



HAL
open science

What's right with the neoclassical legacy? Allais' response

Olivier Baguelin

► **To cite this version:**

| Olivier Baguelin. What's right with the neoclassical legacy? Allais' response. 2020. hal-02541406

HAL Id: hal-02541406

<https://hal.science/hal-02541406>

Preprint submitted on 13 Apr 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

What's right with the neoclassical legacy? Allais' response*

Olivier Baguelin

UEVE-EPEE TEPP (FR-CNRS 3435)

March 2016

Abstract

Most important economic problems such as coordinating individual activities or providing correct incentives arise "far" from economic equilibrium; this is especially true within a context of general interdependence. And yet, the walrasian General equilibrium theory, as dominant legatee of neoclassical economics, is only coherent in the close neighborhood of some equilibrium (Foley, 2010). The marshallian approach is an appealing alternative to deal with "general disequilibrium" situations while incorporating neoclassical concepts. The trouble is that it betrays part of the neoclassical legacy when questioning an ordinal interpretation of utility. This paper draws the attention on Allais' *General theory of surpluses* (1981) as a valuable platform to coherently arrange fundamental neoclassical achievements. It offers a basic but integrative analytical framework: not only does it accommodate disequilibrium situations but it allows connections to such an important development in economic theory as the institutional approach.

Keywords: surplus, loss, general equilibrium, transaction costs.

JEL codes: D3, D5, D6.

1 Introduction

This paper is devoted to Allais' (1981) theory of surpluses: it is shown to capture the fundamental insights of neoclassical economics (substitution principle, marginal reasoning, ordinal utility and general interdependence) while still recognizing the shortcomings of the walrasian General equilibrium theory (GET). Indeed, as shown in Foley (2010) assessing the fundamental claim of walrasian economics, even under strong simplifying assumptions, the list of agent preferences, technologies and stocks of resources describing a free exchange economy is generally not sufficient in itself to predict its *price system* nor *distribution* of its wealth. The reason is that the concept of *exchange equilibrium*¹ which is relevant to study out-of-equilibrium economic behaviors, is path-dependent: the

*I am very grateful to Chantal Marlats, Olivier Tercieux, Duncan Foley and Julia Defendini for insightful comments and suggestions; any errors or omissions are mine. This research received financial support from Labex MME-DII.

¹Allocation where no further mutually advantageous trade is possible.

complete sequence of voluntary exchanges condition the equilibrium eventually realized. The criticism is sharp since it deprives in general the *walrasian equilibrium* concept of any predictive meaning. If even such basic concepts as demand and supply schedules prove flawed, what should be retained of the neoclassical legacy? In this paper, it is argued that Allais' *General theory of surpluses* (GTS), although broadly gone unnoticed at its time, provides a valuable response. First, the methodological concerns expressed in Smith & Foley (2008) are duly taken into account. Allais' concept of surplus applies out of equilibrium, it thus disentangles the theory of preferences and duality from the problems associated with disequilibrium, dynamics, and institutions. Second, it achieves a reconsideration of the neoclassical framework better suited for developing genuine analyses of firm and market, and to which modern developments in microeconomics easily connect. This comes from that, rather than at the level of the market, the GTS considers the economic system at the level of the *transaction* (that is, an infra-institutional scale); economic reasoning can be conducted without any assumption of preexisting price system. Insofar as this allows a position of outwardness with respect to such objects as market or firm, this makes thinking about economic institutions much more natural than in the walrasian framework. And indeed, although Allais did not seem to be aware of it, the GTS appears as a good candidate to constitute the framework for a synthesis of neoclassical and institutional economics; the way Williamson (2005) defines "Transaction costs economics" gives a clear sense of it. That's because, except as regards free exchange and the property right, the analysis does not rely on any specific institutional setting.² Allais' approach distinguishes from other, more recent, efforts to put neoclassical and institutional economics into contact, such as Milgrom & Roberts (1995) or Spulber (2009), with two respects. Allais never departs from a general interdependence perspective, and does not recourse to the case of quasi-linear (QL) economies.³ It is indeed common in microeconomic analysis, so as to avoid the complications arising from *wealth effects*,⁴ to recourse to QL specifications of utility functions; the tradition dates back to Marshall (1921) (who was nevertheless anxious to detail the situations in which this was acceptable).⁵ The QL case is particularly useful to the walrasian GET because it supports the view that the walrasian prediction of the price system is indeed the end of a decentralized process of exchange. However, Smith & Foley (2008) and Foley (2010) show that QL is in fact *the only case* in which the price prediction of the exchange equilibrium concept and that of walrasian equilibrium⁶ indeed concur.⁷ While not pretending to make any definite prediction on prices, the GTS accommodates decentralized out-of-equilibrium transactions with wealth effects improving on other attempts.

²In the development of the GTS, a commodity-money is indeed introduced, which could be regarded as an additional institution, but those considering that market is an intrinsically monetary phenomena should rather put this at the credit of Allais' framework.

³In which all agents have QL utility functions with respect to a common good.

⁴Or "income effects".

⁵This is especially convenient when multilateral transactions are considered since it makes utility transferable.

⁶A collection of intersections of supply and demand schedules

⁷Possibly requiring that some agents end with a *negative* holding of the QL good

The purpose of the present paper is to provide a clarified exposition of the GTS using standard notions. As Grandmont (1989) suggests,⁸ this does not seem superfluous: despite its achievement, some ambiguities or unnecessary complications remain in Allais' original text. This paper further adds some results and formal proofs neglected by Allais, and put key results into perspective. Some developments posterior to the GTS are also included which clarify or extend Allais' analysis. This paper eventually describes how two existing alternatives to the walrasian approach, namely the marshallian and the institutionalist ones, can find their way within the GTS; within this perspective, some elaborations on Allais' vision are proposed.

The GTS is a two-stage rocket. At the bottom, two analytical tools for *welfare analysis* in the absence of any price system: *surplus* and *loss*. At the top, a *positive vision* of economic behaviors and transactional dynamics consistent with the concept of *exchange equilibrium*. Surplus is a measure of the gains resulting from informed voluntary transactions. The GTS provides an operational definition of it with no recourse to any cardinal notion of utility, given price system, nor generalized assumptions of continuity, differentiability, or convexity. Allais' surplus allows to analyze within a general interdependence framework out-of-equilibrium microeconomic behaviors, the economic processes induced by voluntary exchange and cooperation, and the conditions for *Pareto-efficiency*.

1.1 Welfare analysis

The basic framework is neoclassical: a given list of private goods; a given set of agents; given endowments, preferences, and technologies. Starting from an initial allocation, to any reallocation can be associated a *surplus*, as measured in a reference good valued by each agent. Once this reallocation implemented (possibly changing the well-being of each agent), the collective surplus in Allais' definition, is the maximum amount of the reference good that can be removed from the economy (*released* in Allais' terms), all other things equal, bringing at worst each agent back to its initial wellbeing. Allais shows that an allocation belongs to the Pareto set if and only if, whatever the good in which it is measured, surplus is negative or null for any feasible reallocation. The *loss* associated to a given allocation is the *maximal* surplus releasable through a feasible reallocation. Now, assume there exists a perfectly divisible good desired by everyone in the economy (whatever one's endowment in the good under consideration or any other good) and let's call "money" this particular good. If monetary surplus (surplus as measured in money) is negative or null then it is negative or null as measured in any other reference good. It follows that an allocation belongs to the Pareto set if and only if the surplus as measured in money is negative or null for any feasible reallocation. Allais' concept of surplus improves on the welfare measurement literature of his time (see Currie *et al.*, 1971) because it does not confine to partial analysis, nor does it rely on any given price system. These (unnoticed) advances may explain that Allais' surplus was in fact rediscovered in the early 90s by Luenberger, within a dual theory perspective, with the

⁸Grandmont (1989, p. 26): "Allais' arguments are complicated and his *General Theory of Surpluses* has perhaps not been studied and exploited to the extent that it should have."

very close concept of "benefit" (Luenberger, 1992). In a series of papers, the latter provides a rigorous and systematic study of the properties of the "benefit function" as a convenient tool to conduct welfare analysis and study Pareto allocations within a general interdependence framework. But, contrary to Allais, Luenberger's point has not been to use surplus/benefit as the building block of a refreshed abstract economic theory.

1.2 Positive economics

Beyond welfare analysis, Allais' point with the GTS is indeed to incorporate the neo-classical legacy into a representation of the economy which, contrary to the walrasian GET, could reasonably sustain a positive interpretation. Since no unique system of prices is available to agents, they cannot be considered as maximizing utility subject to a budget constraint. As a positive representation, the GTS is based on the assumption that agents are *surplus-seekers*: their interested behaviors feed a loss-reducing process of transactions, in which information on individual dispositions are revealed and exploited.

It follows that, provided a transaction is well-informed, voluntary, and does not give rise to negative externalities to others, it is Pareto-improving. Hence, in the standard case, a decentralized process of voluntary exchange and cooperation can be expected to drive the economy to a "least-loss" allocation, if not to a Pareto allocation (in the absence of any transactional obstacles). The shape of loss-reducing transactions is very open: exchanges may be bilateral or multilateral; there might be integrated organizations or not; in case of market transactions, agents may be price takers or not. In particular, and implicit in the absence of any given price system, there is no assumption about the degree of competition within the economy; in fact, the driving forces are exchange and cooperation rather than competition.

All these features of Allais' vision of the economy obviously reminds other existing alternatives to the walrasian GET. First, as suggested above, the GTS exhibits strong connection to the marshallian approach to markets; second, it provides a framework well-adapted to the integration of institutionalists' concepts and analyses. It is argued that this makes the GTS a valuable analytical infrastructure to economic theory.

1.3 Outlines

The remaining of this paper goes as follows.

Section 2 is devoted to the exposition of the GTS within a familiar basic neoclassical framework with weak assumptions. Concepts of surplus and loss are defined, and their basic properties are set. As compared to Allais' presentation, the present paper adds formal definitions and proofs. It also provides an extensive discussion of the relation between loss-reducing and Pareto-improving reallocations. It is shown that, as an index of inefficiency, loss defines an order on the set of allocations which is "less incomplete" than Pareto-improvement: any P-improving reallocation reduces loss but a reallocation can reduce loss while not being P-improving.

Upon previous basis, Section 3 presents developments considered in the GTS. They first consist in the introduction of a commodity-money which both simplifies the analysis

and allows to relax the assumption of non-satiation in consumption. Explicitly introducing money puts the GTS closer to the marshallian approach. The second development is a rise in abstraction by considering continuous quantities and differentiable functions. This leads to a reformulation of the GTS in terms of subjective marginal valuations (*i.e.* marginal rates of substitution of commodities for money), and to the observation that the loss-reduction process can be seen as involving some unobserved *economic entropy*. A necessary and sufficient condition for P-efficiency is formulated in terms of *decreasing marginal return in collective surplus*.

Section 4 describes the positive economics that can be derived from the GTS. It includes elements explicitly mentioned by Allais regarding the contrast of the GTS with the walrasian GET. But some original elaborations are also proposed as regards the analysis of markets, on the one hand, that of economic interactions beyond markets, on the other hands. The extent to which the GTS can offer a meeting point between neoclassical and institutionalist approaches is finally discussed.

2 Surplus and loss

2.1 The framework

There are N exchangeable goods in the economy indexed by $n \in \mathcal{N} = \{1, \dots, N\}$. A vector of quantities is denoted \mathbf{x} and $\mathbf{x}' = (x_1, \dots, x_N)$.⁹ Agents¹⁰ are indexed by $i \in \mathcal{I} = \{1, \dots, I\}$.

2.1.1 Individual preferences and personal technology

The utility concept is purely ordinal. Agent i 's preferences over $\mathbf{x}_i \in \mathbb{R}^N$ are represented by the utility function $u_i(\cdot)$ defined by $u_i = u_i(\mathbf{x}_i)$ where: $x_{in} > 0$ represents a consumption of good n (drawn from the economy), and $x_{in} < 0$ a production of good n (service provided to the economy), by agent i . Allowing for agent production of services means that agent utility functions do more than representing preferences: they also implicitly represent some personal technology in service providing. In a basic formulation of Allais' approach, the next assumption allows to remove unnecessary complications.

Non-satiation assumption. For all $n \in \mathcal{N}$, $u_i(\mathbf{x}_i)$ is assumed strictly increasing.

This assumption is useful below inasmuch as no exogenous restriction is made on the set of possible plans (no lower bound condition); it is relaxed once a *commodity-money* is introduced.

⁹Sticking to an approach in terms of property rights, $x_{in} > 0$ could be understood as an amount of rights to use (consume) the good n .

¹⁰Consumers and resources holders, including any decision unit whose welfare is considered *per se* in the analysis.

2.1.2 Feasible allocations and reallocations

An allocation a is a list $(\mathbf{x}_i)_{i \in \mathcal{I}}$. It is *feasible* if and only if $\sum_{i \in \mathcal{I}} \mathbf{x}_i \leq \bar{\mathbf{x}}$, where $\bar{\mathbf{x}}$ gives the economy's initial available total quantity of each resource. The set of feasible allocations is denoted \mathcal{A} . A *reallocation* Δa is a list of variations $(\Delta \mathbf{x}_i)_{i \in \mathcal{I}}$. Given $a = (\mathbf{x}_i)_{i \in \mathcal{I}} \in \mathcal{A}$, Δa is *feasible* if and only if $\sum_{i \in \mathcal{I}} \Delta \mathbf{x}_i \leq \bar{\mathbf{x}} - \sum_{i \in \mathcal{I}} \mathbf{x}_i$. For all $n \in \mathcal{N}$, $\sum_{i \in \mathcal{I}} \Delta x_{in}$ is the net variation of the total *allocated* quantity of good n : $\sum_{i \in \mathcal{I}} \Delta x_{in} < 0$ means a reduction (as compared to the initial allocation) in the total quantity of good n allocated to the set \mathcal{I} of agents.

2.2 Allais' concepts of surplus and loss

The main tools of the analysis are now presented under the weakest assumptions: quantities may be continuous or not, preferences convex or not.

2.2.1 Definitions

From an initial allocation, for any subset $\mathcal{I}_\circ \subseteq \mathcal{I}$ of agents, the *surplus* corresponding to a given reallocation Δa , as measured in any reference good n , is the quantity Δv_n of this commodity that can be *released* (made available) from Δa under the threefold condition that:

1. the quantity of each good...
 - (a) used by the group \mathcal{I}_\circ is at most equal to its initial level;
 - (b) provided by the group \mathcal{I}_\circ to others is at least equal to its initial level;
2. each agent in \mathcal{I}_\circ gets a utility at least equal to its initial level.

A formal definition of surplus can be provided distinguishing between the individual and the collective level. Consider a change $\Delta \mathbf{x}_i = (\Delta \mathbf{x}_i^{-n}, \Delta x_{in})$ affecting some agent i where $\Delta \mathbf{x}_i^{-n}$ denotes the list of variations in all quantities except x_{in} .

Definition 1 For any agent $i \in \mathcal{I}$ with initial plan $\mathbf{x}_i \in \mathbb{R}^N$, the individual surplus Δv_{in} , as measured in any reference good n , associated to the change $\Delta \mathbf{x}_i = (\Delta \mathbf{x}_i^{-n}, \Delta x_{in})$ is

$$\Delta v_{in} \equiv \max \{ \Delta \nu_{in} \mid u_i(\mathbf{x}_i^{-n} + \Delta \mathbf{x}_i^{-n}, x_{in} + \Delta x_{in} - \Delta \nu_{in}) \geq u_i(\mathbf{x}_i) \}.$$

Since $u_i(\mathbf{x}_i)$ is strictly increasing with respect to x_{in} , Δv_{in} always exists. The interpretation is familiar:

- if $u_i(\mathbf{x}_i^{-n} + \Delta \mathbf{x}_i^{-n}, x_{in} + \Delta x_{in}) > u_i(\mathbf{x}_i)$, $\Delta v_{in} \geq 0$ is the highest amount of good n agent i would accept to give up in exchange for implementing the change $\Delta \mathbf{x}_i$;

- if $u_i(\mathbf{x}_i^{-n} + \Delta \mathbf{x}_i^{-n}, x_{in} + \Delta x_{in}) < u_i(\mathbf{x}_i)$, $-\Delta v_{in} \geq 0$ is the smallest amount of good n agent i would require so as to accept the change $\Delta \mathbf{x}_i$.

Note the difference in nature between variations $\Delta \mathbf{x}_i$ and the quantity Δv_{in} : $\Delta \mathbf{x}_i$ is an actual change whereas Δv_{in} is a *virtual* quantity measuring the attitude of agent i as regards the implementation of $\Delta \mathbf{x}_i$. In Allais' terms, if $\Delta v_{in} > 0$, the change $\Delta \mathbf{x}_i$ is said to *distribute* a positive individual surplus; if $\Delta v_{in} = 0$, the change $\Delta \mathbf{x}_i$ distributes no surplus.

Fig. 1 illustrates in a two goods case (the quantities of which are denoted x and y) the individual surplus (as measured in good Y) associated to some reallocation Δa . Two distinct agents \hat{i} and \check{i} are considered: starting from a the reallocation Δa is taken beneficial to \hat{i} but disadvantageous to \check{i} . The shape of plotted indifference curves is partly arbitrary: only non-satiation is required, neither continuity nor convexity. The case depicted in Fig. 1 is also special to the extent that the condition defining Δv holds at strict equality which is not required in general.

