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Indranil Biswas, Sorin Dumitrescu

▶ To cite this version:

Indranil Biswas, Sorin Dumitrescu. HOLOMORPHIC BUNDLES TRIVIALIZABLE BY PROPER SURJECTIVE HOLOMORPHIC MAP. International Mathematics Research Notices, inPress, 2022 (5), pp.3636-3650. hal-02540982

HAL Id: hal-02540982 https://hal.science/hal-02540982

Submitted on 12 Apr 2020

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HOLOMORPHIC BUNDLES TRIVIALIZABLE BY PROPER SURJECTIVE HOLOMORPHIC MAP

INDRANIL BISWAS AND SORIN DUMITRESCU

ABSTRACT. Given a compact complex manifold M, we investigate the holomorphic vector bundles E on M such that $\varphi^* E$ is trivial for some surjective holomorphic map φ , to M, from some compact complex manifold. We prove that these are exactly those holomorphic vector bundles that admit a flat holomorphic connection with finite monodromy homomorphism. A similar result is proved for holomorphic principal G-bundles, where G is a connected reductive complex affine algebraic group.

1. INTRODUCTION

Let M be a compact complex manifold. Let E be a holomorphic vector bundle on M with the following property: there is a compact complex manifold X, and a surjective holomorphic map $\varphi : X \longrightarrow M$, such that $\varphi^* E$ is holomorphically trivial. To clarify, the dimension of X is allowed to be larger than that of M. Note that if the assumption that X is compact is removed, then every holomorphic vector bundle satisfies this condition. Indeed, the pullback of E to the total space of the frame bundle for E has a canonical holomorphic trivialization. However, this total space is never compact.

Let E be a holomorphic vector bundle on M admitting a flat connection D whose monodromy homomorphism

$$\rho : \pi_1(M, x_0) \longrightarrow \operatorname{GL}(E_{x_0}),$$

where $x_0 \in M$ is a base point, has finite image. Consider the finite étale Galois covering $f : \widetilde{M} \longrightarrow M$ corresponding to the finite index subgroup kernel $(\rho) \subset \pi_1(M, x_0)$. It is easy to see that f^*E is holomorphically trivial. Therefore, E satisfies the condition stated at the beginning.

Our aim here is to prove a converse of it. More precisely, we prove the following (see Theorem 4.1):

Theorem 1.1. Let E be a holomorphic vector bundle on a compact complex manifold M satisfying the condition that there is a compact complex manifold X, and a surjective map $\varphi : X \longrightarrow M$, such that $\varphi^* E$ is holomorphically trivial. Then E admits a flat holomorphic connection whose monodromy homomorphism has finite image.

It is easy to see that a holomorphic vector bundle E on M admits a flat holomorphic connection with finite monodromy if and only if there is a finite étale Galois covering f: $\widetilde{M} \longrightarrow M$ such that f^*E is holomorphically trivial (the "only if" part was explained above).

²⁰¹⁰ Mathematics Subject Classification. 53C07, 32L10, 70G45.

Key words and phrases. Finite bundle, flat holomorphic connection, finite monodromy, reductive group.

In [No1], Nori characterized such vector bundles in the frame-work of algebraic geometry. When the base field is the field of complex numbers his result gives the following statement. For an algebraic vector bundle V on a complex projective variety Y, the following two conditions are equivalent:

- (1) There is a finite étale Galois covering $f : \widetilde{Y} \longrightarrow Y$ such that f^*V is algebraically trivial.
- (2) There are finitely many algebraic vector bundles W_1, \dots, W_ℓ on Y such that

$$V^{\otimes k} = \bigoplus_{j=1}^{\ell} W_j^{\oplus c_{k,j}}$$

for every $k \geq 1$, where $c_{k,j}$ are nonnegative integers.

Any vector bundle satisfying the second condition is called a finite bundle [No1, p. 35, Definition]. This definition clearly makes sense for holomorphic vector bundles on compact complex manifolds. A holomorphic vector bundle on a compact complex manifold is finite if and only if it admits a flat holomorphic connection with finite monodromy [Bi].

Theorem 1.1 is also extended to holomorphic principal G-bundles over M, where G is a connected reductive complex affine algebraic group (see Lemma 4.2). An application of Lemma 4.2 is given in the context of holomorphic generalized Cartan geometries in the sense of [AM, BD] (see Proposition 5.1).

2. Preliminaries

Let M be a connected complex manifold. The holomorphic tangent and cotangent bundles on M will be denoted by TM and Ω^1_M respectively. The exterior product $\bigwedge^i \Omega^1_M$ will be denoted by Ω^i_M .

Let G be a connected complex Lie group. The Lie algebra of G will be denoted by \mathfrak{g} . Let

$$p: E \longrightarrow M \tag{2.1}$$

be a holomorphic principal G-bundle on M. Therefore, E is equipped with a holomorphic action of G on the right which is both free and transitive on the fibers of p. Consider the holomorphic right action of G on the holomorphic tangent bundle TE induced by the action of G on E. The quotient

$$At(E) := (TE)/G \tag{2.2}$$

is a holomorphic vector bundle over E/G = M; it is called the *Atiyah bundle* for E. The differential

$$dp: TE \longrightarrow p^*TM$$

of the projection p in (2.1) is G-equivariant for the trivial action of G on the fibers of p^*TM . The action of G on E produces a holomorphic homomorphism from the trivial holomorphic bundle

$$E \times \mathfrak{g} \longrightarrow \operatorname{kernel}(dp)$$

which is an isomorphism. Therefore, we have a short exact sequence of holomorphic vector bundles on ${\cal E}$

$$0 \longrightarrow \operatorname{kernel}(dp) = E \times \mathfrak{g} \longrightarrow TE \xrightarrow{dp} p^*TM \longrightarrow 0$$

$$(2.3)$$

in which all the homomorphisms are G-equivariant. The quotient kernel(dp)/G is the adjoint vector bundle ad $(E) = E(\mathfrak{g})$, which is the holomorphic vector bundle over M associated to E for the adjoint action of G on the Lie algebra \mathfrak{g} . Taking quotient of the bundles in (2.3), by the actions of G, the following short exact sequence of holomorphic vector bundles on M is obtained:

$$0 \longrightarrow \operatorname{ad}(E) \longrightarrow \operatorname{At}(E) \xrightarrow{d'p} TM \longrightarrow 0, \qquad (2.4)$$

where d'p is the descent of the homomorphism dp (see [At2]); this exact sequence is known as the *Atiyah exact sequence* for *E*.

A holomorphic connection on E is a holomorphic homomorphism of vector bundles

$$D: TM \longrightarrow \operatorname{At}(E)$$

such that

$$(d'p) \circ D = \mathrm{Id}_{TM}$$

where d'p is the projection in (2.4) (see [At2]).

We note that giving a holomorphic connection on E is equivalent to giving a \mathfrak{g} -valued holomorphic 1-form

$$\omega \in H^0(E, \,\Omega^1_E \otimes_{\mathbb{C}} \mathfrak{g}) \tag{2.5}$$

on E such that

- the homomorphism $\omega : TE \longrightarrow \mathfrak{g}$ is G-equivariant, and
- the restriction of ω to any fiber of p is the Maurer–Cartan form.

The connection homomorphism $D : TM \longrightarrow At(E)$ for the connection defined by ω is uniquely determined by the condition that the image of D corresponds to the kernel of ω .

The curvature of a holomorphic connection D is

$$\mathcal{K}(D) := D \circ D \in H^0(M, \operatorname{ad}(E) \otimes \Omega^2_M).$$

The connection D is called *flat* (or *integrable*) if $\mathcal{K}(D) = 0$. This is equivalent to the Frobenius integrability condition for the distribution on TE defined by the kernel of ω [Eh].

Fix a base point $x_0 \in M$ and also fix a point $z \in p^{-1}(x_0) \subset E$. Given a flat holomorphic connection D on E, by taking parallel translations of z, with respect to D, along loops based at x_0 we obtain the monodromy homomorphism

$$\rho(D,z) : \pi_1(M, x_0) \longrightarrow G.$$

If we replace z by zg, where $g \in G$, then $\rho(D, zg)(\gamma) = g^{-1}\rho(D, z)(\gamma)g$ for all $\gamma \in \pi_1(M, x_0)$. If the image of $\rho(D, z)$ is a finite group, the flat connection D is said to be having *finite monodromy*.

Let E be a holomorphic principal G-bundle over M and D a flat holomorphic connection on E such that the corresponding monodromy homomorphism

$$\rho(D,z) : \pi_1(M, x_0) \longrightarrow G$$

has finite image. Then the subgroup $\operatorname{kernel}(\rho(D, z)) \subset \pi_1(M, x_0)$ defines a finite étale Galois connected covering

$$f: \widetilde{M} \longrightarrow M$$
,

with Galois group $\operatorname{Gal}(f) = \operatorname{image}(\rho(D, z))$, such that

$$f^*E = \widetilde{M} \times E_{x_0} = \widetilde{M} \times G;$$

here G is identified with the fiber E_{x_0} of E using the map $g \longrightarrow zg, g \in G$. In other words, the holomorphic principal G-bundle f^*E is holomorphically trivial.

3. PULLBACK UNDER A SURJECTIVE MAP WITH CONNECTED FIBERS

Let X and M be compact connected complex manifolds and

$$\varphi: X \longrightarrow M \tag{3.1}$$

a surjective holomorphic map. Let G be a connected complex Lie subgroup of $GL(N, \mathbb{C})$ for some $N \ge 1$.

Proposition 3.1. Assume that for every point $x \in M$, the fiber $\varphi^{-1}(x) \subset X$ is connected. Let E be a holomorphic principal G-bundle over M such that φ^*E is holomorphically trivial. Then E is also holomorphically trivial.

Proof. Let

$$\sigma: X \longrightarrow \varphi^* E \tag{3.2}$$

be a holomorphic section of the principal G-bundle $\varphi^* E$ giving a holomorphic trivialization of it. For any point $y \in M$, the fiber $\varphi^{-1}(y)$ will be denoted by X_y . Consider the restriction $(\varphi^* E)|_{X_y}$ of the holomorphic principal G-bundle $\varphi^* E$ to $X_y \subset X$. Note that $(\varphi^* E)|_{X_y}$ is identified with the trivial principal G-bundle $X_y \times E_y$, where E_y is the fiber of E over the point $y \in M$. Using this identification between $(\varphi^* E)|_{X_y}$ and $X_y \times E_y$, the restriction $\sigma|_{X_y}$ of the section in (3.2) to X_y corresponds to a holomorphic map

$$\widehat{\sigma}_y : X_y \longrightarrow E_y. \tag{3.3}$$

We note that E_y is holomorphically isomorphic to G, and G is a complex Lie subgroup of $\operatorname{GL}(N,\mathbb{C})$. On the other hand, X_y is compact and connected, and hence it does not admit any nonconstant holomorphic function. Therefore, the function $\widehat{\sigma}_y$ in (3.3) is a constant map. Consequently, the map σ in (3.2) descends to a holomorphic section of E. In other words, there is a holomorphic section

$$\sigma' \,:\, M \,\longrightarrow\, E$$

such that $\varphi^* \sigma' = \sigma$. This section σ' produces a holomorphic trivialization of E.

It should be mentioned that Proposition 3.1 is not valid if the assumption — that G is a complex Lie subgroup of $\operatorname{GL}(N, \mathbb{C})$ for some $N \geq 1$ — is removed. To see this, let \mathbb{T} be a compact complex torus, and let

$$\phi \,:\, F_{\mathbb{T}} \,\longrightarrow\, M$$

be a nontrivial holomorphic principal \mathbb{T} -bundle; see [Ho] for nontrivial holomorphic torus bundles and their properties. Now, the fibers of ϕ are connected, and the principal \mathbb{T} -bundle $\phi^* E_{\mathbb{T}}$ has a tautological holomorphic trivialization.

4. Pullback of holomorphic principal bundles

4.1. Pullback of holomorphic vector bundles. As before X and M are compact complex manifolds, and φ , as in (3.1), is a surjective holomorphic map. We no longer assume that the fibers of φ are connected.

Theorem 4.1. Let V be a holomorphic vector bundle on M such that the holomorphic vector bundle φ^*V is trivial. Then V admits a flat holomorphic connection of finite monodromy.

Proof. Since φ is surjective, we have a natural inclusion of coherent analytic sheaves

$$\iota : \mathcal{O}_M \hookrightarrow \varphi_* \mathcal{O}_X. \tag{4.1}$$

We will show that $\iota(\mathcal{O}_M)$ is a direct summand of $\varphi_*\mathcal{O}_X$, meaning there is a coherent analytic sheaf S on M such that

$$\varphi_* \mathcal{O}_X = \iota(\mathcal{O}_M) \oplus \mathbb{S}. \tag{4.2}$$

To prove (4.2), let

$$X \stackrel{\beta}{\longrightarrow} Z \stackrel{\gamma}{\longrightarrow} M$$

be the Stein factorization of the map φ (see [GR, p. 213] for Stein factorization). We note that Z is a normal space because X is normal (see (2) of Stein factorization theorem in [GR, p. 213]). Since all the fibers of β are connected, we have

$$\gamma_* \mathcal{O}_Z = \varphi_* \mathcal{O}_X \,. \tag{4.3}$$

There is a Zariski open subset $\mathcal{U} \subset M$ such that

- the restriction $\gamma|_{\gamma^{-1}(\mathcal{U})} : \gamma^{-1}(\mathcal{U}) =: \widetilde{\mathcal{U}} \longrightarrow \mathcal{U}$ of γ to $\widetilde{\mathcal{U}}$ is a finite map, and
- the complex codimension of the complement $M \setminus \mathcal{U} \subset M$ is at least two.

Over \mathcal{U} , we have the trace map

$$\tau' : \gamma_* \mathcal{O}_{\widetilde{\mathcal{U}}} \longrightarrow \mathcal{O}_{\mathcal{U}}$$

$$(4.4)$$

such that the composition of homomorphisms

$$\mathcal{O}_{\mathcal{U}} \longrightarrow \gamma_* \mathcal{O}_{\widetilde{\mathcal{U}}} \xrightarrow{\tau'} \mathcal{O}_{\mathcal{U}},$$
 (4.5)

where $\mathcal{O}_{\mathcal{U}} \longrightarrow \gamma_* \mathcal{O}_{\widetilde{\mathcal{U}}}$ is the natural homomorphism as in (4.1), coincides with multiplication by the degree of the map γ .

Now, since \mathcal{O}_M is locally free, and the complex codimension of the complement $M \setminus \mathcal{U}$ is at least two, using Hartogs' extension theorem, the homomorphism τ' in (4.4) extends uniquely to a homomorphism

$$\tau : \gamma_* \mathcal{O}_Z \longrightarrow \mathcal{O}_M \,. \tag{4.6}$$

As the composition of homomorphisms in (4.5) is multiplication by the degree of γ , and any endomorphism of \mathcal{O}_M is multiplication by a constant function, it follows immediately that the composition of homomorphisms

$$\mathcal{O}_M \longrightarrow \gamma_* \mathcal{O}_Z \xrightarrow{\tau} \mathcal{O}_M,$$
 (4.7)

where $\mathcal{O}_M \longrightarrow \gamma_* \mathcal{O}_Z$ is the natural homomorphism as in (4.1), coincides with multiplication by the degree of the map γ .

Let

$$\mathbb{S} \subset \varphi_* \mathcal{O}_X$$

be the subsheaf that corresponds to $\operatorname{kernel}(\tau) \subset \gamma_* \mathcal{O}_Z$ (see (4.6)) by the isomorphism in (4.3). From the above observation, that the composition of homomorphisms in (4.7) coincides with multiplication by degree(γ), it follows immediately that the isomorphism in (4.2) holds.

Tensoring both sides of (4.2) by V we get

$$V \otimes \varphi_* \mathcal{O}_X = V \oplus (V \otimes \mathbb{S}).$$

$$(4.8)$$

On the other hand, by the projection formula,

$$V \otimes \varphi_* \mathcal{O}_X = \varphi_* \varphi^* V. \tag{4.9}$$

Since $\varphi^* V = \mathcal{O}_X^{\oplus r}$, where r is the rank of V, combining (4.8) and (4.9) we conclude that

$$V \oplus (V \otimes \mathbb{S}) = \varphi_* \mathcal{O}_X^{\oplus r} = (\varphi_* \mathcal{O}_X)^{\oplus r} .$$
(4.10)

In particular, V is a direct summand of $(\varphi_* \mathcal{O}_X)^{\oplus r}$.

We now recall a result of Atiyah in [At1]. Any torsionfree coherent analytic sheaf \mathcal{E} on M can be expressed as

$${\cal E}\,=\,igoplus_{i=1}^\ell {\cal E}_i\,,$$

where each \mathcal{E}_i , $1 \leq i \leq \ell$ is an indecomposable torsionfree coherent analytic sheaf, and \mathcal{E}_i , $1 \leq i \leq \ell$, are unique up to a permutation of $\{1, \dots, \ell\}$ [At1, p. 315, Theorem 2]. We shall apply this theorem of Atiyah to $\varphi_*\mathcal{O}_X$, and we shall separate, for our convenience, the direct summands which are locally free and those which are not locally free.

So $\varphi_* \mathcal{O}_X$ is expressed as

$$\varphi_* \mathcal{O}_X = \left(\bigoplus_{i=1}^m \mathcal{W}_i\right) \bigoplus \left(\bigoplus_{j=1}^n \mathcal{F}_j\right), \qquad (4.11)$$

where each \mathcal{W}_i , $1 \leq i \leq m$, is an indecomposable holomorphic vector bundle on M and each \mathcal{F}_j , $1 \leq j \leq n$, is an indecomposable torsionfree coherent analytic sheaf which is not locally free. In this case, the above theorem of Atiyah says that $\{\mathcal{W}_1, \dots, \mathcal{W}_m\}$ are unique up to a permutation of $\{1, \dots, m\}$, and $\{\mathcal{F}_1, \dots, \mathcal{F}_n\}$ are unique up to a permutation of $\{1, \dots, n\}$.

We noted above that V is a direct summand of $(\varphi_* \mathcal{O}_X)^{\oplus r}$. Using this and (4.11), from the above theorem of Atiyah it can be deduced that V is a direct sum of copies of $\{\mathcal{W}_1, \dots, \mathcal{W}_m\}$,

meaning

$$V = \bigoplus_{i=1}^m \mathcal{W}_i^{\oplus d_i}$$

where d_i are nonnegative integers; by convention, $\mathcal{W}_i^{\oplus 0} = 0$. To see this, first note that from (4.10) and (4.11),

$$V \oplus (V \otimes \mathbb{S}) = \left(\bigoplus_{i=1}^{m} \mathcal{W}_{i}\right)^{\oplus r} \bigoplus \left(\bigoplus_{j=1}^{n} \mathcal{F}_{j}\right)^{\oplus r}$$

Now expressing V as a direct sum of indecomposable holomorphic vector bundles, and $V \otimes \mathbb{S}$ as a direct sum of torsionfree indecomposable coherent analytic sheaves, we conclude from the uniqueness part of the above theorem of Atiyah that V is a direct summand of $\left(\bigoplus_{i=1}^{m} \mathcal{W}_{i}\right)^{\oplus r} = \bigoplus_{i=1}^{m} \mathcal{W}_{i}^{\oplus r}$.

Since φ^*V is a holomorphically trivial vector bundle, it follows that $\varphi^*V^{\otimes k} = (\varphi^*V)^{\otimes k}$ is also a holomorphically trivial vector bundle for every integer $k \geq 1$. Consequently, substituting $V^{\otimes k}$ for V in the above argument we conclude that $V^{\otimes k}$ is a direct sum of copies of $\{W_1, \dots, W_m\}$, for every $k \geq 1$.

We recall from [No1] and [No2] that a holomorphic vector bundle \mathcal{E} on M is called a *finite* bundle if there are finitely many holomorphic vector bundles B_1, \dots, B_ℓ on M such that for every integer $k \geq 1$,

$$\mathcal{E}^{\otimes k} = \bigoplus_{j=1}^{l} B_j^{\oplus c_{k,j}}$$

where $c_{k,j}$ are nonnegative integers ([No1, p. 35, Definition], [No1, p. 35, Lemma 3.1(d)]), [No2, p. 80, Definition], [Bi, (2.1)].

The vector bundle V is finite, because $V^{\otimes k}$ is a direct sum of copies of $\{\mathcal{W}_1, \dots, \mathcal{W}_m\}$, for every $k \geq 1$. Now Theorem 1.1 of [Bi] says that V admits a flat holomorphic connection with finite monodromy.

4.2. Reductive structure group. Let G be a connected reductive complex affine algebraic group. As in (2.1),

$$p : E \longrightarrow M$$

is a holomorphic principal G-bundle over a compact complex manifold M. Take (X, φ) as in Section 4.1.

Lemma 4.2. If the holomorphic principal G-bundle $\varphi^* E$ is holomorphically trivial, then E admits a flat holomorphic connection with finite monodromy.

Proof. Let

$$\rho : G \longrightarrow \operatorname{GL}(\mathbb{V})$$

be a faithful algebraic representation, where \mathbb{V} is a finite dimensional complex vector space. Let $E_{\mathbb{V}} := E(\mathbb{V})$ be the holomorphic principal $\operatorname{GL}(\mathbb{V})$ -bundle obtained by extending the structure group of E using ρ . Since the group G is reductive, the homomorphism of G-modules

$$\rho : \mathfrak{g} \longrightarrow \operatorname{End}(\mathbb{V}) = \operatorname{Lie}(\operatorname{GL}(\mathbb{V}))$$

splits [FH, p. 128, Theorem 9.19]. Fix such a splitting; let

$$\theta : \operatorname{End}(\mathbb{V}) \longrightarrow \mathfrak{g} \tag{4.12}$$

be the projection of G-modules corresponding to the chosen splitting.

Assume that $\varphi^* E$ is holomorphically trivial. Then the holomorphic principal $\operatorname{GL}(\mathbb{V})$ bundle $\varphi^* E_{\mathbb{V}}$ is also holomorphically trivial. Now Theorem 4.1 says that $E_{\mathbb{V}}$ admits a flat holomorphic connection D with finite monodromy. Let

$$\omega \in H^0(E_{\mathbb{V}}, \Omega^1_{E_{\mathbb{V}}} \otimes_{\mathbb{C}} \operatorname{End}(\mathbb{V}))$$
(4.13)

be a homomorphism as in (2.5) giving a flat holomorphic connection on $E_{\mathbb{V}}$ with finite monodromy.

Let

 $f \, : \, E \, \hookrightarrow \, E_{\mathbb{V}}$

be the natural inclusion map. Consider the \mathfrak{g} -valued holomorphic 1-form

$$\theta \circ f^* \omega \in H^0(E, \Omega^1_E \otimes_{\mathbb{C}} \mathfrak{g}),$$

where θ is the homomorphism in (4.12) and ω is the 1-form in (4.13). It is straightforward to check that $\theta \circ f^*\omega$ is a holomorphic connection on E. If $\mathcal{K}(\omega) \in H^0(M, \operatorname{ad}(E_{\mathbb{V}}) \otimes \Omega_M^2)$ is the curvature of the connection given by ω , then the curvature of the connection on E given by $\theta \circ f^*\omega$ is

$$\theta \circ \mathcal{K}(\omega) \in H^0(M, \operatorname{ad}(E) \otimes \Omega^2_M).$$

Therefore, the connection given by $\theta \circ f^* \omega$ is flat, because $\mathcal{K}(\omega) = 0$. The monodromy of the flat connection defined by $\theta \circ f^* \omega$ is finite, because the connection defined by ω has finite monodromy.

5. HOLOMORPHIC GENERALIZED CARTAN GEOMETRY

Let G be a connected complex Lie group and $H \subset G$ a closed connected complex Lie subgroup. Denote by \mathfrak{g} the Lie algebra of G.

There is a standard notion of Cartan geometry [Sh]. Roughly speaking a Cartan geometry with model (G, H) is infinitesimally modeled on the homogeneous space G/H. The Cartan geometry is flat (i.e., has vanishing curvature) if and only if it is locally isomorphic (not just infinitesimally) to the homogeneous space G/H in the sense of Ehresmann [Eh] (see also [Go]).

It is a very stringent condition for a compact complex manifold to admit a holomorphic Cartan geometry. A more flexible notion of *generalized holomorphic Cartan geometry* was introduced in [BD] (see also [AM]). Holomorphic generalized Cartan geometries are stable under pullback by holomorphic maps [BD].

A generalized holomorphic Cartan geometry of model (G, H) on a complex manifold M is given by a holomorphic principal H-bundle E_H over M endowed with a \mathfrak{g} -valued holomorphic one form on E_H such that the following two hold:

- (1) ω is *H*-equivariant with *H* acting on \mathfrak{g} via adjoint representation;
- (2) the restriction of ω to each fiber of E_H coincides with the Maurer-Cartan form associated to the action of H on E_H .

A generalized Cartan geometry is called flat if its curvature $\mathcal{K}(\omega) = d\omega + \frac{1}{2}[\omega, \omega]_{\mathfrak{g}}$ vanishes identically.

Notice that the standard definition of a Cartan geometry requires that ω realizes a pointwise linear isomorphism between TE_H and \mathfrak{g} (which implies that the complex dimension of M coincides with that of G/H).

Denote by E_G the holomorphic principal G-bundle constructed from E_H by extension of the structure group using the inclusion map $H \hookrightarrow G$. Denote by $\operatorname{ad}(E_H)$ and $\operatorname{ad}(E_G)$ the adjoint bundles of E_H and E_G respectively. Let $\operatorname{At}(E_H)$ be the Atiyah bundle for E_H (see (2.2)).

A generalized holomorphic Cartan geometry of model (G, H) is equivalently defined (see [BD]) by a homomorphism Ψ : At $(E_H) \longrightarrow \mathfrak{g}$ such that the following diagram commutes:

Notice that the top row of the diagram is the exact sequence in (2.4) corresponding to the principal bundle E_H and the bottom row is given by the canonical inclusion of $ad(E_H)$ in $ad(E_G)$. The homomorphism t in (5.1) is uniquely defined by Ψ and the commutativity of the diagram.

The homomorphism Ψ in (5.1) defines a canonical connection D_G on the principal Gbundle E_G and the generalized Cartan geometry is flat (i.e., $\mathcal{K}(\omega) = 0$) if and only if the connection D_G is flat (see Proposition 3.4 and Section 3.3 in [BD]). In this case the monodromy of (E_G, D_G) is called the monodromy of the generalized Cartan geometry.

Now let G be a connected reductive complex affine algebraic group and $H \subset G$ a connected closed algebraic subgroup of it. In the context of generalized Cartan geometries, Lemma 4.2 has the following consequence.

Proposition 5.1. Take M, X and φ as (3.1). Let (E_H, ω) be a holomorphic generalized Cartan geometry on M, with model (G, H), such that the pulled back generalized Cartan geometry $(\varphi^* E_H, \varphi^* \omega)$ on X is flat and has trivial monodromy homomorphism. Then Madmits a flat holomorphic generalized Cartan geometry (E_H, ω') , with model (G, H), whose monodromy homomorphism has finite image.

Proof. We apply Lemma 4.2 to the principal G-bundle E_G . Since the generalized Cartan geometry ($\varphi^* E_H$, $\varphi^* \omega$) is flat with trivial monodromy, it follows that $\varphi^* E_G$ is flat with trivial monodromy (note that $\varphi^* E_G$ coincides with the principal G-bundle associated to $\varphi^* E_H$)

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by extension of the structure group). Consequently, $\varphi^* E_G$ is holomorphically trivial. Now Lemma 4.2 implies that E_G admits a holomorphic flat connection D'_G with finite monodromy. Together with the reduction of the structure group $E_H \subset E_G$ to the subgroup H the connection D'_G defines a holomorphic generalized geometry (E_H, ω') on M [BD] (Theorem 3.7, point (3)). Since D'_G is flat, the curvature $\mathcal{K}(\omega')$ of ω' vanishes identically. The monodromy of (E_H, ω') coincides with that of (E_G, D'_G) , so it is finite.

Acknowledgements

This work has been supported by the French government through the UCAJEDI Investments in the Future project managed by the National Research Agency (ANR) with the reference number ANR2152IDEX201. The first-named author is partially supported by a J. C. Bose Fellowship, and school of mathematics, TIFR, is supported by 12-R&D-TFR-5.01-0500. The second-named author wishes to thank TIFR Mumbai for hospitality, while the first-named author thanks Université Côte d'Azur for hospitality.

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

Email address: indranil@math.tifr.res.in

UNIVERSITÉ CÔTE D'AZUR, CNRS, LJAD, FRANCE

Email address: dumitres@unice.fr