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ON LOCAL CONTINUOUS SOLVABILITY OF EQUATIONS ASSOCIATED TO ELLIPTIC AND CANCELING LINEAR DIFFERENTIAL OPERATORS

LAURENT MOONENS AND TIAGO PICON

ABSTRACT. Consider $A(x, D) : C^\infty(\Omega, E) \rightarrow C^\infty(\Omega, F)$ an elliptic and canceling linear differential operator of order ν with smooth complex coefficients in $\Omega \subset \mathbb{R}^N$ from a finite dimension complex vector space E to a finite dimension complex vector space F and $A^*(x, D)$ its adjoint. In this work we characterize the (local) continuous solvability of the partial differential equation $A^*(x, D)v = f$ (in the distribution sense) for a given distribution f ; more precisely we show that any $x_0 \in \Omega$ is contained in a neighborhood $U \subset \Omega$ in which its continuous solvability is characterized by the following condition on f : for every $\varepsilon > 0$ and any compact set $K \subset\subset U$, there exists $\theta = \theta(K, \varepsilon) > 0$ such that the following holds for all smooth function φ supported in K :

$$|f(\varphi)| \leq \theta \|\varphi\|_{W^{\nu-1,1}} + \varepsilon \|A(x, D)\varphi\|_{L^1},$$

where $W^{\nu-1,1}$ stands for the homogenous Sobolev space of all L^1 functions whose derivatives of order $\nu - 1$ belongs to $L^1(U)$.

This characterization implies and extends results obtained before for operators associated to elliptic complex of vector fields (see [16]); we also provide local analogues, for a large range of differential operators, to global results obtained for the classical divergence operator in [4] and [9].

1. INTRODUCTION

Consider $\Omega \subseteq \mathbb{R}^N$ an open set and $A(\cdot, D)$ a linear differential operator of order ν with smooth complex coefficients in Ω denoted by:

$$A(x, D) = \sum_{|\alpha| \leq \nu} a_\alpha(x) \partial^\alpha : C^\infty(\Omega; E) \rightarrow C^\infty(\Omega; F),$$

where E is a complex vector space of dimension n and F is a complex vector space of dimension $n' \geq n$.

A series of results concerning on local L^1 estimates for linear differential operators has been studied by J. Hounie and T. Picon in the setting of elliptic systems of complex vector fields, complexes and pseudocomplexes ([12], [13]). The following characterization of local L^1 estimates for operators $A(x, D)$ was proved in [14], namely:

Theorem 1.1. *Assume, as before, that $A(\cdot, D)$ is a linear differential operator of order ν between the spaces E and F . The following properties are equivalent:*

- (i) *$A(x, D)$ is elliptic and canceling (see below for a definition of those properties);*
- (ii) *every point $x_0 \in \Omega$ is contained in a ball $B = B(x_0, r) \subset \Omega$ such that the a priori estimate*

$$(1) \quad \|u\|_{W^{\nu-1, N/(N-1)}} \leq C \|A(x, D)u\|_{L^1},$$

holds for some $C > 0$ and all smooth functions $u \in C^\infty(B; E)$ having compact support in B .

Here, given $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, $W^{k,p}(\Omega)$ denotes the homogeneous Sobolev space of complex functions in $L^p(\Omega)$ whose weak derivatives of order k belong to $L^p(\Omega)$, endowed with the (semi-)norm $\|u\|_{W^{k,p}} := \sum_{|\alpha|=k} \|\partial_\alpha u\|_{L^p}$.

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It turns out that elliptic linear differential operators that satisfy an *a priori* estimate like (1) can be characterized in terms of properties of their principal symbol $a_\nu(x, \xi) = \sum_{|\alpha|=\nu} a_\alpha(x) \xi^\alpha$. Recall that the *ellipticity* of $A(x, D)$ at $x_0 \in \Omega$ means that for every $\xi \in \mathbb{R}^N \setminus \{0\}$ the map $a_\nu(x_0, \xi) : E \rightarrow F$ is injective.

Definition 1.2. Let $x_0 \in \Omega$. A linear partial differential operator $A(x, D)$ of order ν from E to F with principal symbol $a_\nu(x, \xi)$ that satisfies :

$$(\star) \quad \bigcap_{\xi \in \mathbb{R}^N \setminus \{0\}} a_\nu(x_0, \xi)[E] = \{0\}$$

is said to be *canceling* at x_0 . If (\star) holds for every $x_0 \in \Omega$ we say that $A(x, D)$ is *canceling*.

Examples of canceling operators satisfying (\star) can be founded in [14]; this is the case in particular for operators associated to elliptic system of complex vector fields (see [12], [13]). The canceling property for linear differential operators was originally defined by Van Schaftingen [19] in the setup of homogeneous operators with constant coefficients $A(D)$ and stands out by several applications (and characterizations) in the theory of *a priori* estimates in L^1 norm (see for instance [20] for a brief description).

In this work, we are interested to study the (local) continuous solvability in the weak sense of the equation:

$$(2) \quad A^*(x, D)v = f,$$

where $A(x, D)$ is an elliptic and canceling linear differential operator. We use the notation $A^* := \overline{A^t}$ where \overline{A} denotes the operator obtained from A by conjugating its coefficients and A^t is its formal transpose — namely this means that, for all smooth functions φ and ψ having compact support in Ω and taking values in E and F respectively, we have:

$$\int_{\Omega} A(x, D)\varphi \cdot \bar{\psi} = \int_{\Omega} \varphi \cdot \overline{A^*(x, D)\psi}.$$

Our main result is the following.

Theorem 1.3. Assume $A(x, D)$ is as before. Then every point $x_0 \in \Omega$ admits an open neighborhood $U \subset \Omega$ such that for any $f \in \mathcal{D}'(U)$, the equation (2) is continuously solvable in U if and only if f is an A -charge in U , meaning that for every $\varepsilon > 0$ and every compact set $K \subset\subset U$, there exists $\theta = \theta(K, \varepsilon) > 0$ such that one has:

$$(3) \quad |f(\varphi)| \leq \theta \|\varphi\|_{W^{\nu-1,1}} + \varepsilon \|A(x, D)\varphi\|_{L^1},$$

for any smooth function φ in U vanishing outside K .

One simple argument (see Section 5) shows that the above continuity property on f is a necessary condition for the continuous solvability of equation (2) in U . Theorem 1.3 asserts that the continuity property (3) is also sufficient, under the canceling and ellipticity assumptions on the operator. The proof, which will be presented in Section 6, is based on a functional analytic argument inspired from [9] and already improved in [16] for divergence-type equations associated to complexes of vector fields (observe in particular that one recovers [16, Theorem 1.2] when applying Theorem 1.3 to the latter context). However, it should be mentioned here that by allowing in (2) a much larger class of (higher order) differential operators, that method of proof had to be very substantially refined, leading to the use of new tools. Applications of Theorem 1.3 are presented in the Section 7.

Assume for a moment that $A(x, D)$ be elliptic. We point out the canceling assumption - characterized by inequality (1) - plays a fundamental role in our argument in the proof of Theorem 1.3. However, we should emphasize that this property might not be necessary to obtain a characterization of continuous solutions to the equation (2) formulated along the previous lines. In the context of the Poisson equation with measure data $\Delta u = \mu$ (where the Laplacean operator is *not* a canceling operator), it follows indeed from a result by Aizenman and Simon [2, Theorem 4.14] (see also Ponce [17, Proposition 18.1], where this question is studied in a luminous fashion) that, given a smooth, bounded open set Ω in \mathbb{R}^n and a

measure μ in Ω , the Dirichlet problem associated with $\Delta u = \mu$ in Ω has a continuous solution in $\bar{\Omega}$ if and only if, for every $\varepsilon > 0$, there exists $\theta > 0$ such that for any $\varphi \in C_0^\infty(\Omega)$, one has:

$$\left| \int_{\Omega} \varphi d\mu \right| \leq \theta \|\varphi\|_{L^1} + \varepsilon \|\Delta \varphi\|_{L^1}.$$

This result, however, is proved using very different techniques than the ones developed here.

2. PRELIMINARIES AND NOTATIONS

We always denote by Ω an open set of \mathbb{R}^N , $N \geq 2$. Unless otherwise specified, all functions are complex valued and the notation $\int_A f$ stands for the Lebesgue integral $\int_A f(x)dx$. As usual, $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ are the spaces of complex test functions and distributions, respectively. When $K \subset\subset \Omega$ is a compact subset of Ω , we let $\mathcal{D}_K(\Omega) := \mathcal{D}(\Omega) \cap \mathcal{E}'(K)$, where $\mathcal{E}'(K)$ is the space of all distributions with compact support in K . Since the ambient field is \mathbb{C} , we identify (formally) each $f \in L_{loc}^1(\Omega)$ with the distribution $T_f \in \mathcal{D}'(\Omega)$ given by $T_f(\varphi) = \int_{\Omega} f \bar{\varphi}$. We also consider $C(\Omega)$ the space of all continuous functions in Ω . When working with objects in a function space taking values in a finite-dimensional (normed) vector space E , we shall indicate it as a second argument (e.g. $C(\Omega, E)$ will denote the space of all E -valued continuous vector fields $v : \Omega \rightarrow E$). Finally we use the notation $f \lesssim g$ to indicate the existence of an universal constant $C > 0$, independent of all variables and unmentioned parameters, such that one has $f \leq Cg$.

Some Sobolev spaces. Given a finite-dimensional (complex, normed) vector space E , we denote as before by $W^{k,p}(\Omega; E)$ for $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the homogeneous Sobolev space of functions in $L^p(\Omega; E)$ whose weak derivatives of order k belong to $L^p(\Omega; E)$, endowed with the (semi-)norm $\|u\|_{W^{k,p}} = \sum_{|\alpha|=k} \|\partial_{\alpha} u\|_{L^p}$; we also denote by $W_c^{k,p}(\Omega; E)$ the space of its elements having compact support in Ω . Given $1 < p \leq \infty$ and $k \in \mathbb{N}^*$ we also define the space $W_{loc}^{-k,p}(\Omega; E^*) := (W_c^{k,p'}(\Omega; E))^*$ as the space of distributions $f \in \mathcal{D}'(\Omega; E^*)$ enjoying the following property: for all $K \subset\subset \Omega$, there exists $C_K > 0$ such that for all $\varphi \in \mathcal{D}_K(\Omega; E)$, one has:

$$|\langle f, \varphi \rangle| \leq C_K \|\varphi\|_{W^{k,p'}(\Omega)} = C_K \sum_{|\alpha|=k} \|\partial^{\alpha} \varphi\|_{p'},$$

where $1 \leq p' < \infty$ is defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

Given $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, one also defines the (classical, inhomogeneous) Sobolev space $\mathbf{W}^{k,p}(\Omega; E)$ as the space of complex functions in $L^p(\Omega; E)$ whose weak derivatives up to order k belong to $L^p(\Omega; E)$, endowed with the norm $\|u\|_{\mathbf{W}^{k,p}} := \sum_{|\alpha| \leq k} \|\partial_{\alpha} u\|_{L^p}$. We denote finally by $\mathbf{W}_0^{k,p}(U; E)$ the completion of the space $\mathcal{D}(\Omega; E)$ in $\mathbf{W}^{k,p}(\Omega; E)$. The space $\mathbf{W}_0^{k,p}(\Omega; E)$ is classically reflexive and separable for $1 < p < \infty$ (see e.g. [1, p. 64]).

Lemma 2.1. *Fix $k \in \mathbb{N}^*$ and $1 < p \leq \infty$. Given $f \in W_{loc}^{-k,p}(\Omega; E^*)$ and $K \subset\subset \Omega$ there exists $(g_{\alpha})_{|\alpha|=k} \subseteq L^p(\Omega; E^*)$ for which one has, for all $\varphi \in \mathcal{D}_K(\Omega; E)$:*

$$\langle f, \varphi \rangle = \sum_{|\alpha|=k} \int_{\Omega} g_{\alpha} \overline{\partial^{\alpha} \varphi}.$$

Remark 2.2. In the latter expression, and throughout this paper, one uses, given $e \in E$ and $e^* \in E^*$ (and in any finite-dimensional duality setting), the notation e^*e instead of $\langle e^*, e \rangle$.