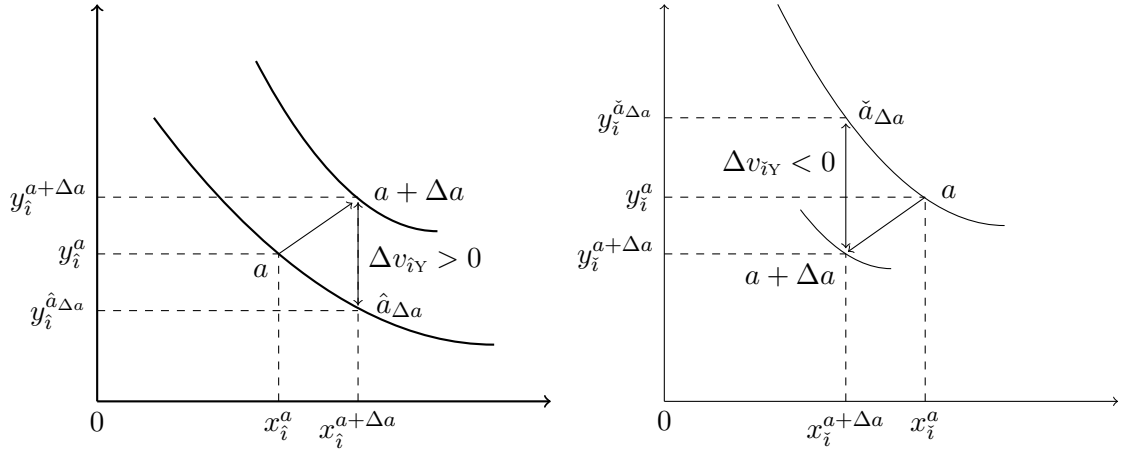


Figure 1: Individual surplus corresponding to some reallocation Δa from a

Note that, although two indifference curves are plotted, only the one passing through the initial allocation is required. It allows to plot agents \hat{i} 's (resp. \check{i} 's) virtual situations denoted $\hat{a}_{\Delta a}$ (resp. $\check{a}_{\Delta a}$) from which surplus measurement derives: the interesting thing about Δv_{iY} and Δv_{iY} is obviously that they are commensurable.

Allais' concept of collective surplus can be defined for any subset of agents $\mathcal{I}_{\circ} \subseteq \mathcal{I}$ (say a group of agents involved in a transaction) while still taking account of general interdependence.

Definition 2 Given $a \in \mathcal{A}$, consider a subset $\mathcal{I}_{\circ} \subseteq \mathcal{I}$ of agents, and let $\Delta a = (\Delta \mathbf{x}_i)_{i \in \mathcal{I}}$ be a feasible reallocation such that for all $i \in \mathcal{I} - \mathcal{I}_{\circ}$: $\Delta \mathbf{x}_i \geq \mathbf{0}$. The surplus to \mathcal{I}_{\circ} associated to Δa , as measured in units of good n , is

$$\Delta v_{\circ n} \equiv \sum_{i \in \mathcal{I}_{\circ}} \Delta v_{in} - \sum_{i \in \mathcal{I}_{\circ}} \Delta x_{in},$$

where for all $i \in \mathcal{I}_\circ : \Delta v_{in} \equiv \max \{ \Delta v_{in} \mid u_i(\mathbf{x}_i^{-n} + \Delta \mathbf{x}_i^{-n}, x_{in} + \Delta x_{in} - \Delta v_{in}) \geq u_i(\mathbf{x}_i) \}$.

Here, $\sum_{i \in \mathcal{I}_\circ} \Delta x_{in}$ is the net change in the total quantity of good n allocated to the group \mathcal{I}_\circ . Three cases can be distinguished:

- if $\sum_{i \in \mathcal{I}_\circ} \Delta x_{in} = 0$, surplus is fully distributed to agents in \mathcal{I}_\circ and $\Delta v_{\circ n} = \sum_{i \in \mathcal{I}_\circ} \Delta v_{in}$;
- if $\sum_{i \in \mathcal{I}_\circ} \Delta x_{in} < 0$ and $\sum_{i \in \mathcal{I}_\circ} \Delta v_{in} \neq 0$, surplus is only partly distributed to the agents in \mathcal{I}_\circ ;
- if $\Delta v_{in} = 0$ for all $i \in \mathcal{I}_\circ$, some surplus may be released by the reallocation but it is not distributed to the agents in \mathcal{I}_\circ , $\Delta v_{\circ n} = - \sum_{i \in \mathcal{I}_\circ} \Delta x_{in}$.

This last case is considered in Smith & Foley (2008) to illustrate the notion of "reversible transformation" (reallocation). They consider agents who cannot trade internally and need the mediation of some external speculator. This speculator is equipped with enough information on each agent's preferences to extract surplus from voluntary exchange. Here, the reallocation impacting the agents in \mathcal{I}_\circ may have been arranged by some agent not in \mathcal{I}_\circ who extract surplus for himself. Maybe some trader in charge of arranging advantageous transactions (reallocations), whose mediation is paid *ex post* in units of good n .

$$\underbrace{\Delta v_{\circ n}}_{Total} \equiv \underbrace{\sum_{i \in \mathcal{I}_\circ} \Delta v_{in}}_{Distributed} - \underbrace{\sum_{i \in \mathcal{I}_\circ} \Delta x_{in}}_{Retained}$$

A transaction advantageous to agents in \mathcal{I}_\circ implies $\Delta v_{\circ n} > 0$: not only is it implementable (through proper transfers of good n from virtual winners to virtual losers) but it may allow some go-between agent to draw a positive amount of good n from \mathcal{I}_\circ ; Al-lais' concept includes the amount of resources possibly devoted to arranging transactions (*transaction costs*) into the calculation of collective surplus. As a consequence, any *resources saving* reallocation leaving each agent to its initial welfare releases surplus. The concept applies at any scale from the bilateral transaction to a reallocation impacting all agents in the economy ($\mathcal{I}_\circ = \mathcal{I}$) and, provided that conditions in definition 2 hold, the total surplus is simply the sum of subsets' surpluses: partial and general analyses lead to consistent measurements.

Below, subscript \circ is removed when the analysis is conducted at the scale of $\mathcal{I}_\circ = \mathcal{I}$. Downscaling the analysis from \mathcal{I} to $\mathcal{I}_\circ \subset \mathcal{I}$ requires to add the specific condition formulated in definition 2: $\forall i \in \mathcal{I} - \mathcal{I}_\circ, \Delta \mathbf{x}_i \geq \mathbf{0}$. Starting from an allocation a and considering a reallocation Δa , collective surplus (as measured in units of good n) at the scale of \mathcal{I} is simply written Δv_n ; the loss attached to a is then denoted $l_n(a)$.

Example 3 Fig. 2 and 3 illustrate the concept of surplus (as measured in commodity Y) for a subset of two agents. The point is to consider reallocations which do not reduce agents' welfare while releasing some positive surplus. Consider Fig. 2 first, which illustrates a reallocation releasing a positive surplus without distributing it. The initial allocation a is such that $x_1^a + x_2^a = \bar{x}(= 16)$, and $y_1^a + y_2^a = \bar{y}(= 16)$, while the final allocation \tilde{a} is such that: $x_1^{\tilde{a}} + x_2^{\tilde{a}} = \bar{x}(= 16)$, $y_1^{\tilde{a}} + y_2^{\tilde{a}}(= 14) < \bar{y}$, and yet, $u_1|_a = u_1|_{\tilde{a}}$ and $u_2|_a = u_2|_{\tilde{a}}$. The released surplus $\Delta v_Y = -\Delta y = \bar{y} - (y_1^{\tilde{a}} + y_2^{\tilde{a}}) = 2$, is fully retained. The Edgeworth diagram of Fig. 3 illustrates the case in which surplus is (fully) distributed. In this case: $y_1^{a+\Delta a} + y_2^{a+\Delta a} = y_1^a + y_2^a = \bar{y}$, $x_1^{a+\Delta a} + x_2^{a+\Delta a} = x_1^a + x_2^a = \bar{x}$, but still $\Delta v_Y = \Delta v_{1Y} + \Delta v_{2Y} = 1 + 1 = 2$. What is done of the surplus released does not impact its amount.

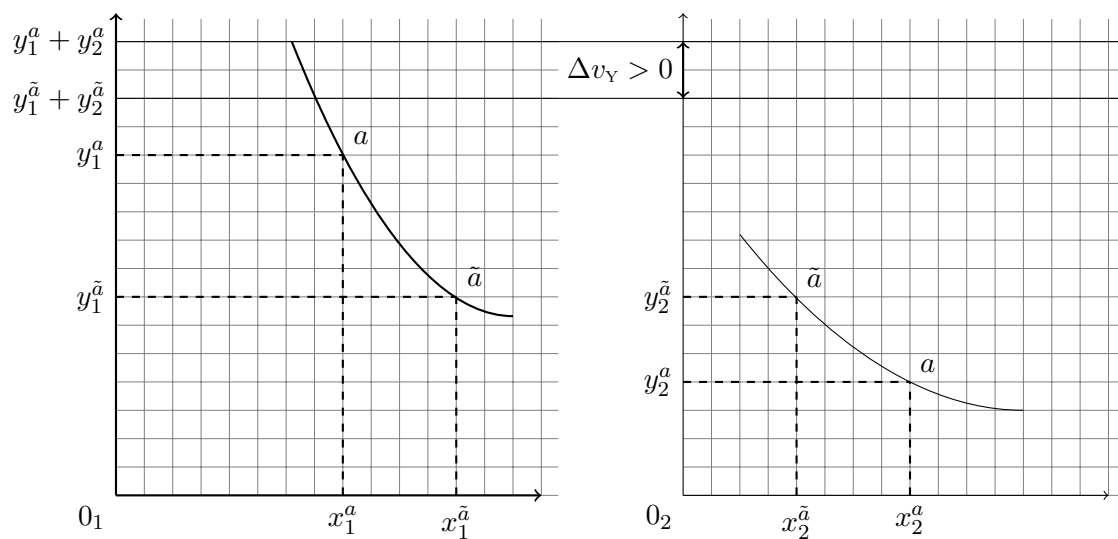


Figure 2: Reallocation releasing a retained surplus

Allais' concept of *loss* derives from that of surplus. The loss associated to some allocation, as measured in some reference good, is the maximal quantity of that good that could be released through a reallocation *i.e.* the maximal "releasable" surplus.

Definition 4 Given $a \in \mathcal{A}$, the loss incurred by the subset $\mathcal{I}_\circ \subseteq \mathcal{I}$, as measured in units of good n , is

$$l_{\circ n}(a) \equiv \max_{\Delta a \text{ feasible}} \Delta v_{\circ n}(a).$$

Note that since *status quo* is always an option, for any $a \in \mathcal{A}$ and $n \in \mathcal{N} : l_{\circ n}(a) \geq 0$.

Fig. 4 illustrates the concept of loss in the 2 agents \times 2 goods case with continuous quantities and indifference curves. In this case, *loss* as measured in good Y (left-hand graph) is the maximum vertical distance between indifference curves. In the special case depicted in Fig. 4 with strictly decreasing marginal rate of substitution (convex

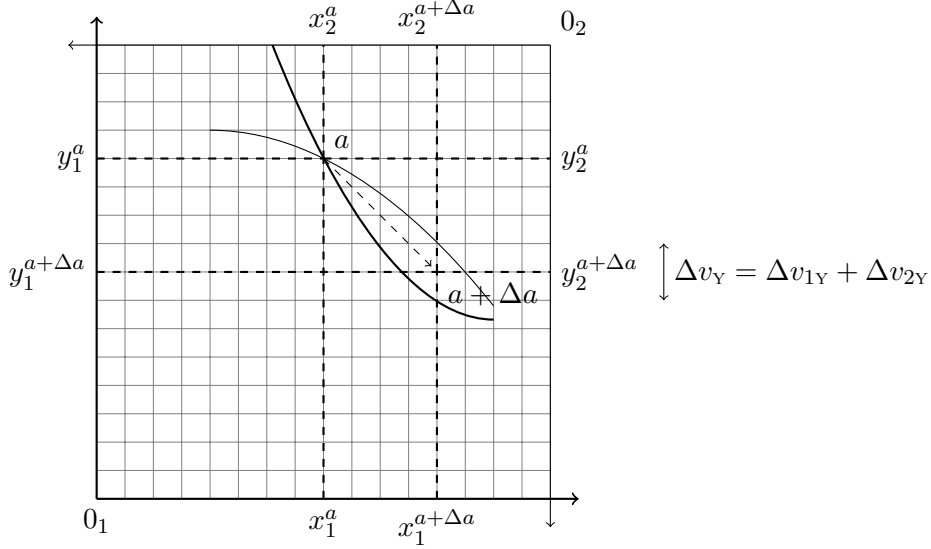


Figure 3: Distributed surplus in the 2 agents \times 2 goods case

preferences) the quantities of good x allocated to each agent corresponding to the maximization of surplus are given by the vertical line equalizing marginal rates of substitution (tangent lines depicted in the figure must be parallel). The right-hand graph of Fig. 4 illustrates the loss as measured in good x .

2.2.2 Properties

As other compensating/equivalent variations used to measure changes in welfare, Allais' surplus respects an ordinal interpretation of utility functions.

Proposition 5 *The surplus attached to some given reallocation is invariant with respect to monotonous strictly increasing transformations of utility functions.*

Proof. See the appendix. ■

Corollary 6 *The loss associated to some given allocation is invariant with respect to monotonous strictly increasing transformations of utility functions.*

2.3 Pareto allocations

The extent to which Allais' concepts of surplus and loss complement the Pareto criterion is of primary interest.

Definition 7 *Given $a = (\mathbf{x}_i)_{i \in \mathcal{I}}$, a reallocation $\Delta a = (\Delta \mathbf{x}_i)_{i \in \mathcal{I}}$ is individually rational at the scale of \mathcal{I}_\circ if it is feasible from a and such that, for all $i \in \mathcal{I}_\circ : u_i(\mathbf{x}_i + \Delta \mathbf{x}_i) \geq u_i(\mathbf{x}_i)$.*

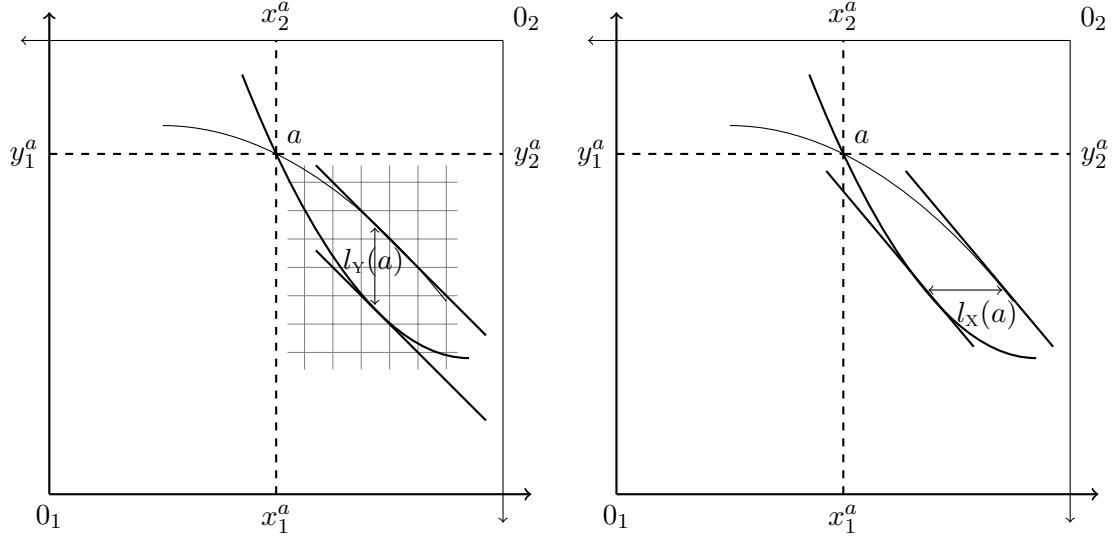


Figure 4: The loss of an exchange economy as measured in good Y (left) or X (right)

Given $a \in \mathcal{A}$, let $\mathcal{R}_\circ(a)$ denotes the set of individually rational reallocations at the scale of \mathcal{I}_\circ that is: $\mathcal{R}_\circ(a) = \{\Delta a \text{ feasible from } a \mid \forall i \in \mathcal{I}_\circ, u_i(\mathbf{x}_i + \Delta \mathbf{x}_i) \geq u_i(\mathbf{x}_i)\}$.

Definition 8 Given $a = (\mathbf{x}_i)_{i \in \mathcal{I}}$, a reallocation $\Delta a = (\Delta \mathbf{x}_i)_{i \in \mathcal{I}}$ is P(areto)-improving at the scale of \mathcal{I}_\circ if and only if:

- $\Delta a \in \mathcal{R}_\circ(a)$;
- $\Delta \mathbf{x}_i \geq \mathbf{0}$ for all $i \in \mathcal{I} - \mathcal{I}_\circ$;
- $u_i(\mathbf{x}_i + \Delta \mathbf{x}_i) > u_i(\mathbf{x}_i)$ for some $i \in \mathcal{I}_\circ$.

