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# ALGORITHMIC COMPLEXITY OF GREENBERG’S CONJECTURE

GEORGES GRAS

ABSTRACT. Let  $k$  be a totally real number field and  $p$  a prime. We show that the “complexity” of Greenberg’s conjecture ( $\lambda = \mu = 0$ ) is of  $p$ -adic nature governed (under Leopoldt’s conjecture) by the finite torsion group  $\mathcal{T}_k$  of the Galois group of the maximal abelian  $p$ -ramified pro- $p$ -extension of  $k$ , by means of images in  $\mathcal{T}_k$  of ideal norms from the layers  $k_n$  of the cyclotomic tower (Theorem 5.2). These images are obtained via the formal algorithm computing, by “unscrewing”, the  $p$ -class group of  $k_n$ . Conjecture 5.4 of equidistribution of these images would show that the number of steps  $b_n$  of the algorithms is bounded as  $n \rightarrow \infty$ , so that Greenberg’s conjecture, hopeless within the sole framework of Iwasawa’s theory, would hold true “with probability 1”. No assumption is made on  $[k : \mathbb{Q}]$ , nor on the decomposition of  $p$  in  $k/\mathbb{Q}$ .

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## 1. INTRODUCTION

Let  $k$  be a totally real number field,  $p \geq 2$  a prime number and  $S$  the set of  $p$ -places  $\mathfrak{p} \mid p$  of  $k$ . Let  $k_\infty$  be the cyclotomic  $\mathbb{Z}_p$ -extension of  $k$  and  $k_n$  the degree  $p^n$  extension of  $k$  in  $k_\infty$ . Let  $\mathcal{C}_k$  and  $\mathcal{C}_{k_n}$  be the  $p$ -class groups of  $k$  and  $k_n$ , respectively. We denote by  $\mathcal{T}_k$  the torsion group of  $\mathcal{A}_k := \text{Gal}(H_k^{\text{pr}}/k)$ , where  $H_k^{\text{pr}}$  is the maximal abelian  $S$ -ramified pro- $p$ -extension of  $k$  (i.e., unramified outside  $S$ ), assuming the Leopoldt conjecture for  $p$  in  $k_\infty$ . The group  $\mathcal{T}_k$  is closely related to the deep Tate–Chafarevich group (same  $p$ -rank):

$$\text{III}_k^2 := \text{Ker}[\text{H}^2(\mathcal{G}_{k,S}, \mathbb{F}_p) \rightarrow \bigoplus_{\mathfrak{p} \mid p} \text{H}^2(\mathcal{G}_{k_{\mathfrak{p}}}, \mathbb{F}_p)],$$

where  $\mathcal{G}_{k,S}$  is the Galois group of the maximal  $S$ -ramified pro- $p$ -extension of  $k$  (hence  $\mathcal{A}_k = \mathcal{G}_{k,S}^{\text{ab}}$ ) and  $\mathcal{G}_{k_{\mathfrak{p}}}$  the local analogue over  $k_{\mathfrak{p}}$ ; but  $\mathcal{T}_k$  is very easily computable and relates the  $p$ -class group and the  $p$ -adic regulator.

We call *Greenberg’s conjecture for  $k$  and  $p$* , the nullity of the Iwasawa invariants  $\lambda, \mu$  (see the origin of the conjecture in [10, Theorems 1 and 2]). The main effective test for this conjecture is the criterion of Jaulent [14, Théorèmes A, B] proving that the conjecture is equivalent to the capitulation in  $k_\infty$  of the logarithmic class group  $\widetilde{\mathcal{C}}_k$  of  $k$  (defined in [12] with PARI/GP package in [1]), an invariant also related to  $S$ -ramification theory.<sup>1</sup> For specific cases of decomposition of  $p$ , as in [10], see [19].

In our opinion, many aesthetic statements, equivalent to Greenberg’s conjecture, are translations of standard formalism of class field and Iwasawa’s theories. In other words, *some “non-algebraic”  $p$ -adic aspects of the “diophantine construction” of the class groups at each layer  $k_n$* , are not taken into account. We show how this construction works and study its arithmetic complexity by means of the number  $b_n$  of steps of the algorithms which become oversized in the tower as soon as  $\lambda$  or  $\mu$  are non-zero, suggesting the triviality of the algorithms for  $n \gg 0$  (i.e.,  $b_n \leq 1$ ).

Our purpose has nothing to do with computational or theoretical approaches in the area of the “main theorem” on abelian fields (analytic formulas, cyclotomic units,  $L_p$ -functions, etc.) as, for instance, the very many contributions (cited in our papers [5, 6]), also giving computations and suggesting that equidistribution results may have striking consequences for the conjecture; our viewpoint is essentially logical and based on the governing group  $\mathcal{T}_k$ , because we have conjectured that  $\mathcal{T}_k = 1$  for all  $p \gg 0$ , due to

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<sup>1</sup>For more information on the main pioneering works about *the practice* of this theory, see “history of abelian  $p$ -ramification” in [9, Appendix] (e.g., Gras: “Crelle’s Journal” (1982/83), Jaulent: “Ann. Inst. Fourier” (1984), Nguyen Quang Do: “Ann. Inst. Fourier” (1986), Movahhedi “Thèse” (1988) and others). For convenience, we mostly refer to our book (2003/2005), which contains all the needed results in the most general statements. For more broad context about the base field and the set  $S$ , see [16] and its bibliography.

properties of  $p$ -adic regulators [8] ( $p$ -rationality of  $k$ , as defined in [17] for such fields), which relativizes Greenberg's conjecture, obvious in that case.

In many papers, as in [10], the decomposition of  $p$  in  $k/\mathbb{Q}$  plays a specific role, which is not necessary for us. We shall not put any assumption on the degree of  $k$  nor on the decomposition of  $p$  in  $k/\mathbb{Q}$ .

**Conventions 1.1.** *Subject to replace  $k$  by a layer  $K = k_{n_0}$  of  $k_\infty = K_\infty$ , one may assume, without any loss of generality, that  $p$  is totally ramified in  $K_\infty/K$  and is such that Iwasawa's formula for  $\#\mathcal{C}_{k_n}$  holds true for all layers above  $K$ ; indeed, we have  $\lambda(K) = \lambda(k)$ ,  $\mu(K) = [K : k]\mu(k)$  and  $\nu(K) = \nu(k) + \lambda(k)n_0$ .*

## 2. MAIN RESULTS

The results of the paper may be described as follows in two parts:

(A) From results of [4, 5, 6]. The formal algorithm, determining  $\#\mathcal{C}_{k_n}$  (whence giving the Iwasawa invariants), computes inductively the classical filtration  $(\mathcal{C}_{k_n}^i)_{i \geq 0}$ , where  $\mathcal{C}_{k_n}^{i+1}/\mathcal{C}_{k_n}^i := (\mathcal{C}_{k_n}/\mathcal{C}_{k_n}^i)^{G_n}$ , for all  $i \geq 0$  ( $\mathcal{C}_{k_n}^0 = 1$ ), where  $G_n = \text{Gal}(k_n/k)$ . We have the decreasing  $i$ -sequence:

$$(2.1) \quad \#(\mathcal{C}_{k_n}^{i+1}/\mathcal{C}_{k_n}^i) = \frac{\#\mathcal{C}_k}{\#\mathbb{N}_{k_n/k}(\mathcal{C}_{k_n}^i)} \cdot \frac{p^{n \cdot (\#S-1)}}{(\Lambda_n^i : \Lambda_n^i \cap \mathbb{N}_{k_n/k}(k_n^\times))},$$

with the increasing  $i$ -sequence of groups  $\Lambda_n^i$ , from  $\Lambda_n^0 = E_k$ :

$$(2.2) \quad \Lambda_n^i := \{x \in k^\times, (x) = \mathbb{N}_{k_n/k}(\mathfrak{A}), \mathfrak{A} \in \mathcal{C}_{k_n}^i\}.$$

Then  $\mathcal{C}_{k_n}^{i+1}/\mathcal{C}_{k_n}^i$  in (2.1) becomes trivial for some minimal  $i =: b_n \geq 0$  (giving  $\mathcal{C}_{k_n}^{b_n} = \mathcal{C}_{k_n}$ ) as soon as the two factors vanish. Thus the length  $b_n$  of the algorithm depends on the decreasing evolution of the ‘‘class factor’’  $\frac{\#\mathcal{C}_k}{\#\mathbb{N}_{k_n/k}(\mathcal{C}_k^i)}$  dividing  $\#\mathcal{C}_k$  and that of the ‘‘norm factor’’  $\frac{p^{n \cdot (\#S-1)}}{(\Lambda_n^i : \Lambda_n^i \cap \mathbb{N}_{k_n/k}(k_n^\times))}$  dividing the order of a suitable quotient  $\mathcal{R}_k^{\text{nr}}$  of the normalized  $p$ -adic regulator  $\mathcal{R}_k$  (defined in [7, §5]), related to the ramification of  $p$  in  $H_k^{\text{pr}}/k_\infty$  (Theorem 3.4, Corollary 4.2). We prove in Theorem 4.3, under Conventions 1.1, the following inequalities (where  $v_p$  is the  $p$ -adic valuation):

$$b_n \leq \lambda \cdot n + \mu \cdot p^n + \nu \leq v_p(\#\mathcal{C}_k \cdot \#\mathcal{R}_k^{\text{nr}}) \cdot b_n,$$

giving  $\mathcal{C}_k = \mathcal{R}_k^{\text{nr}} = 1 \iff \lambda = \mu = \nu = 0 \iff b_n = 0$  for all  $n$ .

