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A characterization of Lipschitz normally embedded surface singularities

Walter D Neumann, Helge Møller Pedersen, Anne Pichon

ABSTRACT

Any germ of a complex analytic space is equipped with two natural metrics: the *outer metric* induced by the hermitian metric of the ambient space and the *inner metric*, which is the associated riemannian metric on the germ. These two metrics are in general nonequivalent up to bilipschitz homeomorphism. We give a necessary and sufficient condition for a normal surface singularity to be Lipschitz normally embedded (LNE), i.e., to have bilipschitz equivalent outer and inner metrics. In a partner paper [17] we apply it to prove that rational surface singularities are LNE if and only if they are minimal.

1. Introduction

If $(X, 0)$ is a germ of a complex variety, then any embedding $\phi: (X, 0) \hookrightarrow (\mathbb{C}^n, 0)$ determines two metrics on $(X, 0)$: the outer metric

$$d_o(x, y) := \|\phi(x) - \phi(y)\| \quad (\text{i.e., distance in } \mathbb{C}^n)$$

and the inner metric

$$d_i(x, y) := \inf\{\text{length}(\phi \circ \gamma) : \gamma \text{ is a rectifiable path in } X \text{ from } x \text{ to } y\},$$

using the riemannian metric on $X \setminus \{0\}$ induced by the hermitian metric on \mathbb{C}^n . For all $x, y \in X$, $d_o(x, y) \leq d_i(x, y)$, and the outer metric determines the inner metric. Up to bilipschitz local homeomorphism both these metrics are independent of the choice of complex embedding. We speak of the (inner or outer) *Lipschitz geometry* of $(X, 0)$ when considering these metrics up to bilipschitz equivalence.

DEFINITION 1.1. A germ of a complex normal variety $(X, 0)$ is *Lipschitz normally embedded* (LNE) if the identity map of $(X, 0)$ is a bilipschitz homeomorphism between inner and outer metrics, i.e., there exists a neighborhood U of 0 in X and a constant $K \geq 1$ such that for all $x, y \in U$

$$\frac{1}{K}d_i(x, y) \leq d_o(x, y).$$

This definition was first introduced by Birbrair and Mostowski in [5], where they just call it *normally embedded*. We prefer adding the word *Lipschitz* to distinguish this notion from that of projective normal embedding (in algebraic geometry) and normality (in local geometry, commutative algebra and singularity theory).

Lipschitz Normal Embedding (LNE) is a very active research area with many recent results giving necessary conditions for LNE in the real and complex setting, e.g., by Birbrair, Fernandes, Kerner, Mendes, Nunõ-Ballesteros, Pedersen, Ruas, Sampaio ([4], [9], [11], [15])

including a characterization of LNE for semialgebraic sets ([3]). In this paper we focus on complex normal surfaces.

It is a classical fact that the topology of a germ of a complex variety $(X, 0) \subset (\mathbb{C}^n, 0)$ is locally homeomorphic to the cone over its link $X^{(\epsilon)} = \mathbb{S}_\epsilon^{2n-1} \cap X$, where $\mathbb{S}_\epsilon^{2n-1}$ denotes the sphere with small radius ϵ centered at the origin in \mathbb{C}^n . If $(X, 0)$ is a curve germ then it is in fact bilipschitz equivalent to the metric cone over its link with respect to the inner metric, while the data of its Lipschitz outer geometry is equivalent to that of the embedded topology of a generic plane projection (see [18], [16]). Therefore, an irreducible complex curve is LNE if and only if it is smooth. This is not true in higher dimension.

The main result of this paper, Theorem 3.8, is a characterization of LNE for normal surface germs based on what we call the *nodal test curve criterion*. The first ingredient of the proof is the *arc criterion* for LNE (Theorem 4.5) of Birbrair and Mendes [3] which says that one can check if a semialgebraic germ $(X, 0)$ is LNE by testing LNE on each pair of real analytic arc $\delta_1, \delta_2 \in (X, 0)$. This criterion is difficult to use effectively since the amount of pairs of curves is incommensurable. Our Theorem 3.8 uses *generic projections* $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ which enable one to reduce drastically the amount of types of pairs of arcs to be tested, namely just certain pairs of arcs in $\ell^{-1}(\delta)$ for certain arcs $\delta \subset (\mathbb{C}^2, 0)$ called *test arcs*. This makes the criterion more efficient to prove LNE. For example, we use it in [17] to prove that rational surface singularities are LNE if and only if they are minimal.

The second ingredient of the proof is the geometric decomposition of a normal surface germ which was introduced in [6] and which is presented in Section 6.

The proof of Theorem 3.8 has two keystones: Proposition 5.3 and its enhancement Proposition 11.2. They consist of two successive reductions of the amount of test arcs to be tested. They are stated in terms of real test arcs which are real slices of complex curves on $(X, 0)$. Then the final part of the proof (Section 14) consists of a reinterpretation of Proposition 11.2 in terms of complex test curves.

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2. Generic projections and Nash modification

In order to state the main Theorem 3.8 in Section 3, we need to introduce the notions of generic projections of a curve and of a surface, and of Nash modification.

Let \mathcal{D} be a $(n-2)$ -plane in \mathbb{C}^n and let $\ell_{\mathcal{D}}: \mathbb{C}^n \rightarrow \mathbb{C}^2$ be the linear projection with kernel \mathcal{D} . Suppose $(C, 0) \subset (\mathbb{C}^n, 0)$ is a complex curve germ. There exists an open dense subset Ω_C of the Grassmanian $\mathbf{G}(n-2, \mathbb{C}^n)$ such that for $\mathcal{D} \in \Omega_C$, \mathcal{D} contains no limit of secant lines to the curve C ([21]).

DEFINITION 2.1. The projection $\ell_{\mathcal{D}}$ is said to be *generic for C* if $\mathcal{D} \in \Omega_C$.

In the sequel, we will use extensively the following result

THEOREM 2.2 [21, pp. 352-354]. *If $\ell_{\mathcal{D}}$ is a generic projection for C , then the restriction $\ell_{\mathcal{D}}|_C: C \rightarrow \ell_{\mathcal{D}}(C)$ is a bilipschitz homeomorphism for the outer metric.*

Let now $(X, 0) \subset (\mathbb{C}^n, 0)$ be a normal surface singularity. We restrict ourselves to those \mathcal{D} in the Grassmanian $\mathbf{G}(n-2, \mathbb{C}^n)$ such that the restriction $\ell_{\mathcal{D}}|_{(X,0)}: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ is finite. The *polar curve* $\Pi_{\mathcal{D}}$ of $(X, 0)$ for the direction \mathcal{D} is the closure in $(X, 0)$ of the singular locus of the restriction of $\ell_{\mathcal{D}}$ to $X \setminus \{0\}$. The *discriminant curve* $\Delta_{\mathcal{D}} \subset (\mathbb{C}^2, 0)$ is the image $\ell_{\mathcal{D}}(\Pi_{\mathcal{D}})$ of the polar curve $\Pi_{\mathcal{D}}$.

Denote by $\check{\mathbb{P}}^{n-1}$ the dual of the projective space \mathbb{P}^{n-1} , so an element of $\check{\mathbb{P}}^{n-1}$ is a hyperplane corresponding to the kernel of a linear projection $\zeta: \mathbb{C}^n \rightarrow \mathbb{C}$ which is not identically zero. Let $C(X)$ be the conormal space of X , i.e., the closure in $X \times \check{\mathbb{P}}^{n-1}$ of the set $\{(x, \zeta): x \in X, T_x X \subset \zeta\}$. Let $\kappa: C(X) \rightarrow X$ be the conormal morphism, defined as the projection on the first factor, and let $\lambda: C(X) \rightarrow \check{\mathbb{P}}^{n-1}$ be the projection on the second factor. Each $\mathcal{D} \in \mathbf{G}(n-2, \mathbb{C}^n)$, can be identified with the pencil of hyperplanes in \mathbb{C}^n which contains \mathcal{D} , i.e., with a line in $\check{\mathbb{P}}^{n-1}$ and $\Pi_{\mathcal{D}} = \kappa(\lambda^{-1}(\mathcal{D}))$.

Now, consider the tautological bundle $E \subset \check{\mathbb{P}}^{n-1} \times \mathbf{G}(n-2, \mathbb{C}^n)$ of lines of $\check{\mathbb{P}}^{n-1}$ and the intersection $\mathcal{P} = (X \times E) \cap (C(X) \times \mathbf{G}(n-2, \mathbb{C}^n)) \subset X \times \check{\mathbb{P}}^{n-1} \times \mathbf{G}(n-2, \mathbb{C}^n)$ and let $e: \mathcal{P} \rightarrow \mathbf{G}(n-2, \mathbb{C}^n)$ be the projection on the third factor, so that for all $\mathcal{D} \in \mathbf{G}(n-2, \mathbb{C}^n)$, $\Pi_{\mathcal{D}}$ is the image of $e^{-1}(\mathcal{D})$ by the projection $X \times \check{\mathbb{P}}^{n-1} \times \mathbf{G}(n-2, \mathbb{C}^n) \rightarrow X$ on the first factor.

PROPOSITION 2.3. *An open dense subset $\Omega \subset \mathbf{G}(n-2, \mathbb{C}^n)$ exists such that:*

- (i) *for each $\mathcal{D} \in \Omega$, $\lambda^{-1}(\mathcal{D}) \cap \kappa^{-1}(0)$ is a point and the restriction $e: (e^{-1}(\Omega))_{red} \rightarrow \Omega$ is a flat analytic family of reduced curves with analytic section the intersection with $\kappa^{-1}(0)$.*
- (ii) *for each $\mathcal{D} \in \Omega$, the restrictions $\kappa: \lambda^{-1}(\mathcal{D}) \rightarrow \Pi_{\mathcal{D}}$ and $\ell_{\mathcal{D}}: \Pi_{\mathcal{D}} \rightarrow \Delta_{\mathcal{D}}$ are finite birational morphisms and the second morphism is a generic projection of the polar curve $\Pi_{\mathcal{D}}$.*
- (iii) *the family $(\Delta_{\mathcal{D}})_{\mathcal{D} \in \Omega}$ is equisingular in terms of strong simultaneous resolution.*

Recall that a strong simultaneous resolution is a morphism which is a simultaneous normalization of the family and a local isomorphism in family. We refer to [20, Definition 3.1.5] for the precise definition.

DEFINITION 2.4. The projection $\ell_{\mathcal{D}}: \mathbb{C}^n \rightarrow \mathbb{C}^2$ is *generic* for $(X, 0)$ if $\mathcal{D} \in \Omega$.

Proof. (i) follows from Kleiman's transversality Theorem [12]. The first part of (ii) follows from the proof of Theorem 2.2.4 in [14] and the fact that $\ell_{\mathcal{D}}: \Pi_{\mathcal{D}} \rightarrow \Delta_{\mathcal{D}}$ is a generic projection of the curve $\Pi_{\mathcal{D}}$ is [21, Lemme-clé V 1.2.2 (ii)]. Item (iii) is [20, Section 4, Proposition 2]. \square

REMARK 2.5. Item (iii) also implies that the family of polar curves $(\Pi_{\mathcal{D}})$, $\mathcal{D} \in \Omega$ is equisingular in terms of strong simultaneous resolution. Indeed, the curves of $(\Pi_{\mathcal{D}})$, $\mathcal{D} \in \Omega$ have same normalization as their images $\Delta_{\mathcal{D}} = \ell_{\mathcal{D}}(\Pi_{\mathcal{D}})$ and this normalization is the fiber over $\mathcal{D} \in \Omega$ of the normalization $\bar{\mathcal{P}} \rightarrow \mathcal{P}$ of \mathcal{P} composed with the morphism $e: \mathcal{P} \rightarrow \mathbf{G}(n-2, \mathbb{C}^n)$ and the simultaneous resolution is then obtained by this morphism.

DEFINITION 2.6 (Nash modification). Let $\lambda: X \setminus \{0\} \rightarrow \mathbf{G}(2, \mathbb{C}^n)$ be the map which sends $x \in X \setminus \{0\}$ to the tangent plane $T_x X$. The closure NX of the graph of λ in $X \times \mathbf{G}(2, \mathbb{C}^n)$ is a reduced analytic surface. By definition, the *Nash modification* of $(X, 0)$ is the induced morphism $\mathcal{N}: NX \rightarrow X$.

LEMMA 2.7 [19, Part III, Theorem 1.2], [10, Section 2]. *A morphism $f: Y \rightarrow X$ factors through Nash modification if and only if it has no base points for the family of polar curves.*

3. Statement of the theorem

Let $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a generic projection, let Π be its polar curve and let $\Delta = \ell(\Pi)$ be its discriminant curve. Denote by $\rho'_\ell: Y_\ell \rightarrow \mathbb{C}^2$ the minimal composition of blow-ups of points starting with the blow-up of the origin which resolves the base points of the family of projections of generic polar curves $(\ell(\Pi_{\mathcal{D}}))_{\mathcal{D} \in \Omega}$.

DEFINITION 3.1. We say Δ -curve for an exceptional curve in $(\rho'_\ell)^{-1}(0)$ intersecting the strict transform of Δ .

Let us blow up all the intersection points between two Δ -curves. We call $\sigma: Z_\ell \rightarrow Y_\ell$ and $\rho_\ell = \rho'_\ell \circ \sigma: Z_\ell \rightarrow \mathbb{C}^2$ the resulting morphisms (if no Δ -curves intersect, $\rho_\ell = \rho'_\ell$).

REMARK 3.2. By ((ii)) of Proposition 2.3, the resolution graph of ρ_ℓ does not depend on ℓ . We denote it by T for the rest of the paper.

DEFINITION 3.3. A Δ -node of T is a vertex (j) of T which represents a Δ -curve. If two Δ -nodes are joined by a string of valency-two vertices which contains neither a Δ -node nor the root vertex, we choose a vertex on that string, and we call it a *separation-node* (in particular, a vertex (j) associated with an exceptional curve C_j resulting from the blow-ups $\sigma: Z_\ell \rightarrow Y_\ell$ is a separation node).

A *node* of T is a vertex (j) of T which is either the root-vertex or a Δ -node or a separation-node or a vertex with at least three incident edges.

Let $E \subset Y$ be a complex curve in a complex surface Y and let E_1, \dots, E_n be the irreducible components of E . We say *curvette* of E_i for any smooth curve germ (β, p) in Y , where p is a point of E_i which is a smooth point of Y and E and such that β and E_i intersect transversely.

If G is a graph, we will denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges.

DEFINITION 3.4. Let C_i be the irreducible component of $\rho_\ell^{-1}(0)$ represented by $(i) \in V(T)$, so we have $\rho_\ell^{-1}(0) = \bigcup_{(i) \in V(T)} C_i$. For $(i) \in V(T)$ we call *test curve at (i)* (of ℓ) any complex curve germ $(\gamma, 0) \subset (\mathbb{C}^2, 0)$ such that

- (i) the strict transform γ^* by ρ_ℓ is a curvette of a C_i ;
- (ii) $\gamma^* \cap \Delta^* = \emptyset$.

A test-curve $\gamma \subset (\mathbb{C}^2, 0)$ at (i) is called a *nodal test curve* if (i) is a node of T .

DEFINITION 3.5. Let $\pi_0: X_0 \rightarrow X$ be the minimal good resolution of X which factors through both the Nash modification and the blow-up of the maximal ideal and let G_0 be its resolution graph. For each vertex (v) of G_0 we denote by E_v the corresponding irreducible component of $\pi_0^{-1}(0)$. A vertex (v) of G_0 such that E_v is an irreducible component of the blow-up of the maximal ideal (resp. an exceptional curve of the Nash transform) is called an \mathcal{L} -node (resp. a \mathcal{P} -node) of G_0 .

DEFINITION 3.6. Consider the graph G'_0 of G_0 defined as the union of all simple paths in G_0 connecting pairs of vertices among \mathcal{L} - and \mathcal{P} -nodes. Let $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a generic projection. Let γ be a test curve for ℓ . A component $\hat{\gamma}$ of $\ell^{-1}(\gamma)$ is called *principal* if its strict transform by π_0 is either a curvette of a component E_v with $v \in V(G'_0)$ or intersects $\pi_0^{-1}(0)$ at an intersection between two exceptional curves E_v and $E_{v'}$ such that both (v) and (v') are in $V(G'_0)$.

We now define the outer and inner contacts between two complex curves on a complex surface germ.

Throughout the paper, we use the “big-Theta” asymptotic notation of Bachman-Landau: given two function germs $f, g: ([0, \infty), 0) \rightarrow ([0, \infty), 0)$ we say f is *big-Theta* of g and we write $f(t) = \Theta(g(t))$ if there exist real numbers $\eta > 0$ and $K \geq 1$ such that for all t with $f(t) \leq \eta$:

$$\frac{1}{K}g(t) \leq f(t) \leq Kg(t).$$

Let $\mathbb{S}_\epsilon^{2n-1} = \{x \in \mathbb{C}^n : \|x\|_{\mathbb{C}^n} = \epsilon\}$. Let $(\gamma_1, 0)$ and $(\gamma_2, 0)$ be two germs of complex curves inside $(X, 0)$. Let $q_{out} = q_{out}(\gamma_1, \gamma_2)$ and $q_{inn} = q_{inn}(\gamma_1, \gamma_2)$ be the two rational numbers ≥ 1 defined by

$$d_o(\gamma_1 \cap \mathbb{S}_\epsilon^{2n-1}, \gamma_2 \cap \mathbb{S}_\epsilon^{2n-1}) = \Theta(\epsilon^{q_{out}}),$$

$$d_i(\gamma_1 \cap \mathbb{S}_\epsilon^{2n-1}, \gamma_2 \cap \mathbb{S}_\epsilon^{2n-1}) = \Theta(\epsilon^{q_{inn}}),$$

where d_i means inner distance in $(X, 0)$ as before. (The existence and rationality of q_{inn} will be a consequence of Proposition 12.1).

