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HOLOMORPHIC BUNDLES TRIVIALIZABLE BY PROPER SURJECTIVE HOLOMORPHIC MAP

INDRANIL BISWAS AND SORIN DUMITRESCU

ABSTRACT. Given a compact complex manifold M , we investigate the holomorphic vector bundles E on M such that φ^*E is trivial for some surjective holomorphic map φ , to M , from some compact complex manifold. We prove that these are exactly those holomorphic vector bundles that admit a flat holomorphic connection with finite monodromy homomorphism. A similar result is proved for holomorphic principal G -bundles, where G is a connected reductive complex affine algebraic group.

1. INTRODUCTION

Let M be a compact complex manifold. Let E be a holomorphic vector bundle on M with the following property: there is a compact complex manifold X , and a surjective holomorphic map $\varphi : X \rightarrow M$, such that φ^*E is holomorphically trivial. To clarify, the dimension of X is allowed to be larger than that of M . Note that if the assumption that X is compact is removed, then every holomorphic vector bundle satisfies this condition. Indeed, the pullback of E to the total space of the frame bundle for E has a canonical holomorphic trivialization. However, this total space is never compact.

Let E be a holomorphic vector bundle on M admitting a flat connection D whose monodromy homomorphism

$$\rho : \pi_1(M, x_0) \rightarrow \mathrm{GL}(E_{x_0}),$$

where $x_0 \in M$ is a base point, has finite image. Consider the finite étale Galois covering $f : \widetilde{M} \rightarrow M$ corresponding to the finite index subgroup $\mathrm{kernel}(\rho) \subset \pi_1(M, x_0)$. It is easy to see that f^*E is holomorphically trivial. Therefore, E satisfies the condition stated at the beginning.

Our aim here is to prove a converse of it. More precisely, we prove the following (see Theorem 4.1):

Theorem 1.1. *Let E be a holomorphic vector bundle on a compact complex manifold M satisfying the condition that there is a compact complex manifold X , and a surjective map $\varphi : X \rightarrow M$, such that φ^*E is holomorphically trivial. Then E admits a flat holomorphic connection whose monodromy homomorphism has finite image.*

It is easy to see that a holomorphic vector bundle E on M admits a flat holomorphic connection with finite monodromy if and only if there is a finite étale Galois covering $f : \widetilde{M} \rightarrow M$ such that f^*E is holomorphically trivial (the “only if” part was explained above).

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In [No1], Nori characterized such vector bundles in the frame-work of algebraic geometry. When the base field is the field of complex numbers his result gives the following statement. For an algebraic vector bundle V on a complex projective variety Y , the following two conditions are equivalent:

- (1) There is a finite étale Galois covering $f : \tilde{Y} \rightarrow Y$ such that f^*V is algebraically trivial.
- (2) There are finitely many algebraic vector bundles W_1, \dots, W_ℓ on Y such that

$$V^{\otimes k} = \bigoplus_{j=1}^{\ell} W_j^{\oplus c_{k,j}}$$

for every $k \geq 1$, where $c_{k,j}$ are nonnegative integers.

Any vector bundle satisfying the second condition is called a finite bundle [No1, p. 35, Definition]. This definition clearly makes sense for holomorphic vector bundles on compact complex manifolds. A holomorphic vector bundle on a compact complex manifold is finite if and only if it admits a flat holomorphic connection with finite monodromy [Bi].

Theorem 1.1 is also extended to holomorphic principal G -bundles over M , where G is a connected reductive complex affine algebraic group (see Lemma 4.2). An application of Lemma 4.2 is given in the context of holomorphic generalized Cartan geometries in the sense of [AM, BD] (see Proposition 5.1).

2. PRELIMINARIES

Let M be a connected complex manifold. The holomorphic tangent and cotangent bundles on M will be denoted by TM and Ω_M^1 respectively. The exterior product $\bigwedge^i \Omega_M^1$ will be denoted by Ω_M^i .