Taking  $k$  high enough in the tower, Greenberg's conjecture is equivalent to  $b_n \leq 1$  for all  $n$  (Corollary 4.4), which constitutes a spectacular algorithmic discontinuity compared to  $b_n \rightarrow \infty$  if  $\lambda$  or  $\mu$  are non-zero. In an heuristic point of view, it is ‘‘necessary’’ that the algorithms become limited, because of the unpredictable behavior of the class and norm factors.

(B) One may replace, in (2.2), the ideal norms  $\mathfrak{a} = \mathbb{N}_{k_n/k}(\mathfrak{A})$  by representatives  $\mathfrak{t} \in I_{k_n} \otimes \mathbb{Z}_p$  ( $I_{k_n}$  is the group of prime-to- $p$  ideals of  $k_n$ ) whose Artin

symbols are in  $\mathcal{T}_k$ , hence finite in number (main Theorem 5.2); so, each step of the algorithm (i.e., the evolution of the class and norm factors) only depends on at most  $\#\mathcal{T}_k$  possibilities, taking the class of the random ideal  $\mathfrak{t}$ , then computing Hasse's symbols on  $S$  of numbers  $\tau \in k_n^\times \otimes \mathbb{Z}_p$  when  $\mathfrak{t} = (\tau)$  is principal, in other words, for this last case a classical situation involving random  $\mathbb{Z}/p^n\mathbb{Z}$ -matrices of symbols for which some equidistribution results are proven [20, Section 6].

Then, under the natural Conjecture 5.4 of independence and randomness of the data obtained, inductively, at each step of the algorithm, one would obtain that Greenberg's conjecture holds true with "probability 1", suggesting possible analytic proof of this fact, using the powerful techniques used in [15, 20] for degree  $p$  cyclic extensions of  $\mathbb{Q}$ , but unfortunately, probably not a complete proof of Greenberg's conjecture.

### 3. ABELIAN $p$ -RAMIFICATION AND GENUS THEORIES

**3.1. Abelian  $p$ -ramification – The torsion group  $\mathcal{T}_k$ .** Recall the data needed for the study of the Galois group  $\mathcal{A}_k$  of the maximal abelian  $p$ -ramified pro- $p$ -extension  $H_k^{\text{pr}}$  of  $k$  and its torsion group  $\mathcal{T}_k$  (under Leopoldt's conjecture). Let  $k'^\times$  be the subgroup of  $k^\times$  of prime-to- $p$  elements:

(a) Let  $E_k$  be the group of  $p$ -principal units  $\varepsilon \equiv 1 \pmod{\prod_{\mathfrak{p} \in S} \mathfrak{p}}$  of  $k$ . Let  $U_k := \bigoplus_{\mathfrak{p} \in S} U_{\mathfrak{p}}$  be the  $\mathbb{Z}_p$ -module of  $p$ -principal local units, where  $U_{\mathfrak{p}}$  is the group of  $\mathfrak{p}$ -principal units of the  $\mathfrak{p}$ -completion  $k_{\mathfrak{p}}$  of  $k$ . Let  $\mu_k$  (resp.  $\mu_{\mathfrak{p}}$ ) be the group of  $p$ th roots of unity of  $k$  (resp.  $k_{\mathfrak{p}}$ ). Put  $W_k := \bigoplus_{\mathfrak{p} \in S} \mu_{\mathfrak{p}}$  and  $\mathcal{W}_k := W_k / \mu_k$ ; thus,  $\mathcal{W}_k = W_k$  for  $p \neq 2$  and  $\mathcal{W}_k = W_k / \langle \pm 1 \rangle$  for  $p = 2$ .

(b) Let  $\iota : k'^\times \otimes \mathbb{Z}_p \rightarrow U_k$  be the canonical surjective diagonal map. Let  $\overline{E}_k$  be the closure of  $\iota E_k$  in  $U_k$  and let  $H_k^{\text{nr}}$  be the  $p$ -Hilbert class field of  $k$ . By class field theory,  $\text{Gal}(H_k^{\text{pr}}/k_\infty H_k^{\text{nr}}) \simeq \text{tor}_{\mathbb{Z}_p}(U_k/\overline{E}_k) = U_k^*/\overline{E}_k$ , where  $U_k^* := \{u \in U_k, N_{k/\mathbb{Q}}(u) \in \langle \pm 1 \rangle\}$ .

(c) Let  $\mathcal{C}_k$  be the  $p$ -class group of  $k$  and let:

$$(3.1) \quad \mathcal{R}_k := \text{tor}_{\mathbb{Z}_p}(\log(U_k)/\log(\overline{E}_k)) = \log(U_k^*)/\log(\overline{E}_k)$$

be the normalized  $p$ -adic regulator [7, § 5].

(d) The sub-group of  $\mathcal{T}_k$  fixing the Bertrandias–Payan field  $H_k^{\text{bp}}$  is isomorphic to  $\mathcal{W}_k$  (the field  $H_k^{\text{bp}}$  is the compositum of all  $p$ -cyclic extensions of  $k$  embeddable in  $p$ -cyclic extensions of arbitrary large degree).

Recall some classical fundamental results (under Leopoldt's conjecture) that may be found in [3, Corollary III.3.6.3], [7, Lemma 3.1, Corollary 3.2], [13, Définition 2.11, Proposition 2.12], then [18, § 1] or [17], via cohomology:

**Proposition 3.1.** *We have the exact sequences:*

$$(3.2) \quad 1 \rightarrow U_k^*/\overline{E}_k \rightarrow \mathcal{T}_k \rightarrow \text{Gal}(k_\infty H_k^{\text{nr}}/k_\infty) \simeq \mathcal{C}_k \rightarrow 1,$$

$$(3.3) \quad 1 \rightarrow \mathcal{W}_k \longrightarrow U_k^*/\overline{E}_k \longrightarrow \log(U_k^*)/\log(\overline{E}_k) \simeq \mathcal{R}_k \rightarrow 0.$$

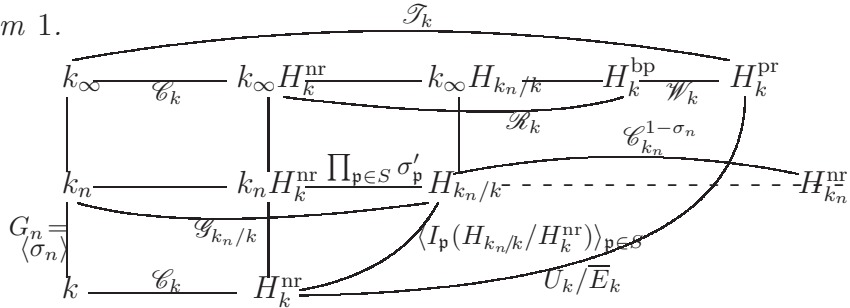
**3.2. Genus theory.** We denote by  $H_{k_n}^{\text{nr}}$  the  $p$ -Hilbert class field of  $k_n$ . Since  $p$  is totally ramified in  $k_n/k$  by convention, the inertia groups  $I_{\mathfrak{p}}(k_n/k)$  in  $k_n/k$ ,  $\mathfrak{p} \in S$ , are isomorphic to  $G_n = \text{Gal}(k_n/k)$ .