Proof. Fix $f \in W_{loc}^{-k,p}(\Omega; E^*)$ and $K \subset\subset \Omega$. Denote by M the number of multi-indices $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$ and let:

$$X := \{(u_{\alpha})_{|\alpha|=k} : \text{there exists } \varphi \in \mathcal{D}_K(\Omega; E) \text{ with } u_{\alpha} = \partial^{\alpha} \varphi \text{ for all } |\alpha| = k\},$$

be endowed with the norm $\|(u_{\alpha})_{|\alpha|=k}\| := \sum_{|\alpha|=k} \|u_{\alpha}\|_{p'}$ — we hence see it as a subspace of $L^{p'}(\Omega, E^M)$.

Now define a linear functional F on X in the following way: if $(u_{\alpha})_{|\alpha|=k} \in X$ is given, there exists a unique $\varphi \in \mathcal{D}_K(\Omega; E)$ with $u_{\alpha} = \partial^{\alpha} \varphi$ for all $|\alpha| = k$; we then let $\langle F, (u_{\alpha}) \rangle := \langle f, \varphi \rangle$. There holds:

$$|\langle F, (u_{\alpha}) \rangle| \leq C_K \sum_{|\alpha|=k} \|u_{\alpha}\|_{p'},$$

for all $(u_\alpha)_{|\alpha|=k} \in X$. By the Hahn-Banach theorem, F extends to a continuous linear functional on $L^{p'}(\Omega, E^M)$ satisfying

$$|\langle F, (u_\alpha) \rangle| \leq C_K \sum_{|\alpha|=k} \|u_\alpha\|_{p'},$$

for all $(u_\alpha) \in L^{p'}(\Omega, E^M)$; there hence exists $(g_\alpha) \in L^p(\Omega; (E^*)^M) = (L^{p'}(\Omega; E^M))^*$ such that one has:

$$\langle F, (u_\alpha) \rangle = \sum_{|\alpha|=k} \int_{\Omega} g_\alpha \bar{u}_\alpha,$$

for all $(u_\alpha) \in L^{p'}(\Omega, E^M)$. We hence get in particular, for $\varphi \in \mathcal{D}_K(\Omega; E)$:

$$\langle f, \varphi \rangle = \langle F, (\partial^\alpha \varphi)_{|\alpha|=k} \rangle = \sum_{|\alpha|=k} \int_{\Omega} g_\alpha \overline{\partial^\alpha \varphi}.$$

The proof is complete. ■

Example 2.3. Assume Ω to be an open set and fix $k \in \mathbb{N}$. Then one has $L_{\text{loc}}^N(\Omega; E^*) \subseteq W_{\text{loc}}^{-k, N}(\Omega; E^*)$.

To see this, observe the statement is obvious in case $k = 0$. Hence assume $k \in \mathbb{N}^*$ and fix $f \in L_{\text{loc}}^N(\Omega; E^*)$. Let $K \subset\subset \Omega$ be compact and compute now for $\varphi \in \mathcal{D}_K(\Omega; E)$:

$$\left| \int_{\Omega} f \bar{\varphi} \right| \leq \|f\|_{L^N(K; E^*)} \|\varphi\|_{L^{N/(N-1)}(K; E)} = \|f\|_{L^N(K; E^*)} \|\varphi\|_{L^{N/(N-1)}(\mathbb{R}^N; E)}.$$

Yet we get, according to the Sobolev-Gagliardo-Nirenberg inequality (which we shall refer to as the “SGN” inequality in the sequel):

$$\|\varphi\|_{L^{N/(N-1)}(\mathbb{R}^N; E)} \leq \kappa(N) \sum_{|\alpha|=1} \|\partial^\alpha \varphi\|_{L^1(\mathbb{R}^N; E)} = \kappa(N) \sum_{|\alpha|=1} \|\partial^\alpha \varphi\|_{L^1(\Omega; E)},$$

and hence we find, for all $\varphi \in \mathcal{D}_K(\Omega)$:

$$\left| \int_{\Omega} f \bar{\varphi} \right| \leq \kappa(N) \|f\|_{L^N(K; E^*)} \|\varphi\|_{W^{1,1}(\Omega; E)}.$$

If $k = 2$, then for all $|\alpha| = 1$ we have, using Hölder’s inequality and SGN inequality again:

$$\begin{aligned} \|\partial^\alpha \varphi\|_{L^1(\Omega; E)} &= \|\partial^\alpha \varphi\|_{L^1(K; E)} \leq |K|^{\frac{1}{N}} \|\partial^\alpha \varphi\|_{L^{N/(N-1)}(\mathbb{R}^N; E)} \\ &\leq |K|^{\frac{1}{N}} \sum_{|\beta|=1} \|\partial^{\alpha+\beta} \varphi\|_{L^1(\mathbb{R}^N; E)} = |K|^{\frac{1}{N}} \sum_{|\beta|=1} \|\partial^{\alpha+\beta} \varphi\|_{L^1(\Omega; E)}, \end{aligned}$$

which implies, for all $\varphi \in \mathcal{D}_K(\Omega)$:

$$\left| \int_{\Omega} f \bar{\varphi} \right| \leq \kappa(N) \|f\|_{L^N(K; E^*)} |K|^{\frac{1}{N}} \sum_{|\alpha|=2} \|\partial^\alpha \varphi\|_{L^1(\Omega; E)} = \kappa(N) \|f\|_{L^N(K; E^*)} |K|^{\frac{1}{N}} \|\varphi\|_{W^{2,1}(\Omega; E)}.$$

One proves inductively for a general $k \in \mathbb{N}^*$ one has:

$$\left| \int_{\Omega} f \bar{\varphi} \right| \leq \kappa(N) \|f\|_{L^N(K; E^*)} |K|^{\frac{k-1}{N-1}} \|\varphi\|_{W^{k,1}(\Omega; E)} \leq \kappa(N) \|f\|_{L^N(K; E^*)} |K|^{\frac{k}{N}} \|\varphi\|_{W^{k, N/(N-1)}(\Omega; E)},$$

where Hölder’s inequality is used again; this means finally that one has $f \in W_{\text{loc}}^{-k, N}(\Omega; E^*)$.

Remark 2.4. Using approximation by smooth functions and a recursive use of the SGN inequality as done in the above example, one shows that, given $K \subset\subset \Omega$ a compact set and an integer $k \in \mathbb{N}$, there exists a constant $C(K, k) > 0$ such that for any $g \in W^{k, N/(N-1)}(\Omega; E)$ satisfying $\text{supp } g \subseteq K$, one has $g \in \mathbf{W}_0^{k, N/(N-1)}(\Omega; E)$ and:

$$\|g\|_{\mathbf{W}^{k, N/(N-1)}(\Omega; E)} \leq C(K, k) \|g\|_{W^{k, N/(N-1)}(\Omega; E)}.$$

3. CANCELING AND ELLIPTIC DIFFERENTIAL OPERATORS

Given $A(x, D)$ as before, the 2ν -order differential operator $\Delta_{\mathcal{A}} := A^*(\cdot, D) \circ A(\cdot, D)$ may be regarded as an elliptic pseudodifferential operator with symbol in the Hörmander class $S^{2\nu}(\Omega) := S_{1,0}^{2\nu}(\Omega)$ ¹, so that there exist properly supported pseudodifferential operators $q, \tilde{q} \in OpS^{-2\nu}(\Omega)$ (parametrixes) and $r, \tilde{r} \in OpS^{-\infty}(\Omega)$ for which one has, for any $f \in C^\infty(\Omega, F)$:

$$(4) \quad \Delta_{\mathcal{A}} q(x, D)f + r(x, D)f = \tilde{q}(x, D)\Delta_{\mathcal{A}} f + \tilde{r}(x, D)f = f.$$

Writing $\Delta_{\mathcal{A}} q(x, D)f = A^*(x, D)u$ for $u = A(x, D)q(x, D)f$ we then get:

$$A^*(x, D)u - f = r(x, D)f$$

for every $f \in C^\infty(\Omega, F)$.

Proposition 3.1. *Assume that $A(x, D)$ is as before. Then for every point $x_0 \in \Omega$ and any $0 < \beta < 1$, there exist an open ball $B = B(x_0, \ell) \subset \Omega$ and a constant $C = C(B) > 0$ such that, for all $\varphi \in \mathcal{D}(B, E)$, one has:*

$$(5) \quad \sum_{|\alpha|=\nu-1} \|\partial^\alpha \varphi\|_{1-\beta, 1} \leq C \|A(x, D)\varphi\|_{L^1}.$$

The previous inequality states the embedding into L^1 of some version of a fractional Sobolev space $W_c^{1-\beta, 1}(B)$ that can be defined according to the following procedure. Given $B = B(x_0, \ell)$ a ball consider $\tilde{B} = B(x_0, 2\ell)$ the ball with the same center as B but twice its radius. Let $\psi \in \mathcal{D}(\tilde{B})$ satisfy $\psi(x) \equiv 1$ on B and define $\Lambda_\gamma := \Lambda_\gamma(x, D)$ the pseudodifferential operator with symbol $\lambda_\gamma(x, \xi) = \psi(x)(1 + 4\pi^2|\xi|^2)^\gamma \in S^\gamma(\mathbb{R}^N)$. Denote then by $W_c^{\gamma, p}(B)$ the set of distributions with compact support $f \in \mathcal{E}'(B)$ such that one has $\Lambda_\gamma f \in L^p(\mathbb{R}^N)$; one endows it with the semi-norm $\|f\|_{\gamma, p} := \|\Lambda_\gamma u\|_p$. Note that the space $W_c^{\gamma, p}(B)$ is independent of the choice of ψ , i.e. that if $\psi_2(x), \psi_1(x) \in \mathcal{D}(\tilde{B})$ satisfy $\psi_1(x) = \psi_2(x) \equiv 1$ on B , then $\|\Lambda_{\gamma, \psi_1} f\|_{L^p} \cong \|\Lambda_{\gamma, \psi_2} f\|_{L^p}$.

Proof. Fix α a multi-index with $|\alpha| = \nu - 1$. Let $h = A(x, D)\varphi$. Thanks to identity (4) and to the calculus of pseudodifferential operators we have

$$\Lambda_{1-\beta} \partial^\alpha \varphi = p(x, D)h + r'(x, D)\varphi,$$

where $p(x, D) := \Lambda_{1-\beta} \partial^\alpha q(x, D)A^*(x, D) \in OpS^{-\beta}$ and $r'(x, D) := \Lambda_{1-\beta} \partial^\alpha r(x, D) \in OpS^{-\infty}$. As a consequence of [16, Theorem 6.1] we have $\|p(x, D)h\|_{L^1} \lesssim \|A(x, D)\varphi\|_{L^1}$, which implies:

$$\|\Lambda_{\beta-1} \partial^\alpha \varphi\|_{L^1} \leq C \|A(x, D)\varphi\|_{L^1} + \|r'(x, D)\varphi\|_{L^1}.$$

As the second term on the right side may be absorbed (see [12, p. 798]), shrinking the neighborhood if necessary, we obtain the estimate (5) after recalling estimate (1.1) in Theorem 1.1 above. \blacksquare

4. FUNCTIONS OF BOUNDED VARIATION ASSOCIATED TO $A(x, D)$

4.1. Basic definitions; approximation and compactness. Let $W_c^{k, p}(\Omega; E)$ be the linear space of all complex functions in $W^{k, p}(\Omega; E)$ whose support is a compact subset of Ω .

The following definition of variation associated to $A(x, D)$ of $g \in W_c^{\nu-1, 1}(\Omega; E)$ recalls the classical definition of variation when $\nu = 1$ and $A(x, D) = \nabla$. It has been formulated for real vector fields by N. Garofalo and D. Nhieu [6] and adapted for complex vector fields in [16].

Definition 4.1. Given $g \in W_c^{\nu-1, 1}(\Omega; E)$ and $U \subseteq \Omega$ an open set, one calls the extended real number:

$$\|D_{\mathcal{A}} g\|(U) := \sup \left\{ \left| \int_\Omega g \overline{A^*(\cdot, D)v} \right| : v \in C_c^\infty(\Omega; F^*), \text{supp } v \subseteq U, \|v\|_\infty \leq 1 \right\},$$

the (total) \mathcal{A} -variation associated to $A(x, D)$ of g in U and we let $\|D_{\mathcal{A}} g\| := \|D_{\mathcal{A}} g\|(\Omega)$ in case there is no ambiguity on the open set Ω . We denote by $BV_{\mathcal{A}, c}(\Omega)$ the set of all $g \in W_c^{\nu-1, 1}(\Omega; E)$ with $\|D_{\mathcal{A}} g\| < +\infty$.

¹Since we will only work with symbols of type (1,0), the type will be omitted in the notation; concerning pseudodifferential operators we refer, for instance, to [11, Chapter 3] and [18].