Lemma 9 Given $a = (\mathbf{x}_i)_{i \in \mathcal{I}}$, let $\Delta a = (\Delta \mathbf{x}_i)_{i \in \mathcal{I}}$ denotes a feasible reallocation fully distributing a strictly positive surplus $\Delta v_n = \sum_{i \in \mathcal{I}} \Delta v_{in} > 0$ where, for all $i \in \mathcal{I}$:

$$\Delta v_{in} = \max \{ \Delta v_{in} \mid u_i(\mathbf{x}_i^{-n} + \Delta \mathbf{x}_i^{-n}, x_{in} + \Delta x_{in} - \Delta v_{in}) \geq u_i(\mathbf{x}_i) \}.$$

Then the reallocation defined as $\Delta_+ a = (\Delta \mathbf{x}_i^{-n}, \Delta x_{in} - \Delta v_{in} + \Delta_+ x_{in})_{i \in \mathcal{I}}$ where, for all $i \in \mathcal{I}$, $\Delta_+ x_{in} > 0$ and $\sum_{i \in \mathcal{I}} \Delta_+ x_{in} = \Delta v_n$ is P-improving.

Proof. Since Δa distributes a strictly positive surplus, $\Delta_+ a$ is feasible. For all $i \in \mathcal{I}$, since $u_i(\mathbf{x}_i)$ is strictly increasing in x_{in} : $u_i(\mathbf{x}_i^{-n} + \Delta \mathbf{x}_i^{-n}, x_{in} + \Delta x_{in} - \Delta v_{in} + \Delta_+ x_{in}) > u_i(\mathbf{x}_i^{-n} + \Delta \mathbf{x}_i^{-n}, x_{in} + \Delta x_{in} - \Delta v_{in}) \geq u_i(\mathbf{x}_i)$, i.e. $\Delta_+ a$ is P-improving. ■

Any P-improving reallocation reduces the loss.

Proposition 10 If Δa is P-improving from $a \in \mathcal{A}$ then, for any $n \in \mathcal{N}$:

$$l_n(a + \Delta a) < l_n(a).$$

Proof. See the appendix ■

The converse is false: $l_n(a + \Delta a) < l_n(a)$ for some $n \in \mathcal{N}$ does not imply Δa P-improving. This can be illustrated in the utility space: a point on the frontier of the utility set ($a + \Delta a$ in Fig. 5) corresponds to a zero-loss allocation (see proposition 12) whereas any interior point (such as a in Fig. 5) corresponds to a strictly positive loss. Starting from a , reallocation Δa reduces loss although it is not P-improving.

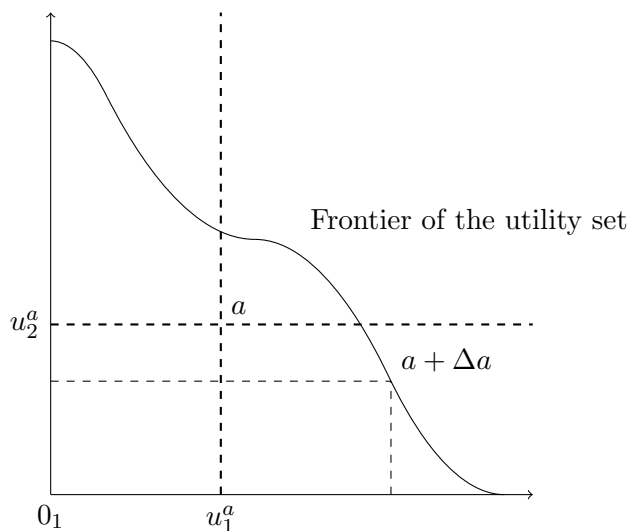


Figure 5: From a , the reallocation Δa is not P-improving but it reduces loss

Let \mathcal{P}_\circ (respectively, \mathcal{P}) denotes the set of P-allocations at the scale of \mathcal{I}_\circ (resp. \mathcal{I}).

Definition 11 $a \in \mathcal{P}_\circ$ if and only if $a \in \mathcal{A}$ and there exists no feasible reallocation (from a) P-improving at the scale of \mathcal{I}_\circ .

It is obviously the case that, for any $\mathcal{I}_\circ \subset \mathcal{I}$, $a \in \mathcal{P} \Rightarrow a \in \mathcal{P}_\circ$. The next proposition is more substantial and anticipates on what Luenberger (1995) calls the first and second zero-maximum theorems.

Proposition 12 For any $\mathcal{I}_\circ \subseteq \mathcal{I}$: $a \in \mathcal{P}_\circ \Leftrightarrow l_{\circ n}(a) = 0$ for all $n \in \mathcal{N}$.

Proof. See in appendix. ■

Among all feasible reallocations from a given a , those maximizing surplus are of special interest. Consider $a \in \mathcal{A} - \mathcal{P}$, and $\Delta^* a \in \arg \max_{\Delta a} \Delta v_n(a)$ for some $n \in \mathcal{N}$ where surplus is strictly positive and *fully* distributed; note that $\Delta^* a$ is far from unique. In general, $a + \Delta^* a \notin \mathcal{P}$. Fig. 6 illustrates this point in the case of a 2 agents \times 2 goods transaction. The Pareto set is the dashed curve joining 0_1 to 0_2 and the contract curve the thick segment of the Pareto set between indifference curves passing through

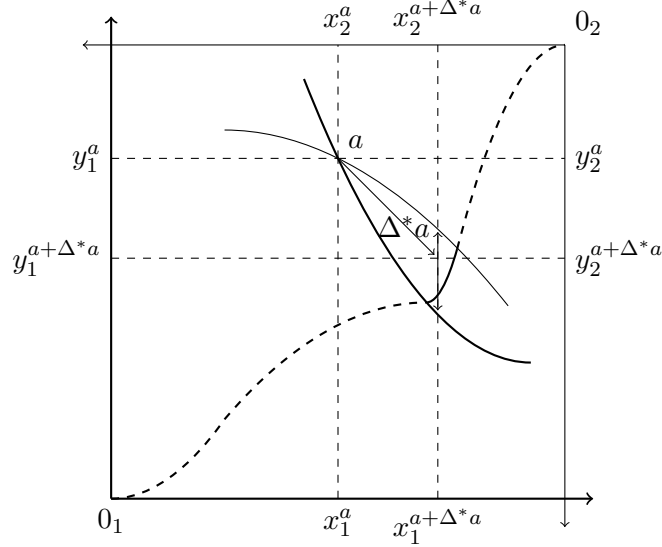


Figure 6: The loss and the contract curve

a. In Fig. 6, there exists a continuum of reallocations maximizing surplus depending on surplus distribution between the agents.

Maximizing surplus demands less information than finding a Pareto allocation. Whereas the surplus-maximizing approach is relative only to the indifference sets corresponding to a , the contract curve incorporates a *global* information on agents' preferences structure. And yet, as Fig. 6 suggests, maximizing surplus brings the economy closer to the Pareto-set. This point is proven rigorously for strictly convex preferences by Courtault & Tallon (2001); their result is reformulated in terms of surplus below. The next remark provides another way to see the connection of surplus maximization with the search for Pareto allocations. It is illustrated with Fig. 7. It comes to consider a "subeconomy" similar to the initial one except in the total amount of the good used as reference to measure surplus.

Remark 13 *Given $a \in \mathcal{A} - \mathcal{P}$, consider $\Delta^*a = (\Delta^*x_i)_{i \in \mathcal{I}} \in \arg \max_{\Delta a} \Delta v_n(a)$ for some $n \in \mathcal{N}$ and $l_n(a) > 0$ the associate surplus. Suppose $l_n(a)$ is fully retained so that: $l_n(a) = -\sum_{i \in \mathcal{I}} \Delta^*x_{in}$. Consider the sub-economy similar to the initial one, except for its total endowment in good n , reduced from \bar{x}_n to $\bar{x}_n - l_n(a)$. The allocation $a + \Delta^*a$ belongs to the Pareto set of this sub-economy.*

In general, one cannot trivially build a P-allocation of the initial economy from that of the subeconomy by distributing the retained surplus. That is because the process of distribution is prone not to maintain agents' marginal rates of substitution. This creates new advantageous transaction opportunities, disrupting previously allocated goods.

The concept of loss provides a quantitative *index of (in-)efficiency* which complement the Pareto criterion. It exhibits several desirable properties. It depends only on the

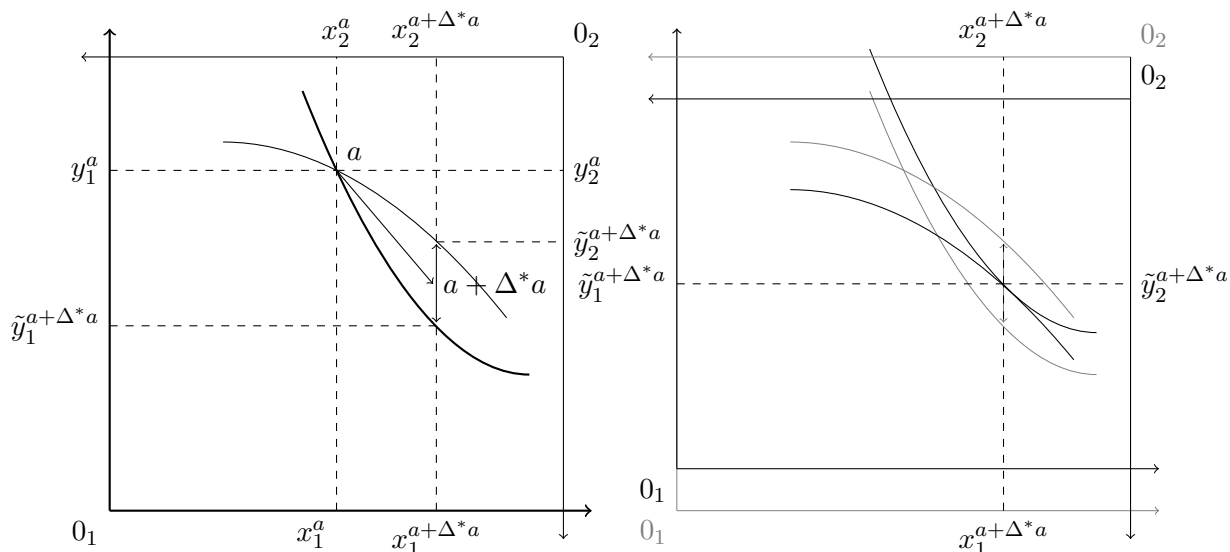


Figure 7: Maximizing and retaining surplus leads to a P-efficient subeconomy

structure of the economy (preferences, technologies, endowments). It respects the ordinal nature of utility functions. It involves all preferences and technologies on a symmetric basis; this symmetry in treatment also holds for all goods except the one chosen for surplus measurement. It is positive for all non-Pareto allocation and equals zero for all P-allocations; it decreases as a result of P-improving reallocations. It can be calculated for any subset of agents and for any allocation. It is independent of any restrictive conditions such as continuity, differentiability or convexity. Finally, it does not depend on any specific system of prices nor on any special economic organization. And yet, with all these desirable properties, surplus calculation still depends on the reference good chosen, a choice which is arbitrary. Furthermore, the concept is built excluding the possibility of satiation which is obviously restrictive. The developments below deal with these limitations.

3 Developments

The developments proposed by Allais take two directions. The first is the introduction of "money" as a natural unit in which measuring surplus, and its use to relax the general assumption of non-satiation formulated above. The second is to make one additional step towards abstraction by assuming continuity of quantities and differentiability of utility functions.

3.1 Commodity-money

The list of goods is extended with an additional one, called "money", a quantity of which is denoted y . Agent i 's preferences are now defined over plans $(\mathbf{x}_i, y_i) \in \mathbb{R}^{N+1}$ and represented by $u_i(\mathbf{x}_i, y_i)$. An allocation rewrites $a = (\mathbf{x}_i, y_i)_{i \in \mathcal{I}}$ and is *feasible* if and only if $\sum_{i \in \mathcal{I}} \mathbf{x}_i \leq \bar{\mathbf{x}}$ and $\sum_{i \in \mathcal{I}} y_i \leq \bar{y}$ where $(\bar{\mathbf{x}}, \bar{y})$ represents preexisting global resources of the economy, including an exogenous and inelastic money supply; the set of feasible allocations is still denoted \mathcal{A} . A reallocation becomes a list of variations $\Delta a = (\Delta \mathbf{x}_i, \Delta y_i)_{i \in \mathcal{I}}$ and it is *feasible* if and only if $\sum_{i \in \mathcal{I}} \Delta \mathbf{x}_i \leq \bar{\mathbf{x}} - \sum_{i \in \mathcal{I}} \mathbf{x}_i$ and $\sum_{i \in \mathcal{I}} \Delta y_i \leq \bar{y} - \sum_{i \in \mathcal{I}} y_i$. So far, the proposed extension does not substantially change the analysis: all novelty comes from specific properties associated to money. It is assumed to be *perfectly divisible* and such that, for all $i \in \mathcal{I}$ and $\mathbf{x}_i \in \mathbb{R}^N$:

1. $u_i(\mathbf{x}_i, y_i)$ is *continuous* in y_i ;
2. $u_i(\mathbf{x}_i, y_i)$ is *strictly increasing* in y_i ;
3. $\lim_{y_i \rightarrow 0} u_{iy}^m(\mathbf{x}_i, y_i) = +\infty$ where $u_{iy}^m(\mathbf{x}_i, y_i)$ denotes the *marginal utility of money* as measured at (\mathbf{x}_i, y_i) .

Previous assumptions mean that agents are always willing to hold money for itself. Although they represents a big deviation from the walrasian doxa, an extensive justification is postponed to the conclusion: the main point here is to draw their analytical advantages.

3.1.1 Definitions

Introducing a commodity-money with previous properties greatly simplifies the analysis. It first allows to relax the assumption that $u_i(\mathbf{x}_i, y_i)$ is strictly increasing in x_{in} for all $(i, n) \in \mathcal{I} \times \mathcal{N}$. In addition, the definition of basic concepts is simplified.

Definition 14 For any agent $i \in \mathcal{I}$ with initial plan $(\mathbf{x}_i, y_i) \in \mathbb{R}^{N+1}$, the individual surplus, as measured in money, associated to the change $(\Delta \mathbf{x}_i, \Delta y_i)$ is the amount $\Delta v_i \in \mathbb{R}$ such that $u_i(\mathbf{x}_i + \Delta \mathbf{x}_i, y_i + \Delta y_i - \Delta v_i) = u_i(\mathbf{x}_i, y_i)$.

The novelty is obviously the strict indifference requirement defining surplus (which was already assumed in Fig. 1 to 7). The assumptions made on money guarantee that Δv_i exists and is well-defined. The interpretation of surplus becomes even more familiar. Starting from $a : \Delta v_i > 0$ is agent i 's willingness-to-pay in exchange for the implementation of the change $(\Delta \mathbf{x}_i, \Delta y_i)$ while $\Delta v_i < 0$ is i 's minimal price for accepting that $(\Delta \mathbf{x}_i, \Delta y_i)$ be implemented.

Note that in the case where $\Delta \mathbf{x}_i' = (0, \dots, 0, \Delta x_{in}, 0, \dots, 0)$ with $\Delta x_{in} > 0$, and $\Delta y_i = 0$, the amount $\frac{\Delta v_i}{\Delta x_{in}}$ is simply agent i 's demand (maximal) price for commodity n (inverse-demand). Furthermore, if $\Delta y_i = -p_n \Delta x_{in}$, where p_n is a given uniform price of commodity $n \in \mathcal{N}$, Δv_i captures the standard marshallian concept of surplus.

The definitions of collective surplus and loss remain the same.

Definition 15 Starting from $a = (\mathbf{x}_i, y_i)_{i \in \mathcal{I}} \in \mathcal{A}$, the collective surplus, as measured in money, associated to a feasible reallocation $\Delta a = (\Delta \mathbf{x}_i, \Delta y_i)_{i \in \mathcal{I}}$ is the amount defined by

$$\Delta v \equiv \sum_{i \in \mathcal{I}} \Delta v_i - \sum_{i \in \mathcal{I}} \Delta y_i$$

where, for all $i \in \mathcal{I} : u_i(\mathbf{x}_i + \Delta \mathbf{x}_i, y_i + \Delta y_i - \Delta v_i) = u_i(\mathbf{x}_i, y_i)$.

Definition 16 Given $a \in \mathcal{A}$, let $l(a)$ denote the loss as measured in money:

$$l(a) \equiv \max_{\Delta a \text{ feasible}} \Delta v(a).$$

3.1.2 Properties

The results in this section highlight why money, as defined above, is a "natural" reference good *i.e.* an appropriate unit in which measuring surplus. First, the distribution of money does not influence collective surplus: only the situation of each agent in "real" terms determines collective surplus. Second, if some surplus, as measured in any other good than money, can be released then some surplus, as measured in money, can be released. This means that it is enough to check that no monetary surplus can be released to make sure that there exists no P-improving reallocation.

Proposition 17 Two reallocations which only differ with respect to individual changes in money balances release the same collective surplus.

Proof. See the appendix. ■

Corollary 18 Starting from an allocation fully distributing the economy money supply, a reallocation which only changes agents' money balances releases no collective surplus.

Releasing surplus is to provide the economy with a reallocation Δa desirable enough so that the total amount of money the direct winners are willing to pay to implement it exceeds what the direct losers call for in order to accept it. The next result considerably simplifies the search for loss-reducing reallocations.

Proposition 19 Let $a \in \mathcal{A}$ be an allocation fully distributing the total money supply. If $l(a) = 0$ then $l_n(a) = 0$ for all $n \in \mathcal{N}$.