Let  $\omega_n$  be the map which associates with  $\varepsilon \in E_k$  the family of Hasse's symbols  $(\frac{\varepsilon, k_n/k}{\mathfrak{p}}) \in G_n$ ,  $\mathfrak{p} \in S$ . This yields the genus exact sequence interpreting the product formula of the Hasse symbols [3, Corollary IV.4.4.1]:

$$1 \rightarrow E_k/E_k \cap N_{k_n/k}(k_n^\times) \xrightarrow{\omega_n} \Omega(k_n/k) \xrightarrow{\pi_n} \text{Gal}(H_{k_n/k}/k_n H_k^{\text{nr}}) \rightarrow 1,$$

where  $\Omega(k_n/k) := \{(\sigma_{\mathfrak{p}})_{\mathfrak{p} \in S} \in G_n^{\#S}, \prod_{\mathfrak{p} \in S} \sigma_{\mathfrak{p}} = 1\} \simeq G_n^{\#S-1}$ , then where  $H_{k_n/k}$  is the  $p$ -genus field of  $k_n/k$  defined as the maximal sub-extension of  $H_{k_n}^{\text{nr}}$ , abelian over  $k$ . The image of  $\omega_n$  is contained in  $\Omega(k_n/k)$  and the map  $\pi_n$  is defined as follows: with  $(\sigma_{\mathfrak{p}})_{\mathfrak{p} \in S} \in G_n^{\#S}$ ,  $\pi_n$  associates the product of the extensions  $\sigma'_{\mathfrak{p}}$  of the  $\sigma_{\mathfrak{p}}$  in the inertia groups  $I_{\mathfrak{p}}(H_{k_n/k}/H_k^{\text{nr}})$  generating  $\text{Gal}(H_{k_n/k}/H_k^{\text{nr}})$ ; from the product formula, if  $(\sigma_{\mathfrak{p}})_{\mathfrak{p} \in S} \in \Omega(k_n/k)$ , then  $\prod_{\mathfrak{p} \in S} \sigma'_{\mathfrak{p}}$  fixes both  $H_k^{\text{nr}}$  and  $k_n$ , whence  $k_n H_k^{\text{nr}}$ . The genus exact sequence shows that the kernel of  $\pi_n$  is  $\omega_n(E_k)$ .

Diagram 1.



We have, using Chevalley's ambiguous class number formula [2, p. 402]:

$$(3.4) \quad \#\mathcal{G}_{k_n/k} = \#\text{Gal}(H_{k_n/k}/k_n) = \frac{\#\mathcal{C}_{k_n}}{\#\mathcal{C}_{k_n}^{1-\sigma_n}} = \#\mathcal{C}_k \cdot \frac{p^{n \cdot (\#S-1)}}{(E_k : E_k \cap N_{k_n/k}(k_n^\times))}$$

In the Diagram, the genus field  $H_{k_n/k}$  is the fixed field of the image of  $\mathcal{C}_{k_n}^{1-\sigma_n}$ , where  $\mathcal{G}_{k_n/k} = \text{Gal}(H_{k_n/k}/k_n)$  is the genus group in  $k_n/k$ .

**3.3. Groups  $\mathcal{R}_k^{\text{nr}}$ ,  $\mathcal{R}_k^{\text{ram}}$  – Ramification in  $H_k^{\text{pr}}/k_\infty$ .** The genus group  $\mathcal{G}_{k_n/k}$  has, in our context, the following main property that will give Theorem 3.4 when  $n$  is large enough:

**Lemma 3.2.** *For all  $n \geq 0$ ,  $k_\infty H_{k_n/k} \subseteq H_k^{\text{bp}}$ . Then  $\#\mathcal{G}_{k_n/k} \mid \#\mathcal{C}_k \cdot \mathcal{R}_k$ , which is equivalent (using formula (3.4)) to  $\frac{p^{n \cdot (\#S-1)}}{(E_k : E_k \cap N_{k_n/k}(k_n^\times))} \mid \#\mathcal{R}_k$ .*

*Proof.* Indeed, using the idelic global reciprocity map (under Leopoldt's conjecture), we have the fundamental diagram [3, § III.4.4.1] of the Galois group of the maximal abelian pro- $p$ -extension  $k^{\text{ab}}$  of  $k$ , with our present notations,



From Lemma 3.2, formula (3.4) and the above study, we can state (a generalization of Taya analytic viewpoint [21, Theorem 1.1]):

**Theorem 3.4.** *Let  $n \gg 0$  be such that  $\mathcal{G}_{k_n/k} := \text{Gal}(H_{k_n/k}/k_n) \simeq \mathcal{G}_k$ . Then  $\#\mathcal{G}_k = \#\mathcal{C}_k \cdot \#\mathcal{R}_k^{\text{nr}} = \mathcal{C}_{k_n}^{G_n}$ , equivalent to  $\frac{p^{n \cdot (\#S-1)}}{(E_k : E_k \cap N_{k_n/k}(k_n^\times))} = \#\mathcal{R}_k^{\text{nr}}$ .*

#### 4. FILTRATION OF $\mathcal{C}_{k_n}$ – CLASS AND NORM FACTORS

Describe now a formal algorithm of computation of  $\#\mathcal{C}_{k_n}$ , for all  $n \geq 0$ , by means of “unscrewing” in  $k_n/k$ . For this, put  $G_n := \text{Gal}(k_n/k) =: \langle \sigma_n \rangle$ . Let  $I_{k_n}$  be the group of prime-to- $p$  ideals of  $k_n$ .

**4.1. Filtration of the class groups.** One uses the filtration of  $M_n := \mathcal{C}_{k_n}$  defined as follows [4, Corollary 3.7]. For  $n \geq 0$  fixed,  $(M_n^i)_{i \geq 0}$  is the  $i$ -sequence of sub- $G_n$ -modules of  $M_n$  defined by  $M_n^0 := 1$  and  $M_n^{i+1}/M_n^i := (M_n/M_n^i)^{G_n}$ , for  $0 \leq i \leq b_n$ , where  $b_n$  is the least integer  $i$  such that  $M_n^i = M_n$  (i.e., such that  $M_n^{i+1} = M_n^i$ ).

If  $\mathcal{C}_k = 1$ ,  $M_0 = M_0^0 = 1$ ,  $b_0 = 0$ ; if  $\mathcal{C}_k \neq 1$ ,  $M_0 = M_0^1 = \mathcal{C}_k$ ,  $b_0 = 1$ .

We will obtain, inductively, ideal groups  $J_n^i \subset I_{k_n}$ , with  $J_n^0 = 1$ , such that:

$$M_n^i =: \mathcal{C}_{k_n}(J_n^i), \text{ for all } i \geq 0.$$

**Proposition 4.1.** *This filtration has the following properties:*

- (i) From  $M_n^0 = 1$ , one gets  $M_n^1 = M_n^{G_n}$  of order  $\#\mathcal{C}_k \cdot \frac{p^{n \cdot (\#S-1)}}{(E_k : E_k \cap N_{k_n/k}(k_n^\times))}$ .
- (ii) One has  $M_n^i = \{c \in M_n, c^{(1-\sigma_n)^i} = 1\}$ , for all  $i \geq 0$ .
- (iii) The  $i$ -sequence  $\#(M_n^{i+1}/M_n^i)$ ,  $0 \leq i \leq b_n$ , is decreasing to 1 and is bounded by  $\#M_n^1$  since  $1 - \sigma_n$  defines the injections  $M_n^{i+1}/M_n^i \hookrightarrow M_n^i/M_n^{i-1}$ .
- (iv)  $\#M_n = \#M_n^{b_n} = \prod_{i=0}^{b_n-1} \#(M_n^{i+1}/M_n^i)$ .

In [4, Formula (29), § 3.2], we established a generalization of Chevalley’s ambiguous class number formula, by means of the norm groups  $N_{k_n/k}(M_n^i) = \mathcal{C}_k(N_{k_n/k}(J_n^i))$  and the subgroups  $\Lambda_n^i := \{x \in k_n^\times, (x) \in N_{k_n/k}(J_n^i)\}$  of  $k_n^\times$ , giving  $\#(M_n^{i+1}/M_n^i) = \frac{\#\mathcal{C}_k}{\#N_{k_n/k}(M_n^i)} \cdot \frac{p^{n \cdot (\#S-1)}}{(\Lambda_n^i : \Lambda_n^i \cap N_{k_n/k}(k_n^\times))}$ , where:

$$(4.1) \quad \frac{\#\mathcal{C}_k}{\#N_{k_n/k}(M_n^i)} \quad \& \quad \frac{p^{n \cdot (\#S-1)}}{(\Lambda_n^i : \Lambda_n^i \cap N_{k_n/k}(k_n^\times))}$$

are integers called *the class factor* and *the norm factor*, respectively, at the step  $i$  of the algorithm in the layer  $k_n$ . These factors are independent of the choice of the ideals defining  $J_n^i$  up to principal ideals of  $k_n$  and the groups  $\Lambda_n^i$  are, therefore, defined up to elements of  $N_{k_n/k}(k_n^\times)$ .