Given $g \in BV_{\mathcal{A},c}(\Omega)$, we denote by $D_{\mathcal{A}}g$ the unique F -valued Radon measure satisfying:

$$(6) \quad \int_{\Omega} g \overline{A^*(\cdot, D)} v = \int_{\Omega} \bar{v} \cdot d[D_{\mathcal{A}}g],$$

for all $v \in C_c^\infty(\Omega, F^*)$. It is clear by definition that $\|D_{\mathcal{A}}g\|$ is also the total variation in Ω of $D_{\mathcal{A}}g$.

Remark 4.2. Given $g \in BV_{\mathcal{A},c}(\Omega)$, one has $\text{supp } D_{\mathcal{A}}g \subseteq \text{supp } g$. Indeed, given $x \in \Omega \setminus \text{supp } g$, find a radius $r > 0$ for which one has $B(x, r) \subseteq \Omega \setminus \text{supp } g$. It is clear according to (6) that for any $v \in C_c^\infty(B(x, r), F^*)$ we then have $D_{\mathcal{A}}g(v) = 0$. Hence we also get $D_{\mathcal{A}}g(v) = 0$ for all $v \in C_c(B(x, r), F^*)$, which ensures that one has $x \notin \text{supp } D_{\mathcal{A}}g$ and finishes to show the inclusion $\text{supp } D_{\mathcal{A}}g \subseteq \text{supp } g$.

Remark 4.3. It follows readily from the previous definition that if $(g_i) \subseteq BV_{\mathcal{A},c}(\Omega)$ converges in L^1 to $g \in W_c^{\nu-1,1}(\Omega)$, one then has $g \in BV_{\mathcal{A},c}(\Omega)$ and:

$$\|D_{\mathcal{A}}g\| \leq \liminf_i \|D_{\mathcal{A}}g_i\|.$$

We shall refer to this in the sequel as the *lower semi-continuity* of the \mathcal{A} -variation.

We say that a sequence $(f_i)_i$ of functions with complex values defined on open set $\Omega \subset \mathbb{R}^N$ is *compactly supported in Ω* if there is a compact set $K \subset \subset \Omega$ such that one has $\text{supp } f_i \subseteq K$ for every i .

We shall make an extensive use of the following concept of convergence.

Definition 4.4. Given $g \in W_c^{\nu-1,1}(\Omega; E)$ and a sequence $(\varphi_i)_i \subseteq \mathcal{D}(\Omega; E)$ we shall write $\varphi_i \rightarrow g$ in case the following conditions hold:

- (i) (φ_i) converges to g in $W^{\nu-1,1}$ norm;
- (ii) (φ_i) is compactly supported in Ω ;
- (iii) $\sup_i \|A\varphi_i\|_1 < +\infty$.

The following lemma is a Friedrich's type lemma; in the case where A is a system of real vector fields, it reminds a result by N. Garofalo and D. Nhieu [6, Lemma A.3]. In order to state it, fix $\eta \in \mathcal{D}(\mathbb{R}^n)$ a radial function with nonnegative values, satisfying $\text{supp } \eta \subseteq B[0, 1]$ and $\int_{\mathbb{R}^n} \eta = 1$, and, for each $\varepsilon > 0$, define $\eta_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ by $\eta_\varepsilon(x) := \varepsilon^{-N} \eta(x/\varepsilon)$.

Lemma 4.5. *For any $g \in BV_{\mathcal{A},c}(\Omega)$, one has:*

$$\lim_{\varepsilon \rightarrow 0} \|A(\eta_\varepsilon * g) - \eta_\varepsilon * (D_{\mathcal{A}}g)\|_{L^1(\Omega)} = 0.$$

Proof. We can assume for simplicity $E = \mathbb{R}^n$ and $F = \mathbb{R}^n$. Let us first assume that $\nu = 1$ and write $A = c + \sum_{j=1}^m a_j \partial_j$ where c and a_j , $1 \leq j \leq m$ are locally Lipschitz functions. Write now, for $x \in \Omega$ and $\varepsilon > 0$ small enough so that one has $|c(y) - c(z)| \leq \delta$ for all $y, z \in \Omega$ with $|y - z| \leq \varepsilon$:

$$|c(x)(\eta_\varepsilon * g)(x) - [\eta_\varepsilon * (cg)](x)| \leq \int_{B(x, \varepsilon)} |c(x) - c(y)| |g(y)| \eta_\varepsilon(x - y) dy \leq \delta \int_{B(x, \varepsilon)} |g(y)| \eta_\varepsilon(x - y) dy.$$

We hence have, for $\varepsilon > 0$ small enough, using Fubini's theorem:

$$\|c(\eta_\varepsilon * g) - \eta_\varepsilon * (cg)\|_{L^1(\Omega)} \leq \delta \|g\|_{L^1(\Omega)}.$$

Writing now $L = \sum_{j=1}^N a_j \partial_j$, it now follows from [6, Lemma A.3] that one also has:

$$\lim_{\varepsilon \rightarrow 0} \|L(\eta_\varepsilon * g) - \eta_\varepsilon * (Lg)\|_{L^1(\Omega)} = 0.$$

This finishes the proof in case $\nu = 1$.

In case $\nu > 1$, it suffices to consider the case $A = a \partial^\alpha$ for some multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq \nu$, and some locally Lipschitz function a . The case $|\alpha| \leq 1$ being already dealt with using the preceding part of the proof, we can assume $A = A' \partial_j$ where $A' = a' \partial^{\alpha'}$ for $|\alpha'| \geq 1$. Since we can now write (using the notation $[\cdot, \cdot]$ for the commutator):

$$[A, \eta_\varepsilon * (\cdot)]g = [A', \eta_\varepsilon * (\cdot)]\partial_j g,$$

it follows inductively from the fact that $\partial_i g \in W_c^{\nu-2,1}(\Omega) \subseteq L_c^1(\Omega)$ and from the preceding part of the proof, that one has:

$$\|[A, \eta_\varepsilon * (\cdot)]g\|_{L^1(\Omega)} \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

The proof is complete. \blacksquare

We now obtain an analogous result, in $BV_{\mathcal{A},c}$, to the standard approximation theorem for BV_c functions.

Lemma 4.6. *For any $g \in BV_{\mathcal{A},c}(U)$, there exists a sequence $\{\varphi_i\}_i \subset \mathcal{D}(U)$ such that one has $\varphi_i \rightharpoonup g$ and, moreover:*

$$\|D_{\mathcal{A}}g\| = \lim_i \|A(\cdot, D)\varphi_i\|_1.$$

Proof. Fix now $g \in BV_{\mathcal{A},c}(\Omega)$ and define for $0 < \varepsilon < \text{dist}(\text{supp } g, \mathbb{C} \setminus \Omega)$ a function $g_\varepsilon \in \mathcal{D}(\Omega)$ by the formula:

$$g_\varepsilon := \eta_\varepsilon * g.$$

It is easy to see that one has $g_\varepsilon \rightarrow g$ in $L^1(\Omega)$ and that there exists a compact set $K \subset\subset \Omega$ such that one has $\text{supp } g_\varepsilon \subseteq K$ for all $\varepsilon > 0$ small enough.

On the other hand, observe that according to the previous lemma, one can write for $\varepsilon > 0$ small enough:

$$(7) \quad A(\eta_\varepsilon * g) = \eta_\varepsilon * (D_{\mathcal{A}}g) + H_\varepsilon(g),$$

where also $\|H_\varepsilon(g)\|_1 \rightarrow 0$, $\varepsilon \rightarrow 0$.

Fix now $v \in C_c^\infty(\Omega, F^*)$ a smooth vector field satisfying $\|v\|_\infty \leq 1$ and compute for $\varepsilon > 0$ small enough:

$$\langle \eta_\varepsilon * (D_{\mathcal{A}}g), \bar{v} \rangle := \langle D_{\mathcal{A}}g, \eta_\varepsilon * \bar{v} \rangle = \int_\Omega \overline{\eta_\varepsilon * \bar{v}} \cdot d[D_{\mathcal{A}}g] = \int_\Omega g \overline{A^*(\cdot, D)(\eta_\varepsilon * v)}.$$

so that one also has:

$$\left| \int_\Omega A(\eta_\varepsilon * g) \cdot \bar{v} \right| \leq \left| \int_\Omega g \overline{A^*(\eta_\varepsilon * v)} \right| + \|H_\varepsilon(g)\|_1 \leq \|D_{\mathcal{A}}g\| + \|H_\varepsilon(g)\|_1.$$

We hence get, by duality:

$$\|A(\eta_\varepsilon * g)\|_1 \leq \|D_{\mathcal{A}}g\| + \|H_\varepsilon(g)\|_1,$$

and the result follows from the aforementioned property of $H_\varepsilon(g)$ when ε approaches 0, and from the lower semicontinuity property already mentioned. \blacksquare

The following proposition is a compactness result in $BV_{\mathcal{A}}$.

Proposition 4.7. *Assume that the open set $U \subseteq \Omega$ supports a SGN inequality of the type appearing in [Theorem 1.1, (1)] as well as an inequality of type (5) for some $0 < \beta < 1$. If $(g_i) \subseteq BV_{\mathcal{A},c}(U)$ is compactly supported in U and if moreover one has:*

$$\sup_i \|D_{\mathcal{A}}g_i\| < +\infty,$$

then there exists $g \in BV_{\mathcal{A},c}(U)$ and a subsequence $(g_{i_k}) \subseteq (g_i)$ converging to g in $W^{\nu-1,1}$.

Proof. Choose a compact set $K \subset\subset U$ for which one has $\text{supp } g_i \subseteq K$ for all i . Choose also, according to Lemma 4.6, a sequence $(\varphi_i) \subseteq \mathcal{D}(U)$ and a compact set $K' \subset\subset U$ satisfying the following conditions for all i :

$$\text{supp } \varphi_i \subseteq K', \quad \|g_i - \varphi_i\|_{W^{\nu-1,1}} \leq 2^{-i} \quad \text{and} \quad \|A\varphi_i\|_1 \leq \|D_{\mathcal{A}}g_i\| + 1.$$

We hence have $\sup_i \|A\varphi_i\|_1 < +\infty$ while it is clear that (φ_i) is compactly supported and satisfies $\|g_i - \varphi_i\|_{W^{\nu-1,1}} \rightarrow 0$, $i \rightarrow \infty$.

Now fix $0 < \beta < 1$, a multi-index $\alpha \in \mathbb{N}^N$ satisfying $|\alpha| = \nu - 1$ and observe that the sequence $(\psi_i)_i$ also satisfies, according to (5):

$$\sup_i \|\partial^\alpha \varphi_i\|_{1-\beta,1} = \sup_i \|\Lambda_{1-\beta} \partial^\alpha \varphi_i\|_1 \leq C \sup_i \|A\varphi_i\|_1 < +\infty.$$

It hence follows from the compactness of the inclusion of $W_c^{1-\beta,1}(U) \subset\subset L^1(U)$ (see [16, Theorem 6.2]) that there exists $h^\alpha \in L_c^1(U)$ and a subsequence $(\varphi_{i_k}^\alpha) \subseteq (\varphi_i)$ such that $\partial^\alpha \varphi_{i_k}^\alpha$ converges to h^α in $L^1(U)$. This yields a subsequence $(\varphi_{i_k}) \subseteq (\varphi_i)$ such that, for all $\alpha \in \mathbb{N}^N$ with $|\alpha| = \nu - 1$, one has $\partial^\alpha \varphi_{i_k} \rightarrow h^\alpha$,

$k \rightarrow \infty$. Using the Rellich-Kondrachov theorem (see *e.g.* Ziemer [22, Theorem 2.5.1]), we can moreover assume that $\varphi_{i_k} \rightarrow g$ in $L_c^1(U)$. According to the closing lemma (see Willem [21, Lemma 6.1.5]), we then have $h^\alpha = \partial^\alpha g$ for all $\alpha \in \mathbb{N}^N$ with $|\alpha| = \nu - 1$. This ensures $g \in W_c^{\nu-1,1}(U)$ and the convergence of (g_{i_k}) to g in $W^{\nu-1,1}(U)$. Moreover, the semicontinuity property of the \mathcal{A} -variation yields $g \in BV_{\mathcal{A},c}(U)$, which terminates the proof. \blacksquare

Remark 4.8. According to Theorem 1.1 and Proposition 3.1, we see that if one assumes A to be elliptic and canceling, each point $x_0 \in \Omega$ is contained a neighborhood $U \subseteq \Omega$ satisfying the hypotheses of the previous proposition.

