



# On essential-selfadjointness of differential operators on closed manifolds

Yves Colin de Verdière, Corentin Le Bihan

## ► To cite this version:

Yves Colin de Verdière, Corentin Le Bihan. On essential-selfadjointness of differential operators on closed manifolds. 2020. hal-02540423v1

**HAL Id: hal-02540423**

**<https://hal.science/hal-02540423v1>**

Preprint submitted on 11 Apr 2020 (v1), last revised 30 Apr 2023 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On essential-selfadjointness of differential operators on closed manifolds

Yves Colin de Verdière\* & Corentin Le Bihan†

April 11, 2020

## 1 Introduction

The goal of this note is to present some arguments leading to the following

**Conjecture 1.1** *Let  $P$  be a formally self-adjoint differential operator on  $C^\infty(X)$  where  $X$  is a closed smooth manifold equipped with a smooth density  $|dx|$ . The completeness of the Hamiltonian flow of the symbol  $p$  of  $P$  is equivalent to the essentially self-adjointness (ESA) (also called quantum completeness) of  $P$ .*

As we will see, this conjecture holds true in the following cases:

1. Differential operators of degree 2 on the circle of the form

$$P = a(x)d_x^2 + \dots$$

where all zeroes of  $a$  are of finite order.

2. Differential operators of degree 1.
3. Generic Lorentzian Laplacians on surfaces.

**Acknowledgements.** *Many thanks to Christophe Bavard, Yves Carrière, Etienne Ghys, Nicolas Lerner and Bernard Malgrange for several discussions helping us.*

---

\*Institut Fourier, Université Grenoble-Alpes, Unité mixte de recherche CNRS-UGA 5582, BP 74, 38402-Saint Martin d'Hères Cedex (France); [yves.colin-de-verdiere@univ-grenoble-alpes.fr](mailto:yves.colin-de-verdiere@univ-grenoble-alpes.fr)

†UMPA, ENS Lyon; [corentin.le-bihan@ens-lyon.fr](mailto:corentin.le-bihan@ens-lyon.fr)

## 2 The context

It is a classical fact that classical completeness and quantum completeness are not equivalent in general; examples of Schrödinger operators on  $\mathbb{R}$  are given in [R-S-75], section X-1, pp 155-157. However the potentials involved are quite complicated near infinity: they do not admit a polynomial asymptotic behaviour. A classical results in this domain is Gaffney's Theorem [Ga-54]: if a Riemannian manifold  $(X, g)$  is complete, the Laplace operator on it is ESA. For a clear proof, see [Dav-89], pp 151–152. A natural question is then to study the case of operators with nice symbols. We will study this question on closed manifolds (i.e. compact manifolds without boundaries) for differential operators with smooth coefficients formally symmetric with respect to a smooth density. Both classical completeness and quantum completeness are independent of the coefficient of degree 0 in the operators. We will see for operators of degree 2 that they can depend of the principal and also of the subprincipal symbols.

## 3 General facts on ESA

Let  $P$  be a differential operator with smooth coefficients on a closed manifold  $X$  equipped with a smooth density  $|dx|$ . In what follows, we denote by  $L^2$  the Hilbert space  $L^2(X, |dx|)$ . We say that  $P$  is formally symmetric if, for all smooth functions  $u, v : X \rightarrow \mathbb{C}$ , we have  $\int_X Pu \bar{v} |dx| = \int_X u \overline{Pv} |dx|$ . The adjoint  $P^*$  of  $P$  has a domain  $D(P^*) = \{u \in L^2 | P'u \in L^2\}$  where  $P'$  is the operator  $P$  acting on Schwartz distributions.

The operator  $P$  with domain  $C^\infty(X, \mathbb{C})$  is then essentially self-adjoint if the graph of  $P^*$  in  $L^2 \times L^2$  is the closure of the graph of  $P$  defined on smooth functions. More explicetely,  $P$  is ESA if, for each  $v = P'u$  with  $u, v \in L^2$ , there exists a sequence  $(u_n, v_n)$  with  $u_n$  smooth,  $v_n = Pu_n$  and  $(u_n, v_n) \rightarrow (u, v)$  in  $L^2 \times L^2$ .