DEFINITION 3.7. We call $q_{out}(\gamma_1, \gamma_2)$ (resp. $q_{inn}(\gamma_1, \gamma_2)$) the *outer contact exponent* (resp. the *inner contact exponent*) between γ_1 and γ_2 .

We now state the main result of the paper.

THEOREM 3.8 (Test curve criterion for LNE of a complex surface). *A normal surface germ $(X, 0)$ is LNE if and only if the following conditions are satisfied for all generic projections $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ and nodal test curves $(\gamma, 0) \subset (\mathbb{C}^2, 0)$:*

- (1*) *for all principal components $\hat{\gamma}$ of $\ell^{-1}(\gamma)$, $\text{mult}(\hat{\gamma}) = \text{mult}(\gamma)$ where mult means multiplicity at 0;*
- (2*) *for all pairs (γ_1, γ_2) of distinct principal components of $\ell^{-1}(\gamma)$, $q_{inn}(\gamma_1, \gamma_2) = q_{out}(\gamma_1, \gamma_2)$.*

REMARK 3.9. The definition of separation node (Definition 3.3) depends on a choice of a vertex along a string joining two Δ -nodes. However, the validity of Theorem 3.8 does not depend on this choice. In fact it follows from the proof that if all the principal components over one test curve at a vertex on such a string satisfy (1*) and (2*) then all principal components over all test curves at vertices on the string satisfy (1*) and (2*).

4. The real arc criterion for LNE of a semialgebraic germ

DEFINITION 4.1 (Real arcs). Let $(X, 0) \subset (\mathbb{R}^n, 0)$ be a semialgebraic germ. A *real arc* on $(X, 0)$ will mean the germ of a semialgebraic map $\delta: [0, \eta) \rightarrow X$ for some $\eta \in \mathbb{R}^+$, such that $\delta(0) = 0$ and $\|\delta(t)\| = t$ (see also Remark 4.3).

When no confusion is possible, we will use the same notation for the arc δ and the germ of its parametrized image $\delta([0, \eta))$.

DEFINITION 4.2. Let $(X, 0) \subset (\mathbb{R}^n, 0)$ be a semialgebraic germ and let $\delta_1: [0, \eta) \rightarrow X$ and $\delta_2: [0, \eta) \rightarrow X$ be two real arcs on X . The *outer contact* of δ_1 and δ_2 is ∞ if $\delta_1 = \delta_2$ and is otherwise the rational number $q_o = q_o(\delta_1, \delta_2)$ defined by:

$$\|\delta_1(t) - \delta_2(t)\| = \Theta(t^{q_o}).$$

The *inner contact* of δ_1 and δ_2 is the rational number $q_i = q_i(\delta_1, \delta_2)$ defined by

$$d_i(\delta_1(t), \delta_2(t)) = \Theta(t^{q_i}).$$

REMARK 4.3. 1) The existence and rationality of q_i comes from the fact that there exists a semialgebraic metric $d_P: X \times X \rightarrow \mathbb{R}$ (the so-called pancake metric) such that d_i and d_P are bilipschitz equivalent ([13]).

2) The inner and outer contacts $q_i(\delta_1, \delta_2)$ and $q_o(\delta_1, \delta_2)$ can also be defined taking reparametrizations by real slices of δ_1 and δ_2 as follows. First note that if δ_1 and δ_2 have different tangent directions then $q_i(\delta_1, \delta_2) = q_o(\delta_1, \delta_2) = 1$, so we may assume they have the same tangent direction. We can choose coordinates (x_1, \dots, x_n) such that the tangent semi-line of δ_1 and δ_2 has $x_1 > 0$ except at 0. For $j = 1, 2$, consider the reparametrization $\tilde{\delta}_j: [0, \eta) \rightarrow \mathbb{R}^n$ defined by $\tilde{\delta}_j(t) = \delta_j \cap \{x_1 = t\}$. Then we have $\|\tilde{\delta}_1(t) - \tilde{\delta}_2(t)\| = \Theta(t^{q_o})$ and $d_i(\tilde{\delta}_1(t), \tilde{\delta}_2(t)) = \Theta(t^{q_i})$.

Indeed, this is an easy consequence of the following standard lemma:

LEMMA 4.4. Let $B \subset \mathbb{C}^n$ be any closed compact convex neighborhood of 0 in \mathbb{C}^n . Let $\phi: B \rightarrow B_1$, where B_1 is the unit ball, be the homeomorphism which maps each ray from 0 to ∂B linearly to the ray with the same tangent, but of length 1. Then the map $\phi: B \rightarrow B_1$ is a bilipschitz homeomorphism.

The following result of Birbrair and Mendes is a characterization of closed semialgebraic germs which are LNE.

THEOREM 4.5 (The arc criterion for LNE, [3]). Let $(X, 0) \subset (\mathbb{R}^m, 0)$ be a closed semi-algebraic germ. It is LNE if and only if for all pairs of real arcs δ_1 and δ_2 in $(X, 0)$, $q_i(\delta_1, \delta_2) = q_o(\delta_1, \delta_2)$.

5. The real test arc criterion for LNE of a normal complex surface germ

DEFINITION 5.1. Let $(\gamma, 0) \subset (\mathbb{C}^n, 0)$ be a complex curve germ. Denote by z_1, \dots, z_n the coordinates of \mathbb{C}^n . We assume that no tangent line to γ is contained in the hyperplane $\{z_1 = 0\}$. Fix $e^{i\alpha} \in \mathbb{S}^1$. We call the intersection $\gamma_\alpha = \gamma \cap \{z_1 = e^{i\alpha}t, t \in \mathbb{R}^+\}$ a *real slice* of γ .

DEFINITION 5.2. We call *test arc* a component of a real slice of a test curve $(\gamma, 0) \subset (\mathbb{C}^2, 0)$ (Definition 3.4).

PROPOSITION 5.3. *A normal surface $(X, 0)$ is LNE if and only if for all generic projections $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ and for all test arcs δ of ℓ , any pair of components δ_1, δ_2 of $\ell^{-1}(\delta)$ satisfies $q_i(\delta_1, \delta_2) = q_o(\delta_1, \delta_2)$.*

Proposition 5.3 will be proved in Section 10. Before this, we introduce the necessary material for the proof in Section 6 and we prove preliminary lemmas in Sections 7 to 9.

We will also prove later an enhanced version of this result (Proposition 11.2) which reduces again drastically the amount of pairs of test arcs and which is a version in terms of real test arcs of the main Theorem 3.8).

6. The geometric decomposition of a normal surface germ

6.1. Pieces

Birbrair, Neumann and Pichon defined in [6] some semialgebraic metric spaces (with inner metric) called *A*-, *B*- and *D*-pieces, which we will need in this paper. We give here the definition and basic facts about these pieces (see [6, Sections 11 and 13] for more details).

The pieces are topologically conical, but usually with metrics that make them shrink non-linearly towards the cone point. We will consider these pieces as germs at their cone-points, but for the moment, to simplify notation, we suppress this.

By D^2 we mean the standard unit disc in \mathbb{C} with \mathbb{S}^1 as its boundary, and I denotes the interval $[0, 1]$.

DEFINITION 6.1 *A*(q, q')-pieces. Let q, q' be rational numbers such that $1 \leq q < q'$. Let A be the euclidean annulus $\{(\rho, \psi) : 1 \leq \rho \leq 2, 0 \leq \psi \leq 2\pi\}$ in polar coordinates and for $0 < r \leq 1$ let $g_{q,q'}^{(r)}$ be the metric on A :

$$g_{q,q'}^{(r)} := (r^q - r^{q'})^2 d\rho^2 + ((\rho - 1)r^q + (2 - \rho)r^{q'})^2 d\psi^2.$$

So A with this metric is isometric to the euclidean annulus with inner and outer radii $r^{q'}$ and r^q . The metric completion of $(0, 1] \times \mathbb{S}^1 \times A$ with the metric

$$dr^2 + r^2 d\theta^2 + g_{q,q'}^{(r)}$$

compactifies it by adding a single point at $r = 0$. We call a metric space which is bilipschitz homeomorphic to this completion an *A*(q, q')-piece or simply an *A*-piece.

DEFINITION 6.2 *B*(q)-pieces. Let F be a compact oriented 2-manifold, $\phi: F \rightarrow F$ an orientation preserving diffeomorphism, and M_ϕ the mapping torus of ϕ , defined as:

$$M_\phi := ([0, 2\pi] \times F) / ((2\pi, x) \sim (0, \phi(x))).$$

Given a rational number $q > 1$, we will define a metric space $B(F, \phi, q)$ which is topologically the cone on the mapping torus M_ϕ .

For each $0 \leq \theta \leq 2\pi$ choose a Riemannian metric g_θ on F , varying smoothly with θ , such that for some small $\delta > 0$:

$$g_\theta = \begin{cases} g_0 & \text{for } \theta \in [0, \delta], \\ \phi^* g_0 & \text{for } \theta \in [2\pi - \delta, 2\pi]. \end{cases}$$

Then for any $r \in (0, 1]$ the metric $r^2 d\theta^2 + r^{2q} g_\theta$ on $[0, 2\pi] \times F$ induces a smooth metric on M_ϕ . Thus

$$dr^2 + r^2 d\theta^2 + r^{2q} g_\theta$$

defines a smooth metric on $(0, 1] \times M_\phi$. The metric completion of $(0, 1] \times M_\phi$ adds a single point at $r = 0$. Denote this completion by $B(F, \phi, q)$. We call a metric space which is bilipschitz homeomorphic to $B(F, \phi, q)$ a $B(q)$ -piece or simply a B -piece.

A $B(q)$ -piece such that F is a disc is called a $D(q)$ -piece or simply a D -piece.

DEFINITION 6.3 Conical pieces. Given a compact smooth 3-manifold M , choose a Riemannian metric g on M and consider the metric $dr^2 + r^2g$ on $(0, 1] \times M$. The completion of this adds a point at $r = 0$, giving a *metric cone on M* . We call a metric space which is bilipschitz homeomorphic to a metric cone a *conical piece*. We will call any conical piece a $B(1)$ -piece (they were called CM -pieces in [6]).

The diameter of the image in $B(F, \phi, q)$ of a fiber $\{r\} \times \{\theta\} \times F$ is $\Theta(r^q)$. Therefore q describes a rate of shrink of the surfaces $\{r\} \times \{\theta\} \times F$ in $B(F, \phi, q)$ with respect to the distance r to the point at $r = 0$. Similarly, the inner and outer boundary components of any $\{r\} \times \{t\} \times A$ in an $A(q, q')$ have rate of shrink respectively q' and q with respect to r .

DEFINITION 6.4 Rate. The rational number q is called the *rate* of $B(q)$ or $D(q)$. The rational numbers q and q' are the two *rates* of $A(q, q')$.

6.2. Classical plane curve theory

Let $(\gamma, 0) \subset (\mathbb{C}^2, 0)$ and $(\gamma', 0) \subset (\mathbb{C}^2, 0)$ be two distinct germs of irreducible complex plane curves. Let $\mathbb{S}_\epsilon^3 = \{x \in \mathbb{C}^2 : \|x\| = \epsilon\}$. Recall that the contact $q_{out} = q_{out}(\gamma, \gamma')$ of γ and γ' is the rational number defined by:

$$d_o(\gamma_1 \cap \mathbb{S}_\epsilon^3, \gamma_2 \cap \mathbb{S}_\epsilon^3) = \Theta(\epsilon^{q_{out}}).$$

Equivalently, $q_{out}(\gamma, \gamma')$ is the largest q for which there exist Puiseux expansions $y = f(x)$ and $y = g(x)$ of γ and γ' which coincide for exponents $< q$ and which have distinct coefficients for x^q .

Let $\rho: Y \rightarrow \mathbb{C}^2$ be a sequence of blow-ups of points starting with the blow-up of the origin of \mathbb{C}^2 . Let C_1, \dots, C_k be the irreducible components in Y of the exceptional divisor of ρ with C_1 being the exceptional curve of the first blow-up.

Let $(\gamma, 0)$ and $(\gamma', 0)$ be two irreducible curve germs whose strict transforms by ρ meet a C_i at two distinct smooth points of $\rho^{-1}(0)$. Then the contact $q_{out}(\gamma, \gamma')$ of γ and γ' in \mathbb{C}^2 does not depend on the choice of γ and γ' .

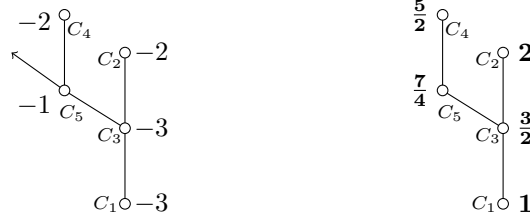
DEFINITION 6.5. We call $q_{out}(\gamma, \gamma')$ the *inner rate* of C_i and we denote it by q_{C_i} , or simply q_i when no confusion is possible.

Let T_0 be the dual tree of ρ , i.e., the vertices $(1), \dots, (k)$ of T_0 are in bijection with the exceptional curves C_1, \dots, C_k and there is an edge between (i) and (j) if and only if $C_i \cap C_j \neq \emptyset$. The vertex (1) , corresponding to the curve C_1 , is the root of T_0 . We weight each vertex (i) by the corresponding inner rate $q_i = q_{C_i}$.

A non-root vertex with one incident edge is called a *leaf*. By classical theory on plane curves, we have:

PROPOSITION 6.6. $q_1 = 1$ and the inner rates along a path from the root to a leaf are strictly increasing.

EXAMPLE 6.7. Consider the curve germ $(\gamma, 0)$ with Puiseux expansion $y = f(x) = x^{3/2} + x^{7/4}$. Its resolution graph T_0 , with exceptional curves labeled in order of blow-up, is pictured on the left. Its arrow represents the strict transform γ^* , which meets C_5 at a smooth point.



By changing the coefficient of $x^{7/4}$ in $f(x)$ one obtains a curve germ γ' whose strict transform γ'^* meets at a different point of C_5 . The contact exponent of γ and γ' , and hence the inner rate q_5 of C_5 , is $7/4$. Similarly, replacing the coefficient of $x^{3/2}$ in $f(x)$ by nonzero coefficients other than 1 gives curves whose strict transforms meet C_3 at distinct smooth points and which have pairwise contact $3/2$. The inner rates q_i for all C_i 's are shown in the picture on the right.

6.3. Geometric decomposition and inner contacts in \mathbb{C}^2

For each $i = 1, \dots, k$ let $N(C_i) \subset Y$ be a disk-bundle neighborhood of C_i and set

$$\mathcal{N}(C_i) := \overline{N(C_i)} \setminus \bigcup_{j \neq i} N(C_j).$$

Let $h: \mathbb{C}^2 \rightarrow \mathbb{C}$ be a linear projection such that the point $h^* \cap \rho^{-1}(0) \in C_1$ is a smooth point of C_1 , where $*$ means strict transform by ρ . In the neighbourhood U of $\rho^{-1}(0)$ given by $|h| < \eta$ with η sufficiently small, we have a decomposition of Y into pieces as follows:

- for each $i = 1 \dots, k$, $\rho(\mathcal{N}(C_i))$ is a $B(q_i)$ -piece B_i ;
- a B -piece corresponding to a leaf of T_0 is a D -piece;
- for each $q_i < q_j$ with $C_i \cap C_j \neq \emptyset$, $A_{ij} := \rho(N(C_i) \cap N(C_j))$ is an $A(q_i, q_j)$ -piece.

We may assume our generic linear form $h: \mathbb{C}^2 \rightarrow \mathbb{C}$ has coordinates chosen so the y -axis is the kernel of h . For $t \in \mathbb{C}$, we set $F_t := \{(x, y) : x = t\}$.

When $q_i \neq 1$, the fibers of the $B(q_i)$ -piece B_i are the intersections $B_i \cap F_t$ and by Proposition 6.6, each connected component of the fiber $B_i \cap F_t$ is a disc with discs inside removed, its diameter is $\Theta(|t|^{q_i})$, and it shrinks uniformly as t tends to 0. An $A(q_i, q_j)$ -piece has annular fibers by intersecting with $F_t, t \in \mathbb{C}$.

This induces a decomposition of F_t as the union

$$F_t = \bigcup_i (B_i \cap F_t) \cup \bigcup (A_{ij} \cap F_t).$$

Consider the graph \mathcal{F} defined as follows: the vertices are in bijection with the connected components of $\bigcup_i (B_i \cap F_t)$ and the edges between two vertices are in bijection with the annuli of $\bigcup (A_{i,j} \cap F_t)$ between the two corresponding components of $\bigcup_i (B_i \cap F_t)$. We weight each vertex by the corresponding inner rate and we then have a natural surjective graph-map

$$\mathcal{E}: \mathcal{F} \rightarrow T_0,$$

i.e., $\mathcal{E}(V(\mathcal{F})) = V(T_0)$ and the image by \mathcal{E} of an edge (v, v') of \mathcal{F} is the edge $(\mathcal{E}(v), \mathcal{E}(v'))$ of T_0 . Moreover, \mathcal{E} preserves the inner rates, i.e., for each vertex (v) of \mathcal{F} the corresponding inner rate q_v satisfies: $q_v = q_{\mathcal{E}(v)}$. If (ν) is a vertex of \mathcal{F} , we denote by $F_{t,\nu}$ the corresponding component of the decomposition of F_t and we denote its inner rate by q_ν .