4.2. A Sobolev-Gagliardo-Nirenberg inequality in $BV_{\mathcal{A}}$. As announced we get the following result:

Proposition 4.9. *Let $A(x, D)$ be as before. Then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$ such that the inequality:*

$$(8) \quad \|g\|_{W^{\nu-1, N/N-1}} \leq C \|D_{\mathcal{A}} g\|,$$

holds for all $g \in BV_{\mathcal{A},c}(U)$, where $C = C(U) > 0$ is a constant depending only on U .

Proof. Fix $x_0 \in \Omega$. It follows from Theorem 1.1 that there exists a neighborhood $U \subset \Omega$ of x_0 and $C = C(U) > 0$ such that, for all $\varphi \in \mathcal{D}(U; E)$, one has:

$$\|\varphi\|_{W^{\nu-1, N/N-1}} \leq C \|A(\cdot, D)\varphi\|_1.$$

Then given $g \in BV_{\mathcal{A},c}(U)$ consider a sequence $\{\varphi_i\} \subset \mathcal{D}(U; E)$ satisfying (i)-(iii) by Lemma 4.6. As a consequence of Fatou's Lemma and the previous estimate we conclude (extracting if necessary a subsequence) that:

$$\|g\|_{W^{\nu-1, N/N-1}} \leq \liminf_{i \rightarrow \infty} \|\varphi_i\|_{W^{\nu-1, N/N-1}} \leq C \lim_{i \rightarrow \infty} \|A(\cdot, D)\varphi_i\|_1 = C \|D_{\mathcal{A}} g\|.$$

The proof is complete. \blacksquare

5. \mathcal{A} -CHARGES AND THEIR EXTENSIONS TO $BV_{\mathcal{A},c}$

We now get back to the original problem of finding, locally, a continuous solution to (2).

5.1. \mathcal{A} -fluxes and \mathcal{A} -charges. Distributions which allow, in an open set Ω , to solve continuously (2), will be called \mathcal{A} -fluxes.

Definition 5.1. A distribution $\mathcal{F} \in \mathcal{D}'(\Omega)$ is called an \mathcal{A} -flux in Ω if the equation (2) has a continuous solution in Ω , *i.e.* if there exists $v \in C(\Omega, F^*)$ such that one has, for all $\varphi \in \mathcal{D}(\Omega; E)$:

$$(9) \quad \mathcal{F}(\varphi) = \int_{\Omega} \bar{v} \cdot A(\cdot, D)\varphi, \quad \forall \varphi \in \mathcal{D}(\Omega; E).$$

\mathcal{A} -fluxes satisfy the following continuity condition.

Lemma 5.2. *If \mathcal{F} is an \mathcal{A} -flux then $\lim_i \mathcal{F}(\varphi_i) = 0$ for every sequence $(\varphi_i)_i \subseteq \mathcal{D}(\Omega; E)$ verifying $\varphi_i \rightarrow 0$.*

Proof. Let \mathcal{F} be an \mathcal{A} -flux and let $v \in C(\Omega, F^*)$ be such that (9) holds. Fix a sequence $(\varphi_i)_i \subseteq \mathcal{D}(\Omega; E)$ verifying $\varphi_i \rightarrow 0$, let $c := \sup_i \|A\varphi_i\|_1 < +\infty$ and choose a compact set $K \subset\subset \Omega$ for which one has $\text{supp } \varphi_i \subseteq K$ for all i .

Fix now $\varepsilon > 0$. According to Weierstrass' approximation theorem, choose a vector field $w \in \mathcal{D}(\Omega, F^*)$ for which one has $\sup_K |v - w| \leq \varepsilon$ and compute, for all i :

$$|\mathcal{F}(\varphi_i)| \leq \left| \int_{\Omega} (\bar{v} - \bar{w}) \cdot A(\cdot, D)\varphi_i \right| + \left| \int_{\Omega} \bar{w} \cdot A(\cdot, D)\varphi_i \right| \leq \varepsilon \|A\varphi_i\|_1 + \left| \int_{\Omega} \varphi_i \overline{A^* w} \right| \leq c\varepsilon + \|A^* w\|_{\infty} \|\varphi_i\|_1.$$

We hence get $\lim_i |\mathcal{F}(\varphi_i)| \leq c\varepsilon$, and the result follows for $\varepsilon > 0$ is arbitrary. \blacksquare

The above property suggests the following definition of linear functionals enjoying some continuity property involving the operator A .

Definition 5.3. A linear functional $\mathcal{F} : \mathcal{D}(\Omega; E) \rightarrow \mathbb{C}$ is called an \mathcal{A} -charge in Ω if $\lim_i \mathcal{F}(\varphi_i) = 0$ for every sequence $(\varphi_i)_i \subseteq \mathcal{D}(\Omega; E)$ satisfying $\varphi_i \rightarrow 0$. The linear space of all \mathcal{A} -charges in Ω is denoted by $CH_{\mathcal{A}}(\Omega)$.

The following characterization of \mathcal{A} -charges will be useful in the sequel.

Proposition 5.4. *If $\mathcal{F} : \mathcal{D}(\Omega; E) \rightarrow \mathbb{C}$ is a linear functional, then the following properties are equivalent*

- (i) \mathcal{F} is an \mathcal{A} -charge,
- (ii) for every $\varepsilon > 0$ and each compact set $K \subset\subset \Omega$ there exists $\theta > 0$ such that, for any $\varphi \in \mathcal{D}_K(\Omega; E)$, one has:

$$(10) \quad |\mathcal{F}(\varphi)| \leq \theta \|\varphi\|_{W^{\nu-1,1}} + \varepsilon \|A(\cdot, D)\varphi\|_1.$$

Proof. We proceed as in [9, Proposition 2.6].

Since (ii) implies trivially (i), it suffices to show that the converse implication holds. To that purpose, assume (i) holds, *i.e.* suppose that \mathcal{F} is an \mathcal{A} -charge. Fix $\varepsilon > 0$ and a compact set $K \subset\subset \Omega$. By hypothesis, there exists $\eta > 0$ such that for every $\varphi \in \mathcal{D}_K(\Omega; E)$ satisfying $\|\varphi\|_{W^{\nu-1,1}} \leq \eta$ and $\|A\varphi\|_1 \leq 1$, we have $|\mathcal{F}(\varphi)| \leq \varepsilon$. We now define $\theta := \varepsilon/\eta$.

Fix $\varphi \in \mathcal{D}_K(\Omega; E)$ and assume by homogeneity that one has $\|A(\cdot, D)\varphi\|_1 = 1$. If moreover one has $\|\varphi\|_{W^{\nu-1,1}} \leq \eta$, then one computes $|\mathcal{F}(\varphi)| \leq \varepsilon = \varepsilon \|A(\cdot, D)\varphi\|_1$. If on the contrary we have $\|\varphi\|_{W^{\nu-1,1}} > \eta$, we define $\tilde{\varphi} = \varphi\eta/\|\varphi\|_{W^{\nu-1,1}}$. We then have $\|\tilde{\varphi}\|_{W^{\nu-1,1}} = \eta$ as well as $\|A(\cdot, D)\tilde{\varphi}\|_1 < 1$, and hence also $|\mathcal{F}(\tilde{\varphi})| \leq \varepsilon$; this yields finally $|\mathcal{F}(\varphi)| = \|\varphi\|_{W^{\nu-1,1}} |\mathcal{F}(\tilde{\varphi})|/\eta \leq \varepsilon \|\varphi\|_{W^{\nu-1,1}}/\eta = \theta \|\varphi\|_{W^{\nu-1,1}}$. ■

As we shall see now, \mathcal{A} -charges can be extended in a unique way to linear forms on $BV_{\mathcal{A},c}$.

Proposition 5.5. *An \mathcal{A} -charge \mathcal{F} in Ω extends in a unique way to a linear functional $\tilde{\mathcal{F}} : BV_{\mathcal{A},c}(\Omega) \rightarrow \mathbb{C}$ satisfying the following property: for any $\varepsilon > 0$ and each compact set $K \subset\subset \Omega$, there exists $\theta > 0$ such that for any $g \in BV_{\mathcal{A},K}(\Omega)$ one has:*

$$(11) \quad |\tilde{\mathcal{F}}(g)| \leq \theta \|g\|_{W^{\nu-1,1}} + \varepsilon \|D_{\mathcal{A}}g\|.$$

Proof. Given $g \in BV_{\mathcal{A},c}(\Omega)$, fix $(\varphi_i)_i \subseteq \mathcal{D}(\Omega; E)$ satisfying $\varphi_i \rightarrow g$ and observe that it follows from (10) that $(\mathcal{F}(\varphi_i))_i$ is a Cauchy sequence of complex numbers whose limit does not depend on the choice of sequence $(\varphi_i)_i \subseteq \mathcal{D}(\Omega; E)$ satisfying $\varphi_i \rightarrow g$. We hence define $\tilde{\mathcal{F}}(g) := \lim_i \mathcal{F}(\varphi_i)$. It now follows readily from (10) and Remark 4.3 that $\tilde{\mathcal{F}}$ satisfies the desired property. ■

Remark 5.6. If $\tilde{\mathcal{F}} : BV_{\mathcal{A},c}(\Omega) \rightarrow \mathbb{C}$ extends the \mathcal{A} -charge \mathcal{F} , it is easy to see from the previous proposition that for any compactly supported sequence $(g_i)_i \subseteq BV_{\mathcal{A},c}(\Omega)$ satisfying $g_i \rightarrow 0$, $i \rightarrow \infty$ in $W^{\nu-1,1}(\Omega; E)$ and $\sup_i \|D_{\mathcal{A}}g_i\| < +\infty$, one has $\mathcal{F}(g_i) \rightarrow 0$, $i \rightarrow \infty$.

From now on, we shall identify any \mathcal{A} -charge with its extension to $BV_{\mathcal{A},c}$ and use the same notation for the two linear forms.

5.2. Examples of \mathcal{A} -charges. Let us define two important classes of \mathcal{A} -charges.

Example 5.7. In case \mathcal{F} is the \mathcal{A} -flux associated to $v \in C(\Omega, F^*)$ according to (9), its unique extension to $BV_{\mathcal{A},c}(\Omega)$ is the \mathcal{A} -charge:

$$\Gamma(v) : BV_{\mathcal{A},c}(\Omega) \rightarrow \mathbb{C}, g \mapsto \int_{\Omega} \bar{v} \cdot d[D_{\mathcal{A}}g].$$

To see this, fix $g \in BV_{\mathcal{A},c}(\Omega)$ together with a sequence $(\varphi_i)_i \subseteq \mathcal{D}(\Omega; E)$ satisfying $\varphi_i \rightarrow g$ and choose a compact set K satisfying $\text{supp } g \subseteq K \subset\subset \Omega$ as well as $\text{supp } \varphi_i \subseteq K$ for all i . Given $\varepsilon > 0$, choose $w \in \mathcal{D}(\Omega, F^*)$ a smooth vector field satisfying $\sup_K |v - w| \leq \varepsilon$ and compute:

$$\left| \Gamma(v)(g) - \int_{\Omega} \bar{v} \cdot d[D_{\mathcal{A}}g] \right| = \lim_i \left| \int_{\Omega} \bar{v} \cdot A(\cdot, D)\varphi_i - \int_{\Omega} \bar{v} \cdot d[D_{\mathcal{A}}g] \right|.$$

On the other hand we have for all i :

$$\begin{aligned} \left| \int_{\Omega} \bar{v} \cdot A(\cdot, D)\varphi_i - \int_{\Omega} \bar{v} \cdot d[D_{\mathcal{A}}g] \right| &\leq \left| \int_{\Omega} (\bar{v} - \bar{w}) \cdot A(\cdot, D)\varphi_i \right| + \left| \int_{\Omega} (\bar{v} - \bar{w}) \cdot d[D_{\mathcal{A}}g] \right| \\ &+ \left| \int_{\Omega} \bar{w} \cdot A(\cdot, D)\varphi_i - \int_{\Omega} \bar{w} \cdot d[D_{\mathcal{A}}g] \right| \leq \varepsilon \|A\varphi_i\|_1 + \varepsilon \|D_{\mathcal{A}}g\| + \left| \int_{\Omega} \varphi_i \overline{A^*w} - \int_{\Omega} \bar{w} \cdot d[D_{\mathcal{L}}g] \right|. \end{aligned}$$

Using the properties of $(\varphi_i)_i$ and Lebesgue's dominated convergence theorem, we thus get:

$$\lim_i \left| \int_{\Omega} \bar{v} \cdot A(\cdot, D)\varphi_i - \int_{\Omega} \bar{v} \cdot d[D_{\mathcal{A}}g] \right| \leq 2\varepsilon \|D_{\mathcal{A}}g\| + \left| \int_{\Omega} g \overline{A^*(\cdot, D)w} - \int_{\Omega} \bar{w} \cdot d[D_{\mathcal{L}}g] \right| = 2\varepsilon \|D_{\mathcal{A}}g\|,$$

according to (6). The result follows, for $\varepsilon > 0$ is arbitrary.

