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A hierarchy of dismantlings in graphs

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Abstract

Given a finite undirected graph X, a vertex is 0-dismantlable if its open neighbourhood is a cone and X is 0-dismantlable if it is reducible to a single vertex by successive deletions of 0-dismantlable vertices. By an iterative process, a vertex is $(k+1)$-dismantlable if its open neighbourhood is k-dismantlable and a graph is k-dismantlable if it is reducible to a single vertex by successive deletions of k-dismantlable vertices. We introduce a graph family, the cubion graphs, in order to prove that k-dismantlabilities give a strict hierarchy in the class of graphs whose clique complex is non-evasive. We point out how these higher dismantlabilities are related to the derivability of graphs defined by Mazurkievicz and we get a new characterization of the class of closed graphs he defined. By generalising the notion of vertex transitivity, we consider the issue of higher dismantlabilities in link with the evasiveness conjecture.

Keywords: dismantlability, flag complexes, collapses, evasiveness, graph derivability.

1 Introduction

The transition from a graph to its clique complex is one of the many ways for associating a simplicial complex to a graph. Through the notion of dismantlability, it is possible to develop homotopic notions adapted to the framework of finite graphs. In this paper we will only discuss the dismantlability of vertices. The principle of dismantlability in graphs is to set a rule that indicates the possibility of adding or removing vertices and two graphs are in the same homotopy class if one can switch from one to the other by a succession of moves (a move being either a vertex addition, or a vertex deletion).

The 0-dismantlability is well known: a vertex x is 0-dismantlable if its open neighbourhood is a cone. This means there is a vertex y adjacent to x such that any neighbour of x is also a neighbour of y (we say that x is dominated by y) and we know that a graph is 0-dismantlable if, and only if, it is cop-win [22, 21]. From a simplicial point of view, the 0-dismantlability of a graph is equivalent to the strong-collapsibility of its clique complex [11]. *Strong collapsibility* is introduced by Barmak and Minian [4] who proved that the strong homotopy type of a simplicial complex can be described in terms of contiguity classes. Assuming that a vertex of a graph is 1-dismantlable if its neighbourhood is 0-dismantlable, we obtain 1-dismantlability for graphs and it is established in [6] that two graphs X and Y have the same 1-homotopy type if, and only if, their clique complexes $\mathtt{cl}(X)$ and $\mathtt{cl}(Y)$ have the same simple homotopy type.

The k-dismantlabilities for $k \geqslant 2$ reproduce this recursive scheme to define increasingly large classes of graphs to which this paper is dedicated. In section 2, the main definitions concerning graphs and simplicial complexes are recalled with the fact (Proposition 4) that the k-dismantlability of a graph X is equivalent to the k-collapsibility of its clique complex $\mathtt{cl}(X)$. While the notions of 0-homotopy and 1-homotopy are very different, it should be noted that the contribution of the higher dismantlabilities is not so much at the homotopy level (Proposition 5) as at the level of dismantlability classes D_k, where D_k is the class of all k-dismantlable graphs. Section 3 is devoted to the presentation of a family of graphs $(\mathfrak{Q}_n)_{n\in\mathbb{N}}$ (called *cubion graphs*) which shows that $(D_k)_{k\in\mathbb{N}}$ is an increasing sequence of strict inclusions (Proposition 8):

$$\forall\, n \geqslant 2, \quad \mathfrak{Q}_n \in D_{n-1} \backslash D_{n-2}\,.$$

We also prove that the existence of a $(k+1)$-dismantlable and non k-dismantlable vertex implies the presence of a clique of cardinal at least $k+3$ (Proposition 11). In Section 4, the introduction of the

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parasol graph shows the very importance of the order in which vertex dismantlings are operated as soon as one leaves the class of 0-dismantlable graphs (Proposition 14). Setting $D_\infty = \bigcup_{k\geqslant 0} D_k$ and considering the 1-skeletons of triangulations of the Dunce Hat and the Bing's House, we explore the question of graphs not in D_∞ but for which it is sufficient to add some 0-dismantlable vertices to get into D_∞. We note that D_∞ is the smallest fixed point of the derivability operator $\triangle$ of Mazurkiewicz [19]. The set of k-collapsible simplicial complexes with varying values of k in $\mathbb{N}$ is the set of non-evasive complexes [4]. Therefore, the elements of D_∞ will be called *non-evasive*. So, the question of whether a k-dismantlable and vertex-transitive graph is necessarily a complete graph is a particular case of the evasiveness conjecture for simplicial complexes, according to which *every vertex homogeneous and non-evasive simplicial complex is a simplex*. In the final section, we introduce the notion of *i-complete-transitive graph* to establish a particular case for which the conjecture is valid.

The study of simplicial complexes appears today in a very wide spectrum of research and applications [8, 14, 27]. Very often, these complexes are constructed from finite data to obtain information on their structure, for instance by the calculus of Betti numbers or homotopy groups. It should be mentioned that the notion of clique complexes (also called *flag complexes* [17]) seems rather general from the homotopic point of view since the barycentric subdivision of any complex is a flag complex (and the 1-homotopy type of a complex and of its barycentric subdivision are the same [6]). From this point of view, the notion of higher dismantlabilities is a contribution to the study of homotopic invariants for simplicial complexes associated to finite data. From another point of view, the notion of higher dismantlabilities extends the list of graph families built by adding or removing nodes with the condition that the neighbourhoods of these nodes check certain properties. The first example is probably the family of finite chordal graphs which is exactly the family of graphs constructed by adding simplicial vertices (i.e. whose neighbourhoods are complete graphs) from the point. They can also be characterized as graphs that can be reduced to a point by a succession of simplicial vertex deletions [10]. Bridged graphs [1] and cop-win graphs [22, 21] are two other examples of graph families that can be iteratively constructed respecting a condition on the neighbourhood of the node added at each step. From the perspective of Topological Data Analysis (TDA), it is worthwhile to identify to what extent a topological structure depends on local constraints. In a complex network for instance, the global topological structure can sometimes be highly explained by local interaction configurations. When they verify certain properties, these local constraints generate a global structure that deviates from classical null models and can thus explain particular global phenomena. Understanding these multi-scale links between local and global structures is now becoming a key element in the modelling of complex networks. Perhaps the best known model is Barabasi's preferential attachment [2] where the attachment of a new node to the network is done preferentially by the nodes of higher degrees. Other examples are hierarchical models obtained for example by a local attachment of each node to a subset of nodes of a maximal clique [23]. These local-global concerns are in line with older issues, but still up-to-date, raised in the context of local computation [25, 12, 18]. So, from an application point of view, the notion of higher dismantlablities could enrich the range of tools available in all these fields.

2 Notations and first definitions

2.1 Graphs

In the following, $X = (V(X), E(X))$ is a finite undirected graph, without multiple edges or loops. The cardinal $|V(X)|$ is equal to the number of vertices of X, at least equal to 1. We denote by $\mathbb{N}$ (resp. $\mathbb{N}^\star$) the set of integers $\{0, 1, 2, \cdots\}$ (resp. $\{1, 2, \cdots\}$).

We write $x \sim y$, or sometimes just xy, for $\{x, y\} \in E(X)$ and $x \in X$ to indicate that $x \in V(X)$. The closed neighbourhood of x is $N_X[x] = \{y \in X\ ,\ x \sim y\} \cup \{x\}$ and $N_X(x) = N_X[x]\backslash\{x\}$ is its open neighbourhood. When no confusion is possible, $N_X[x]$ will also denote the subgraph induced by $N_X[x]$ in X. Let $S = \{x_1, \cdots, x_n\}$ be a subset of $V(X)$, we denote by $X[S]$ or $X[x_1, \cdots, x_n]$ the subgraph induced by S in X. The particular case where $S = V(X)\backslash\{x\}$ will be denoted by $X - x$. In the same way, the notation $X + y$ means that we have added a new vertex y to the graph X and the context must make clear the neighbourhood of y in $X + y$. A *clique* of a graph X is a complete subgraph of X. A *maximal clique* K of X is a clique so that there is no vertex in $V(X)\backslash V(K)$ adjacent to each vertex of K. For $n \geqslant 1$, the complete graph (resp. cycle) with n vertices is denoted by K_n (resp. C_n). The graph K_1 with one vertex will be called *point* and sometimes noted *pt*. The complement $\overline{X}$ of a graph

X has the same vertices as X and two distinct vertices of $\overline{X}$ are adjacent if and only if they are not adjacent in X.

The existence of an isomorphism between two graphs is denoted by $X \cong Y$. We say that a graph X is a *cone* with apex x if $N_X[x] = X$. A vertex a *dominates* a vertex $x \neq a$ in X if $N_X[x] \subset N_X[a]$ and we note $x \vdash a$. Note that a vertex is dominated if, and only if, its open neighbourhood is a cone. Two distinct vertices x and y are *twins* if $N_X[x] = N_X[y]$. We will denote by $\mathrm{Twins}(X)$ the set of *twin vertices* of X.

In a finite undirected graph X, a vertex is 0-dismantlable if it is dominated and X is 0-dismantlable if it exists an order $x_1, \cdots, x_n$ of the vertices of X such that x_k is 0-dismantlable in $X[x_k, x_{k+1}, \cdots, x_n]$ for $1 \leqslant k \leqslant n-1$. In [6], we have defined a weaker version of dismantlability. A vertex x of X is 1-dismantlable[3] if its open neighbourhood $N_X(x)$ is a 0-dismantlable graph.

Generalising the passage from 0-dismantlability to $1-$dismantlability, the higher dismantlabilities in graphs are defined iteratively by:

Definition 1

• *The family C of cones (or conical graphs) is also denoted by D_{-1} and we will say that the cones are the graphs which are (-1)-dismantlable.*

• *For any integer $k \geqslant 0$, a vertex of a graph X is called k-dismantlable if its open neighbourhood is $(k-1)$-dismantlable. The graph X is k-dismantlable if it is reducible to a vertex by successive deletions of k-dismantlable vertices. We denote by $D_k(X)$ the set of k-dismantlable vertices of a graph X and by D_k the set of $k-$dismantlable graphs.*

A cone is a 0-dismantlable graph, that is $D_{-1} \subset D_0$, and by induction on k, we immediately get:

Proposition 2 $\forall k \in \mathbb{N}, D_{k-1} \subset D_k$.