From Lemma 3.2 and Diagram 3, we can state, for any fixed integer  $n$  and for the class and norm factors (4.1):



**Corollary 4.2.** *The class factors divide  $\#\mathcal{C}_k$  and define a decreasing  $i$ -sequence since  $N_{k_n/k}(M_n^i) \subseteq N_{k_n/k}(M_n^{i+1})$  for all  $i \geq 0$ . The norm factors divide  $\#\mathcal{R}_k^{\text{nr}}$  and define a decreasing  $i$ -sequence for all  $i \geq 0$ , due to the injective maps  $E_k/E_k \cap N_{k_n/k}(k_n^\times) \hookrightarrow \cdots \Lambda_n^i/\Lambda_n^i \cap N_{k_n/k}(k_n^\times) \hookrightarrow \Lambda_n^{i+1}/\Lambda_n^{i+1} \cap N_{k_n/k}(k_n^\times) \cdots$*

**4.2. Relation of the algorithms with Iwasawa's theory.** The subgroups  $J_n^i$  of  $I_{k_n}$  are built inductively from  $J_n^0 = 1$ , hence  $\Lambda_n^0 = E_k$ . More precisely the algorithm is the following, for  $n$  and  $i$  fixed [5, §6.2]:

Let  $x \in \Lambda_n^i$ ,  $(x) = N_{k_n/k}(\mathfrak{A})$ ,  $\mathfrak{A} \in J_n^i$ ; thus  $x$  is local norm on the tame places. Suppose that  $x$  is local norm on  $S$ , hence global norm and we can write  $x = N_{k_n/k}(y)$ ,  $y \in k_n^\times$ . The random aspects occur, from the relation  $N_{k_n/k}(y) = N_{k_n/k}(\mathfrak{A})$ , in the mysterious “evolution relation” giving the existence of an ideal  $\mathfrak{B} \in I_{k_n}$  such that  $(y) = \mathfrak{A} \mathfrak{B}^{1-\sigma_n}$ . Remark that for  $N(y) = 1$  and  $y = b^{1-\sigma_n}$ ,  $b$  is given by an *additive Hilbert's resolvent*.

A priori there is no algebraic link with the previous data because of the global solution  $y$  (Hasse's norm theorem) unique up to  $k_n^{\times 1-\sigma_n}$ ; this gives  $\mathfrak{B}$  up to principal ideals. All numbers  $x \in \Lambda_n^i \cap N_{k_n/k}(k_n^\times)$  define the step  $i+1$ :

$$J_n^{i+1} := J_n^i \cdot \langle \dots, \mathfrak{B}, \dots \rangle \text{ and } \Lambda_n^{i+1} := \{x \in k_n^\times, (x) \in N_{k_n/k}(J_n^{i+1})\}.$$

Therefore, for  $i = b_n$  we obtain  $M_n^{b_n} = \mathcal{C}_{k_n}$ ,  $N_{k_n/k}(M_n^{b_n}) = \mathcal{C}_k$  and  $(\Lambda_n^{b_n} : \Lambda_n^{b_n} \cap N_{k_n/k}(k_n^\times)) = p^{n \cdot (\#S-1)}$ , which explains that  $\#\mathcal{C}_{k_n}$  essentially depends on the number of steps  $b_n$  of the algorithm; this is expressed in terms of Iwasawa invariants as follows:

**Theorem 4.3.** *We assume the Conventions 1.1 for the base field  $k$  and recall that  $\mathcal{R}_k^{\text{nr}} := \text{Gal}(H_k^{\text{gen}}/k_\infty H_k^{\text{nr}})$  (Diagram 3), where  $H_k^{\text{gen}}$  is the genus field of  $k_\infty/k$  (Theorem 3.3). Let  $b_n$  be the length of the algorithm in the layer  $k_n$ . Then (where  $v_p$  denotes the  $p$ -adic valuation):*

(i)  $b_n \leq \lambda \cdot n + \mu \cdot p^n + \nu \leq v_p(\#\mathcal{C}_k \cdot \#\mathcal{R}_k^{\text{nr}}) \cdot b_n$ , for all  $n \geq 0$ . So,  $\lambda = \mu = 0 \iff b_n$  bounded.

(ii)  $b_m \geq b_n$ , for all  $m \geq n \geq 0$ .

(iii)  $b_1 = 0 \iff \lambda = \mu = \nu = 0 \iff b_n = 0$  for all  $n \iff \mathcal{C}_k = \mathcal{R}_k^{\text{nr}} = 1$ .

*Proof.* Let  $M_n := \mathcal{C}_{k_n}$ , for all  $n \geq 0$ .

(i) As  $\#(M_n^{i+1}/M_n^i) \geq p$ , for  $0 \leq i \leq b_n - 1$ , Proposition 4.1 (iv) implies  $\#M_n = \#M_n^{b_n} \geq p^{b_n}$ ; whence  $b_n \leq \lambda \cdot n + \mu \cdot p^n + \nu$ .

From the fact that  $\#(M_n^{i+1}/M_n^i) \mid \#\mathcal{C}_k \cdot \#\mathcal{R}_k^{\text{nr}}$  (Corollary 4.2) this yields  $\#(M_n^{i+1}/M_n^i) \leq \#\mathcal{C}_k \cdot \#\mathcal{R}_k^{\text{nr}}$  for  $0 \leq i \leq b_n - 1$ , whence  $\#\mathcal{C}_{k_n} \leq (\#\mathcal{C}_k \cdot \#\mathcal{R}_k^{\text{nr}})^{b_n}$  from Proposition 4.1 (iv); hence the second inequality and the second claim.

(ii) By definition,  $M_m^{b_m} = M_m$  with  $b_m$  minimal. Since  $k_m/k_n$  is totally ramified,  $N_{k_m/k_n}(M_m^{b_m}) = M_n$ , but  $N_{k_m/k_n}(M_m^{b_m}) \subseteq M_n^{b_m}$  (Proposition 4.1 (ii)), whence  $M_n \subseteq M_n^{b_m}$ , thus  $M_n^{b_m} = M_n$ , proving the claim.

(iii) So  $b_1 = 0$  implies  $b_0 = 0$ , whence  $\lambda + \mu p + \nu = \mu + \nu = 0$  yielding  $\lambda = \mu = 0$  and  $\nu = 0$ ; then (i) implies  $b_n = 0$  for all  $n \geq 0$ , in other words,  $\mathcal{C}_{k_n} = 1$  for all  $n \geq 0$ ; thus, taking  $n \gg 0$  to apply Theorem 3.4 yields  $\mathcal{G}_{k_n/k} = \mathcal{C}_{k_n}^{G_n} = 1$ , whence  $\mathcal{C}_k = \mathcal{R}_k^{\text{nr}} = 1$  (reciprocals obvious).  $\square$

**Corollary 4.4.** (a) Under Conventions 1.1,  $\lambda = \mu = 0$  (equivalent to  $b_n$  bounded) is equivalent to each of the following properties:

(i)  $N_{k_n/k} : \mathcal{C}_{k_n} \rightarrow \mathcal{C}_k$  is an isomorphisms for all  $n \geq 0$ .

(ii)  $\#\mathcal{C}_{k_n}^{G_n} = \#\mathcal{C}_{k_n} = \#\mathcal{C}_k$ , for all  $n \geq 0$ .

(iii)  $\mathcal{C}_{k_n}^{G_n} = \mathcal{C}_{k_n}$ , for all  $n \geq 0$  and  $\mathcal{R}_k^{\text{nr}} = 1$ .

(b) Let  $k_{n_1}$ , still denoted  $k$ , be such that  $b_n$  is constant for all  $n \geq n_1$ ;<sup>2</sup> for this new base field  $k$  and the new  $b$ -function,  $b_n \leq 1$ , for all  $n \geq 0$ .

*Proof.* Proof of (a). (i) Under the condition  $\lambda = \mu = 0$ ,  $\#\mathcal{C}_{k_n} = \#\mathcal{C}_k = p^\nu$  for all  $n$ , and all the (surjective) norm maps are isomorphisms.

(ii) Chevalley's formula  $\#\mathcal{C}_{k_n}^{G_n} = \#\mathcal{C}_k \cdot \frac{p^{n \cdot (\#S-1)}}{(E_k : E_k \cap N_{k_n/k}(k_n^\times))} \leq \#\mathcal{C}_{k_n} = \#\mathcal{C}_k$

yields  $\#\mathcal{C}_{k_n}^{G_n} = \#\mathcal{C}_{k_n} = \#\mathcal{C}_k$  and  $\frac{p^{n \cdot (\#S-1)}}{(E_k : E_k \cap N_{k_n/k}(k_n^\times))} = 1$ , for all  $n \geq 0$ .