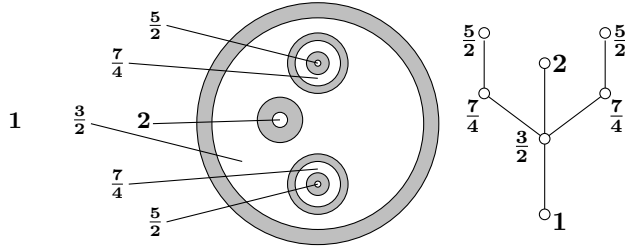
DEFINITION 6.8. We call the decomposition of the germ $(\mathbb{C}^2, 0)$ as the union of germs:

$$(\mathbb{C}^2, 0) = \bigcup_{i \in V(T_0)} B_i \cup \bigcup_{(i,j) \in E(T_0)} A_{i,j}$$

the geometric decomposition of $(\mathbb{C}^2, 0)$ associated to ρ .

We call the decomposition of the real 2-plane F_t as the union of the $F_{t,\nu}$ for $(\nu) \in V(\mathcal{F})$ and intermediate annuli $F_t \cap A_{i,j}$ the geometric decomposition of F_t associated to ρ and we call the graph \mathcal{F} weighted by inner rates its graph.

EXAMPLE 6.9. Using example 6.7 we show below a schematic picture of F_t next to the graph \mathcal{F} . The gray and white pieces are respectively component slices of A - and B -pieces.



We now explain how the geometric decomposition of F_t associated with a suitable ρ enables one to compute the contact between two real arcs $(\delta, 0)$ and $(\delta', 0)$ in $(\mathbb{C}^2, 0)$.

Note that $q_o(\delta, \delta') = q_i(\delta, \delta')$ since we are in \mathbb{C}^2 . If δ and δ' have distinct tangent semi-lines, then $q_i(\delta, \delta') = 1$. Assume that δ and δ' have the same tangent real semi-line and that in suitable coordinates (x, y) of \mathbb{C}^2 , $(\delta, 0)$ and $(\delta', 0)$ have parametrizations of the form $\delta(t) = (t, y(t))$ and $\delta'(t) = (t, y'(t))$. Let ρ be a sequence of blow-ups such that the strict transforms δ^* and δ'^* meet $\rho^{-1}(0)$ at two distinct smooth points of $\rho^{-1}(0)$.

LEMMA 6.10. Let (ν) and (ν') be the two vertices of \mathcal{F} such that $\delta(t)$ and $\delta'(t)$ intersect respectively $F_{t,\nu}$ and $F_{t,\nu'}$. Then $q_i(\delta, \delta') = q_{\nu,\nu'}$, where $q_{\nu,\nu'}$ denotes the maximum among minimum of inner rates along paths from (ν) to (ν') in the graph \mathcal{F} (in particular, if $(\nu) = (\nu')$ then $q_{\nu,\nu'} = q_\nu$).

Proof. Any path p_t between $\delta(t)$ and $\delta'(t)$ corresponds to a path from (ν) to (ν') in \mathcal{F} which describes the sequence of $F_{t,\nu''}$ and intermediate annuli crossed by p_t , and the length of p_t is $\Theta(t^q)$ where q is the minimal inner rate among vertices which are on the path. Since $d_i(\delta(t), \delta'(t))$ is the infimum of $length(p_t)$ among all such paths, this implies $q_i(\delta, \delta') \leq q_{\nu,\nu'}$. Now, choose a path from (ν) to (ν') maximizing q , so with $q = q_{\nu,\nu'}$ and then, a path p_t in F_t from $\delta(t)$ to $\delta'(t)$ realizing it. The path p_t has length $\Theta(t^{q_{\nu,\nu'}})$. Therefore $q_{\nu,\nu'} \leq q_i(\delta, \delta')$. \square

6.4. The Polar Wedge Lemma and the geometric decomposition of a normal surface

DEFINITION 6.11. Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a complex surface and let $\mathcal{N}: NX \rightarrow X$ be the Nash modification of X (Definition 2.6). The lifted Gauss map $\tilde{\lambda}: NX \rightarrow \mathbf{G}(2, \mathbb{C}^n)$ is the restriction to NX of the projection of $X \times \mathbf{G}(2, \mathbb{C}^n)$ on the second factor.

Let $\ell: \mathbb{C}^n \rightarrow \mathbb{C}^2$ be a linear projection such that the restriction $\ell|_X: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ is generic. Let Π and Δ be the polar and discriminant curves of $\ell|_X$.

DEFINITION 6.12. The *local bilipschitz constant* of $\ell|_X$ is the map $K: X \setminus \{0\} \rightarrow \mathbb{R} \cup \{\infty\}$ defined as follows. It is infinite on the polar curve and at a point $p \in X \setminus \Pi$ it is the reciprocal of the shortest length among images of unit vectors in $T_p X$ under the projection $\ell|_{T_p X}: T_p X \rightarrow \mathbb{C}^2$.

Let Π^* denote the strict transform of the polar curve Π by the Nash modification \mathcal{N} . Set $B_\epsilon = \{x \in \mathbb{C}^n: \|x\|_{\mathbb{C}^n} \leq \epsilon\}$.

LEMMA 6.13. Given any neighborhood U of $\Pi^* \cap \mathcal{N}^{-1}(B_\epsilon \cap X)$ in $NX \cap \mathcal{N}^{-1}(B_\epsilon \cap X)$, the local bilipschitz constant K of ℓ is bounded on $B_\epsilon \cap (X \setminus \mathcal{N}(U))$.

Proof. Let $\kappa: \mathbf{G}(2, \mathbb{C}^n) \rightarrow \mathbb{R} \cup \{\infty\}$ be the map sending $H \in \mathbf{G}(2, \mathbb{C}^n)$ to the bilipschitz constant of the restriction $\ell|_H: H \rightarrow \mathbb{C}^2$. The map $\kappa \circ \tilde{\lambda}$ coincides with $K \circ \mathcal{N}$ on $NX \setminus \mathcal{N}^{-1}(0)$ and takes finite values outside Π^* . The map $\kappa \circ \tilde{\lambda}$ is continuous and therefore bounded on the compact set $\mathcal{N}^{-1}(B_\epsilon) \setminus U$. \square

The *polar wedge lemma* will introduce some particular sets $\mathcal{N}(U)$ called *polar wedges* which will be D -pieces.

Consider the resolution $\rho'_\ell: Y_\ell \rightarrow \mathbb{C}^2$ which resolves the base points of the family of projections of generic polar curves $(\ell(\Pi_D))_{D \in \Omega}$ and let ρ be ρ'_ℓ composed with some finite sequence of additional blow-ups of points and T_0 the resolution graph of ρ .

Let Π_0 be a component of the polar curve Π of ℓ and let $\Delta_0 = \ell(\Pi_0)$ be the corresponding component of the discriminant curve $\Delta = \ell(\Pi)$. Let C_i be the irreducible component $(\rho'_\ell)^{-1}(0)$ which intersects Δ_0^* and let (u, v) be local coordinates in Y_ℓ centered at $p = \Delta_0^* \cap (\rho'_\ell)^{-1}(0)$ such that $v = 0$ is the local equation of C .

DEFINITION 6.14. The rate q_i associated with C_i is called the *polar rate* of Π_0 and Δ_0 .

LEMMA 6.15 (Polar Wedge Lemma, [6, Proposition 3.4]). For small $\alpha > 0$, consider the disc $D(\alpha) = \{u \in \mathbb{C}: |u| \leq \alpha\}$ centered at $p = 0$ in C_i and let $D(\alpha) \times D^2$ be the total space of the restriction of the disc bundle $N(C_i)$ over $D(\alpha)$. Set $\mathcal{N}_{\Delta_0} = \rho(D(\alpha) \times D^2)$ and let $A_{\Pi_0}(\alpha)$ be the component of $\ell^{-1}(\mathcal{N}_{\Delta_0})$ which contains Π_0 . Then

- (i) $A_{\Pi_0}(\alpha)$ is a $D(q_i)$ -piece, and when $q_i > 1$, it is fibered by its intersections with the real surfaces $\{h = t\} \cap X$, where $h: \mathbb{C}^n \rightarrow \mathbb{C}$ is a generic linear form;
- (ii) The strict transform $A_{\Pi_0}(\alpha)^*$ by \mathcal{N} is a neighborhood of Π_0^* in NX which has limit Π_0^* as $\alpha \rightarrow 0$.

DEFINITION 6.16. The set

$$A(\alpha) = \bigcup_{\Pi_0 \subset \Pi} A_{\Pi_0}(\alpha)$$

is called a *polar wedge* around Π , and its image $\mathcal{N}_\Delta(\alpha)$ by ℓ is a Δ -wedge.

Notice that \mathcal{N}_{Δ_0} is a $D(q_i)$ -piece inside the q_i -piece $B_i = \rho(\mathcal{N}(C_i))$. We can refine the geometric decomposition of \mathbb{C}^2 associated with ρ by decomposing B_i into the union of \mathcal{N}_{Δ_0} and $\overline{B_i \setminus \mathcal{N}_{\Delta_0}}$, which is still a $B(q_i)$ -piece. As a consequence of Lemma 6.13 and of the Polar Wedge Lemma 6.15, each component $B(q)$ (resp. $A(q, q')$) of this geometric decomposition of \mathbb{C}^2 lifts to components of the same type $B(q)$ (resp. $A(q, q')$) which are fibered by their intersections with the real surfaces $\{h = t\} \cap X$. We obtain a geometric decomposition of the germ $(X, 0)$ as a union of B - and A - pieces. If Δ_0 is a component of Δ , with the same notations as above, we can now amalgamate each $D(q_i)$ -piece component of $\ell^{-1}(\mathcal{N}_{\Delta_0})$ with the adjacent component of $\ell^{-1}(B'_i)$, forming a new $B(q_i)$ -piece which is a component of $\ell^{-1}(B_i)$ (see [6, Lemma 13.1])

Summarizing, we then obtain that each piece of the geometric decomposition of \mathbb{C}^2 associated with ρ lifts to a union of pieces of the same type and rate, giving a geometric decomposition of $(X, 0)$ as a union of A - and B - pieces.

DEFINITION 6.17. We call the decomposition

$$(X, 0) = \bigcup_{i \in V(T_0)} \ell^{-1}(B_i) \cup \bigcup_{(i,j) \in E(T_0)} \ell^{-1}(A_{i,j})$$

the *geometric decomposition* of $(X, 0)$ associated with ρ .

Let \mathcal{G} be the graph whose vertices are in bijection with the component B -pieces of the decomposition and whose edges are in bijection with intermediate A -pieces in such a way that the edge associated with an A -piece joins the two vertices corresponding to the two B -pieces adjacent to it. We weight each vertex by the rate of the corresponding B -piece. We then obtain a natural surjective graph-map $\mathcal{C}: \mathcal{G} \rightarrow T_0$.

6.5. Inner contacts between real arcs on a normal surface

Let $(\delta, 0) \subset (\mathbb{C}^2, 0)$ be a real arc whose strict transform by ρ intersects $\rho^{-1}(0)$ at a smooth point. We now extend the results of Section 6.3 to compute the inner contact in $(X, 0)$ between two real arcs components δ_1 and δ_2 of $\ell^{-1}(\delta)$.

We choose coordinates $\ell: (x_1, \dots, x_n) \rightarrow (x_1, x_2)$ and so that $\delta(t) = (t, x_2(t))$. Intersecting the pieces of the geometric decomposition of $(X, 0)$ associated with ρ with the real surface $\widehat{F}_t = \{x_1 = t\} \cap X$, we get a decomposition of \widehat{F}_t . Let $\widehat{\mathcal{F}}$ be the graph whose vertices are in bijection with the components of the intersections of the B -pieces, the edges are in bijection with intermediate annuli and the vertices are weighted by inner rates (we call it the *fiber-graph*). If (ν) is a vertex of $\widehat{\mathcal{F}}$, we will denote by $\widehat{F}_{t,\nu}$ the corresponding component of the decomposition of \widehat{F}_t .

The following is an extension of Lemma 6.10.

LEMMA 6.18. *Let (ν_1) and (ν_2) be the two vertices of $\widehat{\mathcal{F}}$ such that $\delta_1(t)$ and $\delta_2(t)$ intersect respectively \widehat{F}_{t,ν_1} and \widehat{F}_{t,ν_2} . Then $q_i(\delta_1, \delta_2) = q_{\nu_1, \nu_2}$, where q_{ν_1, ν_2} denotes the maximum among minimum of inner rates along paths from (ν_1) to (ν_2) in the fiber-graph $\widehat{\mathcal{F}}$ (again with $q_{\nu_1, \nu_2} = q_{\nu_1}$ if $(\nu_1) = (\nu_2)$).*

Proof. The proof is a straightforward extension of that of Lemma 6.10. Any path p_t between $\delta_1(t)$ and $\delta_2(t)$ corresponds to a path from (ν) to (ν') in $\widehat{\mathcal{F}}$ which describes the sequence of $\widehat{F}_{t,\nu''}$ and intermediate annuli crossed by p_t , and the length of p_t is $\Theta(t^q)$ where q is the minimal inner rate among vertices which are on the path. This implies $q_i(\delta_1, \delta_2) \leq q_{\nu, \nu'}$. Now, choose

a path from (ν) to (ν') in $\widehat{\mathcal{F}}$ maximizing q , so with $q = q_{\nu,\nu'}$ and then choose a path p_t in \widehat{F}_t from $\delta_1(t)$ to $\delta_2(t)$ realizing it. This path has length $\Theta(t^{q_{\nu,\nu'}})$. Therefore $q_{\nu,\nu'} \leq q_i(\delta_1, \delta_2)$. \square

We also have a natural graph-map respecting inner rates $\widehat{\mathcal{E}}: \widehat{\mathcal{F}} \rightarrow \mathcal{G}$.

To summarize, we have constructed a commutative diagram with four graph-covers respecting inner rates:

$$\begin{array}{ccc} \widehat{\mathcal{F}} & \xrightarrow{\widehat{\mathcal{E}}} & \mathcal{G} \\ \widehat{c} \downarrow & & \downarrow c \\ \mathcal{F} & \xrightarrow{\mathcal{E}} & T_0 \end{array}$$

6.6. *The graph \mathcal{G} and the resolution of $(X, 0)$.*

We use again the notations ρ_ℓ, T and $\pi_0: X_0 \rightarrow X$ in Remark 3.2 and Definitions 3.3 and 3.5. Consider the pull-back X'_ℓ of ρ_ℓ and ℓ and let $\alpha_\ell: X'_\ell \rightarrow X_\ell$ be the minimal good resolution of X_ℓ . This induces a resolution $\pi_\ell: X'_\ell \rightarrow X$ which factors through X_0 and a projection $\tilde{\ell}: X'_\ell \rightarrow Z_\ell$. We will denote by G the resolution graph of π_ℓ . The situation is summarized in the following commutative diagram

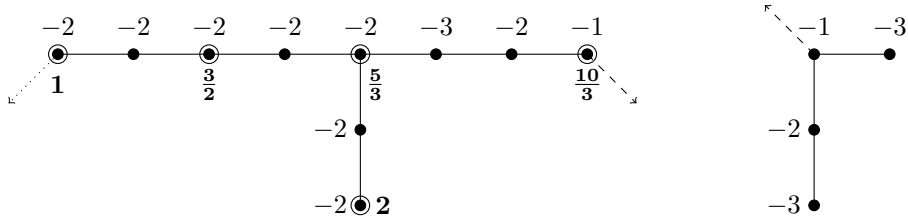
$$\begin{array}{ccc} X'_\ell & \longrightarrow & X_0 \\ \downarrow \alpha_\ell & \searrow \pi_\ell & \downarrow \pi_0 \\ X_\ell & \longrightarrow & X \\ \downarrow \tilde{\ell} & & \downarrow \ell \\ Z_\ell & \xrightarrow{\rho_\ell} & \mathbb{C}^2 \end{array}$$

Let (i) be a vertex of T and C_i the corresponding exceptional curve in $\rho_\ell^{-1}(0)$. The inverse image of C_i by $\tilde{\ell}$ is a union of k_i exceptional curves $E_{i,j}, j = 1 \dots, k_i$ of the exceptional divisor of π_ℓ . So $N(C_i)$ lifts by $\tilde{\ell}$ to k_i connected components $N(E_{i,j}), j = 1 \dots, k_i$ where $N(E_{i,j})$ is a disc neighborhood of $E_{i,j}$ in X'_ℓ . Therefore, the inverse image by ℓ of $B_i = \rho_\ell(N(C_i))$ consists of k_i components of the geometric decomposition of $(X, 0)$. This implies that the graph G is a refinement of \mathcal{G} in the sense that we have an inclusion $\mathcal{I}: V(\mathcal{G}) \rightarrow V(G)$ between the sets of vertices and the graph \mathcal{G} is obtained from G by replacing some strings of valency 2 vertices in G by edges. More precisely, the vertices of \mathcal{G} correspond to the irreducible components E_ν of the exceptional divisor of π_ℓ such that $\tilde{\ell}(E_\nu)$ is a curve in $\rho_\ell^{-1}(0)$. If (ν) is a vertex of \mathcal{G} , we will also denote (ν) the corresponding vertex $\mathcal{I}(\nu)$ in G and E_ν the corresponding component of the exceptional divisor of π_ℓ . For more details, see Sections 13 and 14 in [6].