Example 5.8. Assume that U supports a SGN inequality of type (8) for $BV_{\mathcal{A}}$ functions in U . Then given $f \in W_{\text{loc}}^{-(\nu-1),N}(U; E^*)$, f extends uniquely to an \mathcal{A} -charge in U .

To see this, fix $K \subset\subset U$ and infer from Lemma 2.1 that there exist $(g_{\alpha})_{|\alpha|=\nu-1} \subseteq L^N(U; E^*)$ such that

$$\langle f, \varphi \rangle = \sum_{|\alpha|=\nu-1} \int_U g_{\alpha} \overline{\partial^{\alpha} \varphi}, \quad \forall \varphi \in \mathcal{D}_K(U; E).$$

Now fix $\varepsilon > 0$ and choose $\theta > 0$ large enough for $\left(\int_{K \cap \{|g_{\alpha}| > \theta\}} |g_{\alpha}|^N \right)^{1/N} \leq \varepsilon/C$ to hold for any $|\alpha| = \nu - 1$, where C is a positive constant satisfying (8). We then compute, for $\varphi \in \mathcal{D}_K(U; E)$:

$$\begin{aligned} |\langle f, \varphi \rangle| &\leq \theta \left(\sum_{|\alpha|=\nu-1} \int_{K \cap \{|g_{\alpha}| \leq \theta\}} |\partial^{\alpha} \varphi| \right) + \sum_{|\alpha|=\nu-1} \int_{K \cap \{|g_{\alpha}| > \theta\}} |g_{\alpha} \partial^{\alpha} \varphi|, \\ &\leq \theta \|\varphi\|_{W^{\nu-1,1}} + \sum_{|\alpha|=\nu-1} \left(\int_{K \cap \{|g_{\alpha}| > \theta\}} |g_{\alpha}|^N \right)^{1/N} \|\partial^{\alpha} \varphi\|_{N/(N-1)}, \\ &\leq \theta \|\varphi\|_{W^{\nu-1,1}} + \varepsilon \|A(\cdot, D)\varphi\|_{L^1}. \end{aligned}$$

The conclusion that f extends to an \mathcal{A} -charge follows by approximation.

Example 5.9. Assume that U supports a SGN inequality of type (8) for $BV_{\mathcal{A}}$ functions in U . Given $f \in L_{\text{loc}}^N(U; E^*)$ we know from Example 2.3 that it defines a distribution in $W_{\text{loc}}^{-(\nu-1),N}(U; E^*)$. We then define, for $\varphi \in \mathcal{D}(U; E)$:

$$\langle \Lambda(f), \varphi \rangle := \overline{\langle f, \bar{\varphi} \rangle} = \int_U \bar{f} \varphi.$$

It follows from the previous example and from approximation that $\Lambda(f)$ extends uniquely to an \mathcal{A} -charge in U verifying, for all $g \in BV_{\mathcal{A},c}(U)$:

$$\langle \Lambda(f), g \rangle = \int_U \bar{f} g.$$

Remark 5.10. It is easy to see that for any $x_0 \in \Omega$, there exists an open neighborhood $U \subseteq \Omega$ of x_0 such that one has $\Lambda[\mathcal{D}(U; E)] \subseteq \Gamma[C^{\infty}(U, F^*)]$. Given $\varphi \in \mathcal{D}(U; E)$, thanks to the local solvability of the elliptic operator $\Delta_{\mathcal{A}} = A^*(\cdot, D) \circ A(\cdot, D)$ (see [7, Corollary 4.8]), there exists $u \in C^{\infty}(U; E)$ a smooth solution to $\Delta_{\mathcal{A}} u = \varphi$ in U . Let $v := A(\cdot, D)u$. This yields, for any $g \in BV_{\mathcal{A},c}(U)$:

$$\Lambda(\varphi)(g) = \int_U \bar{\varphi} g = \int_U g \overline{A^*(\cdot, D)v} = \int_U \bar{v} \cdot d[D_{\mathcal{A}}g] = \Gamma(v)(g),$$

for we could, in the computation above, replace v by $v\chi$ where $\chi \in \mathcal{D}(U)$ satisfies $\chi = 1$ in a neighborhood of $\text{supp } g$.

It turns out that a linear functional on $BV_{\mathcal{A},c}$ is an \mathcal{A} -charge if and only if it is continuous with respect to some locally convex topology on $BV_{\mathcal{A},c}$.

5.3. Another characterization of \mathcal{A} -charges. In the sequel, a *locally convex space* means a Hausdorff locally convex topological vector space. For any family \mathcal{O} of sets and any set X we denote $\mathcal{O} \sqsubset X := \{O \cap X : O \in \mathcal{O}\}$. Following [8, Theorem 3.3] we define the following topology on $BV_{\mathcal{A},c}(\Omega)$, called the *localized topology* associated to the family of subspaces $BV_{\mathcal{A},K,\lambda}(\Omega)$.

Definition 5.11. Let $\mathcal{T}_{\mathcal{A}}$ be the unique locally convex topology on $BV_{\mathcal{A},c}(\Omega)$ such that

- (a) $\mathcal{T}_{\mathcal{A}} \sqsubset BV_{\mathcal{A},K,\lambda} \subseteq \mathcal{T}_{W^{\nu-1,1}} \sqsubset BV_{\mathcal{A},K,\lambda}$ for all $K \subset\subset \Omega$ and $\lambda > 0$ where we let:

$$BV_{\mathcal{A},K,\lambda} = \{g \in BV_{\mathcal{A},c}(\Omega) : \text{supp } g \subseteq K, \|D_{\mathcal{A}}g\| \leq \lambda\},$$

and where $\mathcal{T}_{W^{\nu-1,1}}$ is the $W^{\nu-1,1}$ -topology;

- (b) for every locally convex space Y , a linear map $f : (BV_{\mathcal{A},c}; \mathcal{T}_{\mathcal{A}}) \rightarrow Y$ is continuous if and only if $f \upharpoonright BV_{\mathcal{A},K,\lambda}$ is $W^{\nu-1,1}$ continuous for all $K \subset\subset \Omega$ and $\lambda > 0$.

Remark 5.12. Uniqueness of the above topology is easily seen according to property (b). Concerning the existence, one can define the topology $\mathcal{T}_{\mathcal{A}}$ by constructing a basis of neighborhoods $\mathcal{B}_{\mathcal{A}}$ of the origin in the following way: denote by $\mathcal{B}_{\mathcal{A}}$ the collection of all absorbing, balanced and convex subsets $U \subseteq BV_{\mathcal{A}}(\Omega)$ satisfying $U \sqsubset BV_{\mathcal{A},K,\lambda} \in \mathcal{T}_{W^{\nu-1,1}} \sqsubset BV_{\mathcal{A},K,\lambda}$. Calling $\mathcal{T}_{\mathcal{A}}$ the vector topology on $BV_{\mathcal{A}}(\Omega)$ admitting $\mathcal{B}_{\mathcal{A}}$ as a neighborhood basis at the origin, one can see that it satisfies properties (a) and (b) above.

Choosing $(K_k)_{k \in \mathbb{N}}$ an increasing sequence of compact sets exhausting Ω and defining $X_k := BV_{\mathcal{A},K_k}(\Omega)$ for all $k \in \mathbb{N}$, we have:

$$BV_{\mathcal{A},K_k,k} = \{g \in X_k : \|D_{\mathcal{A}}g\| \leq k\}.$$

Since it is straightforward to see that all the vector spaces $BV_{\mathcal{A},K_k}$ are closed in the $W^{\nu-1,1}$ topology, and that $\|D_{\mathcal{A}} \cdot\|$ is a lower semicontinuous norm on $BV_{\mathcal{A},c}(\Omega)$, it now follows readily from [8, Proposition 3.11] that $\mathcal{T}_{\mathcal{A}}$ -continuous linear functionals on $BV_{\mathcal{A}}(\Omega)$ are exactly the \mathcal{A} -charges in Ω .

Proposition 5.13. *A linear functional $\mathcal{F} : BV_{\mathcal{A},c}(\Omega) \rightarrow \mathbb{C}$ is an \mathcal{A} -charge if and only if it is $\mathcal{T}_{\mathcal{A}}$ -continuous.*

The following result will be useful in the sequel.

Corollary 5.14. *Assume that $K \subset\subset \mathbb{R}^n$ is a compact set and that $\lambda > 0$ is a real number. If $(g_i)_{i \in I} \subseteq BV_{\mathcal{A},K,\lambda}(U)$ converges to 0 as distributions, i.e. if one has $\int_U g_i \varphi \rightarrow 0$ for all $\varphi \in \mathcal{D}(U; E^*)$, then the net $(\|g_i\|_{W^{\nu-1,1}})_{i \in I}$ also converges to 0.*

Proof. Proceed towards a contradiction, assume that $(g_i)_{i \in I}$ is as in the statement, and that $(\|g_i\|_{W^{\nu-1,1}})_{i \in I}$ fails to converge to 0, meaning that there is $\varepsilon > 0$ such that for all $i \in I$, one can find $j \in J$ satisfying $j \geq i$ and $\|g_j\|_{W^{\nu-1,1}} \geq \varepsilon$. Define then $J := \{i \in I : \|g_i\|_{W^{\nu-1,1}} \geq \varepsilon\}$, observe J is a directed set and consider the net $(g_j)_{j \in J}$. Since $BV_{\mathcal{A},K,\lambda}(U)$ is compact in the $W^{\nu-1,1}$ topology according to Proposition 4.7, we know that there exists a cluster point $g \in BV_{\mathcal{A},K,\lambda}(U)$ of $(g_j)_{j \in J}$ in the $W^{\nu-1,1}$ topology. It's easy to see from the definition of J that one has $g \neq 0$. On the other hand, fix $\varphi \in \mathcal{D}(U; E^*)$. Since $(g_j)_{j \in J}$ converges to 0 as distributions, we get for $j \in J$:

$$\int_U g_j \varphi \rightarrow 0.$$

Yet we should also get for $j \in J$:

$$\left| \int_U g_j \varphi - \int_U g \varphi \right| \leq C_{\varphi} \|g_j - g\|_{W^{\nu-1,1}(U)} \rightarrow 0,$$

which implies that $\int_U g \varphi = 0$. Since $\varphi \in \mathcal{D}(U; E^*)$ is arbitrary, this means that $g = 0$, which is a contradiction; the proof is complete. \blacksquare

We now turn to proving the key result for obtaining Theorem 1.3.

6. TOWARDS THEOREM 1.3

Throughout this section, we assume that the open set $U \subseteq \Omega$ supports inequalities of type (1) and (5); we also assume that one has $\Lambda[\mathcal{D}(U; E)] \subseteq \Gamma[C(U, F^*)]$.

Remark 6.1. It follows from Theorem 1.1, Proposition 3.1 and Remark 5.10 that for any $x_0 \in \Omega$, one can find an open neighborhood U of x_0 in Ω satisfying all the above assumptions.

Our intention is to prove the following result.

Theorem 6.2. *If $\mathcal{F} : BV_{\mathcal{A},c}(U) \rightarrow F$ is an \mathcal{A} -charge in U , then there exists $v \in C(U, F^*)$ for which one has $\mathcal{F} = \Gamma(v)$, i.e. such that one has, for any $g \in BV_{\mathcal{A},c}(U)$:*

$$\mathcal{F}(g) = \int_U \bar{v} \cdot d[D_{\mathcal{A}}g].$$

To prove this theorem, we have to show that the map

$$\Gamma : C(U, F^*) \longrightarrow CH_{\mathcal{A}}(U), v \mapsto \Gamma(v),$$

is surjective. In order to do this, we endow $C(U, F^*)$ with the usual Fréchet topology of uniform convergence on compact sets, and $CH_{\mathcal{A}}(U)$ with the Fréchet topology associated to the family of seminorms $(\|\cdot\|_K)_K$ defined by:

$$\|\mathcal{F}\|_K := \sup \{|\mathcal{F}(g)| : g \in BV_{\mathcal{A},K}(U), \|D_{\mathcal{A}}g\| \leq 1\},$$

where K ranges over all compact sets $K \subset\subset U$. The surjectivity of Γ will be proven in case we show that Γ is continuous and verifies the following two facts:

- (a) $\Gamma[C(U, F^*)]$ is dense in $CH_{\mathcal{A}}(U)$.
- (b) $\Gamma^*[CH_{\mathcal{A}}(U)^*]$ is sequentially closed in the strong topology of $C(U, F^*)^*$.