An equivalent usefull property is the following one

**Theorem 3.1**  *$P$  is ESA if and only if there no  $L^2$  solution of  $(P' \pm i)u = 0$ .*

The *deficiency indices* of  $P$  are then defined as  $n_\pm = \dim \ker_{L^2}(P' \pm i)$ .

In particular, we see that any elliptic operator  $P$  is ESA: if  $(P' \pm i)u = 0$ ,  $u$  is smooth and the result follows from the symmetry of  $P$ . This is why we are only interested here to non elliptic operators.

We will need also to use compact manifolds with boundary: in this case, we will always consider the Dirichlet boundary conditions and take for domain of  $P$  the space  $C^\infty(\bar{X}, \mathbb{C}) \cap \{u|_{\partial X} = 0\}$ .

## 4 Essential self-adjointness of differential operators of degree 1

In the note [L-19], Nicolas Lerner remarks that, inside the global calculus on  $\mathbb{R}^n$ , any formally self-adjoint pseudo-differential operator of degree 1 is essentially self-adjoint. This property can also be proved for formally self-adjoint pseudo-differential operators of degree 1 on closed manifolds. This follows from Lemma E.45 in the book [D-Z-19]. On a closed manifold, for  $k = 1$  and  $s = 0$ , this lemma says the following

**Lemma 4.1** *Let  $P$  be a differential operator of degree 1 on a closed manifold  $X$  and  $u \in L^2(X)$  so that  $Pu \in L^2(X)$ . There exists a sequence  $u_j \in C^\infty(X)$  so that  $u_j \rightarrow u$  and  $Pu_j \rightarrow Pu$  both in  $L^2(X)$ .*

If  $P$  is formally self-adjoint, this implies that the closure in  $L^2 \oplus L^2$  of the graph of  $P$  restricted to smooth functions is the graph of the adjoint of  $P$ . Hence  $P$  with domain  $C^\infty(X)$  is essentially self-adjoint.

**Theorem 4.1** *Any formally self-adjoint differential operator of degree 1 on a closed manifold is essentially self-adjoint.*

This holds in particular for differential operators of the form  $P := i(V + \frac{1}{2}\text{div}_{|dx|}(V))$  where  $V$  is a vector field.

## Part I

# Sturm-Liouville operators on the circle

We will consider operators  $P$  on the circle  $S^1 = \mathbb{R}/\mathbb{Z}$  of the following form

$$P = d_x a(x) d_x - i b(x) d_x - i \frac{1}{2} b'(x) + \frac{1}{4} a''(x)$$

where  $a, b$  are smooth real valued periodic functions of period 1 and the zeroes of  $a$  are of finite multiplicities. The operator  $P$  is formally symmetric on  $L^2(\mathbb{R}/\mathbb{Z}, |dx|)$ . The operator  $P$  is the Weyl quantization (see [Hor-94], section 18.5) of the symbol  $p$  of  $P$  is defined as

$$p = -a(x)\xi^2 + b(x)\xi$$

The term  $a''/4$  plays no role in the essential self-adjointness, so we will forget it in what follows. Note also that  $-a(x)\xi^2$  is the principal symbol and  $b(x)\xi$  the sub-principal symbol.

Our main result is

**Theorem 4.2** *For operators  $P$  of the previous form, classical completeness of the Hamiltonian flow of  $p$  is equivalent to quantum completeness of  $P$ .*

Our proof consists in describing the properties of  $a$  and  $b$  leading to classical completeness and to study the quantum completeness in the corresponding cases.

## 5 Classical completeness

We have the

**Theorem 5.1** *Let  $p := -a(x)\xi^2 + b(x)\xi$ . Then the Hamiltonian flow of  $h$  is complete on  $T^*S^1$  if and only if the zeroes of  $a$  are not simple and  $b$  does vanish at these zeroes. Moreover, this flow is complete if and only if it is null complete, i.e. complete when restricted to  $p^{-1}(0)$ .*

*Proof.*–

Recall that the Hamiltonian differential equation writes

$$dx/dt = -2a(x)\xi + b(x), \quad d\xi/dt = a'(x)\xi^2 - b'(x)\xi.$$

The momentum  $\xi$  cannot escape to infinity at the points  $x$  where  $a(x) \neq 0$  because  $p(x, \xi)$  is constant on each trajectory. Hence the problem is local near the zeroes of  $a$ .