(iii) From (ii),  $\mathcal{R}_k^{\text{nr}} = 1$ , taking  $n \gg 0$  to apply Theorem 3.4.

In the three cases, the reciprocals are obvious.

Proof of (b). Consider the second step of the algorithm in  $k_n$  (we exclude the case  $b_n = 0$  where all class groups are trivial); the class factor for  $\mathcal{C}_{k_n}^2 / \mathcal{C}_{k_n}^1$  is trivial since  $N_{k_n/k}(\mathcal{C}_{k_n}^{G_n}) = \mathcal{C}_k$  (from (i), (ii)) and the norm factor, as divisor of  $\mathcal{R}_k^{\text{nr}}$ , is also trivial (from (iii)); whence  $b_n = 1$  for all  $n \geq 0$ .  $\square$

Note that under Greenberg's conjecture, in  $\frac{p^{n \cdot (\#S-1)}}{(\Lambda_n^1 : \Lambda_n^1 \cap N_{k_n/k}(k_n^\times))}$ , we have  $\Lambda_n^1 = \{x \in k^\times, (x) = N_{k_n/k}(\mathfrak{A}), \mathfrak{A} \in J_n^1\}$  where  $\mathcal{C}_{k_n}(J_n^1) = \mathcal{C}_{k_n}^{G_n}$ ; thus, norms being isomorphisms,  $(x) = N_{k_n/k}(\mathfrak{A})$  implies that  $\mathfrak{A} = (\alpha)$ ,  $\alpha \in k_n^\times$ , so that  $\Lambda_n^1 = E_k N_{k_n/k}(k_n^\times)$ , showing that the algorithm becomes trivial.

**4.3. The  $n$ -sequences  $(\mathcal{C}_{k_n} / \mathcal{C}_{k_n}^i)^{G_n}$ .** We fix the step  $i$  of the algorithms. For now, we do not assume the Conventions 1.1. For all  $m \geq n \geq 0$ , the norm maps  $N_{k_m/k_n}$  on  $M_m$  and  $M_m^{(1-\sigma_m)^i}$  are surjective (they are, a priori, not injective nor surjective on the kernels  $M_m^i$  of the maps  $M_m \rightarrow M_m^{(1-\sigma_m)^i}$ ). This leads to the following result (see [5, Lemmas 7.1, 7.2] for the details), giving another approach of the conjecture:

**Theorem 4.5.** For all  $i \geq 0$  fixed,  $\{\#\mathcal{C}_{k_n}^{i+1} / \mathcal{C}_{k_n}^i\}_n$  defines an increasing  $n$ -sequence of divisors of  $\#\mathcal{C}_k \cdot \#\mathcal{R}_k^{\text{nr}}$ . Thus  $\lim_{n \rightarrow \infty} \#\mathcal{C}_{k_n}^{i+1} / \mathcal{C}_{k_n}^i =: p^{c^i} p^{\rho^i}$ . The

<sup>2</sup> One must note that for each change of base field in the tower, the Iwasawa invariants are given by Conventions 1.1, and the algorithms are distinct; for instance the parameter  $b_n$  defines a new function of the  $n$ th layer of the new  $k$  (in the meaning  $[k_n : k] = p^n$ ).

$i$ -sequences  $p^{c^i}$  and  $p^{\rho^i}$  are decreasing, stationary at a divisor  $p^c$  of  $\#\mathcal{C}_k$  and  $p^\rho$  of  $\#\mathcal{R}_k^{\text{nr}}$ , respectively. Greenberg's conjecture is equivalent to  $c = \rho = 0$ .

### 5. $\mathcal{T}_k$ AS GOVERNING INVARIANT OF THE ALGORITHMS

The ideals  $\mathfrak{A} \in J_n^i$  may be arbitrarily modified up to principal ideals of  $k_n$ , whence  $N_{k_n/k}(\mathfrak{A})$  defined up to elements of  $N_{k_n/k}(k_n^\times)$ , as well as  $\Lambda_n^i$ . We intend to obtain suitable *finite sets* of representatives of these ideal norms, independently of  $n$ , more precisely of cardinality  $\leq \#\mathcal{T}_k$ .

**5.1. Decomposition of  $N_{k_n/k}(\mathfrak{A})$  – The fundamental ideals  $\mathfrak{t}$ .** Let  $H_k^{\text{pr}}$  and  $H_{k_n}^{\text{pr}}$  be the maximal abelian  $p$ -ramified pro- $p$ -extensions of  $k$  and  $k_n$ , respectively. Let  $F$  be an extension of  $H_k^{\text{nr}}$  such that  $H_k^{\text{pr}}$  be the direct compositum of  $F$  and  $k_\infty H_k^{\text{nr}}$  over  $H_k^{\text{nr}}$  (possible because  $k_\infty \cap H_k^{\text{nr}} = k$  due to the total ramification of  $p$  in  $k_\infty/k$ ); we put  $\Gamma = \text{Gal}(H_k^{\text{pr}}/F) \simeq \mathbb{Z}_p$ .

In the same way, we fix an extension  $F_n$  of  $F$  such that  $H_{k_n}^{\text{pr}}$  be the direct compositum of  $F_n$  and  $H_k^{\text{pr}}$  over  $k_n F$ ; we put  $\Gamma_n = \text{Gal}(H_{k_n}^{\text{pr}}/F_n) \simeq \Gamma^{p^n}$ . We have  $F = F_0 \subset F_1 \subset \dots \subset F_n \subset F_{n+1} \subset \dots$  (see Diagram 4 hereafter).

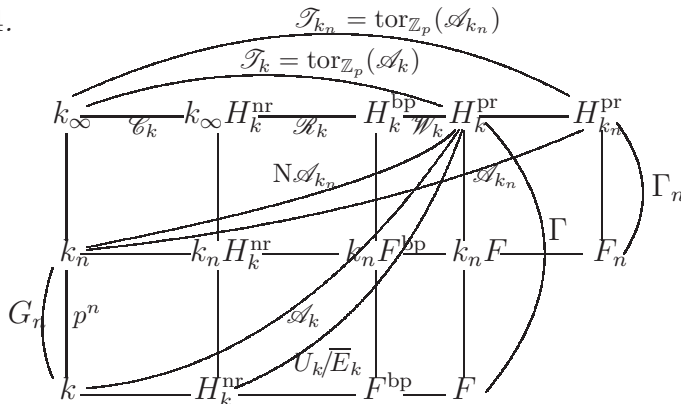
In what follows, we systematically use the flatness of  $\mathbb{Z}_p$ .

Consider the Artin symbols  $\left(\frac{H_k^{\text{pr}}/k}{\cdot}\right)$  and  $\left(\frac{H_{k_n}^{\text{pr}}/k_n}{\cdot}\right)$ , defined on  $I_k \otimes \mathbb{Z}_p$  and  $I_{k_n} \otimes \mathbb{Z}_p$ , respectively. Their images are the Galois groups  $\mathcal{A}_k$  (resp.  $\mathcal{A}_{k_n}$ ); their kernels are the groups of infinitesimal principal ideals  $\mathcal{P}_{k,\infty}$  (resp.  $\mathcal{P}_{k_n,\infty}$ ), where  $\mathcal{P}_{k,\infty}$  is the set of ideals  $(x_\infty)$ ,  $x_\infty \in k'^\times \otimes \mathbb{Z}_p$ , such that  $\iota x_\infty = 1$  in  $U_k$  (idem for  $\mathcal{P}_{k_n,\infty}$ ) [3, Theorem III.2.4, Proposition III.2.4.1].

The arithmetic norm (or restriction of automorphisms), in  $k_n/k$ , leads to  $N_{k_n/k}(\mathcal{A}_{k_n}) = \text{Gal}(H_k^{\text{pr}}/k_n)$  and  $N_{k_n/k}(\mathcal{T}_{k_n}) = \mathcal{T}_k$  since  $k_{n,\infty} = k_\infty$ . The fixed points formula  $\mathcal{T}_{k_n}^{G_n} \simeq \mathcal{T}_k$  ([3, Theorem IV.3.3], [11, Section 2 (c)]), implies  $\text{Ker}(N_{k_n/k}) = \mathcal{T}_{k_n}^{1-\sigma_n} = \text{Gal}(H_{k_n}^{\text{pr}}/H_k^{\text{pr}})$ .

We denote by  $\mathcal{K}_\infty^\times \subset k'^\times \otimes \mathbb{Z}_p$  the subgroup of infinitesimal elements of  $k$  (idem for  $\mathcal{K}_{n,\infty}^\times \subset k_n'^\times \otimes \mathbb{Z}_p$ ). In the sequel, the notations  $x_\infty, y_\infty, \dots$  always denote such infinitesimal elements.