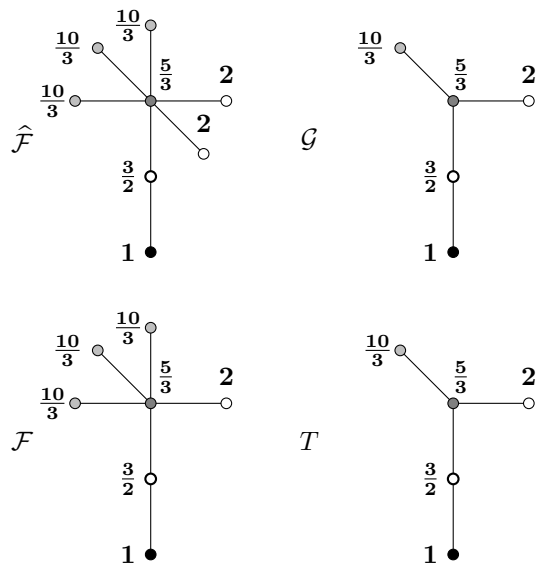
EXAMPLE 6.19. In this example we will give the four graphs for the simple surface singularity E_8 .

We can assume that E_8 has equation $x^2 + y^3 + z^5 = 0$ and then the projection $\ell(x, y, z) = (y, z)$ is generic, so the discriminant is given by $y^3 + z^5 = 0$. Below is the graph G for E_8 on the left and the resolution graph of the discriminant curve Δ discriminant on the right. In this example, the graphs G_0 and G coincide. In G the dashed arrows represent the strict transform of the polar, and the dotted arrows represent the strict transform of a generic hyperplane section. In the graph on the right the dashed arrow represents the strict transform of the

discriminant. The circled vertices of G are the vertices corresponding to vertices of \mathcal{G} by the injection \mathcal{I} , and they are weighted by the corresponding inner rates.



The following picture shows the graphs T , \mathcal{F} , \mathcal{G} and $\widehat{\mathcal{F}}$ of E_8 . The graph-maps \mathcal{C} , \mathcal{E} , $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{E}}$ respect the shape and colour of a vertex, i.e., $\widehat{\mathcal{E}}$ of a white vertex with thick boundary is the white vertex with thick boundary in \mathcal{G} and so on. Notice that in this case the graphs T , and \mathcal{G} are equal, but \mathcal{F} and $\widehat{\mathcal{F}}$ differs from them and from each other. For another example, see 11.5.



7. Vertically aligned arcs

DEFINITION 7.1 (Vertical arc criterion). Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a normal surface. Let $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a generic projection. We say *pair of vertically aligned real arcs* for any pair of distinct real arcs δ_1, δ_2 on $(X, 0)$ such that $\ell(\delta_1) = \ell(\delta_2)$.

We use again the notations introduced at the beginning of Section 5: let $\rho'_\ell: Y_\ell \rightarrow \mathbb{C}^2$ the minimal composition of blow-ups of points which resolves the base points of the family of projections of generic polar curves $(\ell(\Pi_{\mathcal{D}}))_{\mathcal{D} \in \Omega}$. We denote by Δ^* the strict transform of the discriminant curve Δ of ℓ by ρ'_ℓ .

PROPOSITION 7.2. *A normal surface $(X, 0) \subset (\mathbb{C}^n, 0)$ is LNE if and only if for all generic projections $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$, for all real arcs $(\delta, 0) \subset (\mathbb{C}^2, 0)$ such that $\Delta^* \cap \delta^* = \emptyset$, any pair of vertically aligned arcs δ_1, δ_2 in $\ell^{-1}(\delta)$ satisfies the arc criterion, i.e., $q_i(\delta_1, \delta_2) = q_o(\delta_1, \delta_2)$.*

Proof. If $(X, 0)$ is normally embedded, then the vertical arc criterion is a consequence of the arc criterion of Theorem 4.5.

Conversely, assume that any pair of vertically aligned real arcs by a generic projection satisfies the arc criterion. We have to prove that any pair of real arcs also satisfies the arc criterion.

Let δ_1 and δ_2 be a pair of real arcs on X . If their tangent semi-lines are distinct, then $q_i(\delta_1, \delta_2) = q_o(\delta_1, \delta_2) = 1$ and the arc criterion is satisfied. We now assume that δ_1 and δ_2 have the same tangent semi-line.

We choose a generic projection $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ with polar curve Π such that $\delta_i^* \cap \Pi^* = \emptyset$ for $i = 1, 2$, where $*$ means strict transform by the Nash modification \mathcal{N} . Then we can choose $\alpha > 0$ such that the polar wedge $A(\alpha)$ intersects the germs $(\delta_1, 0)$ and $(\delta_2, 0)$ only at 0. Consider the two real arcs $\sigma_i = \ell \circ \delta_i: [0, \eta] \rightarrow \mathbb{C}^2$. Let $P_t: [0, 1] \rightarrow X$ be the path on X defined as the lifting by ℓ with origin $\delta_1(t)$ of the segment S_t joining $\sigma_1(t)$ and $\sigma_2(t)$. Denote by $\delta'_2(t)$ the extremity of the path P_t . Then (δ_2, δ'_2) is a pair of vertically aligned real arcs on $(X, 0)$. By hypothesis, it satisfies the arc criterion so we have $q_i(\delta'_2, \delta_2) = q_o(\delta'_2, \delta_2)$.

Assume first that the path P_t does not intersect $A(\alpha)$. By Lemma 6.13, we can choose a real $K_0 \geq 1$ such that the local bilipschitz constant of ℓ is bounded by K_0 on $B_\epsilon \cap (X \setminus A(\alpha))$.

We then have

$$\|\delta_1(t) - \delta'_2(t)\| \leq d_i(\delta_1(t), \delta'_2(t)) \leq \text{length}(P_t) \leq K_0 \text{length}(S_t) \leq K_0 \|\delta_1(t) - \delta_2(t)\|.$$

Therefore,

$$\|\delta_2(t) - \delta'_2(t)\| \leq \|\delta_2(t) - \delta_1(t)\| + \|\delta_1(t) - \delta'_2(t)\| \leq (1 + K_0) \|\delta_1(t) - \delta_2(t)\|.$$

We then obtain

$$\|\delta_1(t) - \delta_2(t)\| \leq d_i(\delta_1(t), \delta'_2(t)) + \|\delta'_2(t) - \delta_2(t)\| \leq (1 + 2K_0) \|\delta_1(t) - \delta_2(t)\|,$$

which implies

$$q_o(\delta_1, \delta_2) = \min(q_i(\delta_1, \delta'_2), q_o(\delta_2, \delta'_2)).$$

On the other hand, we have $d_i(\delta_1(t), \delta_2(t)) \leq d_i(\delta_1(t), \delta'_2(t)) + d_i(\delta'_2(t), \delta_2(t))$, therefore

$$\min(q_i(\delta_1, \delta'_2), q_i(\delta_2, \delta'_2)) \leq q_i(\delta_1, \delta_2).$$

Since $q_i(\delta'_2, \delta_2) = q_o(\delta'_2, \delta_2)$, we then obtain

$$q_o(\delta_1, \delta_2) = \min(q_i(\delta_1, \delta'_2), q_o(\delta_2, \delta'_2)) \leq q_i(\delta_1, \delta_2),$$

which implies $q_o(\delta_1, \delta_2) = q_i(\delta_1, \delta_2)$ since $q_i(\delta_1, \delta_2) \leq q_o(\delta_1, \delta_2)$. Therefore, the pair δ_1, δ_2 satisfies the arc criterion for normal embedding.

Assume now that the path P_t intersects the polar-wedge $A(\alpha)$.

Let us treat first the case for $\alpha > 0$ sufficiently small. Let $A'(\alpha)$ denote the union of components of $A(\alpha)$ which intersect P_t . If t is sufficiently small, we have $\delta_1(t) \notin A'(\alpha)$ by choice of ℓ , so $\sigma_1(t) \notin \ell(A'(\alpha))$ and also $\delta'_2(t) \notin A'(\alpha)$, so $\sigma_2(t) \notin \ell(A'(\alpha))$. Then we can replace the segment S_t by a path S'_t from $\sigma_1(t)$ to $\sigma_2(t)$ such that $\text{length}(S'_t) \leq \pi \text{length}(S_t)$ and such that the lifting of S'_t by ℓ with extremities $\delta_1(t)$ and $\delta'_2(t)$ does not intersect $A(\alpha)$. Then the previous inequalities are modified by a factor of π and we obtain:

$$\|\delta_1(t) - \delta_2(t)\| \leq d_i(\delta_1(t), \delta'_2(t)) + \|\delta'_2(t) - \delta_2(t)\| \leq (1 + 2K_0\pi) \|\delta_1(t) - \delta_2(t)\|,$$

which leads to $q_o(\delta_1, \delta_2) = q_i(\delta_1, \delta_2)$ by the same arguments as before.

Let us now assume that for all $\alpha > 0$, $\delta'_2 \cap A(\alpha) \neq \{0\}$. Let $A_0(\alpha)$ be the component of $A(\alpha)$ containing δ'_2 . Let $\alpha' > \alpha$ and let $\tilde{\sigma}_2$ be a real arc inside $\ell(A_0(\alpha'))$ such that for all t , $\tilde{\sigma}_2(t) \in S_t$ and the strict transform of $\tilde{\sigma}_2(t)$ by ρ'_ℓ does not intersect Δ^* . Then decreasing α if necessary, we can assume $\tilde{\sigma}_2(t) \notin \ell(A_0(\alpha))$. Consider the path β'_t defined as the lifting by ℓ with origin $\delta_2(t)$ of the segment $\beta_t = [\tilde{\sigma}_2(t), \sigma_2(t)]$. Let $\tilde{\delta}_2(t)$ be the extremity of β'_t and let $\tilde{\delta}'_2(t)$ be the point of P_t such that $\ell(\tilde{\delta}'_2(t)) = \tilde{\sigma}_2(t)$. Then $\tilde{\delta}'_2(t)$ and $\tilde{\delta}_2(t)$ are vertically aligned. Since both $\tilde{\delta}'_2(t)$ and $\tilde{\delta}_2(t)$ are outside $A(\alpha)$, we can apply what we just proved and we obtain $q_i(\delta_1, \tilde{\delta}_2) = q_o(\delta_1, \tilde{\delta}_2)$. Call q this number and let s the polar rate of $A_0(\alpha)$ (Definition 6.14).

Since δ_1 and δ_2 have the same tangent semi-line, then σ_2 and $\tilde{\sigma}_2$ also have the same tangent semi-line L . Therefore, we have $d_o(\sigma_2(t), \tilde{\sigma}_2(t)) = \text{length}(\beta_t) = \Theta(t^s)$.

We then obtain:

$$d_i(\tilde{\delta}_2, \delta_2) \leq \text{length}(\beta'_t) \leq K_0 \text{length}(\beta_t) = \Theta(t^s),$$

which implies:

$$d_i(\delta_1, \delta_2) \leq d_i(\delta_1, \tilde{\delta}_2) + d_i(\tilde{\delta}_2, \delta_2) = \Theta(t^q) + \Theta(t^s).$$

On the other hand, we have

$$\Theta(t^q) + \Theta(t^s) = d_o(\sigma_1, \sigma_2) \leq d_o(\delta_1, \delta_2).$$

Since $d_o(\delta_1, \delta_2) \leq d_i(\delta_1, \delta_2)$, we obtain $d_o(\delta_1, \delta_2) = d_i(\delta_1, \delta_2) = \Theta(t^q) + \Theta(t^s)$, and then $q_i(\delta_1, \delta_2) = q_o(\delta_1, \delta_2)$. \square

REMARK 7.3. We proved Proposition 7.2 because it is what we need to prove Proposition 5.3. But with a little more work one can adapt the proof to get a criterion using just one fixed generic projection $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$, if one consider all real arcs δ of $(\mathbb{C}^2, 0)$, even those such that $\Delta^* \cap \delta^* \neq \emptyset$. It is worth noting that the statement of Proposition 7.2 can also be easily improved by reducing the criterion to pairs of vertically aligned real arcs corresponding to three fixed generic projections $\ell_1, \ell_2, \ell_3: (X, 0) \rightarrow (\mathbb{C}^2, 0)$. Indeed, if ℓ_1 is chosen, it suffices to chose ℓ_2 and ℓ_3 so that $(\ell_1(\Pi_2))^* \cap \Delta_1^* = \emptyset$, $(\ell_1(\Pi_3))^* \cap \Delta_1^* = \emptyset$ and $(\ell_1(\Pi_2))^* \cap (\ell_1(\Pi_3))^* = \emptyset$ where $*$ means strict transform by ρ'_{ℓ_i} and where for $i = 1, 2, 3$, Π_i is the polar curve of ℓ_i and $\Delta_i = \ell_i(\Pi_i)$. Then, for any pair of real arcs δ_1 and δ_2 on $(X, 0)$, at least one of ℓ_1, ℓ_2 or ℓ_3 satisfies $\Delta_i^* \cap \ell_i(\delta_1)^* = \emptyset$ and $\Delta_i^* \cap \ell_i(\delta_2)^* = \emptyset$.

8. Partner pairs

Let $(X, 0)$ be a normal surface germ and let $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a generic projection. Let $(\delta, 0)$ and $(\delta', 0)$ be two real arcs in $(\mathbb{C}^2, 0)$ which have parametrizations of the form $\delta(t) = (t, y(t))$ and $\delta'(t) = (t, y'(t))$ in suitable coordinates and which meet the discriminant curve Δ of ℓ only at 0. Let δ_1, δ_2 be a pair of components of the lifting $\ell^{-1}(\delta)$. Let S_t be the segment in F_t joining $\delta(t)$ and $\delta'(t)$. Let $P_{1,t}$ and $P_{2,t}$ be the liftings of S_t by the cover $\ell|_{\hat{F}_t}: \hat{F}_t \rightarrow F_t$ with origins respectively $\delta_1(t)$ and $\delta_2(t)$. Denote their extremities by $\delta'_1(t)$ and $\delta'_2(t)$. This defines a pair δ'_1, δ'_2 of distinct components of $\ell^{-1}(\delta')$.

DEFINITION 8.1. We say that δ'_1, δ'_2 is the *partner pair* of δ_1, δ_2 over δ' .

LEMMA 8.2. Assume that δ_1, δ_2 has a partner pair δ'_1, δ'_2 such that $q_i(\delta'_1, \delta'_2) < q_i(\delta, \delta')$. If the pair δ_1, δ_2 satisfies the arc criterion $q_i(\delta_1, \delta_2) = q_o(\delta_1, \delta_2)$, then δ'_1, δ'_2 also satisfies it.

Proof. We have $d_i(\delta'_1(t), \delta'_2(t)) \leq \text{length}(P_{1,t}) + d_i(\delta_1(t), \delta_2(t)) + \text{length}(P_{2,t})$.

Since $q_i(\delta'_1, \delta'_2) < q_i(\delta, \delta')$ and since for $k = 1, 2$, we have $\text{length}(P_{k,t}) = \Theta(t^{q_i(\delta, \delta')})$, we then obtain $q_i(\delta'_1, \delta'_2) \geq q_i(\delta_1, \delta_2)$.

We also have $d_i(\delta_1(t), \delta_2(t)) \leq \text{length}(P_{1,t}) + d_i(\delta'_1(t), \delta'_2(t)) + \text{length}(P_{2,t})$, which leads to $q_i(\delta'_1, \delta'_2) \leq q_i(\delta_1, \delta_2)$ with the same argument. Therefore $q_i(\delta'_1, \delta'_2) = q_i(\delta_1, \delta_2)$.

The inequality $d_o(\delta_1(t), \delta_2(t)) \leq \text{length}(P_{1,t}) + d_o(\delta'_1(t), \delta'_2(t)) + \text{length}(P_{2,t})$ gives $q_o(\delta_1, \delta_2) \geq q_o(\delta'_1, \delta'_2)$ and $d_o(\delta'_1(t), \delta'_2(t)) \leq \text{length}(P_{1,t}) + d_o(\delta_1(t), \delta_2(t)) + \text{length}(P_{2,t})$ leads to $q_o(\delta'_1, \delta'_2) \geq q_o(\delta_1, \delta_2)$. Therefore $q_o(\delta'_1, \delta'_2) = q_o(\delta_1, \delta_2)$.

Since $q_o(\delta_1, \delta_2) = q_i(\delta_1, \delta_2)$, then $q_o(\delta'_1, \delta'_2) = q_i(\delta'_1, \delta'_2)$ as desired. \square

LEMMA 8.3. *Let $\rho: Y \rightarrow \mathbb{C}^2$ be a sequence of blow-ups of points which resolves the base points of the family of projected polar curves $(\ell(\Pi_{\mathcal{D}}))_{\mathcal{D} \in \Omega}$ and let T_0 be the resolution graph of ρ . Let δ a real arc in $(\mathbb{C}^2, 0)$ whose strict transform intersects $\rho^{-1}(0)$ at a smooth point p and let δ_1, δ_2 be a pair of vertically aligned arcs over δ . Let δ' be another real arc in $(\mathbb{C}^2, 0)$ and let δ'_1, δ'_2 be the partner pair of δ_1, δ_2 over δ' . Let Δ be the discriminant curve of ℓ . We assume that none of δ^* and δ'^* intersects Δ^* and that the pair δ_1, δ_2 satisfies the arc criterion.*

- (i) *If δ'^* passes through p , then δ'_1, δ'_2 also satisfies the arc criterion;*
- (ii) *Let C_j be the component of $\rho^{-1}(0)$ such that $p \in C_j$ and let (ν_1) and (ν_2) be the two vertices of $\widehat{\mathcal{F}}$ such that $\delta_1(t) \in \widehat{F}_{t, \nu_1}$ and $\delta_2(t) \in \widehat{F}_{t, \nu_2}$, so we have $(\mathcal{E} \circ \widehat{\mathcal{C}})(\nu_1) = (\mathcal{E} \circ \widehat{\mathcal{C}})(\nu_2) = (j)$. Let (ν_0) be a vertex of $\widehat{\mathcal{F}}$ on a path from (ν_1) and (ν_2) such that $q_{\nu_1, \nu_2} = q_{\nu_0}$. Assume $q_{\nu_0} < q_j$ and set $(j_0) = (\mathcal{E} \circ \widehat{\mathcal{C}})(\nu_0)$. Let $T'_0 \subset T_0$ be the connected component of $T_0 \setminus (j_0)$ which contains (j) and set $E' = \bigcup_{k \in V(T'_0)} C_k$. If $\delta'^* \cap E' \neq \emptyset$, then δ'_1, δ'_2 also satisfies the arc criterion.*

Proof. (i) Let C_j be the component of $\rho^{-1}(0)$ such that $p \in C_j$. Since δ and δ' both pass through p , then $q_i(\delta, \delta') > q_j$. By Lemma 6.18, we have $q_i(\delta'_1, \delta'_2) = q_j$. Therefore $q_i(\delta, \delta') > q_i(\delta'_1, \delta'_2)$ and we then get (i) by applying Lemma 8.2.