- Assume that  $a(0) = 0$  and  $b(0) > 0$ . Let us start with  $x(0) > 0$  small enough and  $-a(x(0)\xi(0) + b(x(0))) = 0$ . Then  $-a(x)\xi + b(x) = 0$  along the integral curve and  $dx/dt = -b(x)$ . Hence, there exists  $t_0 > 0$  so that  $x(t_0) = 0$  and  $\xi(t_0) = \pm\infty$ .

- Assume that 0 is a non degenerate zero of  $a$  and  $b(0) = 0$ . Let us start with  $x(0) = 0$  and  $\xi(0) \neq 0$ . We have  $x(t) = 0$  for all  $t$  and  $d\xi/dt = a'(0)\xi^2 - b'(0)\xi$ . The solution of this differential equation is not defined for all  $t$ 's.
- Assume that 0 is a degenerate zero of  $a$  and  $b(0) = 0$ . We have, by conservation of  $h$ ,  $-a(x)\xi^2 + b(x)\xi = E$  and hence  $dx/dt = \pm\sqrt{-4a(x)E - b(x)^2} = O(|x|)$ . If  $x(0) \neq 0$  is close to 0,  $x(t)$  will not reach 0 and  $\xi(t)$  will remain finite. The case  $x(0) = 0$  is even easier.

□

## 6 Localisation

Let us prove the following

**Lemma 6.1** *Let  $Z := a^{-1}(0) = \{x_1, \dots, x_j, \dots, x_N\}$  and let  $U_j, 1 \leq j \leq N$  be disjoint open intervals with  $x_j \in U_j$ . Then  $P$  is ESA iff the operators  $P_{U_j}$  which are the restrictions of  $P$  to each  $U_j$  with the Dirichlet boundary conditions are ESA.*

We will in fact prove a more general result valid in any dimension:

**Lemma 6.2** *Let  $P$  be a formally self-adjoint operator of degree 2 on a closed manifold  $(X, |dx|)$ . Let  $Z \subset X$  be the closed set of points where  $P$  is not elliptic and  $U$  a neighbourhood of  $Z$  with a smooth boundary. Then  $P$  is ESA if and only if the Dirichlet restriction  $P_U$  of  $P$  to  $U$  is ESA.*

By Dirichlet restriction, we mean that the domain is the space of smooth function on  $\bar{U}$  vanishing at the boundary. This operator is known to be ESA if  $P$  is elliptic in  $U$ .

*Proof.*–

Let us first prove that, if  $P_U$  is ESA,  $P$  is ESA: let us take a  $\rho \in C_o^\infty(X \setminus Z)$  with  $\rho \equiv 1$  near  $X \setminus U$ . Then, if  $Pu = v$  with  $u, v \in L^2(X)$ ,  $\rho u \in H^2(X)$  by ellipticity of  $P$  on the support of  $\rho$ . There exists  $(u'_n, v'_n = Pu'_n)$  a sequence of smooth functions converging to  $(\rho u, P\rho u)$  in  $L^2$  by density of  $C^\infty(X)$  in  $H^2(X)$ . We have now  $P((1 - \rho)u) = w$  with  $(1 - \rho)u, w \in L^2$

and  $\text{support}((1 - \rho)u) \subset U$ . ESA of  $P_U$  allows to approximate  $(1 - \rho)u, w$  by smooth functions  $(u''_n, v''_n = P_U u''_n)$  and we can assume that  $u''_n$  vanishes near the boundary because  $(1 - \rho)u$  does.

Let us now prove that if  $P$  is ESA,  $P_U$  is ESA: let us start with  $P_U u = v$  with  $(u, v) \in L^2$  and  $u(\partial U) = 0$ . Similarly to the previous argument, we can approximate  $(\chi_U \rho u, P(\chi_U \rho u))$  by elements of the graph of  $P_U$ . We are left with  $(1 - \rho)u$  which is now in the domain of  $P_U^*$ .

□

## 7 Simple zeroes of $a$

Let 0 be a simple zero of  $a$  and  $I = [-\alpha, \alpha]$  with no other zeroes of  $a$  inside  $I$ . The point 0 is a regular singular point (see Appendix A). of the differential equation  $(P - i)u = 0$ .