Diagram 4.



**Lemma 5.1.** *If  $(x_\infty) \in \mathcal{P}_{k,\infty} \cap N_{k_n/k}(I_{k_n} \otimes \mathbb{Z}_p)$ , then  $x_\infty \in N_{k_n/k}(\mathcal{K}_{n,\infty}^\times)$ .*

*Proof.* The assumption implies that  $x_\infty$  is everywhere local norm in  $k_n/k$ , whence  $x_\infty = N_{k_n/k}(y)$ ,  $y \in k_n'^\times \otimes \mathbb{Z}_p$  (Hasse norm theorem). Thus, we get  $\iota N_{k_n/k}(y) = N_{k_n/k}(\iota_n y) = 1$  and  $\iota_n y = t^{1-\sigma_n}$ ,  $t \in \prod_{\mathfrak{p}_n \in S_n} k_{n,\mathfrak{p}_n}^\times$  (Hilbert's Theorem 90,  $H^1(G_n, \prod_{\mathfrak{p}_n \in S_n} k_{n,\mathfrak{p}_n}^\times) = 1$ ). Consider  $t$  in the profinite completion  $\prod_{\mathfrak{p}_n \in S_n} \widehat{k_{n,\mathfrak{p}_n}^\times}$ ; then one has the exact sequence [11, Chap. 1, § a):

$$1 \rightarrow \mathcal{K}_{n,\infty}^\times \longrightarrow k_n^\times \otimes \mathbb{Z}_p \xrightarrow{\widehat{\iota}_n} \prod_{\mathfrak{p}_n \in S_n} \widehat{k_{n,\mathfrak{p}_n}^\times} \simeq \mathbb{Z}_p^{\#S} \oplus U_k \rightarrow 1.$$

Put  $t = \widehat{\iota}_n z$ ,  $z \in k_n^\times \otimes \mathbb{Z}_p$ ; then  $\widehat{\iota}_n y = \widehat{\iota}_n(z^{1-\sigma_n})$ ,  $y = z^{1-\sigma_n} y_\infty$ ,  $y_\infty \in \mathcal{K}_{n,\infty}^\times$ , then  $x_\infty = N_{k_n/k}(y_\infty)$ . We also have  $H^1(G_n, \mathcal{K}_{n,\infty}^\times) = 1$  [11, Lemme 5].  $\square$

The fundamental link between ideal norms in  $k_n/k$  and the torsion group  $\mathcal{T}_k$  is given, for  $n$  large enough, by the following result where the ‘‘uniqueness’’ are relative to the choices of the  $F_n$ ; we say that some numbers  $a \in k_n'^\times \otimes \mathbb{Z}_p$  (depending on  $n$ ) are ‘‘close to 1’’ if  $\iota a \rightarrow 1$  in  $U_k$  when  $n \rightarrow \infty$ .

**Theorem 5.2.** *Let  $n \gg 0$  fixed and let  $\mathfrak{A} \in I_{k_n} \otimes \mathbb{Z}_p$  (prime-to- $p$  ideal of  $k_n$ ).*

(i) *There exists  $\alpha \in k_n'^\times \otimes \mathbb{Z}_p$  such that  $N_{k_n/k}(\mathfrak{A}(\alpha)) = N_{k_n/k}(\mathfrak{T}) =: \mathfrak{t}$ , with  $(\frac{H_{k_n}^{\text{pr}}/k_n}{\mathfrak{T}}) \in \mathcal{T}_{k_n}$ ,  $(\frac{H_k^{\text{pr}}/k}{\mathfrak{t}}) \in \mathcal{T}_k$  and  $\iota N_{k_n/k}(\alpha)$  close to 1.*

(ii) *The representative  $\mathfrak{t}$  of the class  $N_{k_n/k}(\mathfrak{A}) \cdot N_{k_n/k}(k_n'^\times \otimes \mathbb{Z}_p)$ , does not depend, modulo  $N_{k_n/k}(\mathcal{P}_{k_n,\infty})$ , on the tower  $\bigcup_j F_j$ .*

*Proof.* (i) From Diagram 4 and the properties of Artin symbols, there exist unique ideals  $\mathfrak{T}, \mathfrak{C} \in I_{k_n} \otimes \mathbb{Z}_p$ , modulo  $\mathcal{P}_{k_n,\infty}$ , such that:

$$(5.1) \quad \mathfrak{A} = \mathfrak{T} \cdot \mathfrak{C} \cdot (y_\infty), \text{ with } \left( \frac{H_{k_n}^{\text{pr}}/k_n}{\mathfrak{T}} \right) \in \mathcal{T}_{k_n}, \left( \frac{H_{k_n}^{\text{pr}}/k_n}{\mathfrak{C}} \right) \in \Gamma_n, y_\infty \in \mathcal{K}_{n,\infty}^\times$$

By restriction, the image of  $\Gamma_n$  in  $\Gamma$  is  $\Gamma^{p^n}$ ; thus  $N_{k_n/k}(\mathfrak{C}) = \mathfrak{c}^{p^n} \cdot (x_\infty)$  for  $\mathfrak{c} \in I_k \otimes \mathbb{Z}_p$  such that  $(\frac{H_k^{\text{pr}}/k}{\mathfrak{c}}) \in \Gamma$  and  $x_\infty \in \mathcal{K}_{\infty}^\times$ ; but since  $H_k^{\text{pr}} \subseteq F$ , the ideal  $\mathfrak{c}$  is  $p$ -principal, thus  $\mathfrak{c} = (c)$ ,  $c \in k'^\times \otimes \mathbb{Z}_p$ , and then,  $N_{k_n/k}(\mathfrak{C}) = (c^{p^n}) \cdot (x_\infty)$ . We have from (5.1):

$$N_{k_n/k}(\mathfrak{A}) = N_{k_n/k}(\mathfrak{T}) \cdot (c^{p^n}) \cdot (x_\infty) \cdot N_{k_n/k}(y_\infty) =: \mathfrak{t} \cdot (c^{p^n}) \cdot (x'_\infty),$$

with  $(\frac{H_k^{\text{pr}}/k}{\mathfrak{t}}) \in \mathcal{T}_k$ ,  $x'_\infty \in \mathcal{K}_{\infty}^\times$ . From Lemma 5.1, since  $(x'_\infty)$  is norm of ideal in  $k_n/k$ ,  $x'_\infty = N_{k_n/k}(y'_\infty)$ , whence  $N_{k_n/k}(\mathfrak{A}(c)^{-1}(y'_\infty)^{-1}) = \mathfrak{t}$ .

Let  $\alpha = c^{-1} y'_\infty^{-1}$ ; then  $\iota N_{k_n/k}(\alpha) = \iota(c^{-p^n})$  is close to 1.

(ii) Let  $\bigcup_j F'_j$  be another tower for Diagram 4; with obvious notations (which depend on  $n$ ), put  $u := N_{k_n/k}(\alpha)$ ,  $u' := N_{k_n/k}(\alpha')$ ,  $\iota u, \iota u'$  close to 1, we get  $N_{k_n/k}(\mathfrak{A}) \cdot (u) = \mathfrak{t}$ ,  $N_{k_n/k}(\mathfrak{A}) \cdot (u') = \mathfrak{t}'$ . Whence  $\mathfrak{t}' \mathfrak{t}^{-1} = (a)$ , with  $a$  close to 1. So, if  $p^e$  is the exponent of  $\mathcal{T}_k$ , we obtain  $(a)^{p^e} = (a_\infty) \in \mathcal{P}_{k,\infty}$ ,

which gives  $a^{p^e} = \varepsilon a_\infty$ ,  $\varepsilon \in E_k \otimes \mathbb{Z}_p$  with  $\iota\varepsilon$  close to 1, hence (for  $n \gg 0$ ) of the form  $\varepsilon = \eta^{p^e}$ ,  $\eta \in E_k \otimes \mathbb{Z}_p$ , with  $\iota\eta$  close to 1 (from Leopoldt's conjecture [3, Theorem III.3.6.2 (iv)]). This yields  $(a\eta^{-1})^{p^e} = a_\infty$  and we get  $\iota(a\eta^{-1}) = \xi \in W_k = \text{tor}_{\mathbb{Z}_p}(U_k)$ ; both  $\iota a$  and  $\iota\eta$  are close to 1, thus  $\xi = 1$  and  $a\eta^{-1} = a'_\infty$  giving  $\mathfrak{t}'\mathfrak{t}^{-1} = (a'_\infty) \in N_{k_n/k}(\mathcal{P}_{k_n,\infty})$ , using Lemma 5.1  $\square$

**5.2. Images in  $\mathcal{C}_k$  and  $\mathcal{R}_k$  of the ideals  $\mathfrak{t}$  – Conjecture.** We choose, once for all, a set  $\mathbf{T}_k = \{\mathfrak{t}_\ell\}_{1 \leq \ell \leq \#\mathcal{T}_k}$ , of ideals  $\mathfrak{t}_\ell \in I_k \otimes \mathbb{Z}_p$  whose Artin symbols describe  $\mathcal{T}_k$  isomorphic to  $\mathbf{T}_k \cdot \mathcal{P}_{k,\infty} / \mathcal{P}_{k,\infty}$ .