If $x \in D_k(X)$, we will say that the graph $X - x$ is obtained from the graph X by the k-deletion of the vertex x and that the graph X is obtained from the graph $X - x$ by the k-addition of the vertex x. We write $X \searrow^k Y$ or $Y \overset{k}{\nearrow} X$ when X is k-dismantlable to a subgraph Y, i.e.:

$$X \searrow^k X - x_1 \searrow^k X - x_1 - x_2 \searrow^k \cdots \searrow^k X - x_1 - x_2 - \cdots - x_r = Y$$

with $x_i \in D_k(X - x_1 - x_2 - \cdots - x_{i-1})$. The sequence $x_1, \cdots, x_r$ is called a *k-dismantling sequence*. The notation $X \searrow^k pt$ signifies that $X \in D_k$. A graph X is *k-stiff* when $D_k(X) = \varnothing$. We denote by $D_\infty = \bigcup_{k \geqslant 0} D_k$ the family of graphs which are k-dismantlable for some integer $k \geqslant 0$. Cycles of length greater or equal to 4 and non-connected graphs are two examples of graphs which are not in D_∞. Finally, we write $[X]_k = [Y]_k$ when it is possible to go from X to Y by a succession of additions or deletions of k-dismantlable vertices. Note that $[X]_k$ is an equivalence class. Two graphs X and Y such that $[X]_k = [Y]_k$ will be said *k-homotopic*[3]. We note that for any integers $k \geqslant 0$ and $k' \geqslant 0$, any graph X, any vertex x of X and any vertex y not in X, we have the following *switching property*:

$$(\dagger) \qquad \text{if} \quad X \searrow^k X - x \overset{k'}{\nearrow} (X - x) + y \quad \text{then} \quad X \overset{k'}{\nearrow} X + y \searrow^k (X + y) - x.$$

Actually, since $x \not\sim y$, this property results from $N_X(x) = N_{X+y}(x)$ and $N_{X+y-x}(y) = N_{X+y}(y)$. In particular, this implies that two graphs X and Y are k-homotopic if, and only if, there exists a graph W such that $X \overset{k}{\nearrow} W \searrow^k Y$. Nevertheless, the notion of k-homotopy classes is not so relevant (see Proposition 5).

Remark 3 *Let us also note that the reverse implication of* $(\dagger)$ *is false (see. Fig.1 for a counterexample).*

2.2 Simplicial complexes

For general facts and references on simplicial complexes, see [17]. We recall that a *finite abstract simplicial complex* $\mathcal{K}$ is given by a finite set of vertices $V(\mathcal{K})$ and a collection of subsets $\Sigma(\mathcal{K})$ of $V(\mathcal{K})$ stable by deletion of elements: if $\sigma \in \Sigma(\mathcal{K})$ and $\sigma' \subset \sigma$, then $\sigma' \in \Sigma(\mathcal{K})$. The elements of $\Sigma(\mathcal{K})$ are the *simplices* of $\mathcal{K}$. If σ is a simplex of cardinal $k \geqslant 1$, then its dimension is $k-1$ and the dimension of $\mathcal{K}$

[3] In [6], a *1-dismantlable* vertex was called *s-dismantlable* and *1-homotopy* was called *s-homotopy*.

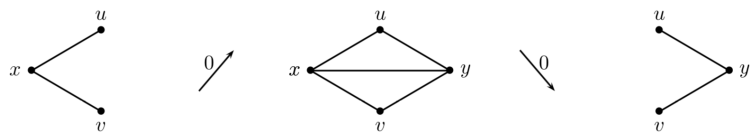

Figure 1: Let Y be the 2-path uxv: $Y \overset{0}{\nearrow} Y+y \overset{0}{\searrow} (Y+y)-x$ but $Y \overset{k}{\searrow} Y-x$ is impossible for any k.

is the maximum dimension of a simplex of $\mathcal{K}$. The j-skeleton of $\mathcal{K}$ consists of all simplices of dimension j or less.

Let us recall that for a simplex σ of a finite simplicial complex $\mathcal{K}$, $\mathrm{link}_{\mathcal{K}}(\sigma) = \{\tau \in \mathcal{K}\,, \sigma \cap \tau = \varnothing \text{ and } \sigma \cup \tau \in \mathcal{K}\}$ is a sub-complex of $\mathcal{K}$ and $\mathrm{star}^0_{\mathcal{K}}(\sigma) = \{\tau \in \mathcal{K}\,, \sigma \subset \tau\}$ is generally not a sub-complex of $\mathcal{K}$. If τ and σ are two simplices of $\mathcal{K}$, we say that τ is a *face* (resp. a *proper face*) of σ if $\tau \subset \sigma$ (resp. $\tau \subsetneq \sigma$). An *elementary simplicial collapse* is the suppression of a pair of simplices (σ, τ) such that τ is a proper maximal face of σ and τ is not the face of another simplex (one says that τ is a *free face* of $\mathcal{K}$). We denote by $\mathcal{K} - x$ the sub-complex of $\mathcal{K}$ induced by the vertices distinct from x. As defined in [4], an *elementary strong collapse* (or 0-collapse) in $\mathcal{K}$ is a suppression of a vertex x such that $\mathrm{link}_{\mathcal{K}}(x)$ is a simplicial cone. There is a strong collapse from $\mathcal{K}$ to $\mathcal{L}$ if there exists a sequence of elementary strong collapses that changes $\mathcal{K}$ into $\mathcal{L}$; in that case we also say that $\mathcal{K}$ 0-collapses to $\mathcal{L}$. A simplicial complex is 0-collapsible or strong collapsible if it 0-collapses to a point. By induction, for any integer $k \geqslant 1$, a vertex of $\mathcal{K}$ is k-collapsible if $\mathrm{link}_{\mathcal{K}}(x)$ is $(k-1)$-collapsible. There is a k-collapse from $\mathcal{K}$ to $\mathcal{L}$ if there exists a sequence of elementary k-collapses that changes $\mathcal{K}$ into $\mathcal{L}$ and, in that case, both complexes have the same simple homotopy type. A simplicial complex is k-collapsible if it k-collapses to a point.

Let also recall that a simplicial complex is *non-evasive* if it is k-collapsible for some $k \geqslant 0$ [4, Definiton 5.3]. A not non-evasive complex is called *evasive*.

When considering graphs, simplicial complexes arise naturally by the way of flag complexes. For any graph X, we denote by $\mathtt{cl}(X)$ the abstract simplicial complex such that $V(\mathtt{cl}(X)) = V(X)$ and whose simplices are the subsets of $V(X)$ which induce a clique of X. The simplicial complex $\mathtt{cl}(X)$ is called the *clique complex* of X and clique complexes are also called *flag complexes* [17]. A flag complex $\mathcal{K}$ is completely determined by its 1-skeleton (in other words, every flag complex is the clique complex of its 1-skeleton) and a simplicial complex $\mathcal{K}$ is a flag complex if, and only if, its minimal non-simplices are of cardinal 2. Remind that a non-simplex of $\mathcal{K}$ is a subset of $V(\mathcal{K})$ which is not a simplex of $\mathcal{K}$ and so a non-simplex $\sigma \subset V(\mathcal{K})$ is minimal if all proper subsets of σ are simplices of $\mathcal{K}$.

Given a vertex x of a graph X, by definition we have $\mathrm{link}_{\mathtt{cl}(X)}(x) = \mathtt{cl}(N_X(x))$. So, it is easy to observe that a graph X is in D_0 if and only if $\mathtt{cl}(X)$ is 0-collapsible [11, Theorem 4.1] and more generally:

Proposition 4 *For all integer $k \geqslant 0$, $X \in D_k$ if, and only if, $\mathtt{cl}(X)$ is k-collapsible.*

So, by Proposition 4, the set of non-evasive flag complexes is in one to one correspondence with D_∞. Before closing this section, it is important to note that since k-collapses don't change the simple homotopy type:

Proposition 5 *For all integer $k \geqslant 1$, $[X]_1 = [X]_k$.*

Proof : Of course, a graph 1-homotopic to X is also k-homotopic to X. Now, let Y be a graph k-homotopic to X. The clique complexes $\mathtt{cl}(X)$ and $\mathtt{cl}(Y)$ have the same simple simplicial homotopy type and, by [6, Theorem 2.10] where $[X]_1$ is denoted by $[X]_s$ and $\mathtt{cl}(X)$ is denoted by $\Delta(X)$, this implies $[X]_1 = [Y]_1$. In particular, Y is 1-homotopic to X and, finally, $[X]_1 = [X]_k$. □

3 A hierarchy of families

3.1 The family of cubion graphs $(\mathfrak{Q}_n)_{n\in\mathbb{N}}$

From Proposition 4, we know that if a graph X is k-dismantlable for some k, then $\mathtt{cl}(X)$ is a non-evasive simplicial complex. It is also known [5, 17] that non-evasive simplicial complexes are collapsible and,

