{m_v, m_{v'}\}^{-1} = \gcd\{m_v, m_{v'}\}/m_v m_{v'}$. Being stable under further blowup by a computation analogous to the one performed in Section 2.1, these metrics on dual graphs induce a metric d on $\mathrm{NL}(X, 0)$. The difference between the two metrics has also been discussed in [Jon15, Section 7.4.10] and, in a more arithmetic setting, in [BN16, Remark 2.3.4]. Recall that $r_\pi \colon \mathrm{NL}(X, 0) \to \Gamma_\pi$ denotes the usual retraction map.

Lemma 5.5. *Let π be a good resolution of $(X, 0)$ which factors through the the blowup of the maximal ideal and through the Nash transform of $(X, 0)$. Then, for every v in $\mathrm{NL}(X, 0)$, we have*

$$\mathcal{I}_X(v) = \mathcal{I}_X\big(r(v)\big) + d\big(v, r_\pi(v)\big).$$

Since $\mathcal{I}_X$ is linear on the edges of Γ_π, this actually proves that $\mathcal{I}_X$ is completely determined by its restriction to the vertices of Γ_π.

Proof. Let π' be the minimal good resolution of $(X, 0)$ that factors through π and such that v is contained in $\Gamma_{\pi'}$. Observe that, if π'' is a good resolution of $(X, 0)$ that is sandwiched in between π' and π, so that $\Gamma_\pi \subset \Gamma_{\pi''} \subset \Gamma_{\pi'}$, then we have $d\big(v, r_\pi(v)\big) = d\big(v, r_{\pi''}(v)\big) + d\big(r_{\pi''}(v), r_\pi(v)\big)$. This equality, together with the fact that π' can be obtained from π as a sequence of point blowups, allows us to reduce ourselves without loss of generality to the case where π' is the composition of π with the blowup of X_π at a smooth point p of the exceptional divisor $\pi^{-1}(0)$. Let v_0 denote the vertex of Γ_π corresponding to the divisor E_{v_0} containing p and let v_1 denote the vertex of $\Gamma_{\pi'}$ corresponding to the exceptional divisor of the blowup at p, so that v is a point of the edge $e = [v_0, v_1]$ of $\Gamma_{\pi'}$. Since $\mathcal{I}_X$ is linear on e, to prove the theorem it is sufficient to prove that $\mathcal{I}_X(v_1) = \mathcal{I}_X(v_0) + d(v_0, v_1)$. This follows immediately from the definition of d and from Lemma 3.6.(i). □

Remark 5.6. Instead of relying on Proposition 5.3, part of the proof of Corollary 5.4 can also be replaced by a purely combinatorial argument. Indeed, one can prove that if F is a piecewise linear map on a metric graph Γ, then the Laplacian $\Delta_\Gamma(F)$ of F determines F uniquely up to an additive constant as follows. Assume that F and F' are two piecewise linear functions on Γ with the same Laplacian, so that $G = F - F'$ is a piecewise linear function such that $\Delta_\Gamma(G) = 0$. In order to show that G is constant one can reason by contradiction as follows. Assume that there exists an oriented segment $[v, v']$ in Γ along which G is strictly increasing. Then, since $\Delta_\Gamma(G)(v') = 0$, there exists a different segment $[v', v'']$ along which G is strictly increasing as well. By iterating the same construction we obtain an infinite chain of segments along which G grows. As Γ is finite, this yields a closed path along which G strictly increases, which is absurd.

Remark 5.7. In earlier papers on the subject the inner rates where always computed by considering a generic projection $\ell\colon (X, 0) \to (\mathbb{C}^2, 0)$ and lifting the inner rates of the components of the exceptional divisor of a suitable resolution of the discriminant curve of ℓ. For this reason, only simple examples have been computed, since it is generally very hard to decide whether a projection is generic, and moreover computing discriminant curves is not simple. Outside of the simplest examples, the calculations were usually done via computational tools such as Maple (see for example [BNP14, Example 15.2] or [SF18, Appendix]). However, it is generally much simpler to compute the data appearing in Corollary 5.4, and particularly so for hypersurfaces in $(\mathbb{C}^3, 0)$. This means that our result often allows for a much simpler computation of the inner rates, as is the case in the next example, where we compute the inner rates for the surface singularity of [BNP14, Example 15.2].

Example 5.8. Consider the hypersurface singularity $(X, 0) \subset (\mathbb{C}^3, 0)$ defined by the equation $(zx^2 + y^3)(x^3 + zy^2) + z^7 = 0$. A good resolution $\pi\colon X_\pi \to X$ of $(X, 0)$ can be computed explicitly, we refer to [BNP14, Example 15.2] for the details. The exceptional divisor of π is a configuration of copies $\mathbb{P}^1$ whose dual graph Γ_π is drawn in Figure 7 represents the dual graph Γ_π. The vertices of Γ_π are decorated with their serf-intersection, with solid arrows representing the components of the polar curve Π of a generic projection of $(X, 0)$, and with dashed arrows representing those of a generic hyperplane section of $(X, 0)$.