The ideals  $N_{k_n/k}(\mathfrak{A}(\alpha)) = \mathfrak{t} \in \mathbf{T}_k$ , well-defined modulo  $N_{k_n/k}(\mathcal{P}_{k_n,\infty})$ , play the following roles in the evolution of the class and norm factors:

(i) *Class factors and  $\mathbf{T}_k$ .* The ideal groups  $N_{k_n/k}(J_n^i)$ , representing the class groups  $N_{k_n/k}(M_n^i)$  as denominator of the class factors, are generated, modulo principal ideals  $(a)$ ,  $a \in N_{k_n/k}(k_n'^{\times} \otimes \mathbb{Z}_p)$ , by ideals  $\mathfrak{t}^i \in \mathbf{T}_k$ .

(ii) *Norm factors and  $\mathbf{T}_k$ .* The groups  $\Lambda_n^i = \{x \in k^\times, (x) \in N_{k_n/k}(J_n^i)\}$ , giving the norm factors, are obtained, modulo elements of  $N_{k_n/k}(k_n'^{\times} \otimes \mathbb{Z}_p)$ , via principal ideals  $(\tau) \in \mathbf{T}_k$  (hence  $\tau$  is local norm at the tame places in  $k_n/k$  and its norm properties only depend on  $S$ ).

Put  $\mathbf{T}_k^{\text{ppl}} := \{\mathfrak{t} \in \mathbf{T}_k, \mathfrak{t} = (\tau)\}$ ; the subgroup  $\mathbf{T}_k^{\text{ppl}} \cdot \mathcal{P}_{k,\infty} / \mathcal{P}_{k,\infty}$  is isomorphic to  $\text{Gal}(H_k^{\text{pr}}/k_\infty H_k^{\text{nr}})$ . Let  $\mathfrak{t} = (\tau) \in \mathbf{T}_k^{\text{ppl}}$ ; so we have  $(\frac{H_k^{\text{pr}}/k}{\mathfrak{t}}) \in \text{Gal}(H_k^{\text{pr}}/k_\infty H_k^{\text{nr}}) \simeq U_k^*/\overline{E}_k$ . This yields  $\tau^{p^e} = \varepsilon x_\infty$ ,  $\varepsilon \in E_k \otimes \mathbb{Z}_p$ ; whence  $\iota N_{k/\mathbb{Q}}(\tau) = \pm 1$  and the image of  $\iota\tau$  modulo  $\overline{E}_k$  is defined in  $U_k^*/\overline{E}_k$ . We consider the image of  $\log(\iota\tau)$  in  $\log(U_k^*)/\log(\overline{E}_k) = \mathcal{R}_k$ , which defines  $\log(\mathfrak{t}) := \log(\iota\tau) \pmod{\log(\overline{E}_k)}$ . We have  $W_k = \text{Ker}(\log)$ , and this gives again the exact sequence (3.3).

*Remark 5.3.* Let  $(\tau) \in \mathbf{T}_k^{\text{ppl}}$ ; choosing a representative of  $\tau$  modulo  $\mathcal{H}_\infty^\times$  one may always assume that  $(\tau) = N_{k_n/k}(\mathfrak{T})$ ,  $\mathfrak{T} \in I_{k_n} \otimes \mathbb{Z}_p$ , since  $N(\mathcal{T}_{k_n}) = \mathcal{T}_k$  (whence  $\tau$  local norm at the tame places). Suppose that the image of  $\iota\tau$  in  $\text{tor}_{\mathbb{Z}_p}(U_k/\overline{E}_k)$  is in the subgroup  $\text{Gal}(H_k^{\text{pr}}/H_k^{\text{gen}})$  generated by the inertia groups  $\text{tor}_{\mathbb{Z}_p}(U_{\mathfrak{p}}\overline{E}_k/\overline{E}_k)$ ,  $\mathfrak{p} \in S$ ; then  $\tau$  is local norm on  $S$ . Indeed, let  $\iota\tau = u = (u_{\mathfrak{p}}, 1, \dots, 1)$ ;  $u$  is local norm at each  $\mathfrak{p}' \neq \mathfrak{p}$ , whence a global norm (product formula). This explains that generators  $\tau$  of ideals  $\mathfrak{t} \in \mathbf{T}_k^{\text{ppl}}$ , whose images are in  $\text{Gal}(H_k^{\text{pr}}/H_k^{\text{gen}})$ , do not modify any norm factor, only depending on the image in  $\mathcal{R}_k^{\text{nr}} = \text{Gal}(H_k^{\text{gen}}/k_\infty H_k^{\text{nr}})$ .

**5.3. The algorithm in terms of fundamental ideals  $\mathfrak{t}$ .** We still assume Conventions 1.1 to the base field  $k$ . In this subsection, we consider the layer  $K = k_n$  (with  $p^n \gg p^e$ , the exponent of  $\mathcal{T}_k$ ) and, to simplify, we delete indices  $n$  (e.g.,  $M_n^i \rightarrow M^i$ ,  $\Lambda_n^i \rightarrow \Lambda^i$ ,  $N_{k_n/k} \rightarrow N$ ,  $b_n \rightarrow b$  (number of steps in  $K$ )); then uppercase (respectively lowercase) letters for ideals are reserved to  $K$  (respectively  $k$ ).

From Theorem 5.2 (i) and for any prime-to- $p$  ideals  $\mathfrak{A} \in J^i$ , defining  $M^i$ , there exist  $\alpha \in K'^{\times} \otimes \mathbb{Z}_p$  and  $\mathfrak{T}$  of finite order modulo  $\mathcal{P}_{K,\infty}$ , such that  $N(\mathfrak{A}(\alpha)) = N(\mathfrak{T}) =: \mathfrak{t} \in \mathbf{T}_k$  with  $N(\alpha)$  close to 1. Denote by  $\Sigma_K^i$ , the set of such representatives  $\mathfrak{T}^i$  and let  $\Sigma_k^i$  be the set of  $\mathfrak{t}^i := N(\mathfrak{T}^i)$ ; so:

$$(5.2) \quad N(\Sigma_K^i) = \Sigma_k^i, \quad N(M^i) = \mathcal{C}_k \langle \Sigma_k^i \rangle, \quad \Sigma_k^i \subseteq \mathbf{T}_k.$$

Replacing  $\mathfrak{A}$  by  $\mathfrak{A}(\alpha)$  does not modify the class and norm factors (4.1) since  $\mathcal{C}_k(N(\mathfrak{A})) = \mathcal{C}_k(\mathfrak{t})$  and, if  $N(\mathfrak{A})$  is principal, then  $\mathfrak{t} = (\tau)$  is equal to  $N(\mathfrak{A})$  up to  $N(K^{\times} \otimes \mathbb{Z}_p)$ , which does not modify the norm properties in  $\Lambda^i$ . Then, in  $\Lambda^i = \{\tau \in k'^{\times} \otimes \mathbb{Z}_p, (\tau) \in \langle \Sigma_k^i \rangle\}$ , one must find all elements  $\tau^i$  (by definition of the form  $N(\mathfrak{T}^i)$ ,  $\mathfrak{T}^i \in \langle \Sigma_K^i \rangle$ ), such that  $\tau^i$  is local norm on  $S$  in  $K/k$ , thus of the form  $N(y^i)$ ,  $y^i \in K'^{\times} \otimes \mathbb{Z}_p$ ; so the algorithm continues, from  $N(y^i) = N(\mathfrak{T}^i)$ , with the following evolution using Theorem 5.2 (i):

$$(5.3) \quad (y^i) = \mathfrak{T}^i \cdot \mathfrak{B}^{1-\sigma}, \quad \text{with } N(\mathfrak{B}(\beta)) = N(\mathfrak{T}^i) = \mathfrak{t}' \in \mathbf{T}_k,$$

for a suitable  $\beta$  such that  $N(\beta)$  is close to 1, and one obtains a new  $\mathfrak{t}'$  to build  $\Sigma_k^{i+1}$ , and so on. If  $\lambda$  or  $\mu$  do not vanish, there exist, when  $[K : k] \rightarrow \infty$ , arbitrary large  $i$ -sequences of sets  $\Sigma_k^i$  such that the class and norm factors are constant, which seems incredible, each new  $\mathfrak{t}'$  being a priori random in  $\mathbf{T}_k$ .