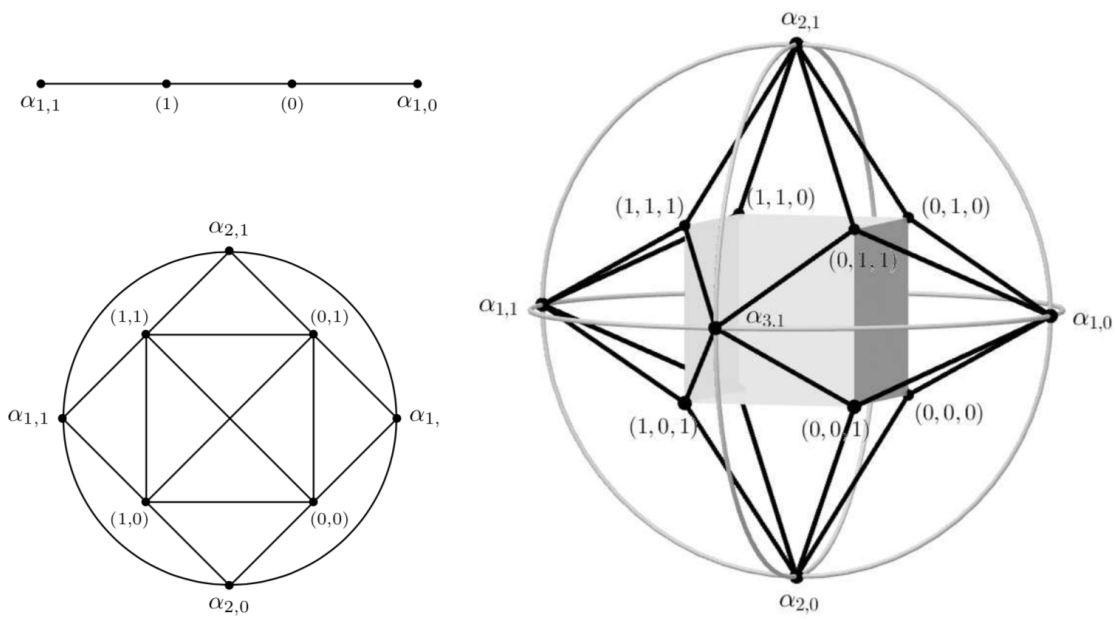

Figure 2: (top left) $\mathfrak{Q}_1 \in D_0 \backslash D_{-1}$, (bottom left) $\mathfrak{Q}_2 \in D_1 \backslash D_0$, (right) $\mathfrak{Q}_3 \in D_2 \backslash D_1$. The drawing of the 3-cubion is a perspective view where the central clique K_8 is symbolized by a cube: edges of the K_8 (ie. between x-type vertices) are not drawn, edges between x-type and α-type vertices are in black, and edges between α-type vertices are in grey.

a fortiori, contractible in the usual topological sense when the simplicial complex is considered as a topological space by the way of some geometrical realisation. In particular, this means that a graph whose clique complex is not contractible cannot be k-dismantlable whatever is the integer k:

Lemma 6 *Given $X_0 \subset X$ and $X \searrow^k X_0$ for $k \geqslant 0$, if $\mathtt{cl}(X_0)$ is non-contractible, so is $\mathtt{cl}(X)$ and $X \notin D_\infty$.*

Let us now show that the inclusions in Proposition 2 are strict.

Definition 7 [n-Cubion] $\forall n \in \mathbb{N}$, *the* n-Cubion *is the graph $\mathfrak{Q}_n$ with vertex set $V(\mathfrak{Q}_n) = \{\alpha_{i,\epsilon}, i = 1, \cdots, n$ and $\epsilon = 0,1\} \cup \{x = (x_1, \cdots, x_n), x_i = 0,1\}$ and edge set $E(\mathfrak{Q}_n)$ defined by:*
- $\forall i \neq j, \forall \epsilon, \epsilon' \in \{0,1\}, \alpha_{i,\epsilon} \sim \alpha_{j,\epsilon'}$
- $\forall x \neq x', x \sim x'$
- $\forall i, \alpha_{i,1} \sim (x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n)$ and $\alpha_{i,0} \sim (x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n)$

The n-Cubion has $2^n + 2n$ vertices partitioned into two sets such that:

$$\mathfrak{Q}_n[\alpha_{1,0}, \alpha_{1,1}, \cdots, \alpha_{n,0}, \alpha_{n,1}] \cong \overline{nK_2} \text{ and } \mathfrak{Q}_n[x, x \in \{0,1\}^n] \cong K_{2^n}.$$

The n-cubion is built from the *n-hypercube* with vertices the n-tuples $x = (x_1, \cdots, x_n) \in \{0,1\}^n$, each one connected to all the others, by adding $2n$ vertices $\alpha_{i,\epsilon}$ which induce an *n-octahedron* $\overline{nK_2}$ and each $\alpha_{i,\epsilon}$ is the apex of a cone whose base is the $(n-1)$-face of the hypercube given by $x_i = \epsilon$. This definition gives an iterative process to construct $\mathfrak{Q}_{n+1}$ from $\mathfrak{Q}_n$.

One sees that $\mathfrak{Q}_1 \cong P_4$ the path of length 3 and, clearly, $\mathfrak{Q}_1 \in D_0 \backslash D_{-1}$. The cubion $\mathfrak{Q}_2$ represented in Fig. 2 is in $D_1 \backslash D_0$. Indeed, $D_0(\mathfrak{Q}_2) = \varnothing$ but $\mathfrak{Q}_2 \searrow^1 \mathfrak{Q}_2 - \alpha_{1,0}$ and $\mathfrak{Q}_2 - \alpha_{1,0} \in D_0$.
More generally, we get:

Proposition 8 $\forall n \geqslant 2, \mathfrak{Q}_n \in D_{n-1} \backslash D_{n-2}$.

Proof :

1. Let us first prove that for any i, ϵ and x, $N_{\mathfrak{Q}_n}(\alpha_{i,\epsilon}) \cong \mathfrak{Q}_{n-1}$ and $N_{\mathfrak{Q}_n}(x) \searrow^0 \overline{nK_2}$.

• $N_{\mathfrak{Q}_n}(\alpha_{i,\epsilon}) \cong \mathfrak{Q}_{n-1}$: on one hand, given i and ϵ, the vertex $\alpha_{i,\epsilon}$ is linked to all the $\alpha_{j,\epsilon'}$ except when $i = j$. Thus, $N_{\mathfrak{Q}_n[\alpha_{1,0},\alpha_{1,1},\cdots,\alpha_{n,0},\alpha_{n,1}]}(\alpha_{i,\epsilon}) \cong \overline{(n-1)K_2}$. On the other hand, within the set of the n-tuples $x = (x_1, \cdots, x_n)$, $\alpha_{i,\epsilon} \sim (x_1, \cdots, x_{i-1}, \epsilon, x_{i+1}, \cdots, x_n)$. These 2^{n-1} vertices x whose i^{th} entry is fixed and equal to ϵ are all linked together and thereby induce a subgraph isomorphic to $K_{2^{n-1}}$ in $\mathfrak{Q}_n$. The edges between $\overline{(n-1)K_2}$ and $K_{2^{n-1}}$ are inherited from $\mathfrak{Q}_n$ and thus $N_{\mathfrak{Q}_n}(\alpha_{i,\epsilon}) \cong \mathfrak{Q}_{n-1}$.
• $N_{\mathfrak{Q}_n}(x) \searrow^0 \overline{nK_2}$: among all the $\alpha_{i,\epsilon}$ the vertex $x = (x_1, \cdots, x_n)$ is linked exactly to the n vertices $\alpha_{1,x_1}, \cdots, \alpha_{n,x_n}$. Let $X = N_{\mathfrak{Q}_n}(x) \backslash \{\alpha_{1,x_1}, \alpha_{2,x_2}, \cdots, \alpha_{n,x_n}\}$, a partition of X is given by $X^0 \cup X^1 \cup \cdots \cup X^{n-1}$ with $X^k = \{y \in X, y \text{ is linked to exactly } k \text{ vertices } \alpha_{i,x_i}\}$. Clearly X^i has $\binom{n}{i}$ elements. For example, we have $X^0 = \{(1-x_1, 1-x_2, \cdots, 1-x_{n-1}, 1-x_n)\}$ and $X^{n-1} = \{\hat{x_i}, i = 1, \cdots, n\}$ with $\hat{x_i} = (x_1, x_2, \cdots, x_{i-1}, 1-x_i, x_{i+1}, \cdots, x_{n-1}, x_n)$. For any $y \in X \backslash X^{n-1}$, there exist $i \neq j$, such that $y_i = 1 - x_i$ and $y_j = 1 - x_j$. Hence, y is dominated by $\hat{x_i}$ and $\hat{x_j}$ both in $N_{\mathfrak{Q}_n}(x)$. By successive 0-dismantlings of the vertices y, we obtain $N_{\mathfrak{Q}_n}(x) \searrow^0 X^{n-1} \cup \{\alpha_{1,x_1}, \alpha_{2,x_2}, \cdots, \alpha_{n,x_n}\}$. Finally, just notice that, between the vertices of $X^{n-1} \cup \{\alpha_{1,x_1}, \alpha_{2,x_2}, \cdots, \alpha_{n,x_n}\}$, all the possible edges exist except the $\hat{x_i}\alpha_{i,x_i}$ and thus $X^{n-1} \cup \{\alpha_{1,x_1}, \alpha_{2,x_2}, \cdots, \alpha_{n,x_n}\} \cong \overline{nK_2}$.

2. By induction on n, $\mathfrak{Q}_{n-1} \in D_{n-2} \backslash D_{n-3}$ and as we have proven that $N_{\mathfrak{Q}_n}(\alpha_{i,\epsilon}) \cong \mathfrak{Q}_{n-1}$, $\alpha_{i,\epsilon} \in D_{n-1}(\mathfrak{Q}_n) \backslash D_{n-2}(\mathfrak{Q}_n)$. Moreover, since the simplicial complex $\mathtt{cl}(\overline{nK_2})$ is non-contractible because it is a triangulation of the sphere S^{n-1}, Lemma 6 implies $N_{\mathfrak{Q}_n}(x) \notin D_\infty$ and thus $x \notin D_{n-2}(\mathfrak{Q}_n)$. Therefore $D_{n-2}(\mathfrak{Q}_n) = \varnothing$ and $\mathfrak{Q}_n \notin D_{n-2}$. Now,

$$\mathfrak{Q}_n \searrow^{n-1} \mathfrak{Q}_n - \{\alpha_{n,0}, \alpha_{n,1}\}$$

since $\alpha_{n,0}$ and $\alpha_{n,1}$ are $(n-1)$-dismantlable and not linked. In $\mathfrak{Q}_n - \{\alpha_{n,0}, \alpha_{n,1}\}$, note that $(x_1, \cdots, x_{n-1}, 0)$ and $(x_1, \cdots, x_{n-1}, 1)$ are twins and therefore

$$\mathfrak{Q}_n - \{\alpha_{n,0}, \alpha_{n,1}\} \searrow^0 \mathfrak{Q}_n - \left\{\alpha_{n,0}, \alpha_{n,1}, (x_1, \cdots, x_{n-1}, 0);\ (x_1, \cdots, x_{n-1}) \in \{0,1\}^{n-1}\right\} \cong \mathfrak{Q}_{n-1}.$$

By induction hypothesis, $\mathfrak{Q}_{n-1} \in D_{n-2} \subset D_{n-1}$. Finally, $\mathfrak{Q}_n \in D_{n-1}$. □

Propositions 2 and 8 now give the following theorem:

Theorem 9 *The sequence $(D_k)_{k \geqslant 0}$ is strictly increasing:*

$$D_{-1} \subsetneq D_0 \subsetneq D_1 \subsetneq D_2 \subsetneq \cdots \subsetneq D_k \subsetneq D_{k+1} \subsetneq \cdots$$

There are no graphs with fewer vertices than $\mathfrak{Q}_1$ in $D_0 \backslash D_{-1}$. One can verify the same result for $\mathfrak{Q}_2$ in $D_1 \backslash D_0$, but there are graphs in $D_1 \backslash D_0$ with fewer edges.