(ii) By composing ρ with an additional sequence of blow-ups of points, we can assume that δ'^* intersects $\rho^{-1}(0)$ at a smooth point. Let $C_{j'}$ be the component of E' such that $\delta'^* \cap C_{j'} \neq \emptyset$. Then we have $q_{j_0} < q_{j'}$. By assumption we also have $q_{j_0} < q_j$. Therefore, by Lemma 6.10, we get $q_i(\delta, \delta') = q_{j, j'} > q_{j_0}$ since (j) and (j') are in the same connected component of $T_0 \setminus (j_0)$. On the other hand, by Lemma 6.18, $q_i(\delta_1, \delta_2) = q_{\nu_1, \nu_2} = q_{\nu_0} = q_{j_0}$. We then have $q_i(\delta, \delta') > q_i(\delta_1, \delta_2)$ and we conclude again by applying Lemma 8.2. \square

9. LNE along strings

Let $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a generic projection and let $\rho: Y \rightarrow \mathbb{C}^2$ be a sequence of blow-ups of points which resolves the base points of the family of projected polar curves $(\ell(\Pi_{\mathcal{D}}))_{\mathcal{D} \in \Omega}$. Let E' be the union of components of $\rho^{-1}(0)$ which are not Δ -curves. Let $(\delta, 0) \subset (\mathbb{C}^2, 0)$ be a real arc such that $\delta^* \cap E' \neq \emptyset$ and such that δ^* intersects $\rho^{-1}(0)$ at a smooth point.

LEMMA 9.1. *Let C be the component of $\rho^{-1}(0)$ such that $C \cap \delta^* \neq \emptyset$ and let q_C be its inner rate. Let δ_1 and δ_2 be two real arc components of $\ell^{-1}(\delta)$ and consider the two points $p_1 = \delta_1^* \cap \mathcal{N}^{-1}(0)$ and $p_2 = \delta_2^* \cap \mathcal{N}^{-1}(0)$ where $*$ means strict transform by the Nash modification \mathcal{N} . Assume that $q_i(\delta_1, \delta_2) = q_C$.*

Then the pair of arcs (δ_1, δ_2) satisfies the arc criterion $q_i(\delta_1, \delta_2) = q_o(\delta_1, \delta_2)$ if and only if $\tilde{\lambda}(p_1) \neq \tilde{\lambda}(p_2)$, where $\tilde{\lambda}$ denotes the lifted Gauss map (Definition 6.11).

Proof. By (i) of Lemma 8.3, it suffices to prove the result when δ is the real slice of a complex curve γ whose strict transform by ρ_ℓ is a curvette of C . We may assume that our coordinates are (x, y) and that δ is parametrized by $\delta(t) = (t, y(t))$. By Lemma 6.18, the assumption $q_i(\delta_1, \delta_2) = q_C$ is equivalent to asking that in the fiber-graph $\widehat{\mathcal{F}}$, the two vertices (ν_1) and (ν_2) such that $\delta_i(t) \in \widehat{F}_{t, \nu_i}$ are joined by a path along which inner rates are $\geq q_C$.

Set $q = q_C$. Then γ has a Puiseux expansion of the form

$$y = \sum_{i=1}^k a_i x^{p_i} + ax^q(1 + x^{q''}b(x)),$$

with $a, a_i \in \mathbb{C}^*$, $1 \leq p_1 < p_2 < \dots < p_k < q$, $1 < q''$ and where the higher order terms $ax^{q+q''}b(x) \in \mathbb{C}\{x^{1/n}\}$ contain only non essential exponents. Let q_1 and q_2 be two rational numbers such that $q_1 < q < q_2$ and such that any branch of the discriminant curve Δ of ℓ with Puiseux expansion of the form $y = \sum_{i=1}^k a_i x^{p_i} + h.o.$, where *h.o.* means higher order terms, satisfies the following property: the first exponent $> p_k$ is not inside the interval $[q_1, q_2]$. Set $\alpha(x) = \sum_{i=1}^k a_i x^{p_i}$. This is equivalent to asking that the strict transforms by ρ_ℓ of the curves with Puiseux expansion $y = \alpha(x) + x^{q'}$ with $q' \in [q_1, q_2]$ intersects $\rho_\ell^{-1}(0)$ along a union of curves $C_{u_1} \cup \dots \cup C_{u_r}$ in E' which corresponds to a string $(u_1) - (u_2) - \dots - (u_r)$ in the resolution tree of ρ_ℓ . Then δ is contained in the real 3-dimensional semialgebraic germ \mathcal{A} with boundary defined by:

$$\mathcal{A} = \{(t, y) \in \mathbb{R}^+ \times \mathbb{C} : t^{q_2} \leq |y - \alpha(t)| \leq t^{q_1}\}.$$

If n is the multiplicity of γ then \mathcal{A} is the union of n real 3-dimensional semialgebraic germs with boundary which are pairwise disjoint outside 0. We write $(A, 0)$ for the one containing δ .

Let m be the multiplicity of $(X, 0)$. Then $\ell^{-1}(A)$ consists of m semialgebraic germs $(A^{(i)}, 0)$, $i = 1, \dots, m$, which are pairwise disjoint outside 0. For each i , the restriction $\ell: A^{(i)} \rightarrow A$ is an inner bilipschitz homeomorphism by the Polar Wedge Lemma 6.15. Let $\ell'_i: (t, y) \mapsto \ell'_i(t, y) \in A^{(i)}$ be the inverse map of $\ell|_{A^{(i)}}$. Assume $\delta_1 \subset A^{(1)}$ and $\delta_2 \subset A^{(2)}$.

For $s \in \mathbb{C}$ such that $t^{q_2} \leq |s| \leq t^{q_1}$, consider the function $g: A \rightarrow (0, +\infty)$ defined by

$$g(t, s) = \|\ell'_1(t, \alpha(t) + as) - \ell'_2(t, \alpha(t) + as)\|.$$

Set $s_0 = t^q(1 + t^{q''}b(t))$. We have in particular:

$$g(t, s_0) = \|\delta_1(t) - \delta_2(t)\|$$

Let us give an estimate of $g(t, s_0)$. Fix a small $t \in \mathbb{R}^+$ and let us write the Taylor formula for $s \mapsto g(t, s)$ at the point s_0 . We get for $s \in \mathbb{C}$ such that $t^{q_2} \leq |s| \leq t^{q_1}$:

$$g(t, s_0) - g(t, s) = (s_0 - s)a_1(t) + (s_0 - s)^2 a_2(t) + (s_0 - s)^3 a_3(t) + \dots,$$

where $a_1(t)$ equals the distance between the lines $L_t^{(1)} = T_{\delta_1(t)}X \cap \{x = t, y \in \mathbb{R}\}$ and $L_t^{(2)} = T_{\delta_2(t)} \cap \{x = t, y \in \mathbb{R}\}$ in the Grassmanian $\mathbf{G}(1, \mathbb{R}^{2n})$.

Set $s = t^{q'}$ where $q < q' \leq q_2$. Then $s_0 - t^{q'} = t^q(1 + t^{q''}b(t)) - t^{q'} = \Theta(t^q)$. We then have:

$$g(t, s_0) - g(t, t^{q'}) = \Theta(t^q)a_1(t) + \Theta(t^{2q})a_2(t) + \Theta(t^{3q})a_3(t) + \dots, \quad (1)$$

By Lemma 6.18, we have $q_i(\delta_1, \delta_2) = q$ and

$$d_i(\ell'_1(t, \alpha(t) + at^{q'}), \ell'_2(t, \alpha(t) + at^{q'})) = \Theta(t^{q'}).$$

The latter equality implies:

$$\begin{aligned} g(t, t^{q'}) &= \|\ell'_1(t, \alpha(t) + at^{q'}) - \ell'_2(t, \alpha(t) + at^{q'})\| \\ &\leq d_i(\ell'_1(t, \alpha(t) + at^{q'}), \ell'_2(t, \alpha(t) + at^{q'})) = \Theta(t^{q'}) \end{aligned}$$

Then dividing the equality (1) by $d_i(\delta_1(t) - \delta_2(t)) = \Theta(t^q)$, we get:

$$\frac{\|\delta_1(t) - \delta_2(t)\|}{d_i(\delta_1(t) - \delta_2(t))} = \Theta(a_1(t)).$$

Set $P_1 = \tilde{\lambda}(p_1)$ and $P_2 = \tilde{\lambda}(p_2)$. As t tends to zero, $a_1(t)$ tends to the distance between the two real lines $L_1 = P_1 \cap \{x = t, y \in \mathbb{R}\}$ and $L_2 = P_2 \cap \{x = t, y \in \mathbb{R}\}$. If $P_1 = P_2$, we then have $L_1 = L_2$ so $\lim_{t \rightarrow 0} a_1(t) = 0$ and we get $q_i(\delta_1, \delta_2) > q_0(\delta_1, \delta_2)$.

Since the components of $\ell^{-1}(A)$ which contain $A^{(1)}$ and $A^{(2)}$ are tangent to the same line L , then by Whitney's Lemma [22, Theorem 22.1], we have $L \subset P_1$ and $L \subset P_2$. Therefore, if $P_1 \neq P_2$, then we also have $L_1 \neq L_2$, which means that $\lim_{t \rightarrow 0} a_1(t) \neq 0$. In that case, we then have $q_i(\delta_1, \delta_2) = q_0(\delta_1, \delta_2)$. \square

10. Proof of Proposition 5.3

DEFINITION 10.1. A \mathcal{P} -node of the fiber-graph $\widehat{\mathcal{F}}$ is a vertex (ν) such that the image of $\widehat{\mathcal{E}}(\nu)$ by the injection $\mathcal{I}: V(\mathcal{G}) \rightarrow V(G)$ is a \mathcal{P} -node of the resolution graph G of π_ℓ (see Section 6.6). Equivalently, (ν) is a \mathcal{P} -node of $\widehat{\mathcal{F}}$ if and only if for all $t \in (0, \eta)$, $\Pi \cap \widehat{F}_{\nu,t} \neq \emptyset$, where Π denotes the polar curve of a generic projection of $(X, 0)$.

The proof will use the relation between the \mathcal{P} -nodes of $\widehat{\mathcal{F}}$ and the lifted Gauss map. Let us explain this first.

Let (ν) be a vertex of $\widehat{\mathcal{F}}$ and let δ be a real arc on $(X, 0)$ such that for all $t \in (0, \eta)$, $\delta(t) \in \widehat{F}_{\nu,t}$. If (ν) is not a \mathcal{P} -node, then the intersection point of the strict transform δ^* of δ by the Nash modification of $(X, 0)$ is a point p which does not depend on the choice of δ . We set $\tilde{\lambda}(\nu) := \tilde{\lambda}(p)$, where $\tilde{\lambda}$ is the lifted Gauss map.

Let now (ν_1) and (ν_2) be two vertices of $\widehat{\mathcal{F}}$ which are in the same connected component of $\widehat{\mathcal{F}}$ minus its \mathcal{P} -nodes and let δ_1 and δ_2 be two real arcs such that for all $t \in (0, \eta)$, $\delta_1(t) \in \widehat{F}_{\nu_1,t}$ and $\delta_2(t) \in \widehat{F}_{\nu_2,t}$. Then the strict transforms of δ_1 and δ_2 by the Nash modification \mathcal{N} of $(X, 0)$ intersect the exceptional divisor at the same point p , and we then have $\tilde{\lambda}(\nu_1) = \tilde{\lambda}(\nu_2)$.

Proof. The ‘‘only if’’ direction of Theorem 5.3 is a direct consequence of Theorem 4.5.

Let us prove the ‘‘if’’ direction. We assume that for all generic projections $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$, any pair of components δ_1, δ_2 over any test arc δ of ℓ satisfies the arc criterion $q_i(\delta_1, \delta_2) = q_o(\delta_1, \delta_2)$.

Let $\delta: [0, \eta) \rightarrow \mathbb{C}^2$ be a real arc such that $\Delta^* \cap \delta^* = \emptyset$ where $*$ means strict transform by ρ_ℓ , and let δ_1 and δ_2 be two components of $\ell^{-1}(\delta)$. We assume that δ is not a test arc. By Proposition 7.2, we have to prove that $q_i(\delta_1, \delta_2) = q_o(\delta_1, \delta_2)$.

Assume first that δ^* intersects the component C_j of $(\rho_\ell)^{-1}(0)$ at a smooth point p of $(\rho_\ell)^{-1}(0)$. Let δ' be a test arc passing through p and let δ'_1, δ'_2 be the partner pair of δ_1, δ_2 over δ' . Since δ' is a test arc, the pair δ'_1, δ'_2 satisfies the arc criterion, then applying (i) of Lemma 8.3, we obtain that the pair δ_1, δ_2 also satisfies it.

Assume now that δ^* intersects $(\rho_\ell)^{-1}(0)$ at an intersection point $p = C_{j'} \cap C_{j''}$. Let us compose ρ_ℓ with an additional sequence of blow-ups of points α so that δ^* intersects $(\rho_\ell \circ \alpha)^{-1}(0)$ at a smooth point and let C_j be the component of $(\rho_\ell \circ \alpha)^{-1}(0)$ which intersects δ^* . In the tree T , this replaces the edge between (j') and (j'') by a string $\mathcal{S} = (j') - \dots - (j) - \dots - (j'')$. We assume $q_{j'} < q_{j''}$. Let δ' and δ'' be test arcs at (j') resp. (j'') . Let δ'_1, δ'_2 be the partner pair of δ_1, δ_2 over δ' , and the same for δ''_1, δ''_2 over δ'' . For $i = 1, 2$, let (ν_i) , (ν'_i) and (ν''_i) be the vertices of $\widehat{\mathcal{F}}$ such that $\delta_i(t) \in \widehat{F}_{t,\nu_i}$, $\delta'_i(t) \in \widehat{F}_{t,\nu'_i}$ and $\delta''_i(t) \in \widehat{F}_{t,\nu''_i}$.

The string \mathcal{S} lifts to two strings \mathcal{S}_1 and \mathcal{S}_2 in $\widehat{\mathcal{F}}$ with extremities respectively (ν'_1) and (ν''_1) and (ν'_2) and (ν''_2) .

Case 1. Assume that (ν'_1) and (ν''_2) can be joined by a path p along which inner rates are $\geq q_{j''}$. We then have $q_{\nu''_1, \nu''_2} = q_{j''}$ and also $q_{\nu'_1, \nu'_2} = q_{j'}$ since one obtains a simple path from (ν'_1) to (ν'_2) by appending \mathcal{S}_1 and \mathcal{S}_2 to p . Since there are no adjacent Δ -nodes in the tree T , one of (j') or (j'') , say (j'') , is not a Δ -node (the arguments will be the same when (j'') is a Δ -node and not (j')). Since $q_{\nu''_1, \nu''_2} = q_{j''}$ and since the pair (δ''_1, δ''_2) satisfies the arc criterion, then Lemma 9.1 implies that $\tilde{\lambda}(\nu''_1) \neq \tilde{\lambda}(\nu''_2)$. Since the vertices (ν_i) and (ν''_i) are in the same connected component of $\widehat{\mathcal{F}}$ minus its \mathcal{P} -nodes, then we have $\tilde{\lambda}(\nu'_1) = \tilde{\lambda}(\nu_1)$ and $\tilde{\lambda}(\nu'_2) = \tilde{\lambda}(\nu_2)$. Therefore, $\tilde{\lambda}(\nu_1) \neq \tilde{\lambda}(\nu_2)$. Applying again Lemma 9.1 in the converse direction, we obtain that δ_1, δ_2 satisfies the arc criterion.