Let G be a connected complex Lie group. The Lie algebra of G will be denoted by \mathfrak{g} . Let

$$p : E \rightarrow M \tag{2.1}$$

be a holomorphic principal G -bundle on M . Therefore, E is equipped with a holomorphic action of G on the right which is both free and transitive on the fibers of p . Consider the holomorphic right action of G on the holomorphic tangent bundle TE induced by the action of G on E . The quotient

$$\text{At}(E) := (TE)/G \tag{2.2}$$

is a holomorphic vector bundle over $E/G = M$; it is called the *Atiyah bundle* for E . The differential

$$dp : TE \rightarrow p^*TM$$

of the projection p in (2.1) is G -equivariant for the trivial action of G on the fibers of p^*TM . The action of G on E produces a holomorphic homomorphism from the trivial holomorphic bundle

$$E \times \mathfrak{g} \rightarrow \text{kernel}(dp)$$

which is an isomorphism. Therefore, we have a short exact sequence of holomorphic vector bundles on E

$$0 \longrightarrow \ker(dp) = E \times \mathfrak{g} \longrightarrow TE \xrightarrow{dp} p^*TM \longrightarrow 0 \quad (2.3)$$

in which all the homomorphisms are G -equivariant. The quotient $\ker(dp)/G$ is the adjoint vector bundle $\mathrm{ad}(E) = E(\mathfrak{g})$, which is the holomorphic vector bundle over M associated to E for the adjoint action of G on the Lie algebra \mathfrak{g} . Taking quotient of the bundles in (2.3), by the actions of G , the following short exact sequence of holomorphic vector bundles on M is obtained:

$$0 \longrightarrow \mathrm{ad}(E) \longrightarrow \mathrm{At}(E) \xrightarrow{d'p} TM \longrightarrow 0, \quad (2.4)$$

where $d'p$ is the descent of the homomorphism dp (see [At2]); this exact sequence is known as the *Atiyah exact sequence* for E .

A holomorphic connection on E is a holomorphic homomorphism of vector bundles

$$D : TM \longrightarrow \mathrm{At}(E)$$

such that

$$(d'p) \circ D = \mathrm{Id}_{TM},$$

where $d'p$ is the projection in (2.4) (see [At2]).

We note that giving a holomorphic connection on E is equivalent to giving a \mathfrak{g} -valued holomorphic 1-form

$$\omega \in H^0(E, \Omega_E^1 \otimes_{\mathbb{C}} \mathfrak{g}) \quad (2.5)$$

on E such that

- the homomorphism $\omega : TE \longrightarrow \mathfrak{g}$ is G -equivariant, and
- the restriction of ω to any fiber of p is the Maurer–Cartan form.

The connection homomorphism $D : TM \longrightarrow \mathrm{At}(E)$ for the connection defined by ω is uniquely determined by the condition that the image of D corresponds to the kernel of ω .

The curvature of a holomorphic connection D is

$$\mathcal{K}(D) := D \circ D \in H^0(M, \mathrm{ad}(E) \otimes \Omega_M^2).$$

The connection D is called *flat* (or *integrable*) if $\mathcal{K}(D) = 0$. This is equivalent to the Frobenius integrability condition for the distribution on TE defined by the kernel of ω [Eh].

Fix a base point $x_0 \in M$ and also fix a point $z \in p^{-1}(x_0) \subset E$. Given a flat holomorphic connection D on E , by taking parallel translations of z , with respect to D , along loops based at x_0 we obtain the monodromy homomorphism

$$\rho(D, z) : \pi_1(M, x_0) \longrightarrow G.$$

If we replace z by zg , where $g \in G$, then $\rho(D, zg)(\gamma) = g^{-1}\rho(D, z)(\gamma)g$ for all $\gamma \in \pi_1(M, x_0)$. If the image of $\rho(D, z)$ is a finite group, the flat connection D is said to be having *finite monodromy*.

Let E be a holomorphic principal G -bundle over M and D a flat holomorphic connection on E such that the corresponding monodromy homomorphism

$$\rho(D, z) : \pi_1(M, x_0) \longrightarrow G$$

has finite image. Then the subgroup $\text{kernel}(\rho(D, z)) \subset \pi_1(M, x_0)$ defines a finite étale Galois connected covering

$$f : \widetilde{M} \longrightarrow M,$$

with Galois group $\text{Gal}(f) = \text{image}(\rho(D, z))$, such that

$$f^*E = \widetilde{M} \times E_{x_0} = \widetilde{M} \times G;$$

here G is identified with the fiber E_{x_0} of E using the map $g \longrightarrow zg, g \in G$. In other words, the holomorphic principal G -bundle f^*E is holomorphically trivial.