3.2 Critical k-dismantlability

Let's complete this section with results on graphs in $D_k \backslash D_{k-1}$ with $k \geqslant 1$. Such a graph X does not always have a vertex in $D_k(X) \backslash D_{k-1}(X)$. Indeed, by duplicating each vertex of a graph in $D_k \backslash D_{k-1}$ with a twin, we get a new graph also in $D_k \backslash D_{k-1}$ in which each vertex is 0-dismantlable and, so, is not in $D_k(X) \backslash D_{k-1}(X)$. However we have the following result:

Lemma 10 *Given $k \in \mathbb{N}^\star$ and $X \in D_k \backslash D_{k-1}$, there exists $x \in V(X)$ and Y an induced subgraph of X such that $x \in D_k(Y) \backslash D_{k-1}(Y)$.*

Proof : Set $V(X) = \{x_1, \cdots, x_n\}$ and suppose that $x_1, \cdots, x_{n-1}$ is a k-dismantling sequence from X to the point x_n. By definition, $\forall i \in \{1, \cdots, n-1\}$, $x_i \in D_k(X[x_i, x_{i+1}, \cdots, x_n])$. Since $X \notin D_{k-1}$, the sequence $x_1, \cdots, x_{n-1}$ is not a $(k-1)$-dismantling sequence of X. Therefore, there exists $i_0 \in \{1, \cdots, n-1\}$ such that $x_{i_0} \notin D_{k-1}(X[x_{i_0}, x_{i_0+1}, \cdots, x_n])$, i.e. $x_{i_0} \in D_k(Y) \backslash D_{k-1}(Y)$ where $Y = X[x_{i_0}, x_{i_0+1}, \cdots, x_n]$. □

We remark that any connected graph with at most three vertices contains at least one apex and therefore any $X \in D_0 \backslash D_{-1}$ has at least four vertices. We recall that the *clique number* $\omega(X)$ of a graph X is the maximum number of vertices of a clique of X.

Proposition 11 *Given $k \in \mathbb{N}^\star$, if $D_k(X) \backslash D_{k-1}(X) \neq \varnothing$, then $\omega(X) \geqslant k+2$. Moreover, if $x \in D_k(X) \backslash D_{k-1}(X)$, there is a clique with $k+2$ vertices and containing x.*

Proof : The proof is by induction on k.

For $k = 1$, if there exists $x_2 \in D_1(X)\backslash D_0(X)$, then $N_X(x_2) \in D_0\backslash C$. Since $N_X(x_2)$ is not a cone but is 0-dismantlable, it contains an edge, so X contains a triangle.

Now, let X be a graph such that $D_{k+1}(X)\backslash D_k(X) \neq \emptyset$ and denote by x_{k+2} a vertex such that $N_X(x_{k+2}) \in D_k\backslash D_{k-1}$. From Lemma 10, there exists $x_{k+1} \in V(N_X(x_{k+2}))$ and Y an induced subgraph of $N_X(x_{k+2})$ such that $x_{k+1} \in D_k(Y)\backslash D_{k-1}(Y)$. The induction hypothesis applied to Y gives that Y contains an induced subgraph $K \cong K_{k+2}$. As $Y \subset N_X(x_{k+2})$, $K + x_{k+2}$ is a complete subgraph of X with $k + 3$ vertices.

And it follows from this proof that any vertex in $D_k(X)\backslash D_{k-1}(X)$ is in a clique of X of cardinal $k + 2$. □

A direct consequence of Lemma 10 and Proposition 11 is:

Corollary 12

(i) Given $k \in \mathbb{N}$, if $X \in D_k\backslash D_{k-1}$, then $\omega(X) \geqslant k + 2$.

(ii) If $X \in D_\infty$ and $|V(x)| = n$, then $X \in D_{n-2}$.

Thus, if a graph of D_∞ contains no triangle, it is in D_0 and it is not hard to prove by induction that a 0-dismantlable graph without a triangle is a tree. So, the only graphs of D_∞ without a triangle are the trees. A more directed proof of this fact is obtained by considering the clique complexes. Indeed, if a graph X is triangle-free, then `cl`(X) is a 1-dimensional complex, and if X is in D_∞, Proposition 4 implies that `cl`(X) is k-collapsible. Thus, `cl`(X) has to be a tree and so is X.

4 Some results on D_∞

4.1 Order in dismantlabilities

For 0-dismantlability, the order of dismantlings does not matter and therefore the 0-stiff graphs to which a graph X is 0-dismantlable are isomorphic ([11, Proposition 2.3], [15, Proposition 2.60]). This property is no longer true for k-dismantlability with $k \geqslant 1$. The graph X of Fig. 3 gives a simple example of a graph that is 1-dismantlable either to C_4, or to C_5 (depending on the choice and order of the vertices to 1-dismantlable) which are non-isomorphic 1-stiff graphs.

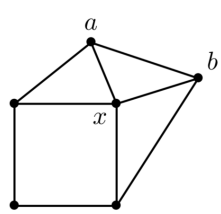

Figure 3: $X \searrow^1 X - a - b \cong C_4$ and $X \searrow^1 X - x \cong C_5$

Actually, there is an important gap between 0-dismantlability and k-dismantlability with $k \geqslant 1$. We have already noted in Proposition 5 that, for any graph X and any $k \geqslant 1$, $[X]_k = [X]_1$ while the inclusion $[X]_0 \subset [X]_1$ of homotopy classes is strict:

- $[C_4]_0 \neq [C_5]_0$ because the cycles C_4 and C_5 are non isomorphic 0-stiff graphs.
- $[C_4]_1 = [C_5]_1$ as it is shown by graph X in Fig. 3.

A major fact concerning the difference between 0-dismantlability and k-dismantlability for $k \geqslant 1$ is that ([11, Corollary 2.1])

$$(X \searrow^0 X'' \ , \ X \searrow^0 X' \text{ and } X'' \subset X') \implies X' \searrow^0 X''$$

while, for $k \geqslant 1$, in general (cf. Fig. 4, (b) and (c)):

$$(X \searrow^k X'' \ , \ X \searrow^0 X' \text{ and } X'' \subset X') \not\Longrightarrow X' \searrow^k X''.$$

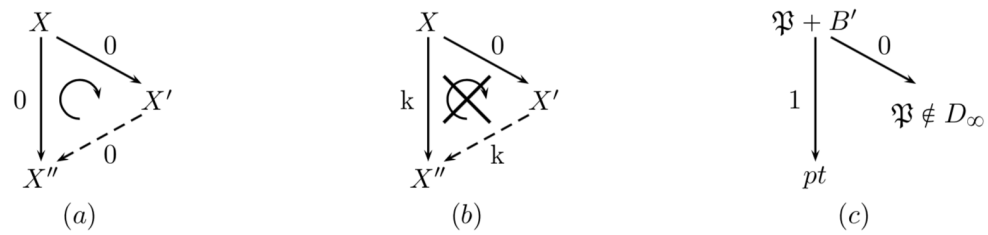

Figure 4: With X'' an induced subgraph of X' and X' an induced subgraph of X: (a) the dashed arrow always exists, (b) the dashed arrow may exist or not, (c) an illustration of (b) where the dashed arrow does not exist.

Actually, one can find graphs X, X' and X'' such that $X'' \subset X' \subset X$, $X \searrow^1 X''$, $X \searrow^0 X'$ and X' is not k-dismantlable to X'' for any integer $k \geqslant 0$. To prove this, we introduce the Parasol graph:

Definition 13 (Parasol graph) *The* Parasol *is the graph* $\mathfrak{P}$ *with* 15 *vertices drawn in Fig. 5. From* $\mathfrak{P}$*, we build a graph* $\mathfrak{P} + B'$ *by adding to* $\mathfrak{P}$ *a vertex* B' *linked to* B_1 *and to the neighbours of* B_1 *except* B_3 *and* B_6*.*

The neighbours of the vertices of $\mathfrak{P}$ are as follows, for all $i \in \{1, \cdots, n\}$:

- $N_{\mathfrak{P}}(B_i)$ is isomorphic to C_4 with two disjoint pendant edges attached to two consecutive vertices of the cycle
- $N_{\mathfrak{P}}(A_i) \cong C_5$
- $N_{\mathfrak{P}}(I) \cong C_7$

Consequently, $D_k(\mathfrak{P}) = \varnothing$ for all positive integer k and:

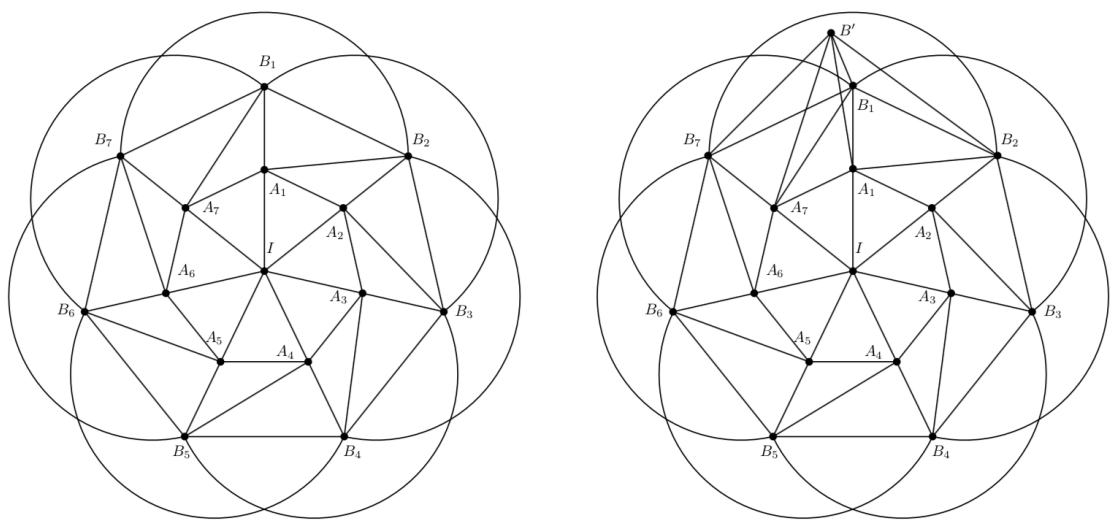

Figure 5: (left) the parasol graph $\mathfrak{P}$ and (right) the graph $\mathfrak{P} + B'$.