Proof. See the appendix. ■

Previous result is useful to question the concern about the choice of a reference good. The dissatisfaction is that surplus calculation may depend on this choice, considered as arbitrary. The assumptions defining money and previous result make money a not so arbitrary choice. It further helps to understand why speaking about welfare in terms of money sounds so natural.

Corollary 20 $a \in \mathcal{P} \Leftrightarrow l(a) = 0$.

3.2 Surplus and marginal valuations

The analysis is now considered assuming continuity and differentiability in all dimensions, that is: for all $i \in \mathcal{I}$ and $n \in \mathcal{N}$, $u_i(\cdot)$ differentiable in x_{in} and y_i . This allows to consider infinitesimal reallocations denoted $da = (d\mathbf{x}_i, dy_i)_{i \in \mathcal{I}}$ in the neighborhood of any allocation $a \in \mathcal{A}$ and to get linearized approximate expressions of surplus.

The model can be restated for all $i \in \mathcal{I} : u'_{in}(\mathbf{x}_i, y_i) \geq 0$ for all $n \in \mathcal{N}$, $u'_{iY}(\mathbf{x}_i, y_i) > 0$ and $\lim_{y_i \rightarrow 0} u'_{iY}(\mathbf{x}_i, y_i) \rightarrow +\infty$.

Definition 21 For all $i \in \mathcal{I}$ and $(\mathbf{x}_i, y_i) \in \mathbb{R}^{N+1}$, let's define agent's i marginal valuation function $v'_{in}(\cdot)$ by

$$v'_{in}(\mathbf{x}_i, y_i) = \frac{u'_{in}(\mathbf{x}_i, y_i)}{u'_{iY}(\mathbf{x}_i, y_i)}.$$

It is expressed in money.

The writing $v'_{in}(\cdot)$ must not be taken as suggesting that a well defined individual value index exists. However, each agent is now described by a *system of (marginal) valuations*. All other things being equal, starting from (\mathbf{x}_i, y_i) and assuming $x_{in} > 0$, $v'_{in}(\mathbf{x}_i, y_i)$ is:

- (i) the *maximum* amount of money agent i would be willing to pay for a one unit increase in his possession of good n ;
- (ii) the *minimum* amount of money agent i would require against a one unit decrease in his possession of good n .

If $x_{in} < 0$, $v'_{in}(\mathbf{x}_i, y_i)$ is:

- (iii) the *maximum* amount of money agent i would agree to give up against a one unit decrease in his supply of service n (e.g. labor) to the economy;
- (iv) the *minimum* amount of money agent i would require against a one unit increase in his supply of service n to the economy.

3.2.1 Infinitesimal reallocations

The calculation of individual surplus directly derives from marginal valuation function.

Proposition 22 Under continuity and differentiability assumptions, for any $i \in \mathcal{I}$ with plan $(\mathbf{x}_i, y_i) \in \mathbb{R}^{N+1}$, the individual surplus, as measured in money, associated to some infinitesimal variations $(d\mathbf{x}_i, dy_i)$ can be written:

$$dv_i = \sum_{n \in \mathcal{N}} v'_{in}(\mathbf{x}_i, y_i) dx_{in} + dy_i.$$

Proof. For any $i \in \mathcal{I}$ and infinitesimal variations $(d\mathbf{x}_i, dy_i)$, the definition of dv_i involves

$$\sum_{n \in \mathcal{N}} u'_{in} dx_{in} + u'_{iY} \cdot (dy_i - dv_i) = 0,$$

where arguments of the utility function are omitted. Since $u'_{iY} > 0$, this can be rewritten

$$dv_i = \sum_{n \in \mathcal{N}} \frac{u'_{in}}{u'_{iY}} dx_{in} + dy_i = \sum_{n \in \mathcal{N}} v'_{in} dx_{in} + dy_i.$$

■

Starting from (\mathbf{x}_i, y_i) , the amount dv_i is the maximum contribution ($dv_i > 0$) or the minimum compensation ($dv_i < 0$) driving i to accept the individual change $(d\mathbf{x}_i, dy_i)$.

The collective surplus associated to an infinitesimal reallocation directly follows.

Corollary 23 *Under continuity and differentiability assumptions, given $a = (\mathbf{x}_i, y_i)_{i \in \mathcal{I}} \in \mathcal{A}$, the collective monetary surplus associated to some infinitesimal reallocation $da = (d\mathbf{x}_i, dy_i)_{i \in \mathcal{I}}$ can be written:*

$$dv = \sum_{i \in \mathcal{I}} \sum_{n \in \mathcal{N}} v'_{in}(\mathbf{x}_i, y_i) dx_{in}.$$

The fact that collective surplus is insensitive to displacement of money is explicit in this expression: this comes from linearization of the expression of surplus. Further points deserve attention. First, due to the symmetry between goods and agents, adding *consumption surpluses* with respect to agents or *individual surpluses* with respect to goods is equivalent. It follows that, for infinitesimal variations, the marshallian partial analysis is adequate: total surplus at the scale of the economy is indeed the sum of the surpluses released on each market... and yet, *wealth effects* are duly allowed in Allais' formulation. Second, previous expression involves the gradient of some unobserved "total valuation function". In the case where a "money" exists and differentiability can be assumed, surplus can duly be thought of in terms of valuations. Note that previous expression makes no assumption as to whether the reallocation da is feasible or not, whether surplus is retained or distributed.

The next proposition presents a remarkable didactic scope as regards the fundamental message of neoclassical economics.

Proposition 24 *Under continuity and differentiability assumptions, given $(\mathbf{x}_i, y_i)_{i \in \mathcal{I}} \in \mathcal{A}$, the collective monetary surplus associated to any infinitesimal reallocation $(d\mathbf{x}_i, dy_i)_{i \in \mathcal{I}}$ can be rewritten*

$$dv = \sum_{n \in \mathcal{N}} \sum_{i \in \mathcal{I}} \sum_{i > i} (v'_{in}(\mathbf{x}_i, y_i) - v'_{in}(\mathbf{x}_i, y_i)) d_i x_{in}.$$

where $d_i x_{in}$ denotes a net flow of good n from \hat{i} to i .

Proof. Starting from $(\mathbf{x}_i, y_i)_{i \in \mathcal{I}}$, the collective surplus associated to some reallocation $(d\mathbf{x}_i, dy_i)_{i \in \mathcal{I}}$ is written $dv = \sum_{i \in \mathcal{I}} \sum_{n \in \mathcal{N}} v'_{in}(\mathbf{x}_i, y_i) dx_{in}$. For any $n \in \mathcal{N}$, and any pair

$(i, \hat{i}) \in \mathcal{I}^2, \hat{i} \neq i$, let $d_i x_{in}$ denote a *net flow* of good n from \hat{i} to i . For all $n \in \mathcal{N} : dx_{in} = \sum_{\hat{i} \neq i} d_i x_{in}$. Omitting functions' arguments, this leads to the rewriting:

$$dv = \sum_{i \in \mathcal{I}} \sum_{n \in \mathcal{N}} \left(v'_{in} \cdot \sum_{\hat{i} \neq i} d_i x_{in} \right) = \sum_{n \in \mathcal{N}} \sum_{i \in \mathcal{I}} \sum_{\hat{i} \neq i} v'_{in} d_i x_{in}.$$

Since one deals with *net flows* of goods, for all $(i, \hat{i}) \in \mathcal{I}^2, \hat{i} \neq i : d_i x_{in} = -d_{\hat{i}} x_{in}$ so that

$$\sum_{i \in \mathcal{I}} \sum_{\hat{i} \neq i} v'_{in} d_i x_{in} = \sum_{i \in \mathcal{I}} \sum_{\hat{i} > i} (v'_{in} - v'_{\hat{i}n}) d_i x_{in}.$$

So the writing of collective surplus as

$$dv = \sum_{n \in \mathcal{N}} \sum_{i \in \mathcal{I}} \sum_{\hat{i} > i} (v'_{in} - v'_{\hat{i}n}) d_i x_{in}.$$

■

This writing expresses the basic but fundamental idea that the possibility to release surplus comes from differences between marginal valuations $v'_{in} - v'_{\hat{i}n}$. It highlights three ways to release surplus: (1) transactions between agents as consumers; (2) transactions between agents as service providers; (3) transactions between a service supplier and a consumer.

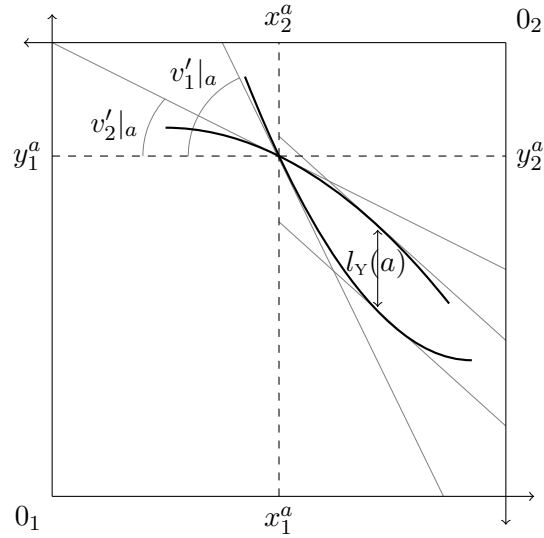


Figure 8: Loss and marginal valuations

The very simplicity of surplus expression under the assumption of continuous quantities and differentiability allow some interesting developments as to the characterization of Pareto allocations.

3.2.2 Surplus variations

Allais extends the specification of a reallocation to include its internal motion. Formally this means to consider lists of infinitesimal variations such as $((d\mathbf{x}_i, d^2\mathbf{x}_i), (dy_i, d^2y_i))_{i \in \mathcal{I}}$ to which one could refer as a "reallocation-in-motion". For any pair $(i, n) \in \mathcal{I} \times \mathcal{N}$, four types of motions ought to be distinguished.

$$\begin{array}{rcc}
 & & d^2x_{in} \\
 & & > 0 & < 0 \\
 dx_{in} & > 0 & \text{convex increase} & \text{concave increase} \\
 & < 0 & \text{convex decrease} & \text{concave decrease}
 \end{array}$$

The point here is to be able to know whether the direction taken by a reallocation induces increasing or decreasing *returns in surplus* that is, whether a further reallocation in the same direction might indeed increase surplus or not. The information required to answer is contained in the variations of the marginal valuations associated to the initial allocation $a \in \mathcal{A}$. An additional assumption is thus required here, namely that for all $i \in \mathcal{I}$ and $n \in \mathcal{N}$, $u_i(\cdot)$ is twice-derivable in x_{in} and y_i . Below, $\nabla v_i(\mathbf{x}_i, y_i) = (v'_{in}(\mathbf{x}_i, y_i))_{n \in \mathcal{N}}$ denotes i 's system of marginal valuations.

Proposition 25 *Under continuity and differentiability assumptions, for any $i \in \mathcal{I}$ with plan $(\mathbf{x}_i, y_i) \in \mathbb{R}^{N+1}$, the variation of individual monetary surplus associated to any infinitesimal exogenous "change-in-motion" $((d\mathbf{x}_i, d^2\mathbf{x}_i), (dy_i, d^2y_i))$ can be written¹¹ (omitting arguments)*

$$d^2v_i = (d\mathbf{x}_i)' \nabla^2 v_i|_{du_i=0} d\mathbf{x}_i + (\nabla v_i)' d^2\mathbf{x}_i + d^2y_i,$$

where $\nabla^2 v_i|_{du_i=0} = (v''_{in\bar{n}} - v'_{i\bar{n}} v''_{iny})_{(n, \bar{n}) \in \mathcal{N}^2}$ and $\nabla v_i = (v'_{in})_{n \in \mathcal{N}}$.

Proof. See the appendix. ■

The return in individual surplus of a given change can be separated into three terms (those of previous addition). The return of a change in money balances (right-hand side, third term of the addition) is always (trivially) increasing. The two other terms involve "real" changes and the agent's situation. One term (right-hand, second term of the addition) captures the variation in surplus assuming that the change does not modify the agent's valuations system: if a change in some direction is desirable (*i.e.* increases surplus) in the first place, pursuing in the same direction keeps increasing surplus. The remaining term captures the fact that the change in i 's vector of possessions deforms his marginal valuations system. This comes from *substitute* or *complement* relationships between goods as well as possibly decreasing or increasing marginal utility functions.

The impact of an infinitesimal change $(d\mathbf{x}_i, dy_i)$ on agent i 's valuation of some good n is written

$$dv'_{in} = \sum_{\bar{n} \in \mathcal{N}} v''_{in\bar{n}} dx_{i\bar{n}} + v''_{iny} dy_i.$$

¹¹Note that allowing for $d^2\mathbf{x}_i$ and d^2y_i different from zero comes from that these are *exogenous* variations

Suppose that the net variation in i 's money balances exactly compensates the impact of $d\mathbf{x}_i$ on agent i 's welfare *i.e.* $dy_i = -\sum_{\bar{n} \in \mathcal{N}} v'_{i\bar{n}} dx_{i\bar{n}}$. Corresponding variation in agent i 's valuation of good n can be written:

$$dv'_{in}|_{du_i=0} = \sum_{\bar{n} \in \mathcal{N}} v''_{in\bar{n}} dx_{i\bar{n}} + v''_{iny} \cdot \left(-\sum_{\bar{n} \in \mathcal{N}} v'_{i\bar{n}} dx_{i\bar{n}} \right) = \sum_{\bar{n} \in \mathcal{N}} (v''_{in\bar{n}} - v'_{i\bar{n}} v''_{iny}) dx_{i\bar{n}}.$$

Given (\mathbf{x}_i, y_i) , $dv'_{in}|_{du_i=0}$ denotes the variation in agent i 's valuation of good n as resulting from the change $d\mathbf{x}_i$, assuming dy_i exactly compensates the impact of $d\mathbf{x}_i$ on i 's welfare.¹² It thus captures *substitute/complement* effects of $d\mathbf{x}_i^{-n}$ on v'_{in} , as well as the direct effect of dx_{in} (possibly, a decreasing marginal utility).

Considering the economy as whole and some infinitesimal "reallocation-in-motion" $(da, d^2a) = ((d\mathbf{x}_i, dy_i), (d^2\mathbf{x}_i, d^2y_i))_{i \in \mathcal{I}}$, a variation of collective surplus is simply $d^2v = \sum_{i \in \mathcal{I}} d^2v_i - \sum_{i \in \mathcal{I}} d^2y_i$ so that

$$d^2v = \sum_{i \in \mathcal{I}} \left((d\mathbf{x}_i)' \nabla^2 v_i|_{du_i=0} d\mathbf{x}_i + (\nabla v_i)' d^2\mathbf{x}_i \right).$$

The first term captures the variation in surplus as resulting from the deformation of marginal valuations due to the reallocation. The second term captures the variation in surplus as resulting from the motion of the reallocation for given initial marginal valuations.

3.3 Pareto allocations

Under continuity and differentiability assumptions, P-allocations can be characterized as maximizing some *unobserved* total value index $v(a)$ without departing from an ordinal interpretation of utility functions.

Proposition 26 *Under continuity and differentiability assumptions, $a \in \mathcal{P} \Leftrightarrow dv(a) = 0$ and $d^2v(a) \leq 0$ for all feasible reallocation $da = (d\mathbf{x}_i, dy_i)_{i \in \mathcal{I}}$ where*

$$dv(a) = \sum_{i \in \mathcal{I}} \nabla v_i(\mathbf{x}_i, y_i) d\mathbf{x}_i,$$

and, for $dv(a) = 0$,

$$d^2v(a) = \sum_{i \in \mathcal{I}} (d\mathbf{x}_i)' \nabla^2 v_i|_{du_i=0}(\mathbf{x}_i, y_i) d\mathbf{x}_i.$$

¹²Obviously: $\nabla^2 v_i|_{du_i=0} d\mathbf{x}_i = \sum_{n \in \mathcal{N}} dv'_{in}|_{du_i=0} dx_{in}$.

Proof. See the appendix. ■

Before providing the economic interpretation of this necessary and sufficient condition for an allocation to be Pareto, two points deserve attention. First, the list of virtual money transfers $(dy_i)_{i \in \mathcal{I}}$ is not involved in this formulation. This comes from the fact that money is the unit of surplus measurement, and obviously that the Pareto criterion is not concerned with equity issues. Second, the motion of virtual reallocations as described by $(d^2 \mathbf{x}_i)_{i \in \mathcal{I}}$ is also absent from the condition. This results from the following property associated to the necessary condition $dv = 0$ for all feasible reallocation.

Proposition 27 *Given $a \in \mathcal{A}$, $dv(a) = 0$ for all feasible reallocation da , if and only if for any pair of agents $(i, \bar{i}) \in \mathcal{I}^2$, $\bar{i} \neq i$, and any good $n \in \mathcal{N}$: $v'_{in}|_a = v'_{\bar{i}n}|_a$.*

Proof. See the appendix. ■

The necessary condition $dv = 0$ for all feasible reallocation thus captures the property that all the agents agree on each good's value. Since no global convexity assumption is imposed, a further condition for an allocation to be Pareto is that there exists no direction in which a marginal reallocation could restore some space for mutually advantageous transactions. This is the essence of the *decreasing return in surplus* additional condition. At the scale of individual agents, at most one can be in a situation of local concavity: in general, previous inequality states that all agents exhibit *local* convexity *i.e.* decreasing marginal valuations. Example 28 and Fig. 9 illustrate how the GTS accommodates possible concavities in the structure of preferences. The process of surplus releasing tends to discard allocations involving concavities so that convexity emerges as a result of the process.