The indicial equation writes  $Ar^2 - iBr = 0$  with  $A := a'(0)$ ,  $B = b(0)$ . Hence the solutions of this equation near 0 writes, for  $x > 0$ ,  $u(x) = f(x) + x_+^{iB/A}g(x)$  if  $B \neq 0$  and  $u(x) = f(x) + g(x) \log x$  if  $B = 0$  with  $f, g$  smooth up to  $x = 0$  (see Appendix A) and similarly for  $x < 0$  with  $x_-$  and  $\log(-x)$ .

Let  $u_+$  be the solution of  $(P - i)u_+ = 0$ ,  $u'_+(\alpha) = 0$  and  $u'_+(\alpha) = 1$  on  $]0, \alpha]$ . And define  $u_-$  similarly. If we extend  $u_+$  by zero for  $x < 0$ , we get a Schwartz distribution  $U_+$  and  $(P' - i)U_+$  is supported by the origine. We have  $P'U_+ = d_x d_x a U_+ + [a, d_x]U_+ - i b d_x U_+$ . We see that  $d_x a U_+$  is in  $L^2_{\text{loc}}$  and so  $P'U_+$  is locally in the Sobolev space  $H^{-1}$ , because  $d_x a U_+ \in L^2$ . The derivatives  $\delta'(0), \dots$  of the Dirac distribution are not in  $H^{-1}$ . It follows that  $(P' - i)U_{\pm} = \mu_{\pm} \delta(0)$ .

Hence there is a non zero linear combination  $U$  of  $U_+$  and  $U_-$  which satisfies  $(P' - i)U = 0$ . This proves that  $P_I$  is not ESA and hence  $P$  is not ESA.

## 8 Degenerate zeroes where $b(0)$ vanishes

In what follows  $k \geq 2$  (resp.  $l \geq 1$ ) is the order of the zero  $x = 0$  of  $a$  (resp.  $b$ ). We denote  $a(x) = x^k A(x)$ ,  $B(x) = x^l B(x)$ . We will show that there exists a solution of  $Pu = 0$  on  $]0, c[$  which is not in  $L^2$ .

If  $k \leq l + 1$ , 0 is regular singular point. with indicial equation

$$A(0)r^2 + ((k - 1)A(0) + iC_1)r + iC_2 = 0$$

with  $C_1$  and  $C_2$  real. The real part of the sum of roots is  $1 - k \leq -1$ . Hence there exists at least a solution which is not in  $L^2$  near 0.

If  $k \geq l + 2$ , then we have an irregular singular point of the form studied in Appendix A. There exists a solution of the form  $u(x) = x^{-l/2}u_1(x)$  with  $u_1$  smooth up to 0 and  $u_1(0) \neq 0$ . This function is not in  $L^2$  near 0.

This proves that we are in the limit point case (see [R-S-75], Theorem X.7): hence, if all zeroes of  $a$  are degenerate and  $b$  vanishes on  $a^{-1}(0)$ ,  $P$  is already ESA on  $C_0^\infty(S^1 \setminus a^{-1}(0))$ . Hence  $P$  is ESA on  $C^\infty(S^1)$ .

## 9 Degenerate zeroes where $b(0)$ does not vanish

We will prove the following

**Lemma 9.1** *Let us choose a smooth function  $E$  on  $I := ]0, c]$  so that  $E' = b/a$ . Two independent solutions  $u_1$  and  $u_2$  of  $(P - i)u = 0$  on  $I$  satisfies  $u_1$  is smooth up to 0,  $u_2 = u_3 e^{iE}$  with  $u_3$  smooth up to 0.*

It follows that the functions  $a(x)d_x u_j$  are in  $L^2$  and that  $P$  is not ESA by the same argument than in Section 7.

*Proof.* –

(of Lemma) We check first the existence of  $u_1$  in an elementary way by showing the existence of a full Taylor expansion directly. Then we make the Ansatz  $u_2 = u_1 v$  and we get the following differential equation for  $v$ :

$$\left( d_x + \frac{a'}{a} + 2 \frac{u_0'}{u_0} - i \frac{b}{a} \right) d_x v = 0$$

It follows that, we can choose

$$d_x v = \frac{1}{a u_0^2} e^{iE}$$

The result follows then by direct identification of the asymptotic expansion of  $d_x(w e^{iE})$ .



□

## Part II

# Lorentzian Laplacians on surfaces

## 10 General facts on Lorentzian tori

We will consider 2-tori with smooth Lorentzian metrics. There is, as in the Riemannian case, an associated geodesic flow (the Hamiltonian flow of the dual metric), a canonical volume form and a Laplace operator, which is an hyperbolic operator.

It is known that Lorentzian metrics on the 2-torus are not always geodesically complete. It is the case for example for the Clifton-Pohl torus:

Let  $T$  be the quotient of  $\mathbb{R}^2 \setminus 0$  by the group generated by the homothety of ratio 2. On  $T$ , the Clifton-Pohl Lorentzian metric is  $g := dx dy / (x^2 + y^2)$ . The associated Laplacian  $\square_g = (x^2 + y^2) \partial^2 / \partial x \partial y$  is formally self-adjoint on  $L^2(T, |dx dy| / (x^2 + y^2))$ .

There is also a much simpler example, namely the quotient on  $(\mathbb{R}_x^+ \times \mathbb{R}_y, dx dy)$  by the group generated by  $(x, y) \rightarrow (2x, y/2)$ . The manifold is not closed, but non completeness sits already in a compact region.

It is known these metric are not geodesically complete. What about ESA of  $\square_g$ ?

## 11 Some results

We will prove a rather general result:

**Theorem 11.1** *1) If the metric  $g$  admits a closed null leave of which the Poincaré section is not tangent to infinite order to the identity, then  $g$  is not geodesically complete and  $\square_g$  is not ESA.*

*2) If  $g$  is conformal to a flat metric with a smooth conformal factor on a 2-torus, then  $\square_g$  is ESA.*

**Remark 11.1** *In the first case, the incompleteness of the geodesic flow is due to Yves Carrière and Luc Rozoy [C-R-94]. We have already null-incompleteness. We will reprove it.*

*If  $X$  is compact, the first assumption is weaker than being Morse-Smale which is already a generic property (see [PdM-82]).*

*Note that the conformal class of a Lorentzian metric is determined by the null foliations; hence ESA is a property of these foliations.*

We will first need some Lemmas:

**Lemma 11.1** *The null-geodesic completeness is invariant by conformal change.*

*Proof.* –

If  $g = e^\phi g_0$ , the dual metric satisfy  $g^\star = e^{-\phi} g_0^\star$  and hence the Hamiltonian dynamics restricted to  $g^\star = 0$  are conformal with a bounded ratio.

□

**Lemma 11.2** *The ESA property is invariant by conformal change.*

*Proof.* –

If  $\square_g u = v$ , we have also  $\square_{g_0} u = e^\phi v$  and  $e^\phi v$  is in  $L^2$  as soon as  $v$  is. Hence, if  $\square_{g_0}$  is ESA, there exists a sequence  $(u_n, w_n = \square_{g_0} u_n)_{n \in \mathbb{N}}$  converging in  $L^2$  to  $(u, e^\phi v)$  and  $e^{-\phi} w_n$  converges to  $v$ .

□

This proves part 2 of Theorem 11.1.

## 12 Normal forms

It is well known and due to Sternberg [St-57] that a smooth germ of map  $(\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$  whose differential at the origine is in  $]0, 1[ \cup ]1, +\infty[$  is smoothly conjugated to  $y \rightarrow \lambda y$  and hence is the time 1 flow of the vector field  $\mu y \partial_y$  with  $\lambda = e^\mu$ .

A similar result hold for more degenerated diffeomorphisms: we assume that  $g$  admits a closed null-leave so that the Poincaré map is of the form  $P = \text{Id} + R$  where  $R$  is exactly of order  $k$  with  $k \geq 2$ . It is proved in [Ta-73] (see Theorem 4)

**Theorem 12.1** *Any such map is the time 1 flow of a vector field  $V = A(y)\partial_y$  with  $A \sim A_0 y^k$ .*

Let  $\gamma$  be closed null-leave of  $g$  and  $U$  a neighbourhood of  $\gamma$  so that the null foliations of  $g$  are  $\mathcal{F}$ , the foliation containing  $\gamma$  as a leave, and  $\mathcal{G}$  the transverse null-foliation. We have the:

**Theorem 12.2** *Let  $\gamma$  be a closed leave whose Poincaré map is  $P = \text{Id} + R$  with  $R$  of order  $k$ . There exists coordinates near  $\gamma$  so that the metric  $g$  is conformal to  $g_0 = dx(dy - a(y)dx)$  with  $a(y) \sim a_0 y^k, a_0 \neq 0$ .*

*Proof.*–

Let us parametrize the closed leave  $\gamma$  by  $x \in \mathbb{R}/\mathbb{Z}$  and extend the coordinate  $x$  in some neighbourhood  $U$  of  $\gamma$  so that the null foliation  $\mathcal{G}$  is given by  $dx = 0$ . Choose then for  $y$  any coordinate in  $U$  so that  $y = 0$  on  $\gamma$ . We introduce the differential equation  $dy/dx = b(x, y)$  associated to the foliation  $\mathcal{F}$  close to  $\gamma$ . Note that  $b(x, 0) = 0$ . Let  $\phi_x(y)$  be the flow of this differential equation. The map  $y \rightarrow \phi_1(y)$  is the Poincaré map of  $\gamma$ . By Theorem 12.1, we can choose a vector field  $a(y)\partial_y$  so that the time one flow is the same Poincaré map; and denote by  $(\phi_0)_x(y)$  this flow. Let us consider the germ of diffeomorphism near  $\gamma$  defined by

$$F : (x, y) \rightarrow (x, y' = (\phi_0)_x \circ \phi_x^{-1}(y)).$$

The map  $F$  sends the integral curves of  $dy - bdx$  onto the integral curves of  $dy - adx$  and is periodic of period 1 because the time 1 flows are the same. Hence the two null foliation are given respectively by  $dx = 0$  and  $dy' - a(y')dx = 0$ . The Theorem follows. □

## 13 Proof of Theorem 11.1, part 1

The idea is to use the normal form which, being invariant by translation in  $x$ , allows a separation of variables and hence application of the results of part I.

Let us first prove the null incompleteness. Using the normal form and the conformal invariance of null completeness, we have to study near  $y = 0$  the Hamiltonian  $h = -\eta(a(y)\eta + \xi)$ . The function  $\xi$  is a constant of the motion. Let us take initial conditions with  $y_0 > 0$ ,  $\xi_0 > 0$ , and  $a(y_0)\eta_0 + \xi_0 = 0$ . We have, using that  $a(y)\eta + \xi_0$  stays at 0,  $dy/dt = 2a(y)\eta = -2\xi_0$ . Hence  $y(t)$  vanishes for a finite time  $t_0$  and, we have then  $\eta(t_0) = \infty$ . Null incompleteness follows.

The Lorentzian Laplacian associated to  $g = dx(dy - a(y)dy)$  is given by

$$\square = \partial_y a(y) \partial_y + \partial_{xy}^2$$

Let us look at solutions of  $\square u = v$  of the form  $u(x, y) = e^{2\pi i x} v(y)$  with  $v$  compactly supported near 0. We have

$$\square u(x, y) = e^{2\pi i x} (\partial_y a(y) \partial_y + 2i\pi \partial_y) v(y)$$

The operator  $P := \partial_y a(y) \partial_y + 2i\pi \partial_y$  is a Sturm-Liouville operator already studied in part I.  $P$  is not ESA. It follows then that there exists  $v$  compactly supported near 0 and  $L^2$  so that  $P'v = w \in L^2$  and there is no sequences  $(v_n, w_n = Pv_n)$  converging in  $L^2 \times L^2$  to  $(v, w)$ . The result follows.

## 14 Further questions

There are still several open problems in this setting; we see at least three of them:

1. Prove our conjecture 1.1.
2. Describe the self-adjoint extensions in the case of Lorentzian tori in a geometrical way.
3. If we choose a self-adjoint extension, are they interesting spectral asymptotics?
4. Extend to higher dimensional Lorentzian manifolds, even to pseudo-differential operators of principal type.

## A Appendix: Basic facts on linear differential equations of order two

For this section, one can look at [Co-Le-55] and [Wa-65].

## A.1 Regular singular points

We consider a linear differential equation

$$Pu := (a(x)d_x^2 + b(x)d_x + c(x))u = 0$$

We assume that  $a(0) = 0$  and 0 is a zero of finite order  $k$  of  $a$ . The singular point  $x = 0$  of  $P$  is said to be *regular* if  $b$  (resp.  $c$ ) vanishes at order at least  $k - 1$  (resp.  $k - 2$ ) at  $x = 0$ . Otherwise 0 is an *irregular* singular point.

If  $x = 0$  is a regular singular point, we introduce the *indicial equation*:

$$a^{(k)}(0)r(r - 1) + kb^{(k-1)}(0)r + k(k - 1)c^{(k-2)}, \quad r \in \mathbb{C}$$

We call  $r_1, r_2$  the two roots of the indicial equation. Then

- If  $\text{Im}(r_1 - r_2) \notin \mathbb{Z}$ , there exists two independent solutions of  $Pu = 0$  on a small interval  $]0, c[$  of the form  $u_j = x_+^{r_j} v_j(x)$
- Similarly, if  $\text{Im}(r_2 - r_1) \in \mathbb{N}$ , we have  $u_1 = x_+^{r_1} v_1(x)$  and  $u_2 = x_+^{r_2} (v_2(x) \log x + v_3(x))$

where the functions  $v_j$  are smooth on  $[0, c[$  and  $v_1(0) = v_2(0) = 1$ .

## A.2 Some irregular singular points

We consider a singular point at  $x = 0$  of the form

$$\left( x^3 \alpha(x) d_x^2 + (x^2 \beta(x) + x) d_x + \left( x \gamma(x) + \frac{l}{2} \right) \right) u = 0$$

with  $\alpha, \beta, \gamma$  smooth. There exists then a solution of the form

$$u(x) = x^{-l/2} v(x)$$

with  $v(0) = 1$  and  $v$  smooth up to 0. The proof is routine: first find a formal solution, then apply Malgrange's Theorem.

## References

- [C-R-94] Yves Carrière & Luc Rozoy. *Complétude des Métriques Lorentziennes sur  $\mathbf{T}^2$  et Difféomorphismes du Cercle*. Bol. Soc. Bras. Mat. **25** (2):223–235 (1994).

- [Co-Le-55] Earl A. Coddington & Norman Levinson. *Theory of ordinary differential equations*. McGraw-Hill (1955).
- [C-SR-19] Yves Colin de Verdière & Laure Saint-Raymond. *Attractors for two dimensional waves with homogeneous Hamiltonians of degree 0*. Comm. Pure Appl. Math. **73**(2):421–462 (2020).
- [Dav-89] Edward Brian Davies. *Heat kernels and spectral theory*. Cambridge U. P. (1989).
- [D-Z-19] Semyon Dyatlov & Maciej Zworski. *Mathematical theory of scattering resonances*. AMS Graduate Studies in Mathematics series **200** (2019).
- [Ga-54] Matthew P. Gaffney. *A special Stokes’s theorem for complete Riemannian manifolds*. Ann. of Math. **60**:140–145 (1954).
- [Hor-94] Lars Hörmander. *The Analysis of Linear Partial Differential Operators III, Pseudo-Differential Operators*. Springer Grundlehren der Mathematischen Wissenschaften **274** (1994).
- [L-19] Nicolas Lerner. *Pseudo-differential operators of degree one are ESA*. Manuscript (2019).
- [Ma-74] Bernard Malgrange. *Sur les points singuliers des équations différentielles*. Enseign. Math. **20**:147–176 (1974).
- [PdM-82] J. Palis & W. de Melo. *Geometric Theory of Dynamical Systems. An Introduction*. Springer-Verlag (1982).
- [R-S-75] Michael Reed & Barry Simon. *Methods of modern mathematical physics*. Academic Press (1975).
- [St-57] S. Sternberg. *Local  $C^n$  transformations of the real line*. Duke Math. J. **24**, 97–102 (1957).
- [Ta-73] Floris Takens. *Normal forms for certain singularities of vector fields*. Ann. Inst. Fourier **23**(2):163–195 (1973).
- [Wa-65] Wolfgang Wasow. *Asymptotic expansions for ordinary differential equations*. Dover Publications (1965).