Proposition 14

(i) $\mathfrak{P} \notin D_\infty$.

(ii) $\mathfrak{P} + B' \searrow^0 \mathfrak{P}$.

(iii) $\mathfrak{P} + B' \searrow^1 (\mathfrak{P} + B') - B_1 \searrow^1 pt$.

Proof : (i) $\mathfrak{P}$ is not in D_∞ by application of Lemma 6 because each vertex has a neighbour which is a cycle of length at least 5 or which is 0-dismantlable to a cycle of length 4.

(ii) As $B' \vdash B_1$ in $\mathfrak{P} + B'$, the vertex B' is 0-dismantlable in $\mathfrak{P} + B'$ and $\mathfrak{P} + B' \searrow^0 \mathfrak{P} \notin D_\infty$.

(iii) It is easy to verify that the neighbourhood of B_1 in $\mathfrak{P} + B'$ is a 0-dismantlable graph; so, $B_1 \in D_1(\mathfrak{P} + B')$ and $\mathfrak{P} + B' \searrow^1 (\mathfrak{P} + B') - B_1$. Then, following the increasing order of the indexes i, all the B_i are successively 1-dismantlable with a path as neighbourhood. The remaining graph induced by I and the vertices A_i is a cone and thus 0-dismantlable. □

This example shows that, for graphs in D_k with $k \geqslant 1$, the dismantling order is crucial: it is possible to reach (resp. quit) D_∞ just by adding (resp. removing) a 0-dismantlable vertex (Fig. 4, (c)).

The parasol graph is not in D_∞ but it is worth noting that the parasol graph is ws-dismantlable:

$$\mathfrak{P} \searrow^{ws} pt$$

Let us recall (cf. [6]) that ws-dismantbility allows not only 1-dismantlability of vertices but also of edges (an edge $\{a, b\}$ of a graph X is 1-dismantlable whenever $N_X(a) \cap N_X(b) \in D_0$)[4]. For example, in the parasol graph, one can 1-delete the edge $\{B_2, B_7\}$ and the remaining graph is 1-dismantlable (beginning by B_1). It is well known ([9, Lemma 3.4],[6, Lemma 1.6]) that the 1-dismantlability of an edge can be obtained by the 0-addition of a vertex followed by the 1-deletion of another vertex. As an illustration, the sequence

$$\mathfrak{P} \overset{0}{\nearrow} \mathfrak{P} + B' \searrow^1 (\mathfrak{P} + B') - B_1$$

can be seen as 1-deletions of the edges $\{B_1, B_6\}$ and $\{B_1, B_3\}$.

Thanks to the switching property (†) which allows to switch 0-expansions and 1-dismantlabilities, we get:

$$X \text{ ws-dismantlable} \Rightarrow \exists W \text{ such that } X \overset{0}{\nearrow} W \searrow^1 pt.$$

The question of which other graphs this property extends to is open. More precisely, while any graph X for which there exists W such that $X \overset{0}{\nearrow} W \searrow^1 pt$ has a contractible clique complex $\mathtt{cl}(X)$, the reverse implication remains open: if X is a graph such that $\mathtt{cl}(X)$ is contractible, does it exist W such that $X \overset{0}{\nearrow} W \searrow^1 pt$? The graphs DH (Fig. 6) and BH (Fig. 7 and [9, Fig. 3, Fig. 4]), 1-skeletons of triangulations of the Dunce Hat and the Bing's House respectively, are interesting cases. Indeed, $\mathtt{cl}(DH)$ and $\mathtt{cl}(BH)$ are known to be contractible but non collapsible and this implies that both graphs are neither in D_∞ nor in the set of ws-dismantlable graphs. However:

Proposition 15 *There exist two graphs W_{DH} and W_{BH} such that:*

$$DH \overset{0}{\nearrow} W_{DH} \searrow^1 pt \quad \textit{and} \quad BH \overset{0}{\nearrow} W_{BH} \searrow^1 pt$$

Proof : For both graphs, the process is the same and consists in successive 0-additions of vertices so as to transform non-0-dismantlable neighbourhoods of some vertices into 0-dismantlable ones. For these two graphs DH and BH, we will transform some cycles into wheels by 0-additions of vertices. Here we give the sequence of 0-additions and 1-deletions only for the Dunce Hat and the details for the Bing's House are given in Appendix. With notations of Fig. 6, we do the following 0-additions and 1-deletions with $G_0 = DH$:

1. Within $N_{G_0}(1)$: 0-additions of vertices $1'$ and $1''$ linked to $1, 2, j, i, h$ and $1, 3, d, e, f$ respectively. Note that $1' \vdash 1$ and $1'' \vdash 1$. Now, since $N_{DH+1'+1''}(1)$ is made of two 4-wheels linked by a path, so is a 0-dismantlable graph, we 1-delete vertex 1. Let us note $G_1 = DH + 1' + 1" - 1$.

2. Within $N_{G_1}(2)$: 0-addition of vertex $2'$ linked to $2, 1', h, 4, j$ and 1-deletion of vertex 2. Let us note $G_2 = G_1 + 2' - 2$.

3. Within $N_{G_2}(3)$: 0-addition of vertex $3'$ linked to $3, 1'', d, 4, f$ and 1-deletion of vertex 3. Let us note $G_3 = G_2 + 3' - 3$.

   Now, observe that $G_3 \in D_1$. Indeed,

[4] In [6], *1-dismantlable edge* was called *s-dismantlable*.

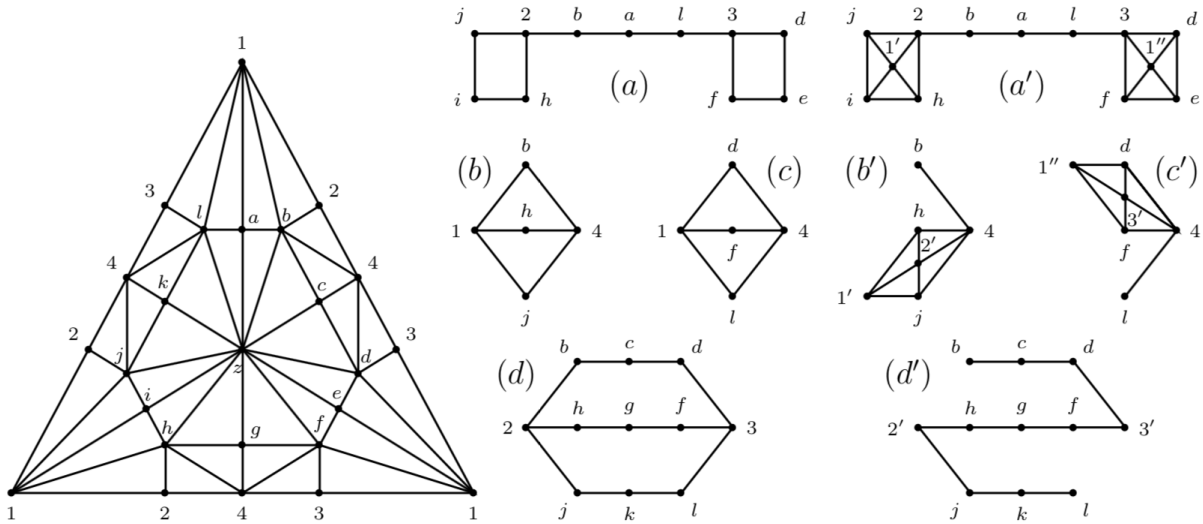

Figure 6: (left) The graph DH, the 1-skeleton of a triangulation of the Dunce Hat. (middle) (a), (b), (c) and (d) are the neighbourhoods of vertices 1, 2, 3 and 4 in $G_0 = DH$, respectively. (right) (a') neighbourhood of vertex 1 in $G_0 + 1' + 1''$ (b') neighbourhood of vertex 2 in $G_1 + 2'$ with $G_1 = G_0 + 1' + 1'' - 1$ (c') neighbourhood of vertex 3 in $G_2 + 3'$ with $G_2 = G_1 + 2' - 2$ (d') neighbourhood of vertex 4 in G_3 with $G_3 = G_2 + 3' - 3$.

4. Vertex 4 is in $D_1(G_3)$ since its neighbourhood is the path $bcd3'fgh2'jkl$. Let us note $G_4 = G_3 - 4$, vertices $2'$ and $3'$ are in $D_1(G_4)$ since $N_{G_4}(2') = h1'j$ and $N_{G_4}(3') = d1''f$ are disjoint 2-paths. Let us note $G_5 = G_4 - 2' - 3'$, vertices $1'$ and $1''$ are in $D_1(G_5)$ since $N_{G_5}(1')$ and $N_{G_5}(1')$ are also disjoint 2-paths.