Example 28 *The case in Fig. 9:*

$$v'_1|_a = v'_2|_a \text{ but } \left(v''_{1x}|_a - v'_1|_a v''_{1y}|_a \right) + \left(v''_{2x}|_a - v'_2|_a v''_{2y}|_a \right) > 0;$$

$$v'_1|_{a^*} = v'_2|_{a^*} \text{ and } \left(v''_{1x}|_{a^*} - v'_1|_{a^*} v''_{1y}|_{a^*} \right) + \left(v''_{2x}|_{a^*} - v'_2|_{a^*} v''_{2y}|_{a^*} \right) < 0.$$

The discussion of the relationships between zero-loss allocations and general equilibrium is postponed; note however that in the case depicted in Fig. 9, no walrasian equilibrium exists (due to non-convexity).

Previous developments show how simple and intuitive the questioning about welfare measurement and Pareto-efficiency become, once formulated in terms of surplus. The analysis nevertheless captures the core of what the marginalist theory tells us about the efficient allocation of resources. An implication of Proposition 27 is of course that, if some allocation a is P-efficient then for any pair of agents $(i, \bar{i}) \in \mathcal{I}^2$, $\bar{i} \neq i$, and any pair of goods $(n, \bar{n}) \in \mathcal{N}^2$, $\bar{n} \neq n$:

$$\frac{u'_{in}}{u'_{i\bar{n}}}\Big|_a = \frac{u'_{\bar{i}n}}{u'_{\bar{i}\bar{n}}}\Big|_a,$$

that is, marginal rates of substitutions are all equal. This shows that explicitly inserting money in the analysis is not such a big sin; at least as far normative analysis is concerned, it does not so dramatically alter the neoclassical message. The next section is devoted to the positive economics Allais derives from his concept of surplus.

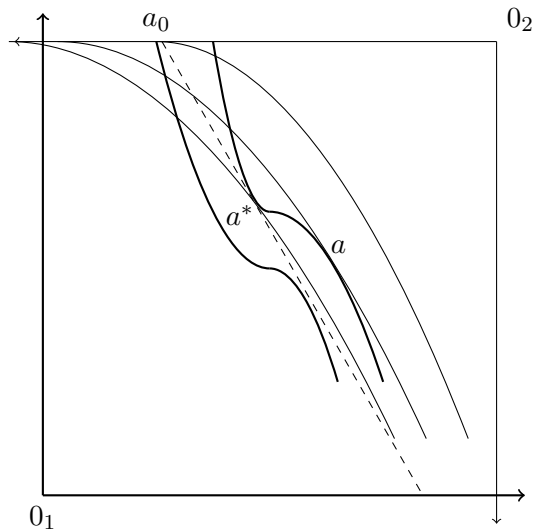


Figure 9: Necessary and sufficient conditions for Pareto-efficiency when global convexity is not assumed (a_0 depicts the initial allocation)

4 Positive economics

So far, surplus and loss are considered as tools for welfare analysis. Allais actually draws from these concepts a vision (rather than a theory) of the functioning of a free-exchange economy. He makes his point through the notion of an "economy of markets" as opposed to the "market economy" modeled by the standard walrasian theory (understood as a "one-market economy"). The expression "surplus-seeking economy" is preferable however since, as already mentioned, except as regards the right to private property and the principle of voluntary transaction, no specific institutional setting such as markets is imposed in the GTS.¹³

4.1 Surplus-seeking

Allais' positive economics all stems from the statement that: "In essence any economic operation, whatever it may be, should be viewed as related to the search, realization and distribution of [...] surplus" (GTS, p. 32, §115). The surplus-seeking process relies on intense (possibly strategic) communication between agents and information collection.

4.1.1 Individual behaviors

Agents' basic impetus is to search for other agents willing to accept bilateral or multi-lateral transactions (exchange or cooperation in production) creating surplus that can

¹³Although Allais does not explicitly highlight this feature of the GTS, some of his remarks (say, GTS, p. 362, §564) suggest he was aware of it.

be distributed. As production is considered, this involves that agents seek technical efficiency. Compared to the walrasian solipsistic view of economic behaviors (anonymous agents facing a price system, and communicating only with a central auctioneer), that of the GTS is relational thus. Agents in Allais' vision have relationships: they are social beings in the sense that they transmit and collect information to/about others, seeking for surplus-releasing opportunities. They do have direct non-anonymous interactions with one another that can take many forms beyond market transactions, the outcome of which is the realization of surplus. In this respect, Allais' approach (although highly abstract) supports a stronger connection of economics to the rest of social sciences.

4.1.2 Equilibrium concept

The GTS primarily describes *out-of-equilibrium* processes of voluntary transactions. These processes are directed by surplus-releasing transactions between agents. Since, in such an economy, surpluses are constantly created and distributed, it follows that, while the utility of one agent grows, the utility levels of others can never decrease (possibly thanks to an increase in money balances). This involves that, for given economic structures (preferences, technology, and endowments), any process of informed voluntary transactions is *loss-reducing*. In this perspective, an allocation is an equilibrium if and only if its associated loss is zero (no feasible reallocation exists releasing surplus) that is, if it satisfies the Pareto criterion. As with the concept of *exchange equilibrium* considered in Foley (2010), the Pareto criterion acquires a positive (predictive) meaning. P-allocations' force of attraction is further enhanced in the GTS by the possibility for an agent to retain surplus from a transaction in which none of his own (tangible) resources have been engaged; only some private information about others' dispositions. Agents' incentives to make P-improving transactions possible come from the prospect of a partial or full appropriation of released surplus. And yet, Allais is aware that some transactional obstacles may stand on the path of some particular loss-reducing process (e.g. that some P-improving transaction may be too difficult to design or require too much information to be implemented). This leads him to consider that the process may end on a "least loss" state, to which he refers as a "second best equilibrium". Adopting the "least loss" or the Pareto criterion as a concept of positive equilibrium obviously involves that the issue of *ex ante* uniqueness is off topic in Allais' vision; confronted to a dilemma between an intuitive vision of the economic process without any theory of value, or a theory of value flawed by a missing theory of price formation, Allais chooses the first option.

4.1.3 A possible explicit process

Courtault and Tallon (2000) provide a possible specification of the process followed by an economy of surplus-seekers, and formally prove that this process indeed ends on a P-allocation. At each stage, starting from a given allocation of resources, agents are assumed to adopt a reallocation maximizing surplus. As illustrated above, such reallocation does not, in general, directly lead to a P-allocation; and yet, each stage

brings the economy closer and closer from a P-allocation. Courtault and Tallon's analysis is conducted in terms of Luenberger's benefit function but it is easily reformulated in terms of surplus. More substantially, they have recourse to a global strict convexity assumption. This is obviously at odds with Allais' reluctance to make this assumption, but the fact that convexity is necessary to get a well defined path converging towards a P-allocation is not surprising if one considers the illustrations provided below. Convexity makes that each stage of the process brings marginal valuations of various agents closer from one another. This is probably not a necessary condition to get that a loss-reducing process converges towards a P-allocation, but it certainly shortens the path.

Proposition 29 (Courtault and Tallon, 2000) *Suppose, for all $i \in \mathcal{I}$, $u_i(\cdot)$ is continuous and strictly quasi-concave. From any $a \in \mathcal{A}$ and for any $n \in \mathcal{N}$, the sequence of reallocations $\{\Delta_t a\}_{t \in \mathbb{N}_+^*}$ defined, for all $t \in \mathbb{N}_+^*$, by $\Delta_t a \in \arg \max_{\Delta a \in \mathcal{R}(a_{t-1})} \Delta v_n(a_{t-1})$, $a_t = a_{t-1} + \Delta_t a$ and $a_0 = a$, leads to a unique allocation $a^* = a + \sum_{t \in \mathbb{N}_+^*} \Delta_t a$ which is Pareto-efficient.*

Proof. See Courtault and Tallon (2000). ■

4.1.4 Relation to walrasian equilibria

Allais' objections to a positive interpretation of the walrasian GET became commonplace. He mainly questions an equilibrium determination relying on virtual behaviors (demand and supply schedules) of agents facing a well-defined single common system of prices, which are presumed to adjust without actual exchanges take place; only at equilibrium do transactions occur, once and for all (although trading partners remain unspecified). He makes no attempt to reconcile his vision with the walrasian equilibrium predictions. And indeed, even in the most favorable cases, there is no reason for the P-allocation selected by some loss-reducing process to precisely be a walrasian equilibrium. This point is illustrated in Fig. 10 which depicts both the results of some surplus-maximizing reallocation, and the walrasian equilibrium of a 2 agents \times 2 goods Cobb-Douglas exchange economy. Three allocations with specific properties are considered as would-be candidates: P-allocations which also result from a surplus-maximizing reallocation (in one dimension or the other), allocation resulting from a reallocation which maximize surplus as expressed in both dimensions. None of these candidates exhibit any specific relation to the walrasian equilibrium; the lens generated by the allocation resulting from the two-dimensions surplus maximization, does not even contain the walrasian equilibrium.

4.2 Markets

Curiously, market and competition are perhaps the economic objects the most poorly represented in the walrasian GET: this point is extensively discussed in Ostroy & Makowski (2001). Difficult to see any relation between what any intuitive description of what a market is or does, on the one hand, and the walrasian "market" with its

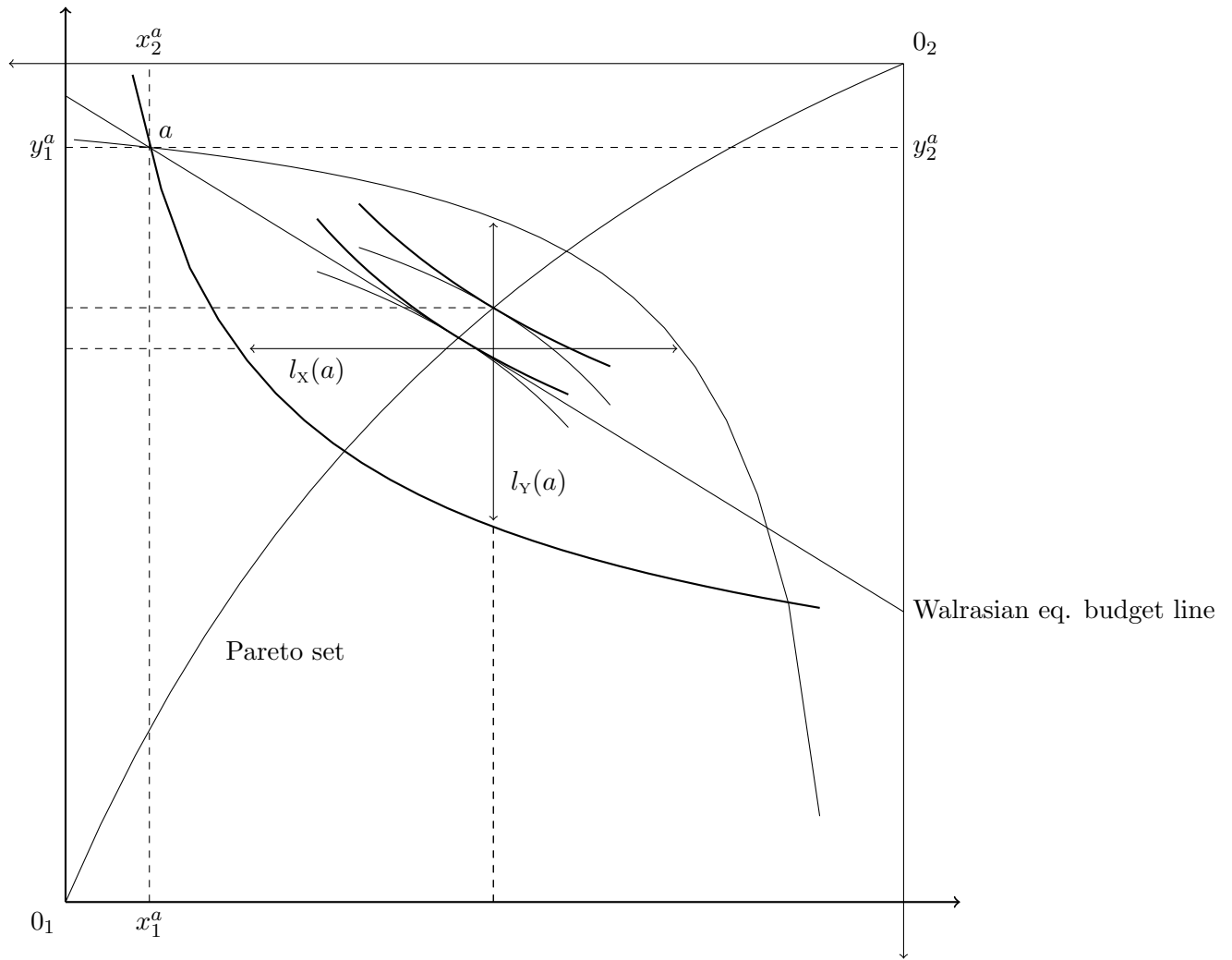


Figure 10: Various general equilibrium concepts

single given price, on the other hand. In an effort, not so common within an economics textbook, to conceptualize market, Phelps (1985, p. 170) depicts it as a phenomenon of information sharing in exchange, an "inherently multilateral mechanism in which arbitrageurs and brokers play the role of linking buyers to one another, and similarly linking sellers, thus bringing all transactions into the market" (p. 180). Markets consist in traders who gather the information on exchange opportunities, and are places where exchanges of a given type are agglomerated. Can a simple formal representation of the economy do a better job than the walrasian one in dealing with the market phenomenon? Colanders & Rothschild (2010) suggest that Marshall's approach remains a valuable alternative within the neoclassical paradigm (see also Plott *et al.*, 2013).

Allais is anxious to contrast his vision of a free-exchange economy, as supported by the GTS, from the one supported by the GET. In his positive interpretation of the GTS, *out of* equilibrium, there is no unique system of prices given to all agents, but a huge variety of prices specific to each transaction. There is no general and centralized single market for all goods in the economy, but a set of partial markets. Each one is associated to the exchange of one particular good against (commodity-)money, and is not the only one where the good in question can be exchanged. In each partial market, a price is set by the confrontation of demands and supplies, and the setting of this price is followed by actual exchanges. Exchanges generally take place between definite agents at prices specific to these agents. All the exchanges involving some particular agent (as well as, possibly, his production decisions) are not realized at once, once and for all at a single price system, and the evolution of the economy toward equilibrium involve a suite of successive exchanges (and related production operations) in successive periods of time, at different prices.

4.2.1 Back to Marshall?

Previous list of claims strikingly recalls a marshallian vision of the economy. The picture is that of a world in which time and space segment markets. Local analysis of the markets can be conducted in terms of partial and temporary equilibria, possibly distinguishing between the short and the long period. Money is required to connect markets the ones to the others. Agents bargain over price, as well as over quantity, and the degree of competition can vary from one market to the other. Some elementary developments can help further understanding the scope of Allais' analysis connection to that of Marshall.

Let's consider the market of some good \bar{n} .

Within a context of general interdependence, Marshall's analysis is formally presented by Friedman and Sakovics (2014). This presentation starts from the standard problem of the price-taking consumer:

$$\begin{cases} \max_{\mathbf{x}} u(x_{\bar{n}}, \mathbf{x}^{-\bar{n}}) \\ p_{\bar{n}}x_{\bar{n}} + (\mathbf{p}^{-\bar{n}})' \mathbf{x}^{-\bar{n}} \leq y \end{cases}$$

where $\mathbf{p} = (p_{\bar{n}}, \mathbf{p}^{-\bar{n}})$ denotes some given price system. To represent Marshall's analysis within a general interdependence framework, Friedman and Sakovics argue that utility

must be separable *i.e.* can be written as: $u(\mathbf{x}) = u_{\bar{n}}(x_{\bar{n}}) + u^{\perp}(\mathbf{x}^{-\bar{n}})$. In this case, the agent's problem can be rewritten: $\max_{x_{\bar{n}}} \{u_{\bar{n}}(x_{\bar{n}}) + \bar{u}^{\perp}(y - p_{\bar{n}}x_{\bar{n}}, \mathbf{p}^{-\bar{n}})\}$, where

$$\bar{u}^{\perp}(y - p_{\bar{n}}x_{\bar{n}}, \mathbf{p}^{-\bar{n}}) \equiv \max_{(\mathbf{p}^{-\bar{n}}) / \mathbf{x}^{-\bar{n}} \leq y - p_{\bar{n}}x_{\bar{n}}} u^{\perp}(\mathbf{x}^{-\bar{n}})$$

is a partial indirect utility function parametrized by $x_{\bar{n}}$. A second assumption, that Marshall makes explicit in the *Principles* is that the spending in good \bar{n} must be small compared to y and $(\mathbf{p}^{-\bar{n}}) / \mathbf{x}^{-\bar{n}}$. The point is obviously to be able to neglect *wealth effects* with the approximation that:

$$\bar{u}^{\perp}(y - p_{\bar{n}}x_{\bar{n}}, \mathbf{p}^{-\bar{n}}) \simeq \bar{u}^{\perp}(y, \mathbf{p}^{-\bar{n}}) - \frac{\partial \bar{u}^{\perp}}{\partial y}(y, \mathbf{p}^{-\bar{n}}) p_{\bar{n}}x_{\bar{n}}.$$

Under these assumptions, the consumer approximately solves:

$$\max_{x_{\bar{n}}} \left\{ u_{\bar{n}}(x_{\bar{n}}) - \frac{\partial \bar{u}^{\perp}}{\partial y}(y, \mathbf{p}^{-\bar{n}}) p_{\bar{n}}x_{\bar{n}} \right\},$$

where $\frac{\partial \bar{u}^{\perp}}{\partial y}(y, \mathbf{p}^{-\bar{n}})$ represents the *marginal utility of money* as measured in $(y, \mathbf{p}^{-\bar{n}})$.¹⁴ With separability and negligible wealth effects, the good \bar{n} market behavior of the agent with endowment $x_{\bar{n}}$ is then simply driven by the sign of $w'_{\bar{n}}(x_{\bar{n}}, y; \mathbf{p}^{-\bar{n}}) - p_{\bar{n}}$ where

$$w'_{\bar{n}}(x_{\bar{n}}, y; \mathbf{p}^{-\bar{n}}) = \frac{\frac{\partial u_{\bar{n}}}{\partial x_{\bar{n}}}(x_{\bar{n}})}{\frac{\partial \bar{u}^{\perp}}{\partial y}(y, \mathbf{p}^{-\bar{n}})}.$$

The rational choice is $dx_{\bar{n}} > 0 \Leftrightarrow w'_{\bar{n}}(x_{\bar{n}}, y; \mathbf{p}^{-\bar{n}}) > p_{\bar{n}}$.