Indeed, it will then follow from the Closed Range Theorem [10, Theorem 8.6.13] together with [8, Proposition 6.8] and (b) that $\Gamma[C(U, F^*)]$ is closed in $CH_{\mathcal{A}}(U)$. Using (a) we shall then conclude that one has:

$$\Gamma[C(U, F^*)] = CH_{\mathcal{A}}(U),$$

i.e. that Γ is surjective.

The strategy of the proof of (b) follows the lines of De Pauw and Pfeffer's proof in [9]. For the proof of (a), however, the proof presented below is slightly different from their approach; we namely manage to avoid an explicit smoothing process and choose instead to use an abstract approach similar to the one used in [15] in order to solve the equation $d\omega = F$.

Let us start by showing that $\Gamma : C(U, F^*) \longrightarrow CH_{\mathcal{A}}(U)$ is linear and continuous. Indeed given a compact set $K \subset\subset U$ and $g \in BV_{\mathcal{A},K}(U)$ we have:

$$|\Gamma(v)(g)| = \left| \int_U \bar{v} \cdot d[D_{\mathcal{A}}g] \right| \leq \|D_{\mathcal{A}}g\| \|v\|_{\infty, K},$$

which implies $\|\Gamma(v)\|_K \leq \|v\|_{\infty, K}$. Next we identify the dual space $CH_{\mathcal{A}}(U)^*$.

6.1. Identifying $CH_{\mathcal{A}}(U)^*$.

Proposition 6.3. *The map $\Phi : BV_{\mathcal{A},c}(U) \longrightarrow CH_{\mathcal{A}}(U)^*$ given by $\Phi(g)(F) := F(g)$ is a linear bijection.*

First let us check that Φ is well defined. In fact, given $K \subset\subset U$ and $g \in BV_{\mathcal{A},K}(U)$ we have

$$|\Phi(g)(F)| = |F(g)| \leq \|D_{\mathcal{A}}g\| \|F\|_K,$$

according to the definition of $\|\cdot\|_K$. Hence $\Phi(g)$ is continuous and $\Phi(g) \in CH_{\mathcal{A}}(U)^*$.

To show that Φ is injective, let $g \in BV_{\mathcal{A},c}(U)$ be such that $\Phi(g) = 0$. Then for any $B \subset U$ measurable and bounded and for any $e^* \in E^*$ we have:

$$\int_B e^* g = \int_U \chi_B e^* g = \Lambda(\chi_B e^*)(g) = \Phi(g)[\Lambda(\chi_B e^*)] = 0.$$

Thus $e^*g = 0$ a.e. in U , which implies that $g = 0$ (and hence that Φ injective) since $e^* \in E^*$ is arbitrary.

The next step is to prove that Φ is surjective. To show this property we shall define a right inverse for Φ , called Ψ .

Let $\Psi : CH_{\mathcal{A}}(U)^* \longrightarrow \mathcal{D}'(U; E^*)$ be defined by:

$$(12) \quad \Psi(\alpha)[\varphi] := \alpha[\Lambda(\varphi)].$$

We claim that Ψ is well defined, *i.e.* that for $\alpha \in CH_{\mathcal{A}}(U)^*$, we have $\Psi(\alpha) \in BV_{\mathcal{A},c}(U)$. Indeed, given $\alpha \in CH_{\mathcal{L}}(U)^*$ there exist $C > 0$ and $K \subset\subset U$ such that for all $F \in CH_{\mathcal{L}}(U)$ we have $|\alpha(F)| \leq C\|F\|_K$. In particular, for every $\varphi \in \mathcal{D}(U; E^*)$ we have:

$$\begin{aligned} |\Psi(\alpha)(\varphi)| &\leq C\|\Lambda(\varphi)\|_K \\ &\leq C \sup \{ |\Lambda(\varphi)(g)| : g \in BV_{\mathcal{A},K}(U), \|D_{\mathcal{A}}g\| \leq 1 \} \leq C \sup \left\{ \int_U |\bar{\varphi}g| : g \in BV_{\mathcal{A},K}(U), \|D_{\mathcal{A}}g\| \leq 1 \right\}, \end{aligned}$$

from which it already follows that one has $\text{supp } \Phi(\alpha) \subseteq K$, since the above inequalities yield $\Phi(\alpha)(\varphi) = 0$ if $\text{supp } \varphi \cap K = \emptyset$. According to Remark 2.4, we see moreover that for any $g \in BV_{\mathcal{A},K}(U)$ satisfying $\|D_{\mathcal{A}}g\| \leq 1$, one has:

$$\begin{aligned} \int_U |\bar{\varphi}g| &\leq \|\varphi\|_{\mathbf{W}^{\nu-1,N/(N-1)}(U;E)^*} \|g\|_{\mathbf{W}^{\nu-1,N/(N-1)}(U;E)} \\ &\leq C(K, \nu) \|\varphi\|_{\mathbf{W}^{\nu-1,N/(N-1)}(U;E)^*} \|g\|_{W^{\nu-1,N/(N-1)}(U;E)} \\ &\leq \tilde{C}(K, \nu) \|\varphi\|_{\mathbf{W}^{\nu-1,N/(N-1)}(U;E)^*}, \end{aligned}$$

where the latter inequality comes from the SGN inequality in $BV_{\mathcal{A}}$ (Proposition 4.9). It follows then from the reflexivity of $\mathbf{W}^{\nu-1,N/(N-1)}(U;E)$ that one has $g \in \mathbf{W}^{\nu-1,N/(N-1)}(U;E) \subseteq W^{\nu-1,N/(N-1)}(U;E)$: using, indeed, Hahn-Banach's theorem we can extend the map $T : \mathcal{D}(U; E^*) \rightarrow \mathbb{C}, \psi \mapsto \int_U g\psi$ into a map $\tilde{T} \in [\mathbf{W}^{\nu-1,N/(N-1)}(U;E)^*]^*$ itself being the evaluation at $h \in \mathbf{W}^{\nu-1,N/(N-1)}(U;E)$, so that one has for $\varphi \in \mathcal{D}(U; E^*)$:

$$\int_U \bar{\varphi}g = T(\bar{\varphi}) = \tilde{T}(\bar{\varphi}) = \langle \bar{\varphi}, h \rangle = \int_U \bar{\varphi}h.$$

This yields $g = h$ a.e., meaning that one has $g \in \mathbf{W}^{\nu-1,N/(N-1)}(U;E)$.

Moreover, for any $v \in C_c^\infty(U, F^*)$ we have:

$$\begin{aligned} |\Psi(\alpha)[A^*(\cdot, D)v]| &= |\alpha[\Lambda(A^*(\cdot, D)v)]|, \\ &\leq C\|\Lambda(A^*(\cdot, D)v)\|_K \\ &\leq C \sup \left\{ \left| \int_U g \overline{A^*(\cdot, D)v} \right| : g \in BV_{\mathcal{A},K}(U), \|D_{\mathcal{A}}g\| \leq 1 \right\} \\ &\leq C \sup \{ \|D_{\mathcal{A}}g\| \|v\|_\infty : g \in BV_{\mathcal{A},K}(U), \|D_{\mathcal{A}}g\| \leq 1 \} \\ &\leq C \sup \|v\|_\infty, \end{aligned}$$

so that one has $\Psi(\alpha) \in BV_{\mathcal{A},c}(U)$.

Lemma 6.4. *The maps Φ and Ψ defined above are inverses, *i.e.* we have:*

- (i) $\Psi \circ \Phi = Id_{BV_{\mathcal{A},c}(U)}$;
- (ii) $\Phi \circ \Psi = Id_{CH_{\mathcal{A}}(U)^*}$ (in particular, Φ is surjective).

In order to prove the previous lemma, we shall need some observations concerning the polar sets of some neighborhoods of the origin in $CH_{\mathcal{A}}(U)$. First, observe that the family of all sets $V(K, \varepsilon)$ (where K ranges over all compact subsets of U , and ε over all positive real numbers) defined by:

$$V(K, \varepsilon) := \{F \in CH_{\mathcal{A}}(U) : \|F\|_K \leq \varepsilon\},$$

is a basis of neighborhoods of the origin in $CH_{\mathcal{A}}(U)$.

Claim 6.5. Fix $K \subset\subset U$ a compact set and a real number $\varepsilon > 0$. For any $\alpha \in V(K, \varepsilon)^\circ$, one has:

- (i) $\text{supp } \Psi(\alpha) \subseteq K$;
- (ii) $\|D_{\mathcal{A}}\Psi(\alpha)\| \leq \frac{1}{\varepsilon}$.

Proof. To prove (i), assume that $\varphi \in \mathcal{D}(U)$ satisfies $K \cap \text{supp } \varphi = \emptyset$. Then, we get for $\lambda > 0$:

$$\|\lambda\Lambda(\varphi)\|_K = \sup \left\{ \lambda \left| \int_U \bar{\varphi} g \right| : g \in BV_{\mathcal{A},K,\lambda}(U), \|D_{\mathcal{A}}g\| \leq 1 \right\} = 0.$$

In particular this yields $\lambda\Lambda(\varphi) \in V(K, \varepsilon)$. We hence obtain:

$$\lambda|\alpha[\Lambda(\varphi)]| = |\alpha[\lambda\Lambda(\varphi)]| \leq 1,$$

for any $\lambda > 0$. Since $\lambda > 0$ is arbitrary, this implies that one has $\alpha[\Lambda(\varphi)] = 0$, *i.e.* that $\Psi(\alpha)(\varphi) = 0$. We may now conclude that $\text{supp } \Psi(\alpha) \subseteq K$. In order to obtain statement (ii), fix $v \in C_c^\infty(U, F^*)$ satisfying $\|v\|_\infty \leq 1$ and compute:

$$\begin{aligned} \|\varepsilon\Lambda[A^*(\cdot, D)v]\|_K &= \varepsilon\|\Lambda[A^*(\cdot, D)v]\|_K, \\ &= \varepsilon \sup \left\{ \left| \int_U g \overline{A^*(\cdot, D)v} \right| : g \in BV_{\mathcal{A},K}, \|D_{\mathcal{A}}g\| \leq 1 \right\}, \\ &= \varepsilon \sup \left\{ \left| \int_U \bar{v} \cdot d[D_{\mathcal{A}}g] \right| : g \in BV_{\mathcal{A},K}, \|D_{\mathcal{A}}g\| \leq 1 \right\}, \\ &\leq \varepsilon \sup \{ \|D_{\mathcal{A}}g\| \cdot \|v\|_\infty : g \in BV_{\mathcal{A},K}, \|D_{\mathcal{A}}g\| \leq 1 \}, \\ &\leq \varepsilon, \end{aligned}$$

so that one has $\varepsilon\Lambda(A^*(\cdot, D)v) \in V(K, \varepsilon)$. It hence follows that:

$$\varepsilon|\Psi(\alpha)(A^*(\cdot, D)v)| = |\alpha[\varepsilon\Lambda(A^*(\cdot, D)v)]| \leq 1,$$

and we thus get:

$$|\Psi(\alpha)[A^*(\cdot, D)v]| \leq \frac{1}{\varepsilon}.$$

Since $v \in \mathcal{D}(U, F^*)$ is an arbitrary vector field satisfying $\|v\|_\infty \leq 1$, this yields $\|D_{\mathcal{A}}\Psi(\alpha)\| \leq \frac{1}{\varepsilon}$, and concludes the proof of the claim. \blacksquare

We now turn to proving Lemma 6.4.

Proof of Lemma 6.4. To prove part (i), fix $g \in BV_{\mathcal{A},c}(U)$ and compute, for $\varphi \in \mathcal{D}(U; E^*)$:

$$\Psi[\Phi(g)](\varphi) := \Phi(g)[\Lambda(\varphi)] = \Lambda(\varphi)(g) = \int_U \bar{\varphi} g,$$

that is, $\Psi[\Phi(g)] = g$ in the sense of distributions.