Case 2. Assume that (ν'_1) and (ν''_2) cannot be joined by a path p along which inner rates are $\geq q_{j''}$. Let (ν_0) be a vertex of $\widehat{\mathcal{F}}$ on a simple path from (ν'_1) to (ν'_2) such that $q_{\nu'_1, \nu'_2} = q_{\nu_0}$ and set $(j_0) = (\mathcal{E} \circ \widehat{\mathcal{C}})(\nu_0)$. Since any path from (ν'_1) to (ν''_2) is obtained by appending the strings \mathcal{S}_1 and \mathcal{S}_2 to a path from (ν_1) to (ν_2) , we have $q_{\nu''_1, \nu''_2} = q_{\nu'_1, \nu'_2} = q_{j_0} < j''$. Then we can apply (ii) of Lemma 8.3 to the vertex (j'') : since the pair (δ''_1, δ''_2) satisfies the arc criterion then the pair (δ_1, δ_2) also satisfies it. \square

11. Enhanced Proposition 5.3

DEFINITION 11.1. Let $\ell: (X, 0): (\mathbb{C}^2, 0)$ be a generic projection and let δ be a real test arc for ℓ . A component $\widehat{\delta}$ of $\ell^{-1}(\delta)$ is *principal* if it is a real slice of a principal component (Definition 3.6) of $\ell^{-1}(\gamma)$. A *nodal test arc* is a test arc δ which is a real slice of a nodal test curve (Definition 3.4).

PROPOSITION 11.2 (Enhanced Proposition 5.3). *A normal surface $(X, 0)$ is LNE if and only if for all generic projections $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ and for all nodal test arcs δ for ℓ , any pair of principal components δ_1, δ_2 of $\ell^{-1}(\delta)$ satisfies the arc criterion $q_i(\delta_1, \delta_2) = q_o(\delta_1, \delta_2)$.*

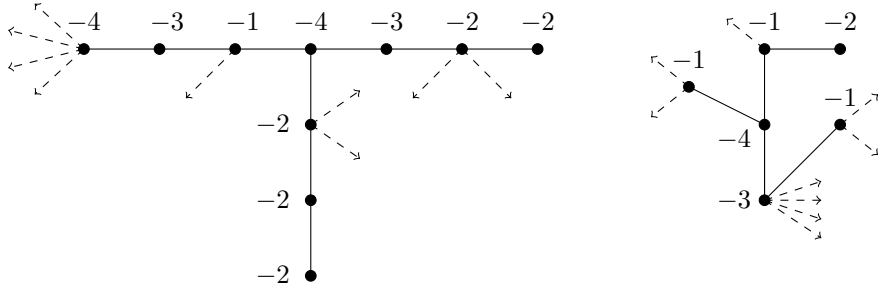
DEFINITION 11.3. An \mathcal{L} -node of the fiber-graph $\widehat{\mathcal{F}}$ is a vertex (ν) such that the image of $\widehat{\mathcal{E}}(\nu)$ by the injection $\mathcal{I}: V(\mathcal{G}) \rightarrow V(G)$ is a \mathcal{L} -node of the resolution graph G of π_ℓ (see Section 6.6). Equivalently, (ν) is an \mathcal{L} -node of $\widehat{\mathcal{F}}$ if and only if for all $t \in (0, \eta)$, $(H \cap X) \cap \widehat{\mathcal{F}}_{\nu, t} \neq \emptyset$ where H is a hyperplane section of X .

The proof of Proposition 11.2 will use the following characterization of principal component in terms of fiber-graph and which follows immediately from the definitions.

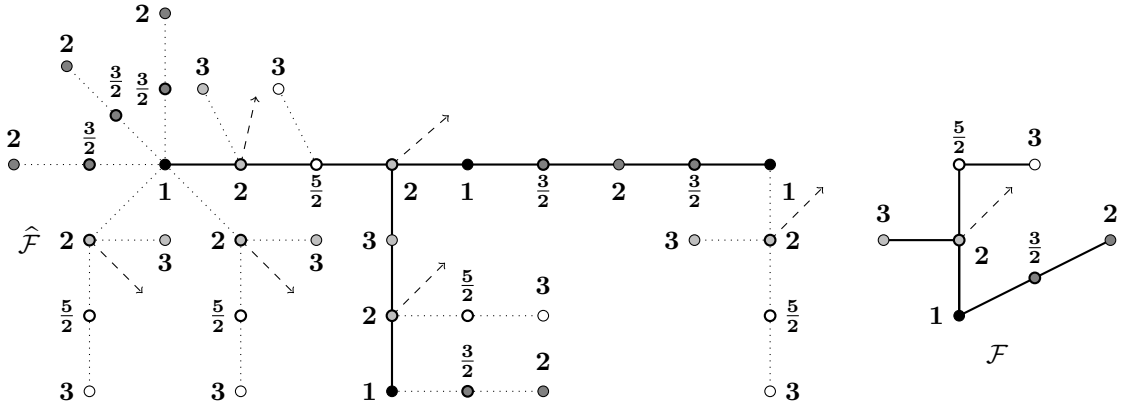
LEMMA 11.4. *Consider the subgraph $\widehat{\mathcal{F}}'$ of $\widehat{\mathcal{F}}$ defined as the union of all simple paths in $\widehat{\mathcal{F}}$ connecting pairs of vertices among \mathcal{L} - and \mathcal{P} -nodes. Let δ be a test arc. A component $\widehat{\delta}$ of $\ell^{-1}(\delta)$ is principal if and only if the vertex (ν) of $\widehat{\mathcal{F}}$ such that $\delta(t) \in \widehat{F}_{\nu, t}$ is a vertex of $\widehat{\mathcal{F}}'$.*

In the twin paper [17], we prove that any minimal surface singularity is LNE by using Theorem 3.8. In the following example, we illustrate the gain by using Proposition 11.2 instead of Proposition 5.3 on a specific minimal singularity.

EXAMPLE 11.5. Let $(X, 0)$ be a minimal surface singularity with dual resolution graph Γ given in Example 1 of [1] and Example 5.6 of [2]. In the picture below we have to the left the dual graph of the minimal resolution of $(X, 0)$ that factors through both the blow-up and the Nash modification, and on the right the resolution tree of the discriminant. The arrows in the left graph indicates the strict transform of the polar and on the right the strict transform of the discriminant.



There are infinitely many minimal singularities with Γ as its dual resolution graph, but as is shown in [17] they all have the same bilipschitz geometry. The discriminant of $(X, 0)$ has equation $(x^4 + y^4)(x^2 + y^5)(x + y^2 + iy^3)(x + y^2 - iy^3)(y^2 + x^4) = 0$ (Example 5.6 of [2]). Notice that the graph on the right above is the graph of the resolution ρ'_ℓ . To get the graph of ρ_ℓ we have to blow up all edges between Δ -nodes. In this case the only edge between Δ -nodes is the edge between the root vertex and adjacent vertex with weight -1 . Hence in the graph above we have blown up this edge to create a separation-node. In the following picture, the left graph is $\widehat{\mathcal{F}}$ and the right graph is \mathcal{F} . The graph-map $\widehat{\mathcal{C}}$ preserves the shape and colour of a vertex, i.e., $\widehat{\mathcal{C}}$ of a white vertex with thick boundary in $\widehat{\mathcal{F}}$ is the white vertex with thick boundary in \mathcal{F} and so on. All the vertices of \mathcal{F} but the white one with thin boundary are nodes. The subgraph of $\widehat{\mathcal{F}}$ consisting of the vertices connected by the thick edges is $\widehat{\mathcal{F}}'$. The graph $\widehat{\mathcal{F}}$ has 36 vertices while $\widehat{\mathcal{F}}'$ has only 12 vertices. Consider a test arc δ at the central vertex of \mathcal{F} with inner rate 2 (its strict transform is represented by an arrow) on the picture. Its lifting by ℓ has 6 components, whose strict transform are represented by the 6 arrows in $\widehat{\mathcal{F}}$. Therefore, to prove LNE by using Proposition 5.3, we would have to test the arc criterion on each of the 15 pairs of arcs among these components. Now, only 3 of the three arrows are attached to $\widehat{\mathcal{F}}'$, i.e., correspond to principal components of $\ell^{-1}(\delta)$. Therefore, to prove LNE by using Proposition 11.2 instead of Proposition 5.3, we just have to test the criterion on pairs of arcs among these 3 principal components, so we just have to test 3 pairs of arcs instead of 15.



Notice that each component of the complementary subgraph $\widehat{\mathcal{F}} \setminus \widehat{\mathcal{F}}'$ is a rooted tree oriented from its root by strictly increasing inner rates and attached to $\widehat{\mathcal{F}}'$ by a single edge adjacent to its root.

The proof of Proposition 11.2 needs the following key Lemma.

LEMMA 11.6. *Let (j') be a vertex of T . Let T' be a maximal connected subgraph of T whose vertices $(k) \neq (j')$ satisfy that the simple path from (k) to the root vertex passes through (j') . Let $\widehat{\mathcal{F}}_1$ and $\widehat{\mathcal{F}}_2$ be two distinct components of $\widehat{\mathcal{F}} \setminus (\mathcal{E} \circ \widehat{\mathcal{C}})^{-1}(j')$ with $(\mathcal{E} \circ \widehat{\mathcal{C}})(\widehat{\mathcal{F}}_1) = (\mathcal{E} \circ \widehat{\mathcal{C}})(\widehat{\mathcal{F}}_2) = T'$.*

Assume that for all pairs of vertices (ν'_1) and (ν'_2) adjacent to $\widehat{\mathcal{F}}_1$ and $\widehat{\mathcal{F}}_2$ respectively, all generic projections $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ and all test arcs δ' at (j') , any pair of components δ'_1, δ'_2 over δ' with $\delta'_i(t) \in \widehat{F}_{t, \nu'_i}$ satisfies the arc criterion. Then for all generic projection $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ and for all test arcs δ at a vertex (j) of T' , any pair of arcs δ_1, δ_2 over δ ~~with $\delta_i(t) \in \widehat{\mathcal{F}}_{t, \nu_i}$~~ ~~with $(\nu_i) \in V(\widehat{\mathcal{F}}_i)$~~ satisfies the arc criterion.

Proof. Let $\widehat{\mathcal{F}}_1$ and $\widehat{\mathcal{F}}_2$ be the two components of $\widehat{\mathcal{F}} \setminus (\mathcal{E} \circ \widehat{\mathcal{C}})^{-1}(j')$ which contain (ν_1) and (ν_2) respectively. First, notice that if (ν'_1) and (ν''_1) are two vertices of $(\mathcal{E} \circ \widehat{\mathcal{C}})^{-1}(j')$ adjacent to $\widehat{\mathcal{F}}_1$, then there is a simple path inside $\widehat{\mathcal{F}}_1$ from (ν'_1) and (ν''_1) and this path then has its inner rates all $\geq q_{j'}$. The same is true for two vertices adjacent to $\widehat{\mathcal{F}}_2$ with a path inside $\widehat{\mathcal{F}}_2$. Therefore, if (ν'_1) and (ν'_2) are two vertices of $(\mathcal{E} \circ \widehat{\mathcal{C}})^{-1}(j')$ adjacent respectively to $\widehat{\mathcal{F}}_1$ and $\widehat{\mathcal{F}}_2$, then $q_{\nu'_1, \nu'_2}$ does not depend on the choice of (ν'_1) and (ν'_2) . We set $q = q_{\nu'_1, \nu'_2}$.

Notice that when $q < q_{j'}$, the proof of the lemma is a direct application of (ii) of Lemma 8.3. We now give the proof in all cases.

We have $q_{\nu_1} = q_{\nu_2} = q_j > q_{j'}$ and for $i = 1, 2$, any simple path from (ν_i) to a vertex of $(\mathcal{E} \circ \widehat{\mathcal{C}})^{-1}(j')$ has strictly decreasing rates. Moreover, any simple path p from (ν_1) to (ν_2) in $\widehat{\mathcal{F}}$ must go through a vertex (ν'_1) (ν_1) adjacent to $\widehat{\mathcal{F}}_1$ and then to a vertex (ν'_2) (ν_2) adjacent to $\widehat{\mathcal{F}}_2$ (maybe $(\nu'_1) = (\nu'_2)$ ($\nu_1 = \nu_2$)). This implies that $q_{\nu_1, \nu_2} = q$ in the graph $\widehat{\mathcal{F}}$.

Let δ' be a test curve at (j') , let δ'_1, δ'_2 be the partner pair of δ_1, δ_2 over δ' and let (ν'_1) and (ν'_2) be the two vertices of $(\mathcal{E} \circ \widehat{\mathcal{C}})^{-1}(j')$ such that $\delta'_1(t) \in \widehat{F}_{t, \nu'_1}$ and $\delta'_2(t) \in \widehat{F}_{t, \nu'_2}$. By Lemma 6.18 we therefore obtain $q_i(\delta'_1, \delta'_2) = q_{\nu'_1, \nu'_2} = q_{\nu_1, \nu_2} = q_i(\delta_1, \delta_2)$, so $q_i(\delta_1, \delta_2) = q$.

Let us now prove that $q_o(\delta'_1, \delta'_2) = q$.[†]

[†]Here misprint : this should be $q_o(\delta_1, \delta_2) = q$

Choose coordinates of \mathbb{C}^2 so that $\ell = (x, y)$ and $F_t = \{x = t\}$ with $t \in \mathbb{R}^+$. For $i = 1, 2$, let $\widehat{S}_{t,i}$ be the surface inside \widehat{F}_t defined as $\bigcup_{v \in V(\widehat{F}_i)} \widehat{F}_{t,v}$ union the intermediate annuli corresponding to the edges of \widehat{F}_i and the edges joining \widehat{F}_i to vertices of $(\mathcal{E} \circ \widehat{\mathcal{C}})^{-1}(j')$. The two surfaces $\widehat{S}_{t,1}$ and $\widehat{S}_{t,2}$ are disjoint. By hypothesis, any test curve δ' at $(\mathcal{E} \circ \widehat{\mathcal{C}})(\nu'_1)$ satisfies the arc criterion, i.e., for any pair of components δ_1, δ_2 of $\ell^{-1}(\delta')$, we have $q_o(\delta'_1, \delta'_2) = q_i(\delta'_1, \delta'_2) = q$. In particular, this holds for any test curve δ' such that $\delta'(t) \in \partial S_t$, where S_t is the disc in F_t defined by $S_t = \ell(\widehat{S}_{t,1}) = \ell(\widehat{S}_{t,2})$ and ∂ means boundary. Then, using the same arguments as in the proof of Proposition 7.2, we obtain that the outer distance between the boundaries $\partial \widehat{S}_{t,1}$ and $\partial \widehat{S}_{t,2}$ is a $\Theta(t^q)$. Therefore, the images $\ell'(\widehat{S}_{t,1})$ and $\ell'(\widehat{S}_{t,2})$ by a generic projection $\ell': (X, 0) \rightarrow (\mathbb{C}^2, 0)$ given by $\ell' = (x, z)$ consist of two discs of diameter $\Theta(t^q)$ at distance $\Theta(t^q)$ from each other inside $\ell'(F_{\nu',t}) \subset \{x = t\}$. Since $\ell'(\delta_1) \subset \ell'(\widehat{S}_{t,1})$ and $\ell'(\delta_2) \subset \ell'(\widehat{S}_{t,2})$, we then have $q_o(\ell'(\delta_1), \ell'(\delta_2)) = q$.

Let γ be a plane curve germ such that δ is a real slice of γ and take for $\ell': (X, 0) \rightarrow (\mathbb{C}^2, 0)$ a projection which is also generic for the complex curve $\gamma_0 = \ell^{-1}(\gamma)$. Then the restriction $\ell'|_{\gamma_0}: \gamma_0 \rightarrow \ell'(\gamma_0)$ is a bilipschitz homeomorphism for the outer metric (Theorem 2.2). In particular, we have $q_o(\delta_1, \delta_2) = q_o(\ell'(\delta_1), \ell'(\delta_2))$. Therefore $q_o(\delta_1, \delta_2) = q$.

Summarizing, we then obtain $q_o(\delta_1, \delta_2) = q = q_i(\delta_1, \delta_2) = q_o(\delta'_1, \delta'_2) = q = q_i(\delta'_1, \delta'_2)$, proving the lemma. \square

Proof of Proposition 11.2. The “only” if direction is a direct consequence of Proposition 5.3. Hence we only need to prove the “if” direction. Assume that for all generic projections ℓ , any pair of principal components over a nodal test arc for ℓ satisfies the arc criterion. By Proposition 5.3 we just need to show that this implies that for all test arcs δ for ℓ , any pair of components δ_1, δ_2 of $\ell^{-1}(\delta)$ also satisfies the arc criterion $q_i(\delta_1, \delta_2) = q_o(\delta_1, \delta_2)$.

By assumption, any pair of principal components over a nodal test arc satisfies the arc criterion. Hence we first assume that δ is a test curve at a vertex (j) of T which is not a node, and that δ_1 and δ_2 are principal components. For $i = 1, 2$, let (ν_i) be the vertex of $\widehat{\mathcal{F}}$ such that $\delta_i \cap \widehat{F}_{t,\nu_i} \neq \emptyset$. Let \mathcal{S} be the string in T minus the nodes which contains (j). Since $\ell^{-1}(\delta)$ contains a principal component, then the two vertices (j') and (j'') adjacent to \mathcal{S} are nodes. We assume $q_{j'} < q_j < q_{j''}$. Let \mathcal{S}_1 and \mathcal{S}_2 be the two strings in $\widehat{\mathcal{F}}$ containing respectively (ν_1) and (ν_2) and such that $(\mathcal{E} \circ \widehat{\mathcal{C}})(\mathcal{S}_i) = \mathcal{S}$. For each $i = 1, 2$, let (ν'_i) and (ν''_i) be the two nodes of $\widehat{\mathcal{F}}$ adjacent to \mathcal{S}_i which map respectively to (j') and (j'') by $\mathcal{C} \circ \widehat{\mathcal{E}}$ (we may have $\nu'_1 = \nu'_2$ or $\nu''_1 = \nu''_2$).