Allais' analysis of simple market behaviors is now considered. Suppose that a surplus-seeker with endowment (\mathbf{x}, y) observes that the good \bar{n} is exchanged on the market at price $p_{\bar{n}}$ (expressed in money). An amount $dx_{\bar{n}}$ of good \bar{n} can be exchanged against $-p_{\bar{n}}dx_{\bar{n}}$. The agent's market behavior is given by $dx_{\bar{n}} > 0 \Leftrightarrow v'_{\bar{n}}(\mathbf{x}, y) > p_{\bar{n}}$. Note that the implicit budget constraint directly influences behavior from the assumption that $\lim_{y \rightarrow 0} u'_Y(\mathbf{x}, y) = +\infty$. More generally, for any subset \mathcal{N}_{\circ} of goods, observing prices $(p_n)_{n \in \mathcal{N}_{\circ}}$, the agent decides on some variations $(dx_n)_{n \in \mathcal{N}_{\circ}}$ so as to increase his individual surplus, simply written as $\sum_{n \in \mathcal{N}_{\circ}} (v'_n(\mathbf{x}, y) - p_n) dx_n$. For any $n \in \mathcal{N}_{\circ}$: $dx_n > 0 \Leftrightarrow v'_n(\mathbf{x}, y) > p_n$.

From previous developments, Allais' and Marshall's analyses appears as remarkably similar in their view of market behaviors if not in their basic assumptions. In both cases, agents are characterized by a system of subjective marginal valuations which drive their economic behaviors through confrontation to some existing prices. The difference rests in the way these subjective marginal valuations are constructed, the difference

¹⁴In this expression, note that only the total expenditure needed to buy the quantity $x_{\bar{n}}$ matters: the price of each unit of good \bar{n} exchanged does not need to be uniform. The amount $p_{\bar{n}}$ can thus be interpreted as the average price paid for each purchased unit of good \bar{n} .

between $v'_n(\mathbf{x}, y)$ and $w'_n(x_{\bar{n}}, y; \mathbf{p}^{-\bar{n}})$. In the former, agents are assumed to draw a direct satisfaction from holding money, not in the latter; and yet, money balances appear as an argument of agent's subjective valuations in both cases. The main difference is thus the requirement of separability in the marshallian approach, which rules out any relation in use between good \bar{n} and the others: neither complements, nor substitutes. This obviously establishes a clear boundary between markets, and ruins the perspective of general interdependence.

This disappointing retreat is avoided in Allais' GTS although a partial equilibrium approach still makes sense. Given any subset \mathcal{I}_\circ of agents and subset \mathcal{N}_\circ of goods, nothing prevents to consider a situation of gains-in-exchange exhaustion as a partial and temporary equilibrium, namely, for all $n \in \mathcal{N}_\circ$, $(i, \bar{i}) \in \mathcal{I}_\circ^2$, $\bar{i} \neq i$:

$$v'_{in}(\mathbf{x}_{\bar{i}}^{-\circ}, \mathbf{x}_{i\circ}^*, y_i^*) = v'_{in}(\mathbf{x}_i^{-\circ}, \mathbf{x}_{i\circ}^*, y_i^*),$$

where $\mathbf{x}_{i\circ}$ denotes agent i 's bundle of the goods in \mathcal{N}_\circ , and $\mathbf{x}_i^{-\circ}$ a given specification of the rest of his consumptions; the star distinguishes (partial) equilibrium values. Allais thus provide a genuinely decentralized approach of markets. The trading path through which partial equilibrium conditions come to prevail is not explicit, and the collection $(\mathbf{x}_{i\circ}^*, y_i^*)_{i \in \mathcal{I}_\circ}$ generally depends on this trading path. In fact a variety of market behaviors is allowed.

4.2.2 A variety of market behaviors

Let's extrapolate on Allais' informal conception of individual economic behaviors. For any $n \in \mathcal{N}$, the transactional behavior of any agent may be active or passive. Active agents are those attempting to initiate transactions; passive ones are those observing existing transactional terms, possibly making take-it or leave-it decisions. Transactional behaviors are grounded on some previously collected information on others' dispositions (preferences and personal technologies). Equipped with such (most likely very partial) information, active agents may formulate private or public transaction proposals. Private proposals are targeted on some specific partner(s) whereas public ones are non-directed. Making transactional proposals may reveal information on active agents' dispositions: privately targeted proposals may be preferred to retain this information as much as possible, but they may lead to bargaining costs and prevent possible benefits (to the author of the initial proposal) from competition between passive agents (bilateral monopoly). The benefits and costs are reversed in the case of a public proposal. The recourse to market comes to publicly formulate a well-defined offer (posted "best" price and fixed transactional conditions): this avoids bargaining costs, possibly increase the likelihood that a transaction takes place, but it imposes to share information on one's valuation with the public. This may further trigger competition between possible partners. Transactional passivity is the choice of information collection (to wait and see) and possibly of bargaining. Effective transactional activity reveal private information which becomes a resource to whom knows how to use it.

Two types of market behaviors are now considered: the behavior of a single price-maker on a perfect (centralized) market, and *market intermediation*.

Let $(\mathbf{x}_i^{-\bar{n}}, x_{i\bar{n}}, y_i)_{i \in \mathcal{I}} \in \mathcal{A}$ and consider agent \bar{i} who is assumed price-maker on the market of good \bar{n} , all other agents being price-takers. The market is assumed *perfect i.e.* the one-price rule applies. With perfect information on other agents valuations, agent \bar{i} behavior is to choose a pair $(dx_{\bar{i}\bar{n}}, p_{\bar{n}})$ such that $v'_{\bar{i}\bar{n}}(\mathbf{x}_i^{-\bar{n}}, x_{i\bar{n}} - dx_{\bar{i}\bar{n}}, y_i + p_{\bar{n}}dx_{\bar{i}\bar{n}}) = p_{\bar{n}}$, $v'_{i\bar{n}}(\mathbf{x}_i^{-\bar{n}}, x_{i\bar{n}} + dx_{\bar{i}\bar{n}}, y_i - p_{\bar{n}}dx_{\bar{i}\bar{n}}) = p_{\bar{n}}$, for all $i \neq \bar{i}$, $\sum_{i \neq \bar{i}} dx_{i\bar{n}} = dx_{\bar{i}\bar{n}}$. It is obviously important that $\mathbf{x}_i^{-\bar{n}}$ remains constant. Note that the sign of $dx_{\bar{i}\bar{n}}$ has not been specified involving that agent \bar{i} is not necessarily a net seller nor other agents net buyers. Setting a price does not mean that you are a monopoly: with a price-signal at hand, some other price-taker agents may use the market to change good \bar{n} against money.

Particularly interesting in a positive interpretation of the GTS, is the possibility for an agent to draw some surplus without engaging any of his own resources, but simply his information about others' valuation. Smith & Foley (2008) refer to this case to operationalize the idea of "reversible transformation", calling "speculator" some external agent drawing wealth from the economy through arbitrage. In the GTS, since money is *per se* an argument of individuals utility, agents have incentive to conduct such arbitrage operation. There is no single benevolent auctioneer but possibly as many brokers as interested agents in the economy. This provide a market making behavior remarkably consistent with what an informal conceptualization of market, such as that mentioned in Phelps (1985), would invoke: the process of market emergence starts with merchants who collect information disseminated all across the economy on agents' valuations and holdings.

Previous elaborations thus provide economic organizing principles which go beyond competition. Analyses are in fact not grounded on any concept of competition: the main focus is on exchange and cooperation; this translates into the use of Pareto efficiency as the main positive concept. Surplus allows to accommodate some space between general equilibrium and (possibly cooperative) game theory.

4.3 Beyond markets

Indeed, the GTS does not confine agents to market transactions. In fact, Luenberger's shows that surplus-seeking behaviors are consistent with externalities: "goods" do not have to be private! This makes possible the extension of the "Coase theorem" to cases where wealth effects can not be ignored. This opens a new door between neoclassical and institutional economics.

4.3.1 Externalities: Luenberger's generalizations

Although Allais confidently asserts that externalities can be taken into account in the GTS (see p. 152, §324), he does not provide any formal presentation of it. Luenberger (1995) fills the gap in terms of benefit function. Corresponding definitions and results can easily be reformulated in terms of surplus. The most general formulation allows

each agent i 's utility to depend on the complete allocation $a = (\mathbf{x}_i, y_i)_{i \in \mathcal{I}}$ that is, for all $i \in \mathcal{I}$, $u_i [(\mathbf{x}_i, y_i)_{i \in \mathcal{I}}]$, but of course, u_i may depend only on a few components of a . The definition of collective surplus can be adapted from Luenberger (1995) as follows.

Definition 30 Starting from $a = (\mathbf{x}_i, y_i)_{i \in \mathcal{I}} \in \mathcal{A}$, the collective surplus, as measured in money, associated to a feasible reallocation $\Delta a = (\Delta \mathbf{x}_i, \Delta y_i)_{i \in \mathcal{I}}$ is the amount defined by

$$\Delta v \equiv \sum_{i \in \mathcal{I}} \Delta v_i - \sum_{i \in \mathcal{I}} \Delta y_i$$

where, for all $i \in \mathcal{I}$: $u_i [(\mathbf{x}_i + \Delta \mathbf{x}_i, y_i + \Delta y_i - \Delta v_i)_{i \in \mathcal{I}}] = u_i [(\mathbf{x}_i, y_i)_{i \in \mathcal{I}}]$.

The list of individual surpluses $(\Delta v_i)_{i \in \mathcal{I}}$ are calculated all together. And yet, the interpretation of Δv remains intuitive: it is the maximum amount of money that the group of all agents would be willing to pay for the reallocation Δa to be implemented. The loss keeps the same definition, that is the maximal surplus which a feasible reallocation can release. Furthermore, Pareto-allocation remain characterized by the same condition.

Proposition 31 $a \in \mathcal{P} \Leftrightarrow l(a) = 0$.

Proof. See Luenberger (1995). ■

Returning to a positive perspective, surplus-seeking behaviors can be motivated by concerns about negative externalities: agents are not *a priori* incapable of dealing with it in a P-improving way. This enlarged transactional ability naturally connects surplus-seeking to institutional economics.

4.3.2 Incorporating (basic) Institutional economics

A better inclusion of the institutional approach within standard microeconomics is on theorists' agenda for decades, most notably with a pedagogical purpose in mind. The trouble is that doing this requires to question the centrality of the GET, a step that one is often reluctant to do. The common view seems to be that neoclassical and institutional economics are two different ways to see the same landscape, which do not need to provide the exact same description.

Williamson (2005) expresses this point by distinguishing between two different lenses: that of choice (neoclassical approach), and that of contract (institutional approach). An illustration of this accommodating position is the analysis of the firm: a "black box" with the neoclassical lens that the institutional one allows "to open". It can be argued that the two lenses metaphor does not do full justice to fundamental disagreements between neoclassical and institutional descriptions of the economy. Take for instance, Coase's famous remark at the root of the transaction cost approach: "The main reason why it is profitable to establish a firm would seem to be that there is cost of using the price mechanism, the most obvious [being] that of discovering what the relevant prices are"

(Coase, 1937, p. 391). How could this issue of "discovering what the relevant prices are" be accommodated within a walrasian description of the economy? Allais' surplus-seeking world appears as much more hospitable to Coase's arguments.

Besides, Allais' vision of the economy exhibits a remarkable concordance to that of the proponents of the institutional approach. Think about Commons defining transaction as the "ultimate unit of activity", Buchanan considering "mutuality of advantage from voluntary exchange" as "the most fundamental of all understandings in economics", or again Williamson to whom economics is about understanding the "ordering" of transactional activity (all quotations in Williamson, 2005).

Milgrom and Roberts (1995) or Spulber (2009) provide two interesting attempts to coherently articulate institutional and neoclassical economics within a single encompassing presentation. To do so, both texts have recourse to quasi-linear specifications of utility functions. This choice is obviously partly grounded on the intention to make the reasoning as simple as possible, but it also happens to be the right framework to give a strong feeling of continuity between a transaction-based (institutional) and a market-based (walrasian) approaches. In fact, as Smith and Foley (2008) have shown, this is the only case where, as for non-linear goods, the price system of any *exchange equilibrium* is the same as that of a *walrasian equilibrium*.

This choice has thus the disadvantage of misleadingly promoting a positive interpretation of the concept of walrasian equilibrium. And yet, it allows Milgrom and Roberts (1995) to give a convincing account of what microeconomics is about: its main objects (organizations and markets), and issues (coordination and incentives). Not surprisingly, this achievement relies on a positive interpretation of efficiency, and on the recourse to an *equivalent value index*.¹⁵ Using this index, Milgrom & Roberts get that "an allocation among a group of people whose preferences display no wealth effects is efficient if and only if it maximizes the total value of the affected parties" (p. 36). Finally, they are able to introduce the *Coase theorem* as: "if the parties bargain to an efficient agreement (for themselves) and if their preferences display no wealth effects, then the value-creating activities that they will agree upon do not depend on the bargaining power of the parties or on what assets each owned when the bargaining began. Rather, efficiency alone determines the activity choice. The other factors can affect only decisions about how the costs and benefits are to be shared" (p. 38).

The surplus-seeking behaviors envisioned in the GTS seem to be the exact tool needed to overcome the limitation of the no wealth effects hypothesis. Surplus appears as an exact counterpart to that of value in Milgrom & Roberts presentation¹⁶ and, as in their formulation of the Coase theorem, reallocations maximizing money-surplus are specified independently from the issue of surplus distribution between agents: a separability result is obtained between "surplus-releasing" ("value-creation") and "surplus distribution".¹⁷

¹⁵Milgrom and Roberts' efficiency principle is stated as: "if people are able to bargain together effectively and can effectively implement and enforce their decisions, then the outcomes of economic activity will tend to be efficient (at least for the parties to the bargain)" (p. 24).

¹⁶The key management concept of "value-creation" itself finds a natural counterpart in that of "surplus-releasing" in Allais' terminology.

¹⁷This point is raised in Guesnerie (1984) who notes: "Although Allais does not seem to be totally

Elaborating further on Allais' vision, even such concepts of *transaction costs* and *asset specificity* could make their way within the GTS. In his Nobel address, Coase observes that "the concept of transaction costs has not been incorporated into a general theory. [...] incorporating transaction costs into standard economic theory, which has been based on the assumption that they are zero, would be very difficult, and economists [...] have not been inclined to attempt it." (Coase, 1992, p. 718). In fact, as suggested above, the "least-loss" equilibrium concept provided by Allais can be regarded as the right criterion to account for transaction costs within a general interdependence framework. Moreover, the GTS inspires a useful difference between transaction cost and "transaction losses". Within a general equilibrium perspective, the cost of arranging some advantageous transaction within a subgroup of agents may be the receipt of some broker: part of the surplus to the subgroup is diverted to the broker but no resources are lost. On the other hand, the time spent by some agent searching for a transactional opportunity, inasmuch as time is of some value to him, is a "transaction loss": it reduces the eventual surplus. More generally, surplus expression as

$$dv = \sum_{n \in \mathcal{N}} \sum_{i \in \mathcal{I}} \sum_{\hat{i} > i} (v'_{in}(\mathbf{x}_i, y_i) - v'_{in}(\mathbf{x}_{\hat{i}}, y_{\hat{i}})) d_i x_{in}.$$

makes explicit that a process of surplus-releasing relies on some structure of possible bilateral transactions, which could be described as a network. For most pairs (\hat{i}, i) , $d_i x_{in} = 0$ for structural reasons, *i.e.* because agents \hat{i} and i are not connected. This represents a structural conception of transaction costs. Unless a third agent intermediates, making the transaction multilateral, there is no way for \hat{i} and i to make mutually advantageous transactions.