In order to prove part (ii), fix $\alpha \in CH_{\mathcal{A}}^*(U)$. We have to show that, for any $F \in CH_{\mathcal{A}}(U)$, we have:

$$\Phi[\Psi(\alpha)](F) = \alpha(F),$$

i.e. that for any $F \in CH_{\mathcal{A}}(U)$, one has:

$$F[\Psi(\alpha)] = \alpha(F).$$

To this purpose, define for any $F \in CH_{\mathcal{A}}(U)$ a map:

$$\Delta_F : CH_{\mathcal{A}}(U)^* \rightarrow \mathbb{C}, \alpha \mapsto \Delta_F(\alpha) := F[\Psi(\alpha)].$$

Claim 6.6. Given $F \in CH_{\mathcal{A}}(U)$, the map Δ_F is weakly*-continuous on $V(K, \varepsilon)^\circ$ for all $K \subset\subset U$ and $\varepsilon > 0$.

To prove this claim, fix $K \subset\subset U$, $\varepsilon > 0$ and assume that $(\alpha_i)_{i \in I} \subseteq V(K, \varepsilon)^\circ$ is a net weak*-converging to 0. In particular one gets:

- (a) for any $\varphi \in \mathcal{D}(U; E^*)$, we have $\Lambda(\varphi) \in CH_{\mathcal{A}}(U)$ and hence the net $(\Psi(\alpha_i)(\varphi))_{i \in I} = (\alpha_i[\Lambda(\varphi)])_{i \in I}$ converges to 0; in other terms, the net $(\Psi(\alpha_i))_{i \in I}$ converges to 0 in the sense of distributions.

According to Claim 6.5, we moreover have:

- (b) $\text{supp } \Psi(\alpha_i) \subseteq K$ for each $i \in I$;
- (c) $c := \sup_{i \in I} \|D_{\mathcal{A}}\Psi(\alpha_i)\| \leq \frac{1}{\varepsilon}$.

We thus have $(\Psi(\alpha_i))_{i \in I} \subseteq BV_{\mathcal{A}, K, 1/\varepsilon}$. It hence follows from Corollary 5.14 that the net $(\|\Psi(\alpha_i)\|_{W^{\nu-1,1}})_{i \in I}$ converges to 0. From the fact that F is an \mathcal{A} -charge we see that the net $(F[\Psi(\alpha_i)])_{i \in I}$ converges to 0 as well. This means, in turn, that $(\Delta_F(\alpha_i))_{i \in I}$ converges to 0, which shows that Δ_F is weak*-continuous on $V(K, \varepsilon)^\circ$.

Claim 6.7. For any $\alpha \in CH_{\mathcal{A}}(U)^*$, we have $\Delta_F(\alpha) = \alpha(F)$.

To prove the latter claim, observe that according to Claim 6.6 and to the Banach-Grothendieck theorem [10, Theorem 8.5.1], there exists $\tilde{F} \in CH_{\mathcal{A}}(U)$ such that for any $\alpha \in CH_{\mathcal{A}}(U)^*$, we have:

$$\Delta_F(\alpha) = \alpha(\tilde{F}).$$

Yet given $g \in BV_{\mathcal{A}, c}(U)$, we then have, according to [Lemma 6.4, (i)]:

$$F(g) = F\{\Psi[\Phi(g)]\} = \Delta_F[\Phi(g)] = \Phi(g)(\tilde{F}) = \tilde{F}(g),$$

i.e. $F = \tilde{F}$, which proves the claim.

It now suffices to observe that Lemma 6.4 is proven for we have established the equality $F[\Psi(\alpha)] = \alpha(F)$ for any $F \in CH_{\mathcal{A}}(U)$ and $\alpha \in CH_{\mathcal{A}}(U)^*$. ■

As a corollary, we get a proof of the density of $\Lambda[\mathcal{D}(U)]$ and $\Gamma[C(U, F^*)]$ in $CH_{\mathcal{A}}(U)$.

Corollary 6.8. *The space $\Lambda[\mathcal{D}(U; E^*)]$ is dense in $CH_{\mathcal{A}}(U)$.*

Proof. Assuming that $\alpha \in CH_{\mathcal{A}}(U)^*$ satisfies $\alpha \upharpoonright \Lambda[\mathcal{D}(U; E^*)] = 0$, we compute for any $\varphi \in \mathcal{D}(U; E^*)$:

$$\Psi(\alpha)(\varphi) := \alpha[\Lambda(\varphi)] = 0.$$

This means that $\Psi(\alpha) = 0$, and implies that $\alpha = \Phi \circ \Psi(\alpha) = \Phi(0) = 0$. The result then follows from the Hahn-Banach theorem. ■

Corollary 6.9. *The space $\Gamma[C(U, F^*)]$ is dense in $CH_{\mathcal{A}}(U)$.*

Proof. It follows from the previous corollary that $\Lambda[\mathcal{D}(U; E^*)]$ is dense in $CH_{\mathcal{A}}(U)$. Since by hypothesis we also have $\Lambda[\mathcal{D}(U; E^*)] \subseteq \Gamma[C(U, F^*)] \subseteq CH_{\mathcal{A}}(U)$, it is clear that $\Gamma(U, F^*)$ is dense in $CH_{\mathcal{A}}(U)$. ■

In order to study the range of Γ^* , we introduce the following linear operator:

$$\Xi : BV_{\mathcal{A}, c}(U) \rightarrow C(U, F^*)^*, g \mapsto \Xi(g),$$

defined by $\Xi(g)(v) := \Gamma(v)(g)$ for any $v \in C(U, F^*)$.

Claim 6.10. We have $\text{im } \Gamma^* = \text{im } \Xi$.

Proof. To prove this claim, fix $\mu \in C(U, F^*)^*$. If one has $\mu = \Gamma^*(\alpha)$ for some $\alpha \in CH_{\mathcal{A}}(U)^*$, then we compute for $v \in C(U, F^*)$:

$$\Xi[\Psi(\alpha)](v) = \Gamma(v)[\Psi(\alpha)] = \Phi[\Psi(\alpha)][\Gamma(v)] = \alpha[\Gamma(v)] = \Gamma^*(\alpha)(v) = \mu(v),$$

so that one has $\mu = \Xi[\Psi(\alpha)] \in \text{im } \Xi$. Conversely, if one has $\mu = \Xi(g)$ for some $g \in BV_{\mathcal{A}, c}(U)$, then we compute for $v \in C(U, F^*)$:

$$\Gamma^*[\Phi(g)](v) = \Phi(g)[\Gamma(v)] = \Gamma(v)(g) = \Xi(g)(v) = \mu(v),$$

so that one has $\mu = \Gamma^*[\Phi(g)] \in \text{im } \Gamma^*$. ■

Consider the set

$$B := \{v \in C(U, F^*) : \|v\|_\infty \leq 1\}.$$

It is clear that B is bounded in $C(U, F^*)$. Hence the seminorm:

$$p : C(U, F^*)^* \rightarrow \mathbb{R}_+, \mu \mapsto p(\mu) := \sup_{v \in B} |\mu(v)|,$$

is strongly continuous (i.e. continuous with respect to the strong topology) on $C(U, F^*)^*$. Observe now that one has, for $g \in BV_{\mathcal{A},c}(U)$:

$$\begin{aligned} p[\Xi(g)] &= \sup_{v \in B} |\Xi(g)(v)|, \\ &= \sup\{|\Gamma(v)(g)| : v \in B\}, \\ &= \|D_{\mathcal{A}}g\|. \end{aligned}$$

Lemma 6.11. *The set $\text{im } \Xi$ is strongly sequentially closed in $C(U, F^*)^*$.*

Proof. Fix a sequence $(\Xi(g_k))_{k \in \mathbb{N}} \subseteq \text{im } \Xi$ and assume that, in the strong topology, one has:

$$\Xi(g_k) \rightarrow \mu \in C(U, F^*)^*, \quad k \rightarrow \infty.$$

The strong continuity of p then yields:

$$c := \sup_{k \in \mathbb{N}} \|D_{\mathcal{A}}g_k\| = \sup_{k \in \mathbb{N}} p[\Xi(g_k)] < +\infty.$$

Claim 6.12. There exists a compact set $K \subset\subset U$ such that one has $\text{supp } g_k \subseteq K$ for each $k \in \mathbb{N}$.

To prove this claim, let us first prove that the sequence $(\text{supp } D_{\mathcal{A}}g_k)_{k \in \mathbb{N}}$ is compactly supported in U (i.e. that there is a compact subset of U containing $\text{supp } D_{\mathcal{A}}g_k$ for all k). To this purpose, we proceed towards a contradiction and assume that it is not the case. Let then $U = \bigcup_{j \in \mathbb{N}} U_j$ be an exhaustion of U by open sets satisfying, for each $j \in \mathbb{N}$, $\bar{U}_j \subseteq U_{j+1}$ and such that \bar{U}_j is a compact subset of U for each $j \in \mathbb{N}$. Since $(\text{supp } D_{\mathcal{A}}g_k)_{k \in \mathbb{N}}$ is not compactly supported, there exist increasing sequences of integers $(j_l)_{l \in \mathbb{N}}$ and $(k_l)_{l \in \mathbb{N}}$ satisfying, for any $l \in \mathbb{N}$:

$$\text{supp}(D_{\mathcal{A}}g_{k_l}) \cap (U_{j_l+1} \setminus \bar{U}_{j_l}) \neq \emptyset.$$

In particular, there exists for each $l \in \mathbb{N}$ a vector field $v_l \in C_c(U_{j_l+1} \setminus \bar{U}_{j_l}, F^*)$ with $\|v_l\|_{\infty} \leq 1$ and:

$$a_l := \left| \int_U \bar{v}_l \cdot d[D_{\mathcal{A}}g_{k_l}] \right| > 0.$$

Let now, for $l \in \mathbb{N}$, $b_l := \max_{0 \leq k \leq l} \frac{1}{a_k}$ and define a bounded set $B' \subseteq C(U, F^*)$ by:

$$B' := \left\{ v \in C(U, F^*) : \|v\|_{\infty, \bar{U}_{j_l+1}} \leq lb_l \text{ for each } l \in \mathbb{N} \right\}.$$

It follows from the construction of B that one has $w_l := lb_l v_l \in B$ for any $l \in \mathbb{N}$. Moreover the seminorm

$$p' := C(U, F^*)^* \rightarrow \mathbb{R}_+, \mu \mapsto \sup_{v \in B'} |\mu(v)|,$$

is strongly continuous. Yet we get for $l \in \mathbb{N}$:

$$p'[\Xi(g_{k_l})] \geq |\Xi(g_{k_l})(w_l)| = |\Gamma(w_l)(g_{k_l})| = lb_l \left| \int_U \bar{v}_l \cdot d(Dg_{k_l}) \right| = lb_l a_l \geq l.$$

Since this yields $p'[\Xi(g_{k_l})] \rightarrow \infty$, $l \rightarrow \infty$, we get a contradiction with the fact that p' is strongly continuous (recall that $(\Xi(g_{k_l}))_{l \in \mathbb{N}}$ converges in the strong topology).

Claim 6.13. Let V be an open set and let $r(x, D) \in S^{-\infty}$ be a regularizing operator. Assume that $g \in L^{N/N-1}(V)$ satisfies $\int_V g[\psi - r(x, D)\psi] = 0$ for all $\psi \in \mathcal{D}(V, F^*)$. Under those assumptions, one has $g = 0$ in V .

