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# Technical Note\*: Performance analysis for time-delay systems and application to the control of an active mass damper

Yassine Ariba<sup>†</sup> and Frédéric Gouaisbaut<sup>‡</sup>

April 10, 2020

## Abstract

This paper proposes a method to analyze, beyond stability, the performances of linear time-delay systems. Using robust analysis techniques, a sufficient condition that analyzes the location of eigenvalues in the complex plane is presented. More precisely, a set of quadratic inequality constraints are designed to define an admissible region for the infinitely many eigenvalues of a time-delay system and the quadratic separation theorem is applied to assess that the eigenvalues are effectively belonging to that stability region. This method is then used for the control of an active mass damper. A standard state feedback control is replaced with a static output feedback plus a static delayed output feedback. This strategy avoids the full measurement of the state and shows that delays in the dynamic may be helpful for stabilization. The closed-loop system is then expressed as a time-delay system and the performance criterion is exploited to analyze the stability and the damping properties. Simulations and experimental tests support the approach.

## 1 Introduction

Analyzing the stability of a linear time delay system is a challenging task since this is an infinite dimensional system and it admits an infinite number of eigenvalues. Assessing the stability by inspecting the root locus is therefore complicated even if some studies have been carried out in this direction as [3], [4]. In the literature many works have been dedicated to find numerically tractable conditions ensuring the stability of time-delay systems by using either Lyapunov approach in [5], [6] or Input-Output approach in [1], [11], [12]. Recently, we are assisting to a renewal of this topic since the use of new integral inequalities,

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\*This is a (little bit) longer version of the conference paper from the World Congress of IFAC 2020.

<sup>†</sup>Icam, School of engineering, Toulouse, France (e-mail: yassine.ariba@icam.fr).

<sup>‡</sup>LAAS - CNRS, Université de Toulouse, CNRS, UPS, France (e-mail: fgouaisb@laas.fr).

see for instance [17], which are at the heart of recent LMI based techniques. Indeed, it generally allows to reduce the conservatism of Lyapunov-Krasovskii techniques. Regarding the performance analysis, results on the topic are scarce. In fact, only the exponential stability (also called  $\alpha$ -stability) have been studied in, for example, [8].

Mainly two approaches can be reported. The first one relies on the classical transformation  $z(t) = e^{\alpha t}x(t)$  (see [14], [9]). Classical tools are then employed to prove that the system expressed in terms of  $z$  is still asymptotically stable, inducing therefore the  $\alpha$ -stability of the original system in  $x(t)$ . The second approach considers specific Lyapunov-Krasovskii functional, incorporating the  $\alpha$  exponential information into their structure. Notice that the way of finding such a pattern is not an easy task and requires intuition. Once the selection has been made, integral inequalities are used and allow to design LMI criteria which depend on  $\alpha$ . To the best of our knowledge, the input-output approach have never been proposed to tackle this problem. In this work, we aim at considering more general regions in which eigenvalues should lie. We make use of an input-output robust approach, namely, the quadratic separation method that has already been used in the context of the stability of delay systems in [2]. Basically, starting from an existing stability condition expressed in the quadratic separation framework, we design a new set of quadratic inequality constraints to redefine the domain of definition of the Laplace variable. More specifically, this new set of inequality constraints enforces the uncertain transfer functions that models the dynamic system to be well defined in the prescribed regions. It thus implies that the eigenvalues do not belong to those regions. Therefore, instead of defining a region where eigenvalues should be located (which is impossible except for  $\alpha$ -stability), we define a region where they should not be.

The second objective of the paper is to use this methodology to stabilize an active mass damper. In that case, following some recent papers [13], we consider that the classical state feedback controller can be approximated by a delayed proportional feedback controller. The closed-loop system becomes thus a time delay system. The practical application<sup>1</sup> considered in this work is a bench-scale building like tall structure as depicted in Figure 1. The mechanical structure being flexible, a highly oscillatory behavior is observed when subjected to some disturbance forces. The principle of the active mass damper is to move an actuated cart on the top of the structure so as to counterbalance the oscillations. Then, the control problem is not only to stabilize the whole system (cart + structure), but also to dampen the vibrations.