Finally, one may draw inspiration from the GTS to sketch a theory of the organization. In the spirit of previous elaborations, organization could be understood as an arrangement of *specific* and *private* transactions allowing some agents to retain information, exploited in surplus appropriation. As mentioned above, market transaction occurs with some degree of publicity, some agents' valuations being revealed in the exchange process. On the contrary, such transactional tools as the *employment contract* let the purpose of the employee-employer relationship unspecified *ex ante*: organizations are places where transactions occur without information revelation. Besides, "asset specificity" easily finds its way in the GTS: since competition has no part, it is not excluded that some goods be of value only for a tiny subgroup of agents,¹⁸ making these goods very "specific" in the neoinstitutional sense of the term. Including the whole apparatus of the asymmetric/incomplete information literature into the analysis, transactions on these goods become hazardous, *i.e.* specificity rises transaction costs. To the concept of "asset specificity" would be associated that of "transaction specificity" to capture what the organization, as opposed to market, is about.

explicit about whether, in the spirit of the Coase theorem, [...] surplus would actually be totally exhausted through economic activity, he emphasizes the role of [...] surplus as the analogue of 'potential' in physics, or [...] as the natural 'Lyapounov function' of non-tâtonnement theory." (p. 782).

¹⁸No atomistic assumption required.

5 Conclusion

The walrasian GET keeps a central role as regards research and teaching in economics: a reference in organizing economic thought; an analytical basis to a multitude of developments. But it is also at the heart of skepticism aroused by economic theory: interrogation as to what it represents (normative theory of value or model of a market economy); occultation of the fundamental issues of markets functioning and price formation; support to ideological reasonings (celebration of competition over cooperation). The present paper is an assessment of Allais' GTS as an alternative to the walrasian GET to handle the neoclassical legacy. The GTS is not meant to make determinate prediction on the system of prices, nor to provide a theory of value. Its purpose is simply to offer a reasonable basic representation of the processes at work within a free exchange economy recognizing private property. It is shown that the most fundamental concepts of neoclassical economics all find their place within Allais' analytical framework.

5.1 Explicit integration of money within the analysis

One big difference compared to the walrasian GET is an explicit integration of money in the analysis. It is not to say that the GTS is intended to provide a theory of money,¹⁹ but that isolating a commodity adequate as a medium of exchange (desired by all agents in any circumstances), a unit of account (perfectly divisible), and a store of value makes sense when thinking about welfare and efficiency.

Money is often considered as an object logically inconsistent with walrasian GET. The usual argument is that, insofar as money has no intrinsic utility, there is no point for agents to retain cash balances: at *equilibrium*, the price of money is zero. The difficulty to make room to money within the walrasian framework is thus closely linked to the emphasis put on equilibrium situations. Two primary features of the GTS make a direct integration of money in (all) utility functions an admissible short-cut: first, it applies out-of-equilibrium; second, it does not rely on any assumption of perfect information. Allais' point is the same as that of Marshall in *The Principles*, that is to analyze the operation of an economy which happens to be monetary. In this respect, the utility of money (something desired by everyone in every circumstances) is its near-universal acceptability in exchange for other commodities,²⁰ which is a concern for any agent who *does not know* whether the economy is at equilibrium. The way money is treated in the GTS is thus internally consistent, provided perfect information is not assumed.

5.2 Thinking about what microeconomic policy is about

Microeconomic policy cannot be reduced to "structural adjustment" *i.e.* the idea that it is all about boosting competition on markets. Rather, it consists of identifying transaction obstacles (imperfect information and/or strategic uncertainty) and the ways to

¹⁹But rather a monetary theory of surpluses.

²⁰Most of the discussion by Shapley and Shubik (1977) on the circumstances under which one can argue for a direct integration of money within utility functions could be repeated here.

overcome them in order to release surplus. It basically sustains the mechanism design agenda on an infra-analytical basis (no specification of information structure): market is one tool appropriate to some class of transactions, firm is another organizing arrangement, and still further consciously designed coordination mechanisms deserve attention.

5.3 Teaching microeconomic theory

Undergraduate microeconomics textbooks typically start by setting efficient allocation of scarce goods as the fundamental issue of economics; it is then argued that a system of competitive markets provides a satisfying response (with usual limitations). The next chapters are devoted to the gradual elaboration of the walrasian GET, supposed to contain a formal proof of what competitive markets achieve.

The trouble experienced by students is that, at the end, coordination does not seem to be realized through a decentralized process of supply and demand adjustments but rather by a central planner... who was supposed to be useless in the first place. To conceal this disconcerting contradiction, the trick is to insert between the analysis of agent price-taking behaviors and the walrasian model, some partial equilibrium analyses (of marshallian nature); with a little skill, it is enough to convince students that the treatment is all consistent. Because, the GTS is in line with the marshallian approach, it brings a more satisfying presentation of what markets are expected to achieve (be they competitive or not). The presentation of the theory of walrasian GET could thus be reserved to chapters devoted to auctions or to normative economics, with the notion that the walrasian equilibrium is *wealth preserving* and involves *equal treatment* - see Foley (2010).

The GTS is more generally well suited to deal with the issues of teaching marginalism. Proposition 24 is a good illustration of this aspect: it sums up the whole marginalist's theory in one equation. More importantly, the GTS allows a position of outwardness as regards markets and helps understanding the institutionalist's point: that economics is about making mutually advantageous transactions possible!

References

- [1] Allais, M. (1981), *La théorie générale des surplus*, Presse universitaire de Grenoble, Grenoble.
- [2] Coase, R. H. (1937), "The nature of the firm", *Economica*, 4, 386-405.
- [3] Coase, R. H. (1992), "The Institutional Structure of Production", *The American Economic Review*, vol. 82, issue 4 (sept.), 713-719.
- [4] Colander, D. (1995), "Marshallian general equilibrium analysis", *Eastern Economic Journal*, vol. 21, 3, 281-293.

- [5] Colander, D. and C. Rothschild (2010), "Sins of the Sons of Samuelson: Vision, pedagogy, and the zig-zag windings of complex dynamics", *Journal of Economic Behavior and Organization*, 74, 277-290.
- [6] Courtault, J.-M. and J.-M. Tallon (2000), "Allais' trading process and the dynamic evolution of a market economy", *Economic Theory*, 16, 477-481.
- [7] Foley, D. K. (2010), "What's wrong with the fundamental existence and welfare theorems?", *Journal of Economic Behavior and Organization*, 75, 115-131.
- [8] Friedman, D. and J. Sakovics (2014), "Tractable Consumer Choice", *Theory and Decision*, 14, 1-26.
- [9] Grandmont, J.-M. (1989), "Report on Maurice Allais' Scientific Work", *Scandinavian Journal of Economics*, 91(1), 17-28.
- [10] Guesnerie, R. (1984), "Review of *La Théorie générale des surplus* by Maurice Allais," *Journal of Political Economy*, vol. 92, 4 (Aug.), 780-783.
- [11] Luenberger, D. G. (1992), "Benefit functions and duality," *Journal of Mathematical Economics*, 21, 461-481.
- [12] Luenberger, D. G. (1994a), "Dual Pareto Efficiency", *Journal of Economic Theory*, vol. 62, 1, 70-85.
- [13] Luenberger, D. G. (1994b), "Optimality and the Theory of Value", *Journal of Economic Theory*, vol. 63, 2, 147-169.
- [14] Luenberger, D. G. (1995), "Externalities and benefits", *Journal of Mathematical Economics*, vol. 24, 159-177.
- [15] Luenberger, D. G. (1996), "Welfare from a benefit viewpoint", *Economic Theory*, 7, 445-462.
- [16] Makowski, L. and J. M. Ostroy (2001), "Perfect Competition and the Creativity of the Market", *Journal of Economic Literature*, vol. 39, 479-535.
- [17] Mas-Colell, A., M. D. Whinston, and J. R. Green (1995), *Microeconomic Theory*, MIT, Princeton.
- [18] Milgrom, P. and J. Roberts (1992), *Economics, organization, and management*, MIT, Princeton.
- [19] Phelps, E. S. (1985), *Political Economy: an introductory Text*, W. W. Norton & Company.
- [20] Plott, C., N. Roy, and B. Tong (2013), "Marshall and Walras, disequilibrium trades and the dynamics of equilibration in continuous double auction market", *Journal of Economic Behavior and Organization*, 94, 190-205.

- [21] Shapley, L. and M. Shubik (1977), "Trade Using One Commodity as a Means of Payment", *Journal of Political Economy*, vol. 85, 5, 937-968.
- [22] Smith, E. and D. K. Foley (2008), "Classical thermodynamics and economic general equilibrium theory", *Journal of Economic Dynamics & Control*, 32, 7-65.
- [23] Spulber, D. (2009), *The Theory of the Firm: Microeconomics with Endogenous Entrepreneurs, Firms, Markets, and Organizations*, Cambridge: Cambridge University Press.
- [24] Williamson, O. E. (2005), "Transaction Cost Economics", in C. Ménard and M. M. Shirley (eds.) *Handbook of New Institutional Economics*, 41-65.

Proof 5. For all $i \in \mathcal{I}$, consider the function $U_i(\cdot)$ defined for all $\mathbf{x}_i \in \mathbb{R}^N$ by $U_i(\mathbf{x}_i) = T_i(u_i(\mathbf{x}_i))$ where $T_i(\cdot)$ is a monotonous, strictly increasing function of u_i . For any $n \in \mathcal{N}$, let ΔV_{in} denote the surplus corresponding to a given reallocation: $\Delta V_{in} = \sum_{i \in \mathcal{I}} \Delta V_{in} - \sum_{i \in \mathcal{I}} \Delta x_{in}$ where, for all $i \in \mathcal{I}$: $\Delta V_{in} = \max \{ \Delta \nu_{in} \in \mathbb{R} \mid U_i(\mathbf{x}_i^{-n} + \Delta \mathbf{x}_i^{-n}, x_{in} + \Delta x_{in} - \Delta \nu_{in}) \geq U_i(\mathbf{x}_i) \}$. Clearly, for all $i \in \mathcal{I}$:

$$\begin{aligned} & \max \{ \Delta \nu_{in} \in \mathbb{R} \mid U_i(\mathbf{x}_i^{-n} + \Delta \mathbf{x}_i^{-n}, x_{in} + \Delta x_{in} - \Delta \nu_{in}) \geq U_i(\mathbf{x}_i) \} \\ &= \max \{ \Delta \nu_{in} \in \mathbb{R} \mid T_i(u_i(\mathbf{x}_i^{-n} + \Delta \mathbf{x}_i^{-n}, x_{in} + \Delta x_{in} - \Delta \nu_{in})) \geq T_i(u_i(\mathbf{x}_i)) \} \\ &= \max \{ \Delta \nu_{in} \in \mathbb{R} \mid u_i(\mathbf{x}_i^{-n} + \Delta \mathbf{x}_i^{-n}, x_{in} + \Delta x_{in} - \Delta \nu_{in}) \geq u_i(\mathbf{x}_i) \} \\ &= \Delta v_{in}, \end{aligned}$$

and $\Delta S_{in} = \sum_{i \in \mathcal{I}} \Delta v_{in} - \sum_{i \in \mathcal{I}} \Delta x_{in} = \Delta v_{in}$. ■

Proof 10. Given $\Delta a = (\Delta \mathbf{x}_i)_{i \in \mathcal{I}}$, for any $n \in \mathcal{N}$, consider the list $(\Delta v_{in})_{i \in \mathcal{I}}$ as defined by

$$\Delta v_{in} = \max \{ \Delta \nu_{in} \in \mathbb{R} \mid u_i(\mathbf{x}_i^{-n} + \Delta \mathbf{x}_i^{-n}, x_{in} + \Delta x_{in} - \Delta \nu_{in}) \geq u_i(\mathbf{x}_i) \}$$

for all $i \in \mathcal{I}$. Since Δa is P-improving, $\Delta v_{in} \geq 0$ for all $i \in \mathcal{I}$ with at least one strict inequality. It follows that the surplus released by Δa is $\Delta v_n(a) = \sum_{i \in \mathcal{I}} \Delta v_{in} > 0$. The

maximal releasable surplus from $a_+ = a + \Delta a$ is $l_n(a_+)$ by definition; let $\Delta^* a_+$ denote the reallocation corresponding to the releasing of $l_n(a_+)$. The maximal surplus released from a through the pair of reallocation $s(\Delta a, \Delta^* a_+)$ is $\Delta v_n(a) + l_n(a_+)$. By definition, the maximal surplus releasable from a is $l_n(a)$ which involves $\Delta v_n(a) + l_n(a_+) \leq l_n(a)$. Since $\Delta v_n(a) > 0$, one concludes that $l_n(a_+) \leq l_n(a)$. ■

Proof 12. The proof goes in two steps.

(1) $a \in \mathcal{P} \Rightarrow l_n(a) = 0, \forall n \in \mathcal{N}$. This restates: $\exists n \in \mathcal{N}$ s.t. $l_n(a) > 0 \Rightarrow a \notin \mathcal{P}$. $\exists n \in \mathcal{N}$ s.t. $l_n(a) > 0 \Rightarrow \exists n \in \mathcal{N}$ and $\Delta a = (\Delta \mathbf{x}_i)_{i \in \mathcal{I}}$ feasible such that $\Delta v_n(a) > 0$. Consider the allocation $\left(\mathbf{x}_i^{-n} + \Delta \mathbf{x}_i^{-n}, x_{in} + \Delta x_{in} - \Delta v_{in} + \frac{\Delta v_n(a)}{I} \right)_{i \in \mathcal{I}}$, it is both feasible and such

that $\forall i \in \mathcal{I}$:

$$u_i \left(\mathbf{x}_i^{-n} + \Delta \mathbf{x}_i^{-n}, x_{in} + \Delta x_{in} - \Delta v_{in} + \frac{\Delta v_n(a)}{I} \right) \geq u_i(\mathbf{x}_i^{-n} + \Delta \mathbf{x}_i^{-n}, x_{in} + \Delta x_{in} - \Delta v_{in}) = u_i(\mathbf{x}_i),$$

by definition of $\Delta v_n(a)$. Furthermore, provided that preferences for good n are non-satiated for at least one individual, $\exists i \in \mathcal{I}$ for whom

$$u_i \left(\mathbf{x}_i^{-n} + \Delta \mathbf{x}_i^{-n}, x_{in} + \Delta x_{in} - \Delta v_{in} + \frac{\Delta v_n(a)}{I} \right) > u_i(\mathbf{x}_i^{-n} + \Delta \mathbf{x}_i^{-n}, x_{in} + \Delta x_{in} - \Delta v_{in}) = u_i(\mathbf{x}_i).$$

The allocation under consideration is thus P-improving as compared to a i.e. $a \notin \mathcal{P}$. As a consequence, $a \in \mathcal{P} \Rightarrow l_n(a) = 0, \forall n \in \mathcal{N}$.

(2) $l_n(a) = 0$ for all $n \in \mathcal{N} \Rightarrow a \in \mathcal{P}$. This restates: $a \notin \mathcal{P} \Rightarrow \exists n \in \mathcal{N}$ s.t. $l_n(a) > 0$. Since $a \notin \mathcal{P}$, $\exists \hat{a} = (\hat{\mathbf{x}}_i)_{i \in \mathcal{I}}$ P-improving as compared to a i.e. such that: $\forall i \in \mathcal{I}$, $u_i(\hat{\mathbf{x}}_i) \geq u_i(\mathbf{x}_i)$ with at least one strict inequality. Let \hat{i} index an individual for whom $u_{\hat{i}}(\hat{\mathbf{x}}_{\hat{i}}) > u_{\hat{i}}(\mathbf{x}_{\hat{i}})$ and consider the reallocation $\Delta a = (\Delta \mathbf{x}_i)_{i \in \mathcal{I}}$ defined by: $\Delta \mathbf{x}_i = \hat{\mathbf{x}}_i - \mathbf{x}_i$ for all $i \in \mathcal{I} - \{\hat{i}\}$ and $\Delta \mathbf{x}_{\hat{i}} = (\hat{\mathbf{x}}_{\hat{i}}^{-n} - \mathbf{x}_{\hat{i}}^{-n}, \hat{x}_{\hat{i}n} - x_{\hat{i}n} - \Delta v_{\hat{i}n})$ where $\Delta v_{\hat{i}n} > 0$ is defined by $u_{\hat{i}}(\hat{\mathbf{x}}_{\hat{i}}^{-n}, \hat{x}_{\hat{i}n} - \Delta v_{\hat{i}n}) = u_{\hat{i}}(\mathbf{x}_{\hat{i}})$. Such a reallocation is feasible and releases the surplus $\Delta v_n = \Delta v_{\hat{i}n} > 0$. By definition, $l_n(a) \geq \Delta v_n$ and thus $l_n(a) > 0$. As a consequence, $l_n(a) = 0$ for all $n \in \mathcal{N} \Rightarrow a \in \mathcal{P}$. ■

Proof 17. For any reallocation $\Delta a = (\Delta \mathbf{x}_i, \Delta y_i)_{i \in \mathcal{I}}$, define $\bar{\Delta} a = (\bar{\Delta} \mathbf{x}_i, \bar{\Delta} y_i)_{i \in \mathcal{I}}$ by $\bar{\Delta} \mathbf{x}_i = \Delta \mathbf{x}_i$ for all $i \in \mathcal{I}$ but $\bar{\Delta} y_i \neq \Delta y_i$ for some $i \in \mathcal{I}$. For all $i \in \mathcal{I}$, by definition of individual surplus Δv_i and $\bar{\Delta} v_i$:

$$u_i(\mathbf{x}_i + \Delta \mathbf{x}_i, y_i + \Delta y_i - \Delta v_i) = u_i(\mathbf{x}_i + \Delta \mathbf{x}_i, y_i + \bar{\Delta} y_i - \bar{\Delta} v_i) = u_i(\mathbf{x}_i, y_i),$$

that is $\Delta y_i - \Delta v_i = \bar{\Delta} y_i - \bar{\Delta} v_i$. As a consequence

$$\bar{\Delta} v = \sum_{i \in \mathcal{I}} \bar{\Delta} v_i - \sum_{i \in \mathcal{I}} \bar{\Delta} y_i = \sum_{i \in \mathcal{I}} \Delta v_i - \sum_{i \in \mathcal{I}} \Delta y_i = \Delta v,$$