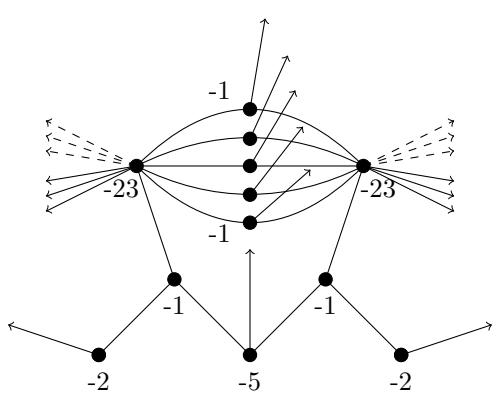

FIGURE 7.

We deduce from the data contained in Figure 7 the vectors $\underline{m} = -M^{-1}\underline{L}$ and $\underline{b} = \underline{K} + \underline{L} - \underline{P}$. We thus obtain the graph on the left of Figure 8, whose vertices are decorated by the pairs (m_v, b_v). Finally, by computing $\underline{a} = {}^t(q_{v_1}m_{v_1}, \dots, q_{v_n}m_{v_n}) = M^{-1}\underline{b}$ and dividing each entry by the corresponding multiplicity m_{v_i}, we deduce the inner rates q_{v_i}, which are inscribed on the graph on the right of Figure 8.

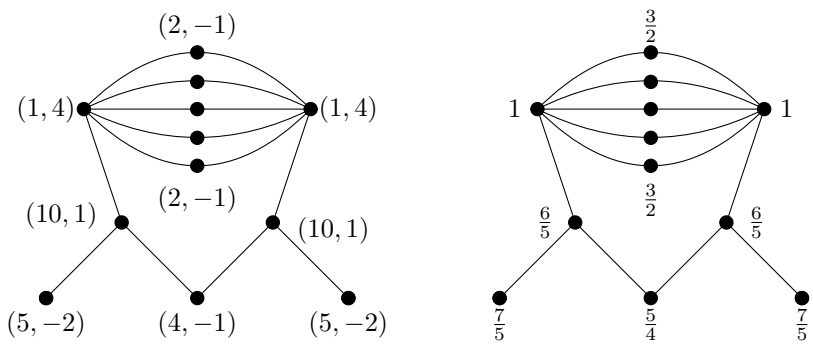

FIGURE 8. On the left: pairs (m_v, b_v). On the right: inner rates q_v.

5.4. **Restrictions on generic hyperplane sections and generic polar curves.** Proposition 5.3 gives fairly strong restrictions on the possible values of $\underline{L} - \underline{P}$ and so on the possible relative positions of the $\mathcal{L}$ and $\mathcal{P}$-nodes, as a consequence of the fact that the coefficients of the vectors $\underline{L}$ and $\underline{P}$ must be positive and that q_v must be always at least 1, and precisely 1 at the $\mathcal{L}$-nodes of $\mathrm{NL}(X, 0)$. This can be interpreted as a first step towards the famous question of D. T. Lê (see [LÔ0, Section 4.3] or [BL02, Section 8]) inquiring about the existence of a duality between the two main algorithms of resolution of a complex surface, via normalized blowups of points ([Zar39]) or via normalized Nash transforms ([Spi90]).

Example 5.9. To illustrate this, let us consider the graph of Figure 9. This graph is the minimal dual resolution graph of any member of the classical Briançon–Speder family introduced in [BS75], the family of hypersurfaces in $(\mathbb{C}^3, 0)$ defined by the equations $x^5 + z^{15} + y^7z + txy^6 = 0$, which depend on the parameter t (it is a μ-constant family which is not Whitney equisingular). The number 8 between brackets means that the corresponding exceptional component has genus 8.

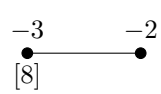

FIGURE 9.

As shown in [BNP14, Example 15.7], the configurations of $\mathcal{L}$-nodes differ for $t \neq 0$ and $t = 0$. The dual graphs of the minimal resolutions factoring through the blowup of the maximal ideal are depicted in Figure 10. The numbers between parenthesis are the multiplicities m_v and the arrows represent the components of a generic hyperplane section of $(X, 0)$. We label the vertices of both configurations by $v_1, \dots, v_4$.