5. Now the resulting graph $G_5 - 1' - 1''$ is a 12-wheel centered in z. Like any cone, it is 0-dismantlable.

The switching property (†) finishes the proof and $W_{DH} = DH + 1' + 1'' + 2' + 3'$. □

Remark 16 *The strategy used in the previous proof is based on the removal of the vertices* $1, 2, 3$ *and* 4 *corresponding to the gluing data of the Dunce Hat in order to get the* 0*-dismantlable* 12*-wheel centered in* z*. However, to get a* 1*-dismantlable graph, the* 0*-additions of vertices* $1'$ *and* $1''$ *are enough, as shown by the* 1*-dismantling sequence* $1, a, b, c, d, e, f, g, h, i, j, k, z, l, 1', 2, 4, 3, 1''$ *of* $DH + 1' + 1''$ *which alternates* 0*- and* 1*-dismantlings.*

4.2 A link with graph derivability

In [19], Mazurkiewicz introduces the following notion of *locally derivable graphs*. For any family $\mathcal{R}$ of non-empty graphs, $\triangle(\mathcal{R})$ is the smallest family of graphs containing the point graph pt and such that

$$(X - x \in \triangle(\mathcal{R}) \text{ and } N_X(x) \in \mathcal{R}) \Rightarrow X \in \triangle(\mathcal{R}).$$

Graphs in $\triangle(\mathcal{R})$ are called *locally derivable by* $\mathcal{R}$. By definition, the graphs of $\triangle(\mathcal{R})$ are non-empty and connected graphs and $\triangle$ is monotone: $\mathcal{R} \subset \mathcal{R}'$ implies $\triangle(\mathcal{R}) \subset \triangle(\mathcal{R}')$. By an inductive proof on the cardinals of the vertex sets, it is easy to see that $D_0 = \triangle(C)$ and more generally (recall that C is also denoted D_{-1}):

Proposition 17 *For all* $k \in \mathbb{N}$*,* D_k *is locally derivable by* D_{k-1}*, i.e.* $\triangle(D_{k-1}) = D_k$*.*

It is worth noting the following fact:

Proposition 18 D_∞ *is the smallest fixed point of* $\triangle$*.*

Proof : Following the notations of [19] and given a family $\mathcal{R}$ of graphs, we denote by $\triangle^\star(\mathcal{R})$ the set $\bigcup_{n\geqslant 0}\triangle^n(\mathcal{R})$ with the convention $\triangle^0(\mathcal{R})=\{pt\}$ and $\triangle^1(\mathcal{R})=\triangle(\mathcal{R})$. Let us note that, if $\mathcal{F}$ is a fixed family by $\triangle$, then $\mathcal{F}=\triangle^\star(\mathcal{F})$. Now, for any family $\mathcal{R}$, the inclusion $\{pt\}\subset\triangle(\mathcal{R})$ and the monotony of $\triangle$ imply $\triangle^\star(\{pt\})\subset\triangle^\star(\mathcal{R})$. Consequently, $\triangle^\star(\{pt\})$ is the smallest fixed point of $\triangle$ and we have to prove that $D_\infty=\triangle^\star(\{pt\})$.

From Proposition 17, $D_\infty=\triangle^\star(C)$ and $\triangle^\star(\{pt\})\subset D_\infty$. For the reverse inclusion, by induction on n, observe that any cone with n vertices is in $\triangle^{n-1}(\{pt\})$. Consequently, $C\subset\triangle^\star(\{pt\})$ and so $D_\infty=\triangle^\star(C)\subset\triangle^\star(\{pt\})$. □

In [19], the set D_∞ is denoted by F (and the elements of $D_\infty=F$ are called *closed graphs*) and the author states that for any $X\in F$, if $x\in V(X)$ with $N_X(x)\in F$, then $X-x\in F$. The graph $\mathfrak{P}+B'$ is a counter-example. Indeed, $\mathfrak{P}\notin D_\infty$ while, by Proposition 14, $\mathfrak{P}+B'\in D_1\subset D_\infty$ and $N_{\mathfrak{P}+B'}(B')\in D_\infty$ because it is a cone with apex B_1.

5 Vertex-transitive graphs, k-dismantlability and evasivity

The relation $\mathscr{E}$ defined on the set $V(X)$ of vertices of a graph X by $x\,\mathscr{E}\,y\iff N_X[x]=N_X[y]$ is an equivalence relation whose equivalence classes are maximal sets of twin vertices. With notations of [26], we denote by $X_\star$ the graph obtained from this equivalence relation: $V(X_\star)$ is the set of equivalence classes of $\mathscr{E}$ with adjacencies $x_\star\sim y_\star$ if, and only if, $x\sim y$.

Proposition 19 *[26, Lemma 6.4] Let X be a graph.*

(i) There is a subgraph of X isomorphic to $X_\star$.

(ii) $(X_\star)_\star=X_\star$ (i.e., $\mathrm{Twins}(X_\star)=\varnothing$).

(iii) $X\cong X_\star$ if, and only if, $x_\star=\{x\}$ for every vertex x of X.

(iv) If $X\cong Y$, then $X_\star\cong Y_\star$.

The following lemma is easy to prove:

Lemma 20 *$X\in D_0$ if, and only if, $X_\star\in D_0$.*

We recall that a graph X is *vertex-transitive* if its automorphism group $\mathrm{Aut}(X)$ acts transitively on $V(X)$ (i.e., for any vertices x,y, there is an automorphism φ of X such that $\varphi(x)=y$). In a vertex-transitive graph, all vertices have isomorphic neighbourhoods and:

Lemma 21 *Let X be a vertex-transitive graph.*

(i) $\mathrm{Twins}(X)=D_0(X)$.

(ii) $X_\star$ is vertex-transitive.

(iii) Let $x\in X$ such that $N_X[x]$ is a clique, $x_\star$ is equal to $N_X[x]$ and is a connected component of X.

Proof : (i) The inclusion $\mathrm{Twins}(X)\subset D_0(X)$ is obvious. Now, let a and b be two vertices with $a\vdash b$. The inclusion $N_X[a]\subset N_X[b]$ becomes $N_X[a]=N_X[b]$ in a vertex-transitive graph. This proves that a and b are twin vertices and that $D_0(X)\subset\mathrm{Twins}(X)$.

(ii) This follows directly from the fact that every automorphism $\varphi:X\to X$ induces an automorphism $\varphi_\star:X_\star\to X_\star$ defined by $\varphi_\star(x_\star)=(\varphi(x))_\star$ for every vertex x of X.

(iii) Since $N_X[x]$ is a clique of X, $N_X[x]\subset N_X[y]$ for any vertex y adjacent to x. So, by vertex transitivity, $N_X[x]=N_X[y]$ and $x_\star=N_X[x]$. Now, as $N_X[x]=N_X[y]$ whenever $y\sim x$, we get that $z\sim y$ and $y\sim x$ implies $z\sim x$ for all vertices y and z and this proves that the connected component containing x is equal to $N_X[x]$. □

Proposition 22 *If X is a 0-dismantlable and vertex-transitive graph, then X is a complete graph.*

Proof : Let X be a 0-dismantlable and vertex-transitive graph. We prove that X is a complete graph by induction on $k=|V(X)|$. If $|V(X)|=1$, $X\cong pt=K_1$ and there is nothing to prove. Let $k\geqslant 1$ and let us suppose that any 0-dismantlable and vertex-transitive graph with at most k vertices

is a complete graph. Let X be a 0-dismantlable and vertex-transitive graph with $k+1$ vertices. By Lemma 21(ii), the graph $X_\star$ is a vertex-transitive graph and, by Lemma 20, $X_\star \in D_0$. As $X \in D_0$ and $|V(X)| \geqslant 2$, $D_0(X) \neq \varnothing$ and, by Lemma 21(i), $\mathrm{Twins}(X) = D_0(X) \neq \varnothing$. So, $|V(X_\star)| < |V(X)|$ and, by induction hypothesis, $X_\star$ is a complete graph. As $\mathrm{Twins}(X_\star) = \varnothing$, by Proposition 19(ii), we conclude that $X_\star \cong pt$ and this proves that X is a complete graph. □

Given the equivalence between 0-dismantlability for graphs and strong collapsibility for clique complexes (case $k = 0$ of Proposition 4), Proposition 22 is nothing but [4, Corollary 6.6] in the restricted case of flag complexes. But the proof given here doesn't refer to the fixed points scheme and can be generalised by introducing the notion of i-complete-transitive graphs. In what follows, if $(v_1, \cdots, v_k) \in V(X)^k$, then the subgraph of X induced by $(v_1, \cdots, v_k)$ refers to $X[v_1, \cdots, v_k]$, the subgraph induced by $\{v_1, \cdots, v_k\}$.

Definition 23 *Given $i \geqslant 1$, a graph X will be called* i-complete-transitive *if for all $1 \leqslant k \leqslant i$ and all pairs*

$$\{(x_1, \cdots, x_k), (x'_1, \cdots, x'_k)\}$$

of k-tuples of pairwise distinct vertices inducing a complete subgraph of X, there exists $f \in \mathrm{Aut}(X)$ such that $f(x_j) = x'_j$ for all $j \in \{1, \cdots, k\}$.

The set of i-complete-transitive graphs contains the set of *i-transitive graphs* previously introduced in [7, 20]. We note that 1-complete-transitive graphs are just vertex-transitive graphs and a 2-complete-transitive graph is a vertex-transitive and arc-transitive graph. Complete-transitive graphs are a generalisation of arc-transitive graphs but to complete subgraphs and not to paths, as are the i-arc-transitive graphs [13]. Kneser graphs are examples of i-complete-transitive graphs for all integers i. We now have the following generalisation of Proposition 22.

Proposition 24 *Let X be a graph and $k \in \mathbb{N}$. If $X \in D_k$ and if X is $(k+1)$-complete-transitive, then X is a complete graph.*

Proof : We prove it by induction on $k \geqslant 0$.

For $k = 0$, the claimed assertion is given by Proposition 22.