QED. ■

Proof 19. One has to show that: $l(a) = 0 \Rightarrow l_n(a) = 0$ for all $n \in \mathcal{N}$. This is equivalent to: $\exists n \in \mathcal{N}, l_n(a) > 0 \Rightarrow l(a) > 0$. $l_n(a) > 0$ means that there exists $\Delta^* a = (\Delta^* \mathbf{x}_i, \Delta^* y_i)_{i \in \mathcal{I}}$ feasible

$$\sum_{i \in \mathcal{I}} \Delta^* \mathbf{x}_i \leq \bar{\mathbf{x}} - \sum_{i \in \mathcal{I}} \mathbf{x}_i \text{ et } \sum_{i \in \mathcal{I}} \Delta^* y_i \leq \bar{y} - \sum_{i \in \mathcal{I}} y_i$$

such that $\Delta^* v_n = \sum_{i \in \mathcal{I}} (\Delta^* v_{in} - \Delta^* x_{in}) = l_n(a) > 0$ where , for all $i \in \mathcal{I}$: $\Delta^* v_{in} = \max \{ \Delta v_{in} \mid u_i(\mathbf{x}_i^{-n} + \Delta^* \mathbf{x}_i^{-n}, x_{in} + \Delta^* x_{in} - \Delta v_{in}, y_i + \Delta^* y_i) \geq u_i(\mathbf{x}_i, y_i) \}$. If $\bar{y} - \sum_{i \in \mathcal{I}} y_i > 0$, one trivially gets, $l(a) \geq \bar{y} - \sum_{i \in \mathcal{I}} y_i > 0$. Let's focus on the case where $\bar{y} - \sum_{i \in \mathcal{I}} y_i = 0$ that is, where the money supply is fully distributed between agents. $l_n(a) > 0$ involves there exists \hat{i} for which $\Delta^* v_{\hat{i}n} - \Delta^* x_{\hat{i}n} > 0$ or $u_{\hat{i}}(\mathbf{x}_{\hat{i}}^{-n} + \Delta^* \mathbf{x}_{\hat{i}}^{-n}, x_{\hat{i}n} + \Delta^* x_{\hat{i}n}, y_{\hat{i}} + \Delta^* y_{\hat{i}}) >$

$u_i(\mathbf{x}_i, y_i)$. From the properties attached to money, there exists an amount of money $m > 0$ defined by

$$u_i(\mathbf{x}_i^{-n} + \Delta^* \mathbf{x}_i^{-n}, x_{in} + \Delta^* x_{in}, y_i + \Delta^* y_i - m) = u_i(\mathbf{x}_i, y_i).$$

Let $\Delta a = (\Delta \mathbf{x}_i, \Delta y_i)_{i \in \mathcal{I}}$ be the reallocation defined by $(\Delta \mathbf{x}_i, \Delta y_i) = (\Delta^* \mathbf{x}_i^{-n}, \Delta^* x_{in} - \Delta^* v_{in}, \Delta^* y_i)$ for all $i \neq \hat{i}$ but $(\Delta \mathbf{x}_{\hat{i}}, \Delta y_{\hat{i}}) = (\Delta^* \mathbf{x}_{\hat{i}}^{-n}, \Delta^* x_{in}, \Delta^* y_{\hat{i}})$. Δa is feasible. For all $i \neq \hat{i}$, let Δv_i be defined by $u_i(\mathbf{x}_i^{-n} + \Delta^* \mathbf{x}_i^{-n}, x_{in} + \Delta^* x_{in} - \Delta v_{in}, y_i + \Delta^* y_i - \Delta v_i) = u_i(\mathbf{x}_i, y_i) : \Delta v_i \geq 0$. The total monetary surplus associated to Δa is defined by

$$\Delta v = \sum_{i \in \mathcal{I} - \{\hat{i}\}} (\Delta v_i - \Delta^* y_i) + (m - \Delta^* y_{\hat{i}}) = m + \sum_{i \in \mathcal{I} - \{\hat{i}\}} \Delta v_i - \sum_{i \in \mathcal{I}} \Delta^* y_i.$$

$\sum_{i \in \mathcal{I} - \{\hat{i}\}} \Delta v_i \geq 0$ and, $\Delta^* a$ being feasible, $\sum_{i \in \mathcal{I}} \Delta^* y_i \leq 0$ so that $\Delta v \geq m > 0$. But by definition of the loss $l(a) \geq \Delta v : l(a) > 0$. QED. ■

Proof 25. The definition $v'_{in} \equiv \frac{u'_{in}}{u'_{iy}} \Rightarrow dv'_{in} = \frac{du'_{in} u'_{iy} - u'_{in} du'_{iy}}{(u'_{iy})^2} = \frac{du'_{in}}{u'_{iy}} - v'_{in} \frac{du'_{iy}}{u'_{iy}}$ or $\frac{du'_{in}}{u'_{iy}} = dv'_{in} + v'_{in} \frac{du'_{iy}}{u'_{iy}}$. Hence:

$$\begin{aligned} d^2 u_i &= \sum_{n \in \mathcal{N}} du'_{in} dx_{ni} + \sum_{n \in \mathcal{N}} u'_{in} d^2 x_n + du'_{iy} \cdot (dy_i - dv_i) + u'_{iy} \cdot (d^2 y_i - d^2 s_i), \\ \frac{d^2 u_i}{u'_{iy}} &= \sum_{n \in \mathcal{N}} \frac{du'_{in}}{u'_{iy}} dx_{in} + \frac{du'_{iy}}{u'_{iy}} \cdot (dy_i - dv_i) + \sum_{n \in \mathcal{N}} v'_{in} d^2 x_{in} + (d^2 y_i - d^2 s_i), \\ \frac{d^2 u_i}{u'_{iy}} &= \sum_{n \in \mathcal{N}} \left(dv'_{in} + v'_{in} \frac{du'_{iy}}{u'_{iy}} \right) dx_{in} + \frac{du'_{iy}}{u'_{iy}} \cdot (dy_i - dv_i) + \sum_{n \in \mathcal{N}} v'_{in} d^2 x_{in} + (d^2 y_i - d^2 s_i), \\ \frac{d^2 u_i}{u'_{iy}} &= \sum_{n \in \mathcal{N}} dv'_{in} dx_{in} + \left(\sum_{n \in \mathcal{N}} v'_{in} dx_{in} + dy_i - dv_i \right) \frac{du'_{iy}}{u'_{iy}} + \sum_{n \in \mathcal{N}} v'_{in} d^2 x_{in} + d^2 y_i - d^2 s_i, \end{aligned}$$

where $\frac{du'_{iy}}{u'_{iy}} = \sum_{n \in \mathcal{N}} \frac{u''_{iny}}{u'_{iy}} dx_{in} + \frac{u''_{iy^2}}{u'_{iy}} (dy_i - dv_i)$ and $v''_{iny} = \frac{u''_{iny}}{u'_{iy}} - v'_{in} \frac{u''_{iy^2}}{u'_{iy}} \Rightarrow \frac{du'_{iy}}{u'_{iy}} =$

$$\sum_{n \in \mathcal{N}} \left(v''_{iny} + v'_{in} \frac{u''_{iy^2}}{u'_{iy}} \right) dx_{in} + \frac{u''_{iy^2}}{u'_{iy}} (dy_i - dv_i) = \sum_{n \in \mathcal{N}} v''_{iny} dx_{in} + \left(\sum_{n \in \mathcal{N}} v'_{in} dx_{in} + dy_i - dv_i \right) \frac{u''_{iy^2}}{u'_{iy}}$$

so that

$$\begin{aligned}
\frac{d^2 u_i}{u'_{iy}} &= \sum_{n \in \mathcal{N}} \left(\sum_{\bar{n} \in \mathcal{N}} v''_{in\bar{n}} dx_{i\bar{n}} + v''_{iny} \cdot (dy_i - dv_i) \right) dx_{in} \\
&+ \left(\sum_{n \in \mathcal{N}} v'_{in} dx_{in} + dy_i - dv_i \right) \left(\sum_{n \in \mathcal{N}} v''_{iny} dx_{in} + \left(\sum_{n \in \mathcal{N}} v'_{in} dx_{in} + dy_i - dv_i \right) \frac{u''_{iy^2}}{u'_{iy}} \right) \\
&+ \sum_{n \in \mathcal{N}} v'_{in} d^2 x_{in} + d^2 y_i - d^2 s_i, \\
\frac{d^2 u_i}{u'_{iy}} &= \sum_{n \in \mathcal{N}} \sum_{\bar{n} \in \mathcal{N}} v''_{in\bar{n}} dx_{i\bar{n}} dx_{in} + \sum_{n \in \mathcal{N}} v''_{iny} \cdot (dy_i - dv_i) dx_{in} \\
&+ \left(\sum_{n \in \mathcal{N}} v'_{in} dx_{in} + dy_i - dv_i \right) \sum_{n \in \mathcal{N}} v''_{iny} dx_{in} + \left(\sum_{n \in \mathcal{N}} v'_{in} dx_{in} + dy_i - dv_i \right)^2 \frac{u''_{iy^2}}{u'_{iy}} \\
&+ \sum_{n \in \mathcal{N}} v'_{in} d^2 x_{in} + d^2 y_i - d^2 s_i, \\
\frac{d^2 u_i}{u'_{iy}} &= \sum_{n \in \mathcal{N}} \sum_{\bar{n} \in \mathcal{N}} v''_{in\bar{n}} dx_{i\bar{n}} dx_{in} + \sum_{n \in \mathcal{N}} v'_{in} dx_{in} \sum_{n \in \mathcal{N}} v''_{iny} dx_{in} + 2(dy_i - dv_i) \sum_{n \in \mathcal{N}} v''_{iny} dx_{in} \\
&+ \left(\sum_{n \in \mathcal{N}} v'_{in} dx_{in} + dy_i - dv_i \right)^2 \frac{u''_{iy^2}}{u'_{iy}} \\
&+ \sum_{n \in \mathcal{N}} v'_{in} d^2 x_{in} + d^2 y_i - d^2 s_i,
\end{aligned}$$

with $dv_i = \sum_{n \in \mathcal{N}} v'_{in} dx_{in} + dy_i$ so that

$$\begin{aligned} \frac{d^2 u_i}{u'_{iy}} &= \sum_{n \in \mathcal{N}} \sum_{\bar{n} \in \mathcal{N}} v''_{in\bar{n}} dx_{i\bar{n}} dx_{in} + \sum_{n \in \mathcal{N}} v'_{in} dx_{in} \sum_{n \in \mathcal{N}} v''_{iny} dx_{in} - 2 \sum_{n \in \mathcal{N}} v'_{in} dx_{in} \sum_{n \in \mathcal{N}} v''_{iny} dx_{in} \\ &\quad + \sum_{n \in \mathcal{N}} v'_{in} d^2 x_{in} + d^2 y_i - d^2 s_i, \\ \frac{d^2 u_i}{u'_{iy}} &= \sum_{n \in \mathcal{N}} \sum_{\bar{n} \in \mathcal{N}} v''_{in\bar{n}} dx_{i\bar{n}} dx_{in} - \sum_{n \in \mathcal{N}} v'_{in} dx_{in} \sum_{n \in \mathcal{N}} v''_{iny} dx_{in} \\ &\quad + \sum_{n \in \mathcal{N}} v'_{in} d^2 x_{in} + d^2 y_i - d^2 s_i, \\ \frac{d^2 u_i}{u'_{iy}} &= \sum_{n \in \mathcal{N}} \sum_{\bar{n} \in \mathcal{N}} v''_{in\bar{n}} dx_{i\bar{n}} dx_{in} - \sum_{n \in \mathcal{N}} \sum_{\bar{n} \in \mathcal{N}} v'_{in} v''_{iny} dx_{i\bar{n}} dx_{in} \\ &\quad + \sum_{n \in \mathcal{N}} v'_{in} d^2 x_{in} + d^2 y_i - d^2 s_i, \\ \frac{d^2 u_i}{u'_{iy}} &= \sum_{n \in \mathcal{N}} \sum_{\bar{n} \in \mathcal{N}} (v''_{in\bar{n}} - v'_{in} v''_{iny}) dx_{i\bar{n}} dx_{in} + \sum_{n \in \mathcal{N}} v'_{in} d^2 x_{in} + d^2 y_i - d^2 s_i, \end{aligned}$$

Conclusion $\frac{d^2 u_i}{u'_{iy}} = 0$ involves

$$d^2 s_i = \sum_{n \in \mathcal{N}} \sum_{\bar{n} \in \mathcal{N}} (v''_{in\bar{n}} - v'_{in} v''_{iny}) dx_{i\bar{n}} dx_{in} + \sum_{n \in \mathcal{N}} v'_{in} d^2 x_{in} + d^2 y_i,$$

QED. ■

Proof 26. $a \in \mathcal{P} \Leftrightarrow \Delta v(a) \leq 0$ for all feasible reallocation. This means that the total (unobserved) value $v(a)$ is maximal in a . Provided that functions $(u_i(\cdot))_{i \in \mathcal{I}}$ are continuous and derivable in all directions, this occurs under the necessary and sufficient condition that $dv(a) = 0$ and $d^2 v(a) \leq 0$. ■

Proof 27. If for any pair of agents $(i, \hat{i}) \in \mathcal{I}^2$, $\hat{i} \neq i$, and any good $n \in \mathcal{N} : v'_{in}|_a = v'_{\hat{i}n}|_a$ then $dv(a) = 0$. Since

$$dv = \sum_{n \in \mathcal{N}} \sum_{i \in \mathcal{I}} \sum_{\hat{i} > i} (v'_{in} - v'_{\hat{i}n}) d_i x_{in},$$

and $v'_{in} = v'_{\hat{i}n}$ for all n , $dv = 0$. Let's consider a feasible infinitesimal reallocation $(dx_i, dy_i)_{i \in \mathcal{I}}$ involving no loss increase, that is such that

$$\sum_{i \in \mathcal{I}} dy_i = 0 \text{ and } \sum_{i \in \mathcal{I}} dx_{in} = 0 \text{ for all } n \in \mathcal{N}.$$

If $dv = 0$ then for any pair of agents $(\check{i}, \hat{i}) \in \mathcal{I}^2$, $\hat{i} \neq \check{i}$, and any good $n \in \mathcal{N} : v'_{\check{i}n} = v'_{\hat{i}n}$. This restates: if $dv = 0$ then $\nexists (\check{i}, \hat{i}) \in \mathcal{I}^2$, $\hat{i} \neq \check{i}$, and $\bar{n} \in \mathcal{N}$ such that $v'_{\check{i}\bar{n}} \neq v'_{\hat{i}\bar{n}}$. It is shown that if $\exists (\check{i}, \hat{i}) \in \mathcal{I}^2$, $\hat{i} \neq \check{i}$, and $\bar{n} \in \mathcal{N}$ such that $v'_{\check{i}\bar{n}} \neq v'_{\hat{i}\bar{n}}$ then $dv \neq 0$. With no

loss in generality, let's consider $(\check{i}, \hat{i}) \in \mathcal{I}^2$, $\hat{i} \neq \check{i}$, and $\bar{n} \in \mathcal{N}$ such that $v'_{\hat{i}\bar{n}} > v'_{\check{i}\bar{n}}$, as well as the reallocation $da = (dx_i, dy_i)_{i \in \mathcal{I}}$ defined by:

$$dx_{in} = dy_i = 0 \text{ for all } i \in \mathcal{I} - \{\hat{i}, \check{i}\} \text{ and } n \in \mathcal{N}$$

but $dx_{i\bar{n}} = -dx_{\check{i}\bar{n}} > 0$ and $dy_{\hat{i}} = -dy_{\check{i}} (= -v'_{\hat{i}\bar{n}} dx_{i\bar{n}} < 0)$. One can check that this reallocation is feasible:

$$\begin{aligned} \sum_{i \in \mathcal{I}} dx_{in} &= 0 \text{ for all } n \in \mathcal{N} - \{\bar{n}\} \\ \sum_{i \in \mathcal{I}} dx_{i\bar{n}} &= dx_{\hat{i}\bar{n}} + dx_{\check{i}\bar{n}} = 0 \\ \sum_{i \in \mathcal{I}} dy_i &= dy_{\hat{i}} + dy_{\check{i}} = 0 \end{aligned}$$

and yet:

$$\begin{aligned} dv &= \sum_{i \in \mathcal{I}} \sum_{n \in \mathcal{N}} v'_{in} dx_{in} = v'_{\hat{i}\bar{n}} dx_{i\bar{n}} + v'_{\check{i}\bar{n}} dx_{i\bar{n}}, \\ dv &= v'_{\hat{i}\bar{n}} dx_{i\bar{n}} - v'_{\check{i}\bar{n}} dx_{i\bar{n}} = \underbrace{(v'_{\hat{i}\bar{n}} - v'_{\check{i}\bar{n}})}_{>0} \underbrace{dx_{i\bar{n}}}_{>0} > 0. \end{aligned}$$

■