Assume first that $q_{\nu_1, \nu_2} = q_j$. Since the strings \mathcal{S}_1 and \mathcal{S}_2 are disjoint except perhaps at their extremities, this means that there is a simple path from (ν'_1) to (ν''_2) with rates $\geq q_{j''}$ and that $q_{\nu'_1, \nu'_2} = q_{j'}$ and $q_{\nu''_1, \nu''_2} = q_{j''}$. By definition of ρ_ℓ , at least one of (j') and (j'') is not a Δ -node. Assume that (j') is not a Δ -node. Let δ' be a test curve at (j') and let (δ'_1, δ'_2) be the partner pair of (δ_1, δ_2) over δ' . Since δ_1 and δ_2 are principal it follows that δ'_1 and δ'_2 are principal and hence by hypothesis (δ'_1, δ'_2) satisfies the arc criterion. Then, by Lemma 9.1, the values of the Gauss map $\tilde{\lambda}$ at (ν'_1) and (ν'_2) are distinct. Since for each $i = 1, 2$, the vertices (ν'_i) and (ν_i) are in the same connected component of $\widehat{\mathcal{F}}$ minus its \mathcal{P} -nodes, then they correspond to the same value of the Gauss-map. Summarizing, we obtain that $\tilde{\lambda}$ has distinct values at (ν_1) and (ν_2) . Applying Lemma 9.1 in the other direction, we obtain that (δ_1, δ_2) satisfies the arc criterion. When (j') is a Δ -node, then (j'') is not a Δ -node and one proves that (δ_1, δ_2) satisfies the arc criterion by the same arguments as in the previous case.

Assume now that $q_{\nu_1, \nu_2} < q_j$, so we have $q_{\nu_1, \nu_2} \leq q_{j'}$. Since (j) and (j'') are in the same connected component of $T \setminus (j')$ and since the pair (δ'_1, δ'_2) satisfies the arc criterion, then (δ_1, δ_2) also satisfies the arc criterion by Lemma 8.3 (ii).

Since we have proved that all pairs of principal components satisfy the arc criterion, we must prove that if one of δ_1 and δ_2 , say δ_1 , is not principal, then the pair δ_1, δ_2 also satisfies

$q_i(\delta_1, \delta_2) = q_o(\delta_1, \delta_2)$. Let (ν_i) be the vertex of $\widehat{\mathcal{F}}$ such that $\delta_i \cap \widehat{F}_{t, \nu_i} \neq \emptyset$. Since δ_1 is not principal, (ν_1) is in the complementary subgraph $\widehat{\mathcal{F}} \setminus \widehat{\mathcal{F}}'$. Let $\widehat{\mathcal{F}}_1$ be connected component of $\widehat{\mathcal{F}} \setminus \widehat{\mathcal{F}}'$ containing (ν_1) . It is a tree and there is a unique vertex (ν'_1) of $\widehat{\mathcal{F}}'$ adjacent to $\widehat{\mathcal{F}}_1$. Then (ν'_1) is a node of $\widehat{\mathcal{F}}$ and the vertex $(j') = (\mathcal{C} \circ \widehat{\mathcal{E}})(\nu'_1)$ is a node of T . We use again the notations introduced in the proof of Lemma 11.6: let $\widehat{S}_{t,1}$ be the part of \widehat{F}_t associated with $\widehat{\mathcal{F}}_1$ and set $S_t = \ell(\widehat{S}_{t,1})$. Let Π be the polar curve of ℓ . Since $\Pi \cap \widehat{S}_{t,1} = \emptyset$, the restriction $\ell|_{\widehat{S}_{t,1}} : \widehat{S}_{t,1} \rightarrow S_t$ is a regular connected covering over a disc, and hence, an isomorphism. This implies that (ν_2) is not a vertex of $\widehat{\mathcal{F}}_1$, so it is in a connected component $\widehat{\mathcal{F}}_2$ of $\widehat{\mathcal{F}} \setminus (\mathcal{E} \circ \widehat{\mathcal{C}})^{-1}(j')$ distinct from $\widehat{\mathcal{F}}_1$.

Assume first that there is a vertex (ν'_2) adjacent to $\widehat{\mathcal{F}}_2$ which is in $\widehat{\mathcal{F}}'$. Then all vertices adjacent to $\widehat{\mathcal{F}}_2$ also are in $\widehat{\mathcal{F}}'$. Since (ν'_1) is in $\widehat{\mathcal{F}}'$, then all vertices adjacent to $\widehat{\mathcal{F}}_1$ also are in $\widehat{\mathcal{F}}'$. Therefore, by hypothesis, any pair of components δ'_1, δ'_2 at such vertices over any test curve at (j') satisfies the arc criterion. Then, applying Lemma 11.6 we obtain that (δ_1, δ_2) also satisfies the arc criterion.

Assume now that all vertices (ν'_2) adjacent to $\widehat{\mathcal{F}}_2$ are in $\widehat{\mathcal{F}} \setminus \widehat{\mathcal{F}}'$, i.e., δ'_2 is not principal. The component $\widetilde{\mathcal{F}}_2$ of $\widehat{\mathcal{F}} \setminus \widehat{\mathcal{F}}'$ containing (ν_2) is a tree and there is a unique vertex (ν''_2) of $\widehat{\mathcal{F}}'$ adjacent to $\widetilde{\mathcal{F}}_2$. Set $(j'') = (\mathcal{E} \circ \widehat{\mathcal{C}})(\nu''_2)$. Then (ν''_2) is a node of $\widehat{\mathcal{F}}$. Let $\widetilde{\mathcal{F}}_1$ be the connected component of $\widehat{\mathcal{F}} \setminus (\mathcal{E} \circ \widehat{\mathcal{C}})^{-1}(j'')$ containing (ν_1) . Since (ν'_1) is in $\widehat{\mathcal{F}}'$ and since $(\nu'_1) \in \widetilde{\mathcal{F}}_1$, then any vertex (ν''_1) adjacent to $\widetilde{\mathcal{F}}_1$ is also in $\widehat{\mathcal{F}}'$.

Therefore, by hypothesis, any pair of components δ'_1, δ'_2 at such vertices (ν''_1) and (ν''_2) over any test curve at (j'') satisfies the arc criterion. Then, applying twice Lemma 11.6 we obtain first that any pair (δ'_1, δ'_2) at vertices (ν'_1) and (ν'_2) over any test curve δ' at (j') satisfies the arc criterion, and then that (δ_1, δ_2) also satisfies the arc criterion.

We then have proved that for all test curves δ , any pair of components δ_1, δ_2 of $\ell^{-1}(\delta)$ where at least one of δ_1, δ_2 is not principal satisfies the arc criterion. \square

12. Inner contacts between complex curves on a normal surface

We need the following proposition in the proof of Theorem 3.8.

PROPOSITION 12.1. *Let $(\gamma, 0)$ and $(\gamma', 0)$ be two irreducible complex curves on $(X, 0)$ tangent to the same complex line parametrized by $x \in \mathbb{C}$, let $\delta_1, \dots, \delta_r$ be the components of the real slice $\gamma \cap \{x = t \in \mathbb{R}^+\}$ and let $\delta'_1, \dots, \delta'_s$ be those of $\gamma' \cap \{x = t \in \mathbb{R}^+\}$. Then we have:*

$$q_{out}(\gamma, \gamma') = \max_{k,l} q_o(\delta_k, \delta'_l) \quad \text{and} \quad q_{inn}(\gamma, \gamma') = \max_{k,l} q_i(\delta_k, \delta'_l).$$

Proof. Assume $(X, 0) \subset (\mathbb{C}^n, 0)$ and call (x, x_2, \dots, x_n) the coordinates of \mathbb{C}^n . We can assume that the tangent lines at 0 to γ and γ' are not included in the hyperplane $\{x = 0\}$. Therefore the outer contact $q_{out}(\gamma, \gamma')$ can be computed by taking intersection with balls with corners $\mathbb{S}_t^1 \times B_\alpha^{2n-2}$ with $\alpha > 0$ sufficiently large instead of standard spheres \mathbb{S}_t^{2n-1} , i.e., we have $q_{out}(\gamma, \gamma') = \text{ord}_t d_o(\gamma \cap \{|x| = t\}, \gamma' \cap \{|x| = t\})$.

Moreover, γ and γ' admit Puiseux expansions $(x, x_2, \dots, x_n) = (x, f_2(x), \dots, f_n(x))$ with $f_i \in \mathbb{C}\{x^{1/r}\}$ and $(x, x_2, \dots, x_n) = (x, g_2(x), \dots, g_n(x))$ with $g_i \in \mathbb{C}\{x^{1/s}\}$ respectively. Then the intersection $\gamma \cap \{|x| = t\}$ is a uniform braid obtained as the trajectory of the r points $\gamma \cap \{x = t\}$ over the circle $\mathbb{S}_t^1 = \{x = te^{i\theta} : \theta \in \mathbb{R}\}$ through the Puiseux expansion, and we have the same for γ' . Therefore, $q_{out}(\gamma, \gamma')$ can be computed by measuring distances inside the hyperplane $\{x = t\}$, i.e., $q_{out}(\gamma, \gamma') = \text{ord}(t) d_o(\gamma \cap \{x_1 = t\}, \gamma' \cap \{x = t\})$. Since by definition $\gamma \cap \{x = t\} = \{\delta_k(t), k = 1, \dots, r\}$ and $\gamma' \cap \{x = t\} = \{\delta'_l(t), l = 1, \dots, s\}$, we then obtain $q_{out}(\gamma, \gamma') = \max_{k,l} q_{out}(\delta_k, \delta'_l)$.

When working in \mathbb{C}^2 , inner and outer distances coincide and we then have $q_{inn}(\gamma, \gamma') = \max_{k,l} q_i(\delta_k, \delta'_l)$. When $n > 2$, then we can extend the above argument working with a geometric decomposition of $(X, 0)$. Consider a generic projection $\ell: (X, 0) \rightarrow (\mathbb{C}^2)$ which is also generic for the curve $\gamma \cup \gamma'$ and let $\rho: Y \rightarrow \mathbb{C}^2$ be the minimal sequence of blow-ups of points which resolves the basepoints of the family of projected polar curves $(\ell(\Pi_{\mathcal{D}}))_{\mathcal{D} \in \Omega}$ and which resolves the curve $\ell(\gamma) \cup \ell(\gamma')$. Then consider the geometric decomposition of $(X, 0)$ obtained by lifting by ℓ the pieces of the geometric decomposition associated with ρ . Then γ and γ' are trajectories of the r points $\gamma \cap \{x = t\}$ and $\gamma' \cap \{x = t\}$ inside the B -pieces containing them, and since the pieces of the geometric decomposition are fibered by their intersections with $\{x = t\}$, which shrink faster than linearly when t tends to 0, then $q_{out}(\gamma, \gamma')$ can be computed by measuring inner distance between γ and γ' inside the fibers $\widehat{F}_t = X \cap \{x = t\}$. This proves $q_{inn}(\gamma, \gamma') = \max_{k,l} q_i(\delta_k, \delta'_l)$. \square

13. The LNE_{test} -resolution

In this section, we construct a resolution $\mu_0: W_0 \rightarrow X$ which will be used in the proof of the “only if” direction of Theorem 3.8. It has the following property: it is a good resolution for every principal component over every nodal test curve for every generic projection.

We use again the notations introduced in Section 6.6. Let $\xi_\ell: X'_\ell \rightarrow W_\ell$ be the morphism obtained by blowing down iteratively the exceptional (-1) -curves which are not on simple paths joining vertices of G'_0 (Definition 3.6). We then obtain a resolution $\mu_\ell: W_\ell \rightarrow X$ which factors through $\pi_0: X_0 \rightarrow X$ by a morphism $\beta_\ell: W_\ell \rightarrow X_0$.

LEMMA 13.1. *The morphism β_ℓ does not depend on ℓ . We set $\beta_0 = \beta_\ell$, $W_0 = W_\ell$ and $\mu_0 = \mu_\ell$, so we have a commutative diagram:*

$$\begin{array}{ccccc}
 X'_\ell & \xrightarrow{\xi_\ell} & W_0 & \xrightarrow{\beta_0} & X_0 \\
 \downarrow \tilde{\ell} & & \searrow \mu_0 & & \downarrow \pi_0 \\
 & & & & X \\
 & & & & \downarrow \ell \\
 Z_\ell & \xrightarrow{\rho_\ell} & & & \mathbb{C}^2
 \end{array}$$

Proof. β_ℓ is a sequence of blow-ups of points which are all double points of the successive exceptional divisors. Since the families of curves $(\Pi_{\mathcal{D}})_{\mathcal{D} \in \Omega}$ and $(\ell_{\mathcal{D}'}(\Pi_{\mathcal{D}}))_{\mathcal{D}, \mathcal{D}' \in \Omega \times \Omega}$ are equisingular in terms of strong simultaneous resolution, this sequence of double points does not depend on the choice of the generic ℓ . \square

DEFINITION 13.2. We call $\mu_0: W_0 \rightarrow X$ the LNE_{test} -resolution of $(X, 0)$. We denote by Γ_0 the graph of μ_0 and by Γ'_0 the subgraph of Γ_0 which consists of the union of simple path joining \mathcal{L} - or \mathcal{P} -nodes.

By construction, we have the following characterization of the principal components over nodal test curves:

LEMMA 13.3. *Let ℓ be a generic projection and let γ be a nodal test curve for ℓ . A component $\hat{\gamma}$ of $\ell^{-1}(\gamma)$ is principal if and only if its strict transform by μ_0 is a curvette of a component E_v of $\mu_0^{-1}(0)$ such that $(v) \in V(\Gamma'_0)$.*

14. Proof of Theorem 3.8

Proof of the “only if” direction of Theorem 3.8.

Assume $(X, 0)$ is LNE. Let $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a generic projection. Let γ_1 and γ_2 be a pair of complex curves on $(X, 0)$ such that $\ell(\gamma_1) = \ell(\gamma_2)$ and $\ell(\gamma_i)^* \cap \Pi^* = \emptyset$. If γ_1 and γ_2 do not have a common tangent line, then $q_{out}(\gamma_1, \gamma_2) = 1$ and then $q_{inn}(\gamma_1, \gamma_2) = q_{out}(\gamma_1, \gamma_2) = 1$. If γ_1 and γ_2 have a common tangent line, let $\delta^{(1)}$ (resp. $\delta^{(2)}$) be a component of the real slice of γ_1 (resp. γ_2) as in Lemma 12.1. Since $(X, 0)$ is LNE, then $q_i(\delta^{(1)}, \delta^{(2)}) = q_o(\delta^{(1)}, \delta^{(2)})$ by Theorem 4.5, and then by Lemma 12.1 we get $q_{inn}(\gamma_1, \gamma_2) = q_{out}(\gamma_1, \gamma_2)$ and Condition (2*) is satisfied.

We now consider a nodal test curve $(\gamma, 0)$ and a principal component $\hat{\gamma}$ of $\ell^{-1}(\gamma)$. Let us prove that Condition (1*) is satisfied, i.e., that $mult(\hat{\gamma}) = mult(\gamma)$.

Let (i) be the node of T such that γ^* is a curvette of C_i . The strict transform $\hat{\gamma}^*$ of $\hat{\gamma}$ by μ_0 is a curvette of an irreducible component E of $(\mu_0)^{-1}(0)$ (Lemma 13.3). Let us again denote by E the irreducible curve in X'_ℓ which maps surjectively on E by ξ_ℓ . Then we have $\tilde{\ell}(E) = C_i$.

Let $\ell': (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be another generic projection for $(X, 0)$ which is also generic for the curve $\ell^{-1}(\gamma)$ (Definition 2.1). By Lemma 13.1, $\tilde{\ell}'(E)$ is the component C'_i of $\rho_{\ell'}^{-1}(0)$ corresponding to the node (i) of T . We then have $q_{C_i} = q_{C'_i}$.

The strict transform of $\ell'(\hat{\gamma})$ by $\rho_{\ell'}$ intersects C'_i at a smooth point p of $(\rho_{\ell'})^{-1}(0)$. Let γ' be the $\rho_{\ell'}$ -image of a curvette of C'_i which meets C'_i at a point distinct from p . We then have $q_{out}(\ell'(\hat{\gamma}), \gamma') = q_{C'_i} = q_{C_i}$, so there are Puiseux expansions of $\ell'(\hat{\gamma})$ and γ' which coincide for exponents $< q_{C_i}$ and which have distinct coefficients for $x^{q_{C_i}}$.

Assume that $mult(\hat{\gamma}) \neq mult(\gamma)$, i.e., $\hat{\gamma}$ does not satisfy Condition (1*). Then $mult(\hat{\gamma}) = k mult(\gamma)$ where k is the degree of the restriction $\ell|_{\hat{\gamma}}: (\hat{\gamma}, 0) \rightarrow (\gamma, 0)$. Since ℓ' is a generic projection for $\hat{\gamma}$, then it is a bilipschitz homeomorphism from $\hat{\gamma}$ to $\ell'(\hat{\gamma})$ for the outer metric (Theorem 2.2) so these two curves have same multiplicity ([18]). Therefore we have $mult(\ell'(\hat{\gamma})) = k mult(\gamma)$. Since the strict transforms of γ and γ' by ρ_ℓ and $\rho_{\ell'}$ are curvettes of C_i and C'_i respectively and since these exceptional curves correspond to the same node (i) of T , then $mult(\gamma) = mult(\gamma')$. We therefore obtain $mult(\ell'(\hat{\gamma})) = k mult(\gamma')$. Since γ'^* is a curvette of C'_i , then all the characteristic Puiseux exponents of γ' are $\leq q_{C_i}$. Since the Puiseux expansions of γ' and $\ell'(\hat{\gamma})$ coincide up to exponent q_{C_i} , we then obtain that $\ell'(\hat{\gamma})$ admits a characteristic exponent $q > q_{C_i}$. After change of coordinates if necessary, we can assume that $\ell = (x, y)$, $\ell' = (x, z)$ and that γ is tangent to the x -axis. We consider the real slices of the curves by intersecting them with $\{x = t \in \mathbb{R}^+\}$. Since q is a characteristic exponent of $\ell'(\hat{\gamma})$, there exists two real arcs $t \mapsto p_1(t)$ and $t \mapsto p_2(t)$ in the real slice of $\ell'(\hat{\gamma})$ such that $d_o(p_1(t), p_2(t)) = \Theta(t^q)$. Lifting p_1 and p_2 by ℓ' , we obtain two real arcs $t \mapsto \delta_1(t)$ and $t \mapsto \delta_2(t)$ inside the real slice of $\hat{\gamma}$. Since $\ell'|_{\hat{\gamma}}: \hat{\gamma} \rightarrow \ell'(\hat{\gamma})$ is a bilipschitz map for the outer metric, then $q_o(\delta_1, \delta_2) = q_o(p_1, p_2) = q$.