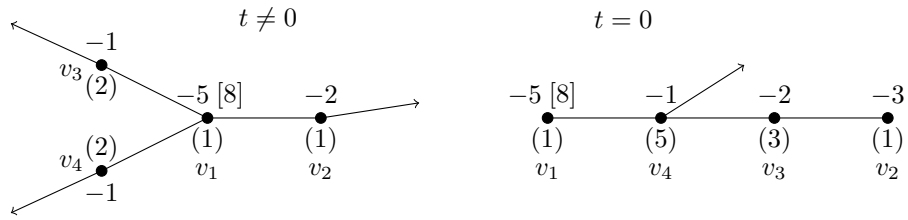

FIGURE 10. Dual graphs of the minimal resolutions factoring through the blowup of the maximal ideal for the generic element ($t \neq 0$) and for the central fiber ($t = 0$) of the Briançon–Speder family.

For each of the two graphs of Figure 10 we will now examine the possible values for $\underline{P}_\pi$ and for the inner rates, by looking at the linear equations coming from the equality $M\underline{a} = \underline{K} + \underline{L} - \underline{P}$.

Let us treat the case of the graph on the left of Figure 10 first. At v_3 the equation is $-2q_{v_3} + q_{v_1} = -1 + 1 - p_\pi(v_3)$. Since v_3 is an $\mathcal{L}$-node, we have $q_{v_3} = 1$, and so $q_{v_1} = 2 - p_\pi(v_3)$. Since q_{v_1} is strictly greater than 1 and $p_\pi(v_3)$ is a positive integer, this implies that $p_\pi(v_3) = 0$, and therefore $q_{v_1} = 2$. By symmetry, we also have $p_\pi(v_4) = 0$ (and of course $q_{v_4} = 1$). The equation corresponding to v_1 gives $-5q_{v_1} + 2q_{v_3} + 2q_{v_4} + q_{v_2} = 17 - p_\pi(v_1)$. Since $q_{v_3} = q_{v_4} = q_{v_2} = 1$ and $q_{v_1} = 2$, we then obtain $p_\pi(v_1) = 22$. Finally, the equation for v_2 is: $-2q_{v_2} + q_{v_1} = -1 + 1 - p_\pi(v_2)$. Replacing $q_{v_2} = 1$ and $q_{v_1} = 2$, we then obtain $p_\pi(v_2) = 0$. Therefore, this data of $\underline{L}$ determines a unique possible value for $\underline{P}$ and for the inner rates on Γ_π (which corresponds then to that of the singularity $x^5 + z^{15} + y^7 z + txy^6 = 0$ with $t \neq 0$). We have obtained $(q_{v_1}, q_{v_2}, q_{v_3}, q_{v_4}) = (2, 1, 1, 1)$ and $\underline{P} = {}^t(22, 0, 0, 0)$

Let us now treat the graph on the right of Figure 10. The equation for v_1 is $-5q_{v_1} + 5q_{v_4} = 15 - p_\pi(v_1)$. Since $q_{v_4} = 1$, this gives $p_\pi(v_1) = 10 + 5q_{v_1}$. For v_4, we get: $-5q_{v_4} + q_{v_1} + 3q_{v_3} = 1 - p_\pi(v_4)$, that is $qv_1 + 3q_{v_3} = 6 - p_\pi(v_4)$. Since $m_{v_3} = 3$, then $q_{v_3} \in \frac{1}{3}\mathbb{N}$. Since $q_{v_3} > 1$, we then obtain $q_{v_3} \geq \frac{4}{3}$ and since $m_{v_1} = 1$, then $q_{v_1} \in \mathbb{N}_{\geq 2}$. We then get $q_{v_1} + 3q_{v_3} \geq 6$. Since $qv_1 + 3q_{v_3} = 6 - p_\pi(v_4)$, this implies $q_{v_3} = \frac{4}{3}$, $q_{v_1} = 2$ and $p_\pi(v_4) = 0$. Moreover, $p_\pi(v_1) = 10 + 5q_{v_1} = 20$. Now, the equation for v_3 is: $-2.3.q_{v_3} + 5q_{v_4} + q_{v_2} = -p_\pi(v_3)$, therefore $q_{v_2} = 3 - p_\pi(v_3)$. In the other hand, the last equation is $-3q_{v_2} + 3q_{v_3} = -1 - p_\pi(v_2)$, which gives $p_\pi(v_2) = 3q_{v_2} - 5$. Since $q_{v_2} \in \mathbb{N}_{>1}$, we then obtain two cases: either $p_\pi(v_3) = 1$, $q_{v_2} = 2$ and $p_\pi(v_2) = 1$, or $p_\pi(v_3) = 0$, $q_{v_2} = 3$ and $p_\pi(v_2) = 4$. So there are exactly two possible configurations for the inner rates on Γ_π and for the vector $\underline{P}$:

Case 1. $(q_{v_1}, q_{v_2}, q_{v_3}, q_{v_4}) = (2, 2, 4/3, 1)$ and $\underline{P} = {}^t(20, 1, 1, 0)$

Case 2. $(q_{v_1}, q_{v_2}, q_{v_3}, q_{v_4}) = (2, 3, 4/3, 1)$ and $\underline{P} = {}^t(20, 4, 0, 0)$

Computing explicitly the polar curve from the equations from the total transform of the three functions x, y and z computed in [BNP14, Example 15.2] (as is done in [BNP14, Example 3.5]), it is easy to see that the singularity $x^5 + z^{15} + y^7 z = 0$ corresponds to the first case. We do not know whether the second case is realized by a surface singularity.

5.5. **Localization of the polar curve of a morphism.** Proposition 5.3 can also be related to the approach of [Mic08], where the following natural question is considered. Given a complex surface germ $(X, 0)$ and a finite morphism

$(f,g)\colon (X,0)\to(\mathbb{C}^2,0)$, let $\pi\colon X_\pi\to X$ be the minimal good resolution of the pair $\big(X,(fg)^{-1}(0)\big)$ (that is, π is the minimal good resolution of $(X,0)$ such that the total transform $(fg)^{-1}(0)$ is a normal crossing divisor) and let $\Gamma_{\pi,f,g}$ be the dual resolution graph of π decorated with two sets of arrows corresponding to the strict transforms of f and g. Is it then possible to localize the strict transform of the polar curve Π of the morphism (f,g) in the decorated graph $\Gamma_{\pi,f,g}$?

A partial answer to this question is given in [Mic08] and goes as follows. For each vertex v of $\Gamma_{\pi,f,g}$, consider the corresponding Hironaka quotient $m_v(f)/m_v(g)\in\mathbb{Q}_{>0}$, where $m_v(f)$ and $m_v(g)$ denote the multiplicities of f and g along the corresponding irreducible component E_v of the exceptional divisor of π (that is, the valuations $\mathrm{ord}_{E_v}(f)$ and $\mathrm{ord}_{E_v}(g)$). For each Hironaka quotient $q\in\mathbb{Q}$, let $E(q)$ be the union of those irreducible components E_v of $\pi^{-1}(0)$ that have Hironaka quotient q, and let $E^1(q)\dots,E^{n_q}(q)$ be the connected components of $E(q)$. Finally, for every $k=1,\dots,n_q$, denote by $\Gamma^k(q)$ the union of those components of Π whose strict transform via π intersects the curve $E^k(q)$. The curve $\Gamma^k(q)$ is called a *bunch* in [Mic08].

Theorem 5.10. *([Mic08, Theorem 4.9]) The bunch $\Gamma^k(q)$ is not empty if and only if $E^k(q)$ contains a component E_v such that $\chi(E'_v)<0$, where E'_v is the the subset of E_v consisting of the points that are smooth in the total transform $\big((fg)\circ\pi\big)^{-1}(0)$. Moreover,*

$$V_f\big(\Gamma^k(q)\big)=-\sum_{E_v\subset E^k(q)} m_f(v)\chi(E'_v),$$

where $V_f(\gamma)$ denotes the multiplicity of f relatively to a curve germ $(\gamma,0)\subset(X,0)$ which has no components in common with $f=0$ (see [Mic08, Definition 7.1]).

Now, let us consider the case where f,g is a pair of generic linear forms, so that $\ell=(f,g)\colon(X,0)\to(\mathbb{C}^2,0)$ is a generic projection of $(X,0)$. Then π is the minimal good resolution of $(X,0)$ which factors through the blowup of the maximal ideal and the data of the graph $\Gamma_{\pi,f,g}$ is equivalent to that consisting of the dual graph Γ_π and of the vector $\underline{L}$. In this case, the Hironaka quotient $m_v(f)/m_v(g)$ is equal to 1 at each vertex v of Γ_π and therefore there is only one bunch, the whole polar curve Π of the projection ℓ. In this case, the result [Mic08, Theorem 4.9] says nothing more than the Lê–Greuel–Teissier formula stated in Proposition 5.1. However, our Proposition 5.3 gives strong restrictions on the possible localization of the polar curve Π. More precisely, it decomposes the polar curve Π into smaller bunches Π_v than in [Mic08], one for each vertex v of Γ_π, and it gives strong restrictions on the possible values of $p_\pi(v)$, as illustrated in Example 5.9. Observe that in this case we have $V_f(\Pi_v)=\sum_{\delta\subset\Pi_v}\mathrm{mult}_0(\delta)$, and therefore $p_\pi(v)$ is nothing else than $V_f(\Pi_v)/m_v$.

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