Let $k \geqslant 0$ and suppose that any k-dismantlable and $(k+1)$-complete-transitive graph is a complete graph. Let X be a $(k+1)$-dismantlable and $(k+2)$-complete-transitive graph and $x \in D_{k+1}(X)$. We will verify that the k-dismantlable graph $N_X(x)$ is a $(k+1)$-complete-transitive graph. Let $\{(x_1, \cdots, x_{k+1}), (x'_1, \cdots, x'_{k+1})\}$ be a pair of sets of vertices of cardinal $k+1$, each of them inducing a clique of $N_X(x)$, the pair of sets $\{(x, x_1, \cdots, x_{k+1}), (x, x'_1, \cdots, x'_{k+1})\}$ is of cardinal $(k+2)$, each of them inducing a clique of X. By $(k+2)$-complete-transitivity of X, there exists $f \in \mathrm{Aut}(X)$ such that $f(x) = x$ and $f(x_i) = x'_i$ for all $i \in \{1, \cdots, k+1\}$. In particular, $\varphi = f_{|N_X(x)}$ verifies $\varphi \in \mathrm{Aut}(N_X(x))$ and $f(x_i) = x'_i$ for all $i \in \{1, \cdots, k+1\}$. So, $N_X(x)$ is a k-dismantlable and $(k+1)$-complete-transitive graph. By induction hypothesis, $N_X(x)$ is a complete graph. As X is vertex-transitive, by Lemma 21(iii), it means that, for any vertex x of X, the connected component of X containing x is a complete subgraph. Now, X is connected since it is in D_∞ and so, X is a complete graph. □

Let us recall the notion of evasivity for simplicial complexes [5, 17]. One can present it as a game: given a (known) simplicial complex $\mathcal{K}$ with vertex set V of cardinal n, through a series of questions, a player has to determine if a given (unknown) subset A of V is a simplex of $\mathcal{K}$. The only possible questions for the player are, for every vertex x of V, « is x in A ? ». The complex $\mathcal{K}$ is called *non-evasive* if, whatever is the chosen subset A of V, the player can determine if A is a simplex of $\mathcal{K}$ in at most $(n-1)$ questions. By restriction to flag complexes, we get the notion of non-evasiveness for graphs:

Definition 25 *A graph X is called* non-evasive *if $\mathtt{cl}(X)$ is a non-evasive simplicial complex.*

In other terms, a graph X is called *non-evasive* if for any $A \subset V(X) = \{x_1, \cdots, x_n\}$ one can guess if A is a complete subgraph of X in at most $n-1$ questions of the form « is x in A ? ». In [4], the authors note that a complex $\mathcal{K}$ is non-evasive if, and only if, there is an integer n such that $\mathcal{K}$ is n-collapsible. The equivalence between k-dimantlability of graphs and k-collapsibility of flag complexes (cf. Proposition 4) gives:

Proposition 26 *A graph X is non-evasive if, and only if, X is in D_∞.*

The Evasiveness Conjecture for simplicial complexes states that *every non-evasive vertex homogeneous simplicial complex is a simplex* [16]. Again, its restriction to clique complexes can be formulated in terms of graphs:

Conjecture 1 (Evasiveness conjecture for graphs) *Let X be a graph, if X is in D_∞ and vertex-transitive, then X is a complete graph.*

This formulation should not be confused with the evasiveness conjecture for monotone graph properties [5, 16, 17]. Let's note that Proposition 24 is a direct consequence of the conjecture, if that one is true. Following a remark due to Lovász, Rivest and Vuillemin [24] pointed out that a positive answer to the evasiveness conjecture implies that a finite vertex-transitive graph with a clique which intersects all its maximal cliques is a complete graph. Actually, they prove that a graph with a clique which intersects all its maximal cliques is non-evasive, i.e. is in D_∞ by Proposition 26. Remark 3.3 of [3] is another formulation of this result. Indeed, the 1-skeletons of star clusters are exactly the graphs which contain a clique intersecting all maximal cliques. Theorem 28 will give a stronger result.

We recall that if Y and Z are two subgraphs of a graph X, $Y \cap Z$ will denote the subgraph $(V(Y) \cap V(Z), E(Y) \cap E(Z))$ and one says that Y intersects Z if $V(Y \cap Z) \neq \emptyset$.

Lemma 27 *If X is a graph with a clique A which intersects all maximal cliques of X and x is in $V(X)\backslash V(A)$, then:*

(i) A intersects all maximal cliques of $X - x$.
(ii) $A \cap N_X(x)$ is a complete graph which intersects all maximal cliques of $N_X(x)$.

Proof :
(i) Let K be a maximal clique of $X - x$. If K is a maximal clique of X, then, by property of A, $K \cap A \neq \emptyset$. Otherwise, $K + x$ is a maximal clique of X and, by property of A, $(K + x) \cap A \neq \emptyset$. Since $x \notin A$, it implies $K \cap A \neq \emptyset$.
(ii) If K is a maximal clique in $N_X(x)$, then $K + x$ is a maximal clique of X and, by property of A, $(K + x) \cap A \neq \emptyset$. As $x \notin A$, $K \cap A \neq \emptyset$ and also $K \cap \big(A \cap N_X(x)\big) \neq \emptyset$ since $K \subset N_X(x)$. □

Theorem 28 *Let X be a graph. If A is a clique which intersects all maximal cliques of X, then $X \in D_{a-2}$ with $a = |V(A)| \geqslant 1$. Moreover, $X \searrow^{a-2} A$ if $|V(A)| \geqslant 2$.*

Proof : Let X be a graph with n vertices and A a clique of X which intersects all maximal cliques of X with $a = |V(A)| \geqslant 1$.

If $a = 1$, then X is a cone whose apex is the vertex of A, that is $X \in D_{-1}$.

If $a = 2$, let us denote by u and v the vertices of A. If $x \in V(X)\backslash V(A)$ with $x \sim u$ and $x \nsim v$, then x is dominated by u. Indeed, let $y \sim x$ and K a maximal clique of X containing x and y, by property of $A = \{u, v\}$, either $u \in K$ or $v \in K$. As $v \in K$ contradicts $x \nsim v$, we conclude that $u \in K$ and $u \sim y$. In conclusion, a vertex not in $V(A)$ is dominated by u or v or both together. So, $X \searrow^0 A$ and $X \in D_0$.

Let us suppose that $a \geqslant 3$, we will prove that $X \searrow^{a-2} A$ by induction on $n = |V(X)| \geqslant 3$. For $n = 3$, we have $X = A$ and X is a complete graph. Now, suppose that the assertion of the theorem is true for some $n \geqslant 3$ and let us consider a graph X with $n + 1$ vertices and a clique A which intersects all its maximal cliques. If $V(A) \subset N_X(x)$ for every $x \in V(X)\backslash V(A)$, then every vertex of A is an apex of X and $X \searrow^0 A$. If $V(A) \not\subset N_X(x)$ for some $x \in V(X)\backslash V(A)$, then $|V(A) \cap N_X(x)| \leqslant a - 1$. By Lemma 27(ii), $A \cap N_X(x)$ is a complete subgraph of $N_X(x)$ which intersects all its maximal cliques and, by induction hypothesis applied to $N_X(x)$, we get $N_X(x) \searrow^{a-3} A \cap N_X(x) \searrow^0 pt$ as $|A \cap N_X(x)| - 2 \leqslant (a-1) - 2 = a - 3$. This proves that $x \in D_{a-2}(X)$, that is $X \searrow^{a-2} X - x$. Moreover, by Lemma 27(i), the induction hypothesis implies that $X - x \searrow^{a-2} A$. The composition $X \searrow^{a-2} X - x \searrow^{a-2} A$ proves that $X \searrow^{a-2} A$.

Of course, we conclude that $X \in D_{a-2}$ because $A \searrow^0 pt$. □

6 Appendix: the Bing's house

The graph BH (Fig. 7), given in [9] where it is denoted by G_b, is the 1-skeleton of the topological Bing's House, a space which is known to be contractible but non collapsible. In [9], the authors give an explicit sequence of deformations of BH, by using additions and deletions of edges, in order to prove that the

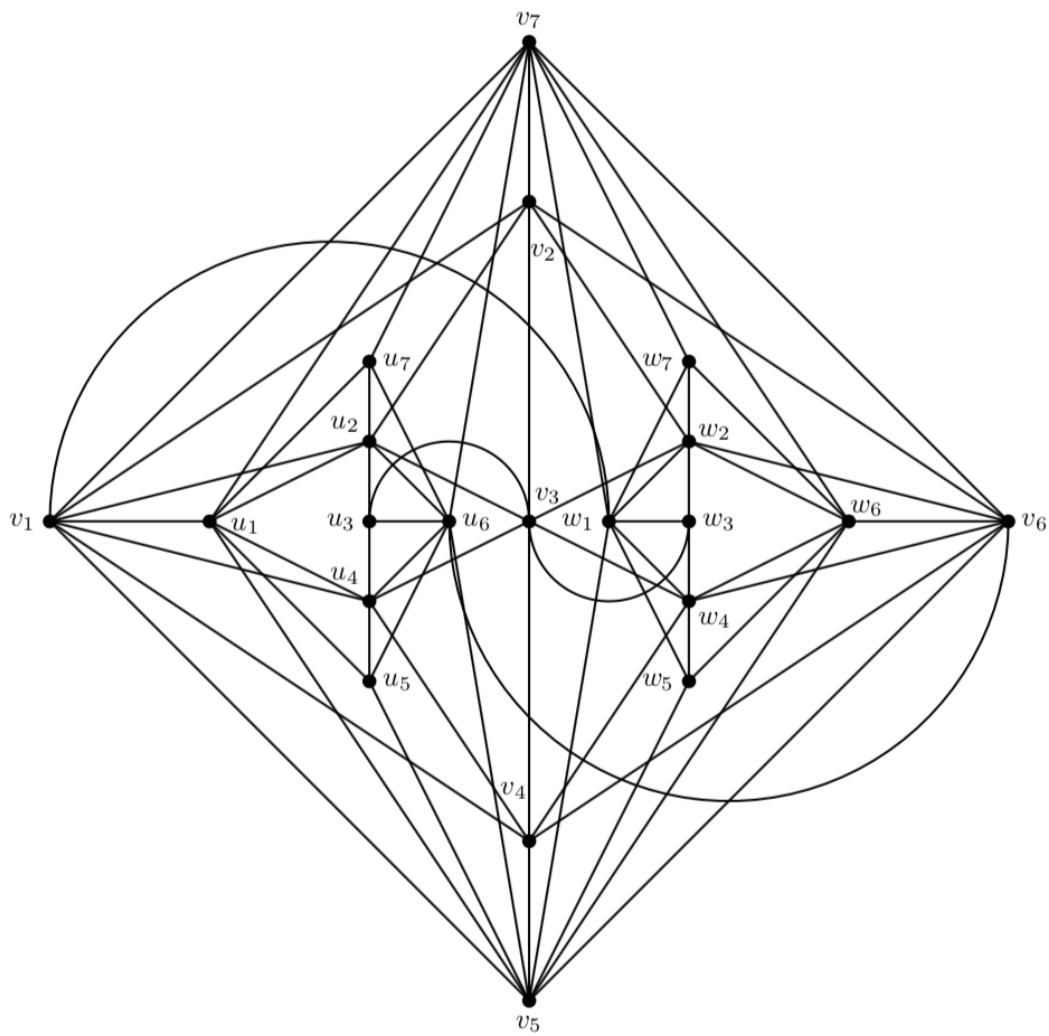

Figure 7: The graph BH, the 1-skeleton of a triangulation of the Bing's House, given in [9].