Proof. Fix $x_0 \in V$ and let $\ell > 0$ be such that $B(x_0, \ell) \subseteq V$. Given $\psi \in \mathcal{D}(B(x_0, \ell), F^*)$, begin by observing that, for all $\phi \in \mathcal{D}(B(x_0, \ell), F)$, one has:

$$\left| \int_{B(x_0, \ell)} \langle \phi, r(x, D)\psi \rangle \right| \leq C \|\phi\|_{L^1} \|\psi\|_{L^N} \leq C' \ell \|\phi\|_{L^{N/N-1}} \|\psi\|_{L^N},$$

where the first inequality follows from [12, inequality (3.3)]. It hence follows by duality that one has:

$$\|r(x, D)\psi\|_{L^N(B(x_0, \ell))} = \sup_{\substack{\phi \in \mathcal{D}(B(x_0, \ell), F) \\ \|\phi\|_{L^{N/N-1}} \leq 1}} \int_{B(x_0, \ell)} \langle \phi, r(x, D)\psi \rangle \leq C \ell \|\psi\|_{L^N}.$$

Assuming hence that ℓ is small enough (say $\ell = \ell_0$), this yields $\|r(x, D)\|_{L^N(B(x_0, \ell_0)) \rightarrow L^N(B(x_0, \ell_0))} < 1$. Now writing $\psi - r(x, D)\psi = (I - r(x, D))\psi$, and using [5, Exercice 6.14] to infer that $I - r(x, D)$ is a linear isomorphism of $L^N(B(x_0, \ell_0))$, we see that $[I - r(x, D)](\mathcal{D}(B(x_0, \ell_0)))$ is dense in $L^N(B(x_0, \ell_0))$, which is sufficient to conclude that $g = 0$ in $B(x_0, \ell_0)$, and hence that $g = 0$ on V since x_0 is arbitrary. ■

Now choose $K \subset\subset U$ a compact set for which one has $\text{supp}(D_{\mathcal{A}}g_k) \subseteq K$ for all $k \in \mathbb{N}$ and fix $k \in \mathbb{N}$. Fix also $x_0 \in U \setminus K$, choose $\ell > 0$ so that $V := B(x_0, \ell) \subseteq U$ and fix $f \in \mathcal{D}(V, F^*)$. Define $u = A(\cdot, D)q(\cdot, D)f \in \mathcal{D}(V, F^*)$ satisfying (4). One then has:

$$\int_V g_k \overline{f + r(x, D)f} = \int_V g_k \overline{A^*(x, D)u} = \int_V \bar{u} \cdot d[D_{\mathcal{A}}g_k] = 0,$$

since one knows that $V \cap \text{supp}[D_{\mathcal{A}}g_k] = \emptyset$. Applying the previous claim to g_k , V and $-r(\cdot, D)$ we hence get $g_k = 0$ on V . It then follows that one has $\text{supp } g_k \subseteq K$, for x_0 is an arbitrary point in $U \setminus K$.

Getting back to the proof of Lemma 6.11, observe that, according to Proposition 4.7, there exists a subsequence $(g_{k_l}) \subseteq (g_k)$, $W^{\nu-1,1}$ -converging to $g \in BV_{\mathcal{A},c}(U)$. Using the fact that $\Gamma(v)$ is an \mathcal{A} -charge, we compute:

$$\mu(v) = \lim_{l \rightarrow \infty} \Xi(g_{k_l})(v) = \lim_{l \rightarrow \infty} \Gamma(v)(g_{k_l}) = \Gamma(v)(g) = \Xi(g)(v),$$

and hence we get $\mu = \Xi(g) \in \text{im } \Xi$. ■

We hence proved the following theorem.

Theorem 6.14. *We have $CH_{\mathcal{A}}(U) = \Gamma[C(U, F^*)]$.*

7. APPLICATION: ELLIPTIC COMPLEXES OF VECTOR FIELDS

Consider n complex vector fields L_1, \dots, L_n , $n \geq 1$, with smooth coefficients defined on an open set $\Omega \subset \mathbb{R}^N$, $N \geq 2$. Naturally, we assume that the vector fields L_j , $1 \leq j \leq n$ do not vanish in Ω ; in particular, they may be viewed as non-vanishing sections of the vector bundle $\mathbb{C}T(\Omega)$ as well as first order differential operators of principal type.

We impose two fundamental properties on those vector fields in our context; namely, we require that:

- (a) L_1, \dots, L_n are everywhere linearly independent;
- (b) the system $\mathcal{L} := \{L_1, \dots, L_n\}$ is *elliptic*.

The latter means for any 1-form ω , i.e. any section of $T^*(\Omega)$, the equality $\langle \omega, L_j \rangle = 0$ for $1 \leq j \leq n$ implies that one has $\omega = 0$ — which is equivalent to require that the second order operator: $\Delta_{\mathcal{L}} := L_1^* L_1 + \dots + L_n^* L_n$ is elliptic. We use the notation $L_j^* := \overline{L_j^t}$ where $\overline{L_j^t}$ denotes the vector field obtained from L_j by conjugating its coefficients and let L_j^t denote the formal transpose of L_j for $j = 1, \dots, n$ — namely this means that, for all (complex valued) $\varphi, \psi \in \mathcal{D}(\Omega)$, we have:

$$\int_{\Omega} (L_j \varphi) \bar{\psi} = \int_{\Omega} \varphi \overline{(L_j^* \psi)}.$$

Consider the gradient $\nabla_{\mathcal{L}} : C^\infty(\Omega) \longrightarrow C^\infty(\Omega)^n$ associated to the system \mathcal{L} defined by $\nabla_{\mathcal{L}} u := (L_1 u, \dots, L_n u)$ and its formal complex adjoint operator, defined for $v \in C^\infty(\Omega, \mathbb{C}^n)$ by:

$$\text{div}_{\mathcal{L}^*} v := L_1^* v_1 + \dots + L_n^* v_n.$$

The following local continuous solvability result is known for divergence-type operators of the previous type; it is borrowed from [16, Theorem 1.2].

Corollary 7.1. *Assume that the system of vector fields \mathcal{L} satisfies (i) and (ii). Then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$ such that for any $f \in \mathcal{D}'(U)$, the equation:*

$$\text{div}_{\mathcal{L}^*} v = f$$

is continuously solvable in U if and only if f is an \mathcal{L} -charge in U , meaning that for every $\varepsilon > 0$ and every compact set $K \subset\subset U$, there exists $\theta = \theta(K, \varepsilon) > 0$ such that one has, for every $\varphi \in \mathcal{D}_K(U)$:

$$(13) \quad |f(\varphi)| \leq \theta \|\varphi\|_1 + \varepsilon \|\nabla_{\mathcal{L}} \varphi\|_1.$$

This result can be seen as a direct consequence of Theorem 1.3 applied to the first order operator $A(\cdot, D) := \nabla_{\mathcal{L}}$, which is elliptic and canceling. Indeed, from the fact that \mathcal{L} is elliptic we easily see that $\nabla_{\mathcal{L}}$ is elliptic as well. Furthermore, [14, Lemma 4.1] together with the assumption that the system \mathcal{L} be linearly independent, shows that $\nabla_{\mathcal{L}}$ is canceling.

Let $C^\infty(\Omega, \Lambda^k \mathbb{R}^n)$ denote the space of k -forms on \mathbb{R}^n , $0 \leq k \leq n$, with smooth, complex coefficients defined on Ω . Each $f \in C^\infty(\Omega, \Lambda^k \mathbb{R}^n)$ may be written as :

$$f = \sum_{|I|=k} f_I dx_I, \quad dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where one has $f_I \in C^\infty(\Omega)$ and where $I = \{i_1, \dots, i_k\}$ is a set of strictly increasing indices with $i_l \in \{1, \dots, n\}$, $l = 1, \dots, k$. Consider the differential operators :

$$d_{\mathcal{L},k} : C^\infty(\Omega, \Lambda^k \mathbb{R}^n) \rightarrow C^\infty(\Omega, \Lambda^{k+1} \mathbb{R}^n)$$

defined by: $d_{\mathcal{L},0}f := \sum_{j=1}^n (L_j f) dx_j$ for $f \in C^\infty(\Omega)$, and, for $f = \sum_{|I|=k} f_I dx_I \in C^\infty(\Omega, \Lambda^k \mathbb{R}^n)$, $1 \leq k \leq n-1$, by:

$$d_{\mathcal{L},k}f := \sum_{|I|=k} (d_{\mathcal{L},0}f_I) dx_I = \sum_{|I|=k} \sum_{j=1}^n (L_j f_I) dx_j \wedge dx_I.$$

We also define the dual pseudo-complex $d_{\mathcal{L},k}^* : C^\infty(\Omega, \Lambda^{k+1} \mathbb{R}^n) \rightarrow C^\infty(\Omega, \Lambda^k \mathbb{R}^n)$, $0 \leq k \leq n-1$, determined by the following relation for any $u \in C_c^\infty(\Omega, \Lambda^k \mathbb{R}^n)$ and $v \in C_c^\infty(\Omega, \Lambda^{k+1} \mathbb{R}^n)$:

$$\int d_{\mathcal{L},k} u \cdot \bar{v} = \int u \cdot \overline{d_{\mathcal{L},k}^* v},$$

where the dot indicates the standard pairing on forms of the same degree. This is to say that given $f = \sum_{|J|=k} f_J dx_J$, one has:

$$d_{\mathcal{L},k}^* f = \sum_{|J|=k} \sum_{j \in J} L_j^* f_J dx_j \vee dx_J,$$

where, for each $j_l \in J = \{j_1, \dots, j_k\}$ and $l \in \{1, \dots, k\}$, $dx_{j_l} \vee dx_J$ is defined by: $dx_{j_l} \vee dx_J := (-1)^{l+1} dx_1 \wedge \cdots \wedge dx_{j_{l-1}} \wedge dx_{j_{l+1}} \wedge \cdots \wedge dx_{j_k}$.

Suppose first that \mathcal{L} is involutive, *i.e.* that each commutator $[L_j, L_\ell]$, $1 \leq j, \ell \leq n$ is a linear combination of L_1, \dots, L_n . Then the chain $\{d_{\mathcal{L},k}\}_k$ defines a complex of differential operators associated to the structure \mathcal{L} , which is precisely the *de Rham complex* when $n = N$ and $L_j = \partial_{x_j}$ (see [3] for more details). In the non-involutive situation, we do not get a complex in general, and the fundamental complex property $d_{\mathcal{L},k+1} \circ d_{\mathcal{L},k} = 0$ might not hold. On the other hand, this chain still satisfies a “pseudo-complex” property in the sense that $d_{\mathcal{L},k+1} \circ d_{\mathcal{L},k}$ is a differential of operator of order one rather than two, as it is generically expected. We will refer to $(d_{\mathcal{L},k}, C^\infty(\Omega, \Lambda^k \mathbb{R}^n))$ as the pseudo-complex $\{d_{\mathcal{L}}\}$ associated with \mathcal{L} on Ω .

Consider the operator

$$A(\cdot, D) = (d_{\mathcal{L},k}, d_{\mathcal{L},k-1}^*) : C_c^\infty(\Omega, \Lambda^k \mathbb{R}^n) \rightarrow C_c^\infty(\Omega, \Lambda^{k+1} \mathbb{R}^n) \times C_c^\infty(\Omega, \Lambda^{k-1} \mathbb{R}^n),$$

for $0 \leq k \leq n$. Here the operator $d_{\mathcal{L},-1} = d_{\mathcal{L},-1}^*$ is understood to be zero. The operator $A(\cdot, D)$ is elliptic and canceling for $k \notin \{1, n-1\}$ (see Section 4 [14] for details), so that for each $x_0 \in \Omega$ there exists an neighborhood $U \subset \Omega$ of x_0 and $C > 0$ such that the inequality:

$$\|u\|_{L^{N/N-1}} \leq C(\|d_{\mathcal{L},k} u\|_{L^1} + \|d_{\mathcal{L},k-1}^* u\|_{L^1}),$$

holds for any $u \in \mathcal{D}(U, \Lambda^k \mathbb{R}^n)$ (see [13, Theorem B]).

Now we consider the equation 2 associated to operator $A(\cdot, D)$, *i.e.* the equation:

$$(14) \quad d_{\mathcal{L},k}^* u + d_{\mathcal{L},k-1} v = f.$$

The following local continuous solvability result for (14) is a consequence of our main theorem.

Corollary 7.2. *Consider a system of complex vector fields $\mathcal{L} = \{L_1, \dots, L_n\}$, $n \geq 2$ satisfying hypotheses (i)-(ii) above, and the pseudo-complex $\{d_{\mathcal{L},k}\}_k$ associated with \mathcal{L} on Ω with $k \notin \{1, n-1\}$. Then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$ such that for any $f \in \mathcal{D}'(U, \Lambda^k \mathbb{R}^n)$, the equation*

(14) is continuously solvable in U if and only if for every $\varepsilon > 0$ and every compact set $K \subset U$, there exists $\theta = \theta(K, \varepsilon) > 0$ such that one has, for every $\varphi \in \mathcal{D}_K(U, \Lambda^k \mathbb{R}^n)$:

$$|f(\varphi)| \leq \theta \|\varphi\|_1 + \varepsilon (\|d_{\mathcal{L},k}\varphi\|_1 + \|d_{\mathcal{L},k-1}^*\varphi\|_1).$$

Theorem 7.1 is a direct consequence of the previous result, taking $k = 0$ (recall that one has $d_{\mathcal{L},0} = \nabla_{\mathcal{L}}$ and $d_{\mathcal{L},0}^* = \operatorname{div}_{\mathcal{L}}^*$). We emphasize that the operator is not canceling when $k = 1$ or $k = n - 1$ (see [14, Section 4]).

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