We claim $\ell(\delta_1(t)) = \ell(\delta_2(t))$. Indeed, assume the contrary and let q' be the rational number defined by $d_o(\ell(\delta_1(t)), \ell(\delta_2(t))) = \Theta(t^{q'})$. Since ℓ is the restriction of a linear projection, we have $q' \geq q$ and then $q' > q_{C_i}$. On the other hand, $\ell(\delta_1)$ and $\ell(\delta_2)$ are distinct components of the real slice $\gamma \cap \{x = t\}$, therefore q' is one of the characteristic exponents of γ . So we have $q' \leq q_{C_i}$. Contradiction.

Since $\ell(\delta_1) = \ell(\delta_2)$, then δ_1 and δ_2 are tangent to the same semi-line and we get $q_i(\delta_1, \delta_2) = q_{C_i}$ by Lemma 6.18. Since $q > q_{C_i}$, then $q_i(\delta_1, \delta_2) < q_o(\delta_1, \delta_2)$ so the pair (δ_1, δ_2) does not satisfy the arc criterion (Theorem 4.5) and $(X, 0)$ is not LNE. Contradiction. \square

The commutative diagram $\mathcal{E} \circ \widehat{\mathcal{C}} = \widehat{\mathcal{E}} \circ \mathcal{C}$ has the following remarkable property which will be used in the proof of the “if” direction of Theorem 3.8

LEMMA 14.1. *Let m be the multiplicity of $(X, 0)$. There exists an integer $k > 0$, such that*

- (i) $\mathcal{E}: \mathcal{F} \rightarrow T$ is the quotient map of a cyclic action of $\mathbb{Z}/k\mathbb{Z}$ on the graph \mathcal{F} ;
- (ii) $\widehat{\mathcal{E}}: \widehat{\mathcal{F}} \rightarrow \mathcal{G}$ is the quotient map of a cyclic action of $\mathbb{Z}/km\mathbb{Z}$ on the graph $\widehat{\mathcal{F}}$;
- (iii) the graph-map $\widehat{\mathcal{C}}: \widehat{\mathcal{F}} \rightarrow \mathcal{F}$ is equivariant for these actions.

Proof. Set $\ell = (x, y)$ where $h = x$ is a generic linear form on $(X, 0)$. By the method described in [7, Section 1], one can construct, using the resolution ρ_ℓ , a quasi-periodic representative $\phi: F_t \rightarrow F_t$ of the monodromy of the Milnor fibration of h which has the following property: for each vertex (v) of T , ϕ exchanges cyclically the connected components of $B_v \cap F_t$, for each edge $(v) - (v')$ of T , ϕ exchanges cyclically the annular connected components of $A_{v,v'} \cap F_t$, and there is an integer $k > 0$ such that ϕ^k is the identity on $B_v \cap F_t$ and a Dehn twist on each annulus component of $A_{v,v'} \cap F_t$. This action induces a cyclic action of $\mathbb{Z}/k\mathbb{Z}$ on the graph \mathcal{F} whose quotient map is $\mathcal{E}: \mathcal{F} \rightarrow T$.

This method to construct ϕ is presented in [7] in the case of the Milnor fibration of a holomorphic function germ $f: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$, but the same method applies in the more general setting of a holomorphic function $(X, 0) \rightarrow (\mathbb{C}, 0)$ where $(X, 0)$ is a normal surface germ. Applying this method to the holomorphic function $h \circ \ell: (X, 0) \rightarrow (\mathbb{C})$ and to the resolution $\pi_\ell: X'_\ell \rightarrow X$ (where $h = x$), we construct a quasi-periodic representative $\phi: \widehat{F}_t \rightarrow \widehat{F}_t$ with order km of the monodromy of $h \circ \ell$ which induces a cyclic action of $\mathbb{Z}/km\mathbb{Z}$ on the graph $\widehat{\mathcal{F}}$ whose quotient map is $\widehat{\mathcal{E}}: \widehat{\mathcal{F}} \rightarrow \mathcal{G}$.

By construction, the graph-map $\widehat{\mathcal{C}}: \widehat{\mathcal{F}} \rightarrow \mathcal{F}$ is equivariant for these actions. □

Proof of the “if” direction of Theorem 3.8. Assume that Conditions (1*) and (2*) are satisfied (we will prove at the end that Condition (2*) is not needed for test curves at Δ -nodes). First, notice that Condition (1*) implies that for all nodal test curve γ and all principal component $\widehat{\gamma}$ of $\ell^{-1}(\gamma)$, the restriction $\ell|_{\widehat{\gamma}}: \widehat{\gamma} \rightarrow \gamma$ has degree 1, i.e., it is bijective. Let δ be a nodal test arc and let δ_1 and δ_2 be two distinct principal components of $\ell^{-1}(\delta)$. Let γ be the nodal test curve such that δ is a real slice of γ and let γ_1 and γ_2 be the two components of $\ell^{-1}(\gamma)$ containing respectively δ_1 and δ_2 . Let (ν_1) and (ν_2) be the two vertices of $\widehat{\mathcal{F}}$ such that $\delta_i(t) \in \widehat{F}_{t, \nu_i}$.

Case 1. Assume that $q_{\nu_1, \nu_2} = q_j$. This implies $q_i(\delta_1, \delta_2) = q_j$ by Lemma 6.18. Let δ'_1 (resp. δ'_2) be a component of the real slice $\gamma_1 \cap \{x = t\}$ (resp. $\gamma_2 \cap \{x = t\}$).

Assume that $\ell(\delta'_1) = \ell(\delta'_2)$ and let (ν'_1) and (ν'_2) be the two vertices of $\widehat{\mathcal{F}}$ such that $\delta'_i(t) \in \widehat{F}_{t, \nu'_i}$. By condition (1*), if δ'_1 is a real slice $\gamma_1 \cap \{x = t\}$, there is a unique component δ'_2 of the real slice $\gamma_2 \cap \{x = t\}$ such that $\ell(\delta'_1) = \ell(\delta'_2)$, and by Lemma 14.1, we have $q_{\nu_1, \nu_2} = q_{\nu'_1, \nu'_2}$. Therefore, $q_i(\delta'_1, \delta'_2) = q_i(\delta_1, \delta_2) = q_j$ by Lemma 6.18.

Assume now that $\ell(\delta'_1) \neq \ell(\delta'_2)$. Let S_t be the segment in \mathbb{C}^2 joining $\ell(\delta'_1)$ to $\ell(\delta'_2)$ and let $\delta''_2(t)$ be the extremity of the lifting of S_t with origin $\delta'_1(t)$. Then δ''_2 is a component of the real slice $\gamma_2 \cap \{x = t\}$ such that $\ell(\delta''_2) = \ell(\delta'_2)$, and using the arguments of the proof of Lemma 7.2, we have $q_o(\delta'_1, \delta'_2) = \min(q_i(\delta'_1, \delta''_2), q_o(\delta''_2, \delta'_2))$. Then a fortiori $q_i(\delta'_1, \delta'_2) \leq q_o(\delta'_1, \delta'_2) \leq q_i(\delta'_1, \delta''_2)$. Since $\ell(\delta'_1)$ and $\ell(\delta''_2) = \ell(\delta'_2)$ are distinct component of a real slice of γ , then $q_i(\ell(\delta'_1), \ell(\delta''_2))$ equals one of the essential Puiseux exponents of γ , which are all $\leq q_j$. Since $q_i(\delta'_1, \delta''_2) \leq q_i(\ell(\delta'_1), \ell(\delta''_2))$, then $q_i(\delta'_1, \delta'_2) \leq q_i(\ell(\delta'_1), \ell(\delta'_2))$, so we then obtain $q_i(\delta'_1, \delta'_2) \leq q_j$. Therefore, $q_{inn}(\gamma_1, \gamma_2) = q_i(\delta_1, \delta_2)$ (cf. Proposition 12.1). Using Condition (2*), we then have $q_o(\delta_1, \delta_2) \leq q_{out}(\gamma_1, \gamma_2) = q_{inn}(\gamma_1, \gamma_2) = q_i(\delta_1, \delta_2)$. Therefore $q_o(\delta_1, \delta_2) \leq q_i(\delta_1, \delta_2)$, and then $q_o(\delta_1, \delta_2) = q_i(\delta_1, \delta_2)$ since the converse inequality is always true.

Case 2. Assume now that $q_{\nu_1\nu_2} < q_j$ and consider a simple path in $\widehat{\mathcal{F}}$ from (ν_1) to (ν_2) such that q_{ν_1,ν_2} is the minimal inner rate along it and let (ν) be the vertex on it such that $q_\nu = q_{\nu_1,\nu_2}$. Set $(j') = (\mathcal{E} \circ \widehat{\mathcal{C}})(\nu)$. For any test curve γ' at (j') and any test arc $\delta' \subset \gamma'$, we can apply Case 1 to any pair of δ'_1, δ'_2 over δ' such that $\delta'_1(t)$ and $\delta'_2(t)$ are in $\widehat{\mathcal{F}}_{t,\nu}$ and we get $q_i(\delta'_1, \delta'_2) = q_o(\delta'_1, \delta'_2)$. Then using Lemma 11.6, we conclude $q_i(\delta_1, \delta_2) = q_o(\delta_1, \delta_2)$. \square

15. Inner contacts on a normal surface and resolution

In this last section, we state and prove an analog of Lemma 6.18 in terms of complex curves and resolution which is very useful to compute inner rates between complex curves on a normal surface germ using resolution and then to check condition (2*) of Theorem 3.8. First, let us introduce a generalization of the inner rates of Definition 6.5.

LEMMA 15.1. *Let $\pi: X' \rightarrow X$ be a resolution of X and let E be an irreducible component of the exceptional divisor $\pi^{-1}(0)$. Let γ and γ' be two complex curve germs in $(X, 0)$ whose strict transforms by π are curvettes of E meeting E at two distinct points. Then $q_{inn}(\gamma, \gamma')$ is independent of the choice of γ and γ' .*

DEFINITION 15.2. We set $q_E = q_{inn}(\gamma, \gamma')$ and we call q_E the inner rate of E .

Proof. Let $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a generic projection for $(X, 0)$ which is also generic for the curve $\gamma \cup \gamma'$ and let $\rho: Y \rightarrow \mathbb{C}^2$ be the minimal sequence of blow-ups defined as in the proof of Proposition 12.1. There is a component C of $\rho^{-1}(0)$ which intersect both $\ell(\gamma)^*$ and $\ell(\gamma')^*$. Let q_C be its inner rate. Then the piece $\rho(\mathcal{N}(C))$ of the associated geometric decomposition of \mathbb{C}^2 lifts by ℓ to a union of $B(q_C)$ -pieces, one of them being of the form $\pi(\mathcal{N}(E))$. Therefore, $q_{inn}(\gamma, \gamma') = q_C$. \square

We then have the following consequence of Proposition 12.1 and Lemma 6.18.

PROPOSITION 15.3. *Let γ and γ' be two complex curves on $(X, 0)$. Consider a resolution $\pi: X' \rightarrow X$ which factors through the Nash modification and through the blow-up of the maximal ideal and which is a resolution of the complex curve $\gamma \cup \gamma'$ and set $\pi^{-1}(0) = \bigcup_v E_v$. Let \tilde{G} be the resolution graph of π whose vertices (v) are weighted by the inner rates introduced in Definition 15.2. Let (v) and (v') be the vertices of G such that $\gamma^* \cap E_v \neq \emptyset$ and $\gamma'^* \cap E_{v'} \neq \emptyset$. Then $q_{inn}(\gamma, \gamma') = q_{v,v'}$ where $q_{v,v'}$ is the maximum among minimum of inner rates along paths from (v) to (v') in the graph \tilde{G} .*

Proof. By Lemma 12.1, we have $q_{inn}(\gamma, \gamma') = \max_{k,l} q_i(\delta_k, \delta'_l)$ where the δ_k (resp. δ'_l) are the components of a real slice of γ (resp. γ').

First, notice that the statements and proofs of Lemmas 6.18 and 14.1 stay the same if one replaces the resolution ρ_ℓ and the associated geometric decompositions of $(\mathbb{C}^2, 0)$, $(X, 0)$, \mathcal{F} and $\widehat{\mathcal{F}}$ by any sequence of blow-ups of points $\rho: Y \rightarrow \mathbb{C}^2$ which resolves the basepoints of the family of projected polar curves $(\ell(\Pi_{\mathcal{D}}))_{\mathcal{D} \in \Omega}$. As in the proof of Proposition 12.1, consider a generic projection $\ell: (X, 0) \rightarrow (\mathbb{C}^2)$ which is also generic for the curve $\gamma \cup \gamma'$ and let $\rho: Y \rightarrow \mathbb{C}^2$ be the minimal sequence of blow-ups of points which resolves the basepoints of the family of projected polar curves $(\ell(\Pi_{\mathcal{D}}))_{\mathcal{D} \in \Omega}$ and which resolves the curve $\ell(\gamma) \cup \ell(\gamma')$. We will use the same notations as before for the graphs of the associated geometric decompositions, in

particular the graph-map $\hat{\mathcal{E}}: \hat{\mathcal{F}} \rightarrow \mathcal{G}$. If (v) and (v') are two vertices of \mathcal{G} , we will denote by $q_{v,v'}$ the maximal among minimal rates along simple paths in \mathcal{G} between the vertices (v) and (v') . Since ρ resolves the curve $\ell(\gamma) \cup \ell(\gamma')$, then there are two vertices (v) and (v') in \mathcal{G} such that γ is contained in the B -piece B_v and γ' is contained in the B -piece $B_{v'}$. As a consequence of Lemma 14.1, the map $\hat{\mathcal{E}}: \hat{\mathcal{F}} \rightarrow \mathcal{G}$ satisfies the following property: if p is a simple path in \mathcal{G} from the vertices (v) to (v') and if (ν) and (ν') are two vertices in $\hat{\mathcal{F}}$ such that $\hat{\mathcal{E}}(\nu) = (v)$ and $\hat{\mathcal{E}}(\nu') = (v')$, then there exists $m \in \mathbb{N}^*$ such that the lifting by $\hat{\mathcal{E}}$ of mp with origin (ν) is a simple path \hat{p} from (ν) to (ν') in $\hat{\mathcal{F}}$.

Let p be a path in \mathcal{G} between the vertices (v) and (v') such that the minimal inner rate along p equals $q_{v,v'}$. Consider two components $\delta \subset \gamma$ and $\delta' \subset \gamma'$ of the real slices. Let (ν) and (ν') be the two vertices of $\hat{\mathcal{F}}$ such that $\delta(t) \in \hat{F}_{t,\nu}$ and $\delta'(t) \in \hat{F}_{t,\nu'}$. Lemma 6.18 extends with the same statement and the same proof to a computation of inner contacts using the new graph $\tilde{\mathcal{F}}$ instead of \mathcal{F} : $q_i(\delta, \delta')$ equals $q_{\nu,\nu'}$ where $q_{\nu,\nu'}$ equals the maximum of minimum inner rates among all simple paths in $\tilde{\mathcal{F}}$ from (ν) to (ν') . Using the remark above, we obtain $q_{v,v'} \leq q_{\nu,\nu'}$.

Now, take a pair δ, δ' such that $q_i(\delta, \delta') = q_{inn}(\gamma, \gamma')$ (Lemma 12.1) and take a path \hat{p} in $\tilde{\mathcal{F}}$ which realizes $q_{\nu,\nu'}$. Then \hat{p} projects by $\hat{\mathcal{E}}$ to a path p whose support is a simple path from (v) to (v') . Therefore $q_{\nu,\nu'} \leq q_{v,v'}$, and the previous inequality gives then $q_{\nu,\nu'} = q_{v,v'}$. Then applying Lemma 12.1, we obtain $q_{inn}(\gamma, \gamma') = q_{v,v'}$, where this number is computed in the graph \mathcal{G} .

As in Section 6.6, there is again a natural injection $\mathcal{I}: V(\mathcal{G}) \rightarrow V(\tilde{\mathcal{G}})$. We denote again by (v) and (v') the images $\mathcal{I}(v)$ and $\mathcal{I}(v')$. By construction, the sequence of inner rates along a string in $\tilde{\mathcal{G}}$ between two consecutive vertices $\mathcal{I}(v)$ and $\mathcal{I}(v')$ is strictly monotone. Therefore, $q_{v,v'}$ computed in the graph \mathcal{G} equals the number $q_{v,v'}$ computed in the graph $\tilde{\mathcal{G}}$. This proves the proposition. \square

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