Bing's House is deformable to the simplicial complex reduced to a point by a sequence of expansions or reductions (proving that the Bing's House has the simple homotopy type of a point). We give here a more precise result with the proof (Proposition 15 for the Bing's House) of the existence of a graph W_{BH} such that:

$$BH \overset{0}{\nearrow} W_{BH} \overset{1}{\searrow} pt$$

Let be $G_0 = BH$, we do the following transformations (illustrated in Fig. 8):

1. Within $N_{G_0}(u_1)$: 0-additions of vertices u_1' and u_1'' linked to u_1, v_1, u_2, u_7, v_7 and u_1, v_1, u_4, u_5, v_5 respectively. Note that $u_1' \vdash u_1$ and $u_1'' \vdash u_1$. After that, since $N_{G_0+u_1'+u_1''}(u_1)$ is made of two 4-wheels glued in vertex v_1, and thus is a 0-dismantlable graph, it is possible to 1-delete u_1. Let us note $G_1 = G_0 + u_1' + u_1'' - u_1$.

2. Within $N_{G_1}(v_1)$: 0-additions of vertices v_1' and v_1'', linked to v_1, u_1', u_2, v_2, v_7 and $v_1, u_1'', u_4, v_4, v_5$ respectively, and 1-deletion of vertex v_1. Let us note $G_2 = G_1 + v_1' + v_1'' - v_1$.

3. Within $N_{G_2}(w_6)$: 0-additions of vertices w_6' and w_6'', linked to w_6, v_6, v_5, w_5, w_4 and w_6, v_6, w_2, w_7, v_7 respectively, and 1-deletion of vertex w_6. Let us note $G_3 = G_2 + w_6' + w_6'' - w_6$.

4. Within $N_{G_3}(v_6)$: 0-additions of vertices v_6' and v_6'', linked to v_6, w_6', v_5, v_4, w_4 and $v_6, w_6'', w_2, v_2, v_7$ respectively, and 1-deletion of vertex v_6. Let us note $G_4 = G_3 + v_6' + v_6'' - v_6$.

Now, observe that $G_4 \in D_1$. Indeed (see Fig. 9):

5. Vertices v_5 and v_7 are in $D_1(G_4)$ since their neighbourhoods are paths $w_1w_5w'_6v'_6v_4v''_1u''_1u_5u_6$ and $u_6u_7u'_1v'_1v_2v''_6w''_6w_7w_1$ respectively. Let us note $G_5 = G_4 - v_5 - v_7$.

6. The graph G_5 is a planar triangulated graph which is 0-dismantlable with 22 successive 0-deletions.

The switching property (†) finishes the proof and $W_{BH} = BH + u'_1 + u''_1 + v'_1 + v''_1 + w'_6 + w''_6 + v'_6 + v''_6$.

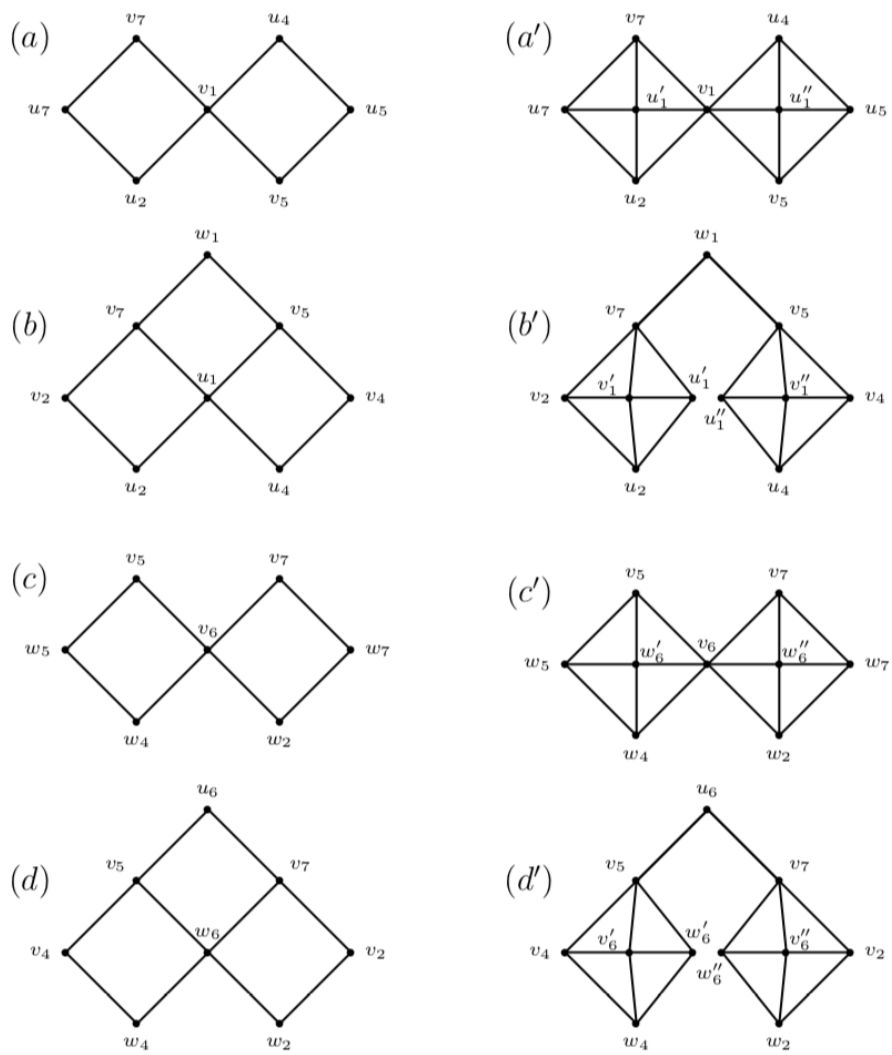

Figure 8: Eight 0-additions and four 1-deletions of vertices. (left) Neighbourhoods in $BH = G_0$ of vertices u_1 (a), v_1 (b), w_6 (c) and v_6 (d). (right) (a') neighbourhood of vertex u_1 in $G_0 + u'_1 + u''_1$, (b') neighbourhood of vertex v_1 in $G_1 + v'_1 + v''_1$ with $G_1 = G_0 + u'_1 + u''_1 - u_1$, (c') neighbourhood of vertex w_6 in $G_2 + w'_6 + w''_6$ with $G_2 = G_1 + v'_1 + v''_1 - v_1$, (d') neighbourhood of vertex v_6 in $G_3 + v'_6 + v''_6$ with $G_3 = G_2 + w'_6 + w''_6 - w_6$

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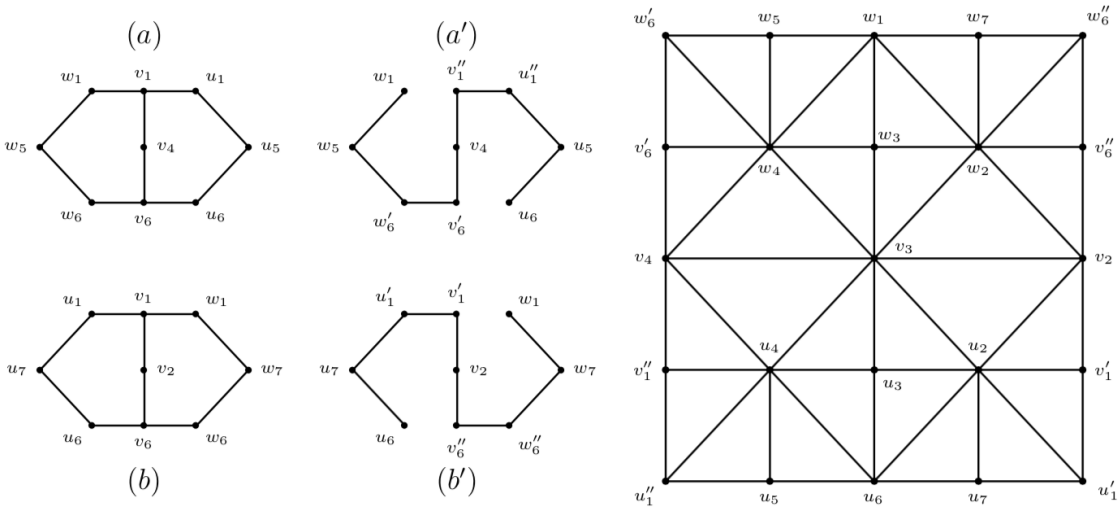

Figure 9: The graph G_4 obtained from BH by 0-addition of vertices u_1', u_1'', v_1', v_1'', w_6', w_6'', v_6', v_6'' and 1-deletion of vertices u_1, v_1, w_6, v_6 is 1-dismantlable. (left) Neighbourhoods in BH of vertices v_5 (a) and v_7 (b), (center) Neighbourhoods in G_4 of vertices v_5 (a') and v_7 (b') (right) The graph $G_4 - v_5 - v_7$ is 0-dismantlable.

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