## 2 Problem statement

We consider in this study a standard linear time-delay system of the form:

$$\dot{x}(t) = Ax(t) + A_d x(t - h) \tag{1}$$

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<sup>1</sup>URL: <https://www.quanser.com/products/active-mass-damper/>



Figure 1: *Active mass damper experiment*

where  $x \in \mathbb{R}^n$  is the state vector,  $A$  and  $A_d \in \mathbb{R}^{n \times n}$  are constant matrices, and  $h$  is a constant delay. The eigenvalues of the system (1) are solutions of the characteristic equation:

$$\det(s1_n - A - A_d e^{-sh}) = 0.$$

It is well known (see for instance [6]) that, this system admits an infinite number of solutions called characteristics roots or also eigenvalues, which satisfies the following properties:

1. There exists a finite number of eigenvalues in any set of the form  $|s| \leq \alpha, \alpha > 0$ .
2. There exists a finite number of eigenvalues for which the real part is greater than a prescribed  $\beta > 0$ .
3. The solutions of the characteristic equation satisfy

$$\lim_{|s| \rightarrow +\infty} \operatorname{Re}(s) = -\infty.$$

It means also that the time-delay system is asymptotically stable if all the eigenvalues have their real part strictly negative. At this stage, analyzing the performances of such a system is rather complicated as stated in [8]. Indeed, for LTI systems, there is only a finite number of poles and their placement in the complex plane indicates the dynamic performances. Thus, the concept of  $\mathcal{D}$ -stability was developed (see [7] and references therein) and allowed the development of methods to ensure that the poles of a finite dimensional linear

systems belong to some given regions. Convex regions of the complex plane can be easily expressed as an inequality of the form:

$$d_1 s s^* + d_2 s^* + d_2^* s + d_3 \leq 0. \quad (2)$$

Depending on the choice of parameters  $\{d_1, d_2, d_3\}$ , various regions  $\mathcal{D}$  can be defined [7]. Typical regions are:

- half-plane,  $\mathcal{D} = \{s = x + jy \in \mathbb{C} : ax + by + c < 0\}$ , where  $a, b, c$  are real scalars, obtained with the set  $\{d_1, d_2, d_3\} = \{0, a + jb, 2c\}$ .
- disk of center  $s_0 \in \mathbb{C}$  and radius  $r > 0$ ,  $\mathcal{D} = \{s \in \mathbb{C} : |s - s_0| < r\}$ , obtained with the set  $\{d_1, d_2, d_3\} = \{1, -s_0, s_0 s_0^* - r^2\}$ .

In the case of time-delay systems, due to the infinite dimensional nature of such a system, it is impossible to impose the eigenvalues to belong to certain bounded set. At least, in the literature of time delay systems, one can find the concept of  $\alpha$ -stability, meaning that the eigenvalues belong to the set for which  $Re(s) \leq \alpha$ . If  $\alpha < 0$ , it means therefore that the system is exponentially stable with a convergence rate  $\alpha$  (see [8] and references therein). The idea in this paper is to define regions that are, in this case, forbidden for the eigenvalues. Knowing the specific properties of eigenvalues of time delay systems of the form (1), suitable regions can be chosen so as to state a certain level of performances.

### 3 Main result

#### 3.1 A first stability result

Two approaches are generally considered to study the stability of the time delay system (1): the Lyapunov method and the robust analysis. This latter is employed in this paper with the quadratic separation method (see [11, 15] and references therein). It consists in modeling the time-delay system (1) as an uncertain feedback system as shown in Figure 2. The uncertainty  $\nabla$  embeds the delay dynamics in the feedback block (