A philosophy should be that it is the  $\mathfrak{t}'$  which govern (numerically) the  $G$ -structure of the class groups in  $K/k$  and not the inverse (see also [6, Remarques 11, §6]).

Let's give a more precise description of the numerical possibilities, assuming to simplify the comments that  $1 \rightarrow \mathcal{R}_k \rightarrow \mathcal{T}_k \rightarrow \mathcal{C}_k \rightarrow 1$  is an exact sequence of  $\mathbb{F}_p$ -vector spaces; we compute the filtration  $\{M^i\}_{i \geq 0}$  for  $M = \mathcal{C}_K$  ( $K = k_n$  fixed) with the following exact sequence at the step  $i$  (see (5.2)):

$$1 \longrightarrow \Lambda^i / E_k \xrightarrow{(\cdot)} \langle \Sigma_k^i \rangle \xrightarrow{\mathcal{C}_k} \mathcal{C}_k \langle \Sigma_k^i \rangle = N(M^i) \longrightarrow 1,$$

where  $\Lambda^i = \{\tau \in k'^{\times} \otimes \mathbb{Z}_p, (\tau) \in \langle \Sigma_k^i \rangle\}$ , and let  $\mathfrak{t}^{i+1}$  (obtained as above). Various cases may arrive to get the  $(i+1)$ th exact sequence

$$1 \rightarrow \Lambda^{i+1} / E_k \xrightarrow{(\cdot)} \langle \Sigma_k^{i+1} \rangle \xrightarrow{\mathcal{C}_k} \mathcal{C}_k \langle \Sigma_k^{i+1} \rangle = N(M^{i+1}) \rightarrow 1 :$$

(a)  $\mathcal{C}_k(\mathfrak{t}^{i+1}) \notin \mathcal{C}_k \langle \Sigma_k^i \rangle$ . Thus  $N(M^{i+1}) \not\supseteq N(M^i)$  and this decreases the class factor; but there is no new relation of principality between ideals, so  $\Lambda^{i+1} = \Lambda^i$  (norm factor unchanged).

(b)  $\mathcal{C}_k(\mathfrak{t}^{i+1}) \in \mathcal{C}_k \langle \Sigma_k^i \rangle$ . Thus  $N(M^{i+1}) = N(M^i)$  (class factor unchanged); but  $\mathfrak{t}^{i+1} = (\tau) \cdot \prod_j \mathfrak{t}_j^{a_j}$  gives, possibly, some  $\tau \notin \Lambda^i$ . Then two cases arise:

(i)  $\tau \notin \Lambda^i N(K^{\times})$ , therefore  $(\Lambda^{i+1} : \Lambda^{i+1} \cap N(K^{\times})) > (\Lambda^i : \Lambda^i \cap N(K^{\times}))$ , which decreases the norm factor.

(ii)  $\tau \in \Lambda^i N(K^\times)$  (class and norm factors unchanged). This is the “bad case” occurring, roughly,  $O(\lambda n + \mu p^n)$  times if Greenberg’s conjecture falls (see Remark 5.3 for more enlightenment).

We have given, in [6, Section 6], some heuristics about the “equation  $(y) = \mathfrak{A} \mathfrak{B}^{1-\sigma}$ ” in cyclic extensions  $L/K$  when  $N_{L/K}(y) = N_{L/K}(\mathfrak{A})$  and its “additive aspects”, which applies to  $(\tau) = N(\mathfrak{T}) = N(y)$  and  $(y) = \mathfrak{T} \mathfrak{B}^{1-\sigma}$ .

Assuming that the ideals  $\mathfrak{t}$ , given by the algorithm, are random,  $\mathcal{C}_k(\mathfrak{t})$  (resp.  $\log(\iota\tau) \pmod{\log(\overline{E}_k)}$ ) are random in  $\mathcal{C}_k$  (resp.  $\mathcal{R}_k$ ). This is likely to avoid unbounded algorithms and suggests the following conjecture:

**Conjecture 5.4.** *For  $n \gg 0$  fixed, let  $\mathfrak{t}_j$  (or  $\tau_j$ , when  $\mathfrak{t}_j = (\tau_j)$  is principal), be the fundamental ideals encountered by the algorithm computing inductively the successive class and norm factors, in  $b_n$  steps; then:*

(i) *The classes  $\mathcal{C}_k(\mathfrak{t}_j)$  are uniformly distributed in  $\mathcal{C}_k$ .*

(ii) *When  $\mathfrak{t}_j = (\tau_j)$ , the images  $\log(\mathfrak{t}_j) := \log(\iota\tau_j) \pmod{\log(\overline{E}_k)}$  are uniformly distributed in the normalized regulator  $\mathcal{R}_k = \log(U_k^*) / \log(\overline{E}_k)$ .*

**5.4. Conclusion and possible methods.** Recall that  $b_n$  is the length of the algorithm for the layer  $n$ . We observe the huge discontinuity between the case  $b_n$  bounded, which characterizes Greenberg’s conjecture (Theorem 4.3 and Corollary 4.4) and the case where  $\lambda$  or  $\mu$  are non-zero, giving  $b_n \rightarrow \infty$ . In other words, there is a conflict between the “random aspect” of the algorithm, when  $\lambda$  or  $\mu$  are non-zero, and the smooth algebraic form given by Iwasawa’s theory. We indeed have, under Conventions 1.1,  $\#\mathcal{C}_{k_n} = p^{\lambda n + \mu p^n + \nu}$  for all  $n \geq 0$ , so that the algorithm must obtain rigorously these formulas, for all  $n$ , which seems to be an excessive requirement in contradiction with Conjecture 5.4.

To give a logical way, the sole “solution”, where  $b_n$  does not tend to infinity, is  $b_n$  constant for all  $n \geq n_1$ , giving, from the new base field  $k_{n_1}$ , that we still denote  $k$ , the well-known properties when Greenberg’s conjecture holds. In that case,  $\mathcal{C}_{k_n}^2 / \mathcal{C}_{k_n}^1 = 1$  and  $b_n \leq 1$  for all  $n$ . In other words, in this situation, the “unpredictable” evolution relation (5.3) is not needed. The quotient  $\mathcal{C}_{k_n}^2 / \mathcal{C}_{k_n}^1$  does appear (written instead  $(1 - \sigma)\mathcal{C}_{k_n}[(1 - \sigma)]$ ) in works of Koymans–Pagano–Smith [15, 20], where deep distribution results are proved for the degree  $p$  cyclic case.

We believe that these techniques can be successful for Greenberg’s conjecture since the general algorithm of “unscrewing” in  $k_n/k$  is identical and is essentially based on random values of classical norm symbols. In other words, Greenberg’s conjecture would be, for  $k_\infty/k$  ( $k$  taken high enough in the cyclotomic tower), an extreme version (of the degree  $p$  cyclic case) giving the non-existence of “exceptional  $p$ -classes” (i.e., non-invariant  $p$ -classes) in the tower, that is to say,  $b_n \leq 1$  for all  $n \geq 0$  (to be compared with  $b_n \rightarrow \infty$  if  $\lambda$  or  $\mu$  do not vanish).

*Remark 5.5.* For a base field which does not fulfill the previous conditions, the algorithms may need several steps and (under Greenberg's conjecture) they regularize at some layer such that the above trivialization holds; for instance, the case of  $k = \mathbb{Q}(\sqrt{6559})$ ,  $p = 3$ , computed in [6, §7.2], yields  $\mathcal{C}_k \simeq \mathbb{Z}/9\mathbb{Z}$ ,  $\mathcal{R}_k \simeq \mathbb{Z}/27\mathbb{Z}$ ,  $\mathcal{C}_{k_1} \simeq \mathbb{Z}/27\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$  (whence  $b_1 = 2$ ) and  $\mathcal{C}_{k_2} \simeq \mathbb{Z}/27\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$ ; we compute with [1] that  $\tilde{\mathcal{C}}_k \simeq \mathbb{Z}/3\mathbb{Z}$ ,  $\tilde{\mathcal{C}}_{k_1} \simeq \tilde{\mathcal{C}}_{k_2} \simeq \mathbb{Z}/9\mathbb{Z}$ .

All this shows how classical arguments of *algebraic number theory* seem insufficient to prove unconditionally Greenberg's conjecture (among others), but that density results may be accessible, giving that the conjecture holds except, possibly, for pathological families of zero density (probably none).

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