3. PULLBACK UNDER A SURJECTIVE MAP WITH CONNECTED FIBERS

Let X and M be compact connected complex manifolds and

$$\varphi : X \longrightarrow M \tag{3.1}$$

a surjective holomorphic map. Let G be a connected complex Lie subgroup of $\text{GL}(N, \mathbb{C})$ for some $N \geq 1$.

Proposition 3.1. *Assume that for every point $x \in M$, the fiber $\varphi^{-1}(x) \subset X$ is connected. Let E be a holomorphic principal G -bundle over M such that φ^*E is holomorphically trivial. Then E is also holomorphically trivial.*

Proof. Let

$$\sigma : X \longrightarrow \varphi^*E \tag{3.2}$$

be a holomorphic section of the principal G -bundle φ^*E giving a holomorphic trivialization of it. For any point $y \in M$, the fiber $\varphi^{-1}(y)$ will be denoted by X_y . Consider the restriction $(\varphi^*E)|_{X_y}$ of the holomorphic principal G -bundle φ^*E to $X_y \subset X$. Note that $(\varphi^*E)|_{X_y}$ is identified with the trivial principal G -bundle $X_y \times E_y$, where E_y is the fiber of E over the point $y \in M$. Using this identification between $(\varphi^*E)|_{X_y}$ and $X_y \times E_y$, the restriction $\sigma|_{X_y}$ of the section in (3.2) to X_y corresponds to a holomorphic map

$$\widehat{\sigma}_y : X_y \longrightarrow E_y. \tag{3.3}$$

We note that E_y is holomorphically isomorphic to G , and G is a complex Lie subgroup of $\text{GL}(N, \mathbb{C})$. On the other hand, X_y is compact and connected, and hence it does not admit any nonconstant holomorphic function. Therefore, the function $\widehat{\sigma}_y$ in (3.3) is a constant map. Consequently, the map σ in (3.2) descends to a holomorphic section of E . In other words, there is a holomorphic section

$$\sigma' : M \longrightarrow E$$

such that $\varphi^*\sigma' = \sigma$. This section σ' produces a holomorphic trivialization of E . \square

It should be mentioned that Proposition 3.1 is not valid if the assumption — that G is a complex Lie subgroup of $\text{GL}(N, \mathbb{C})$ for some $N \geq 1$ — is removed. To see this, let \mathbb{T} be a compact complex torus, and let

$$\phi : F_{\mathbb{T}} \longrightarrow M$$

be a nontrivial holomorphic principal \mathbb{T} -bundle; see [Ho] for nontrivial holomorphic torus bundles and their properties. Now, the fibers of ϕ are connected, and the principal \mathbb{T} -bundle $\phi^*E_{\mathbb{T}}$ has a tautological holomorphic trivialization.

4. PULLBACK OF HOLOMORPHIC PRINCIPAL BUNDLES

4.1. Pullback of holomorphic vector bundles. As before X and M are compact complex manifolds, and φ , as in (3.1), is a surjective holomorphic map. We no longer assume that the fibers of φ are connected.

Theorem 4.1. *Let V be a holomorphic vector bundle on M such that the holomorphic vector bundle φ^*V is trivial. Then V admits a flat holomorphic connection of finite monodromy.*

Proof. Since φ is surjective, we have a natural inclusion of coherent analytic sheaves

$$\iota : \mathcal{O}_M \hookrightarrow \varphi_*\mathcal{O}_X. \quad (4.1)$$

We will show that $\iota(\mathcal{O}_M)$ is a direct summand of $\varphi_*\mathcal{O}_X$, meaning there is a coherent analytic sheaf \mathbb{S} on M such that

$$\varphi_*\mathcal{O}_X = \iota(\mathcal{O}_M) \oplus \mathbb{S}. \quad (4.2)$$

To prove (4.2), let

$$X \xrightarrow{\beta} Z \xrightarrow{\gamma} M$$

be the Stein factorization of the map φ (see [GR, p. 213] for Stein factorization). We note that Z is a normal space because X is normal (see (2) of Stein factorization theorem in [GR, p. 213]). Since all the fibers of β are connected, we have

$$\gamma_*\mathcal{O}_Z = \varphi_*\mathcal{O}_X. \quad (4.3)$$

There is a Zariski open subset $\mathcal{U} \subset M$ such that

- the restriction $\gamma|_{\gamma^{-1}(\mathcal{U})} : \gamma^{-1}(\mathcal{U}) =: \tilde{\mathcal{U}} \rightarrow \mathcal{U}$ of γ to $\tilde{\mathcal{U}}$ is a finite map, and
- the complex codimension of the complement $M \setminus \mathcal{U} \subset M$ is at least two.

Over \mathcal{U} , we have the trace map

$$\tau' : \gamma_*\mathcal{O}_{\tilde{\mathcal{U}}} \rightarrow \mathcal{O}_{\mathcal{U}} \quad (4.4)$$

such that the composition of homomorphisms

$$\mathcal{O}_{\mathcal{U}} \rightarrow \gamma_*\mathcal{O}_{\tilde{\mathcal{U}}} \xrightarrow{\tau'} \mathcal{O}_{\mathcal{U}}, \quad (4.5)$$

where $\mathcal{O}_{\mathcal{U}} \rightarrow \gamma_*\mathcal{O}_{\tilde{\mathcal{U}}}$ is the natural homomorphism as in (4.1), coincides with multiplication by the degree of the map γ .

Now, since \mathcal{O}_M is locally free, and the complex codimension of the complement $M \setminus \mathcal{U}$ is at least two, using Hartogs' extension theorem, the homomorphism τ' in (4.4) extends uniquely to a homomorphism

$$\tau : \gamma_*\mathcal{O}_Z \rightarrow \mathcal{O}_M. \quad (4.6)$$

As the composition of homomorphisms in (4.5) is multiplication by the degree of γ , and any endomorphism of \mathcal{O}_M is multiplication by a constant function, it follows immediately that the composition of homomorphisms

$$\mathcal{O}_M \longrightarrow \gamma_*\mathcal{O}_Z \xrightarrow{\tau} \mathcal{O}_M, \quad (4.7)$$

where $\mathcal{O}_M \longrightarrow \gamma_*\mathcal{O}_Z$ is the natural homomorphism as in (4.1), coincides with multiplication by the degree of the map γ .

Let

$$\mathbb{S} \subset \varphi_*\mathcal{O}_X$$

be the subsheaf that corresponds to $\text{kernel}(\tau) \subset \gamma_*\mathcal{O}_Z$ (see (4.6)) by the isomorphism in (4.3). From the above observation, that the composition of homomorphisms in (4.7) coincides with multiplication by $\text{degree}(\gamma)$, it follows immediately that the isomorphism in (4.2) holds.

Tensoring both sides of (4.2) by V we get

$$V \otimes \varphi_*\mathcal{O}_X = V \oplus (V \otimes \mathbb{S}). \quad (4.8)$$

On the other hand, by the projection formula,

$$V \otimes \varphi_*\mathcal{O}_X = \varphi_*\varphi^*V. \quad (4.9)$$

Since $\varphi^*V = \mathcal{O}_X^{\oplus r}$, where r is the rank of V , combining (4.8) and (4.9) we conclude that

$$V \oplus (V \otimes \mathbb{S}) = \varphi_*\mathcal{O}_X^{\oplus r} = (\varphi_*\mathcal{O}_X)^{\oplus r}. \quad (4.10)$$

In particular, V is a direct summand of $(\varphi_*\mathcal{O}_X)^{\oplus r}$.

We now recall a result of Atiyah in [At1]. Any torsionfree coherent analytic sheaf \mathcal{E} on M can be expressed as

$$\mathcal{E} = \bigoplus_{i=1}^{\ell} \mathcal{E}_i,$$

where each \mathcal{E}_i , $1 \leq i \leq \ell$ is an indecomposable torsionfree coherent analytic sheaf, and \mathcal{E}_i , $1 \leq i \leq \ell$, are unique up to a permutation of $\{1, \dots, \ell\}$ [At1, p. 315, Theorem 2]. We shall apply this theorem of Atiyah to $\varphi_*\mathcal{O}_X$, and we shall separate, for our convenience, the direct summands which are locally free and those which are not locally free.

So $\varphi_*\mathcal{O}_X$ is expressed as

$$\varphi_*\mathcal{O}_X = \left(\bigoplus_{i=1}^m \mathcal{W}_i \right) \oplus \left(\bigoplus_{j=1}^n \mathcal{F}_j \right), \quad (4.11)$$

where each \mathcal{W}_i , $1 \leq i \leq m$, is an indecomposable holomorphic vector bundle on M and each \mathcal{F}_j , $1 \leq j \leq n$, is an indecomposable torsionfree coherent analytic sheaf which is not locally free. In this case, the above theorem of Atiyah says that $\{\mathcal{W}_1, \dots, \mathcal{W}_m\}$ are unique up to a permutation of $\{1, \dots, m\}$, and $\{\mathcal{F}_1, \dots, \mathcal{F}_n\}$ are unique up to a permutation of $\{1, \dots, n\}$.

We noted above that V is a direct summand of $(\varphi_*\mathcal{O}_X)^{\oplus r}$. Using this and (4.11), from the above theorem of Atiyah it can be deduced that V is a direct sum of copies of $\{\mathcal{W}_1, \dots, \mathcal{W}_m\}$,

meaning

$$V = \bigoplus_{i=1}^m \mathcal{W}_i^{\oplus d_i},$$

where d_i are nonnegative integers; by convention, $\mathcal{W}_i^{\oplus 0} = 0$. To see this, first note that from (4.10) and (4.11),

$$V \oplus (V \otimes \mathbb{S}) = \left(\bigoplus_{i=1}^m \mathcal{W}_i \right)^{\oplus r} \oplus \left(\bigoplus_{j=1}^n \mathcal{F}_j \right)^{\oplus r}.$$

Now expressing V as a direct sum of indecomposable holomorphic vector bundles, and $V \otimes \mathbb{S}$ as a direct sum of torsionfree indecomposable coherent analytic sheaves, we conclude from the uniqueness part of the above theorem of Atiyah that V is a direct summand of $(\bigoplus_{i=1}^m \mathcal{W}_i)^{\oplus r} = \bigoplus_{i=1}^m \mathcal{W}_i^{\oplus r}$.

Since φ^*V is a holomorphically trivial vector bundle, it follows that $\varphi^*V^{\otimes k} = (\varphi^*V)^{\otimes k}$ is also a holomorphically trivial vector bundle for every integer $k \geq 1$. Consequently, substituting $V^{\otimes k}$ for V in the above argument we conclude that $V^{\otimes k}$ is a direct sum of copies of $\{\mathcal{W}_1, \dots, \mathcal{W}_m\}$, for every $k \geq 1$.

We recall from [No1] and [No2] that a holomorphic vector bundle \mathcal{E} on M is called a *finite bundle* if there are finitely many holomorphic vector bundles B_1, \dots, B_ℓ on M such that for every integer $k \geq 1$,

$$\mathcal{E}^{\otimes k} = \bigoplus_{j=1}^{\ell} B_j^{\oplus c_{k,j}},$$

where $c_{k,j}$ are nonnegative integers ([No1, p. 35, Definition], [No1, p. 35, Lemma 3.1(d)], [No2, p. 80, Definition], [Bi, (2.1)]).

The vector bundle V is finite, because $V^{\otimes k}$ is a direct sum of copies of $\{\mathcal{W}_1, \dots, \mathcal{W}_m\}$, for every $k \geq 1$. Now Theorem 1.1 of [Bi] says that V admits a flat holomorphic connection with finite monodromy. \square

4.2. Reductive structure group. Let G be a connected reductive complex affine algebraic group. As in (2.1),

$$p : E \longrightarrow M$$

is a holomorphic principal G -bundle over a compact complex manifold M . Take (X, φ) as in Section 4.1.

Lemma 4.2. *If the holomorphic principal G -bundle φ^*E is holomorphically trivial, then E admits a flat holomorphic connection with finite monodromy.*

Proof. Let

$$\rho : G \longrightarrow \mathrm{GL}(\mathbb{V})$$

be a faithful algebraic representation, where \mathbb{V} is a finite dimensional complex vector space. Let $E_{\mathbb{V}} := E(\mathbb{V})$ be the holomorphic principal $\mathrm{GL}(\mathbb{V})$ -bundle obtained by extending the structure group of E using ρ .

Since the group G is reductive, the homomorphism of G -modules

$$\rho : \mathfrak{g} \longrightarrow \text{End}(\mathbb{V}) = \text{Lie}(\text{GL}(\mathbb{V}))$$

splits [FH, p. 128, Theorem 9.19]. Fix such a splitting; let

$$\theta : \text{End}(\mathbb{V}) \longrightarrow \mathfrak{g} \tag{4.12}$$

be the projection of G -modules corresponding to the chosen splitting.

Assume that φ^*E is holomorphically trivial. Then the holomorphic principal $\text{GL}(\mathbb{V})$ -bundle $\varphi^*E_{\mathbb{V}}$ is also holomorphically trivial. Now Theorem 4.1 says that $E_{\mathbb{V}}$ admits a flat holomorphic connection D with finite monodromy. Let

$$\omega \in H^0(E_{\mathbb{V}}, \Omega_{E_{\mathbb{V}}}^1 \otimes_{\mathbb{C}} \text{End}(\mathbb{V})) \tag{4.13}$$

be a homomorphism as in (2.5) giving a flat holomorphic connection on $E_{\mathbb{V}}$ with finite monodromy.

Let

$$f : E \hookrightarrow E_{\mathbb{V}}$$

be the natural inclusion map. Consider the \mathfrak{g} -valued holomorphic 1-form

$$\theta \circ f^*\omega \in H^0(E, \Omega_E^1 \otimes_{\mathbb{C}} \mathfrak{g}),$$

where θ is the homomorphism in (4.12) and ω is the 1-form in (4.13). It is straightforward to check that $\theta \circ f^*\omega$ is a holomorphic connection on E . If $\mathcal{K}(\omega) \in H^0(M, \text{ad}(E_{\mathbb{V}}) \otimes \Omega_M^2)$ is the curvature of the connection given by ω , then the curvature of the connection on E given by $\theta \circ f^*\omega$ is

$$\theta \circ \mathcal{K}(\omega) \in H^0(M, \text{ad}(E) \otimes \Omega_M^2).$$

Therefore, the connection given by $\theta \circ f^*\omega$ is flat, because $\mathcal{K}(\omega) = 0$. The monodromy of the flat connection defined by $\theta \circ f^*\omega$ is finite, because the connection defined by ω has finite monodromy. \square

5. HOLOMORPHIC GENERALIZED CARTAN GEOMETRY

Let G be a connected complex Lie group and $H \subset G$ a closed connected complex Lie subgroup. Denote by \mathfrak{g} the Lie algebra of G .

There is a standard notion of Cartan geometry [Sh]. Roughly speaking a Cartan geometry with model (G, H) is infinitesimally modeled on the homogeneous space G/H . The Cartan geometry is flat (i.e., has vanishing curvature) if and only if it is locally isomorphic (not just infinitesimally) to the homogeneous space G/H in the sense of Ehresmann [Eh] (see also [Go]).

It is a very stringent condition for a compact complex manifold to admit a holomorphic Cartan geometry. A more flexible notion of *generalized holomorphic Cartan geometry* was introduced in [BD] (see also [AM]). Holomorphic generalized Cartan geometries are stable under pullback by holomorphic maps [BD].

A generalized holomorphic Cartan geometry of model (G, H) on a complex manifold M is given by a holomorphic principal H -bundle E_H over M endowed with a \mathfrak{g} -valued holomorphic one form on E_H such that the following two hold:

- (1) ω is H -equivariant with H acting on \mathfrak{g} via adjoint representation;
- (2) the restriction of ω to each fiber of E_H coincides with the Maurer–Cartan form associated to the action of H on E_H .

A generalized Cartan geometry is called flat if its curvature $\mathcal{K}(\omega) = d\omega + \frac{1}{2}[\omega, \omega]_{\mathfrak{g}}$ vanishes identically.

Notice that the standard definition of a Cartan geometry requires that ω realizes a point-wise linear isomorphism between TE_H and \mathfrak{g} (which implies that the complex dimension of M coincides with that of G/H).

Denote by E_G the holomorphic principal G -bundle constructed from E_H by extension of the structure group using the inclusion map $H \hookrightarrow G$. Denote by $\text{ad}(E_H)$ and $\text{ad}(E_G)$ the adjoint bundles of E_H and E_G respectively. Let $\text{At}(E_H)$ be the Atiyah bundle for E_H (see (2.2)).

A generalized holomorphic Cartan geometry of model (G, H) is equivalently defined (see [BD]) by a homomorphism $\Psi : \text{At}(E_H) \rightarrow \mathfrak{g}$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{ad}(E_H) & \longrightarrow & \text{At}(E_H) & \longrightarrow & TM & \longrightarrow & 0 \\
 & & \parallel & & \downarrow \Psi & & \downarrow t & & \\
 0 & \longrightarrow & \text{ad}(E_H) & \longrightarrow & \text{ad}(E_G) & \longrightarrow & \text{ad}(E_G)/\text{ad}(E_H) & \longrightarrow & 0
 \end{array} \tag{5.1}$$

Notice that the top row of the diagram is the exact sequence in (2.4) corresponding to the principal bundle E_H and the bottom row is given by the canonical inclusion of $\text{ad}(E_H)$ in $\text{ad}(E_G)$. The homomorphism t in (5.1) is uniquely defined by Ψ and the commutativity of the diagram.

The homomorphism Ψ in (5.1) defines a canonical connection D_G on the principal G -bundle E_G and the generalized Cartan geometry is flat (i.e., $\mathcal{K}(\omega) = 0$) if and only if the connection D_G is flat (see Proposition 3.4 and Section 3.3 in [BD]). In this case the monodromy of (E_G, D_G) is called the monodromy of the generalized Cartan geometry.

Now let G be a connected reductive complex affine algebraic group and $H \subset G$ a connected closed algebraic subgroup of it. In the context of generalized Cartan geometries, Lemma 4.2 has the following consequence.

Proposition 5.1. *Take M, X and φ as (3.1). Let (E_H, ω) be a holomorphic generalized Cartan geometry on M , with model (G, H) , such that the pulled back generalized Cartan geometry $(\varphi^*E_H, \varphi^*\omega)$ on X is flat and has trivial monodromy homomorphism. Then M admits a flat holomorphic generalized Cartan geometry (E_H, ω') , with model (G, H) , whose monodromy homomorphism has finite image.*

Proof. We apply Lemma 4.2 to the principal G -bundle E_G . Since the generalized Cartan geometry $(\varphi^*E_H, \varphi^*\omega)$ is flat with trivial monodromy, it follows that φ^*E_G is flat with trivial monodromy (note that φ^*E_G coincides with the principal G -bundle associated to φ^*E_H

by extension of the structure group). Consequently, φ^*E_G is holomorphically trivial. Now Lemma 4.2 implies that E_G admits a holomorphic flat connection D'_G with finite monodromy. Together with the reduction of the structure group $E_H \subset E_G$ to the subgroup H the connection D'_G defines a holomorphic generalized geometry (E_H, ω') on M [BD] (Theorem 3.7, point (3)). Since D'_G is flat, the curvature $\mathcal{K}(\omega')$ of ω' vanishes identically. The monodromy of (E_H, ω') coincides with that of (E_G, D'_G) , so it is finite. \square

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REFERENCES

- [AM] D. V. Alekseevsky and P. W. Michor, Differential geometry of Cartan connections, *Publ. Math. Debrecen*, **47** (1995), 349–375.
- [At1] M. F. Atiyah, On the Krull-Schmidt theorem with application to sheaves, *Bull. Soc. Math. Fr.* **84** (1956), 307–317.
- [At2] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* **85** (1957), 181–207.
- [Bi] I. Biswas, Étale triviality of finite vector bundles over compact complex manifolds, *Adv. Math.* (in press), arXiv:2004.04089v1.
- [BD] I. Biswas and S. Dumitrescu, Generalized holomorphic Cartan geometries, arxiv.org/abs/1902.06652, to be published in *European Journ. Math.*, DOI: 10.1007/s40879-019-00327-6
- [Eh] C. Ehresmann, Sur les espaces localement homogènes, *L'Enseign. Math.* **35** (1936), 317–333.
- [FH] W. Fulton and J. Harris, *Representation theory. A first course*, Graduate Texts in Mathematics, 129, Readings in Mathematics. Springer-Verlag, New York, 1991.
- [Go] W. Goldman, Locally homogeneous geometric manifolds, *Proceedings of the International Congress of Mathematicians* Volume II, 717–744, Hindustan Book Agency, New Delhi, 2010.
- [GR] H. Grauert and R. Remmert, *Coherent analytic sheaves*, Grundlehren der Mathematischen Wissenschaften, 265. Springer-Verlag, Berlin, 1984.
- [Ho] T. Höfer, Remarks on torus principal bundles, *Jour. Math. Kyoto Univ.* **33** (1993), 227–259.
- [No1] M. V. Nori, On the representations of the fundamental group scheme, *Compos. Math.* **33** (1976), 29–41.
- [No2] M. V. Nori, The fundamental group–scheme, *Proc. Ind. Acad. Sci. (Math. Sci.)* **91** (1982), 73–122.
- [Sh] R. W. Sharpe, *Differential geometry. Cartan's generalization of Klein's Erlangen program*, Graduate Texts in Mathematics, 166, Springer-Verlag, New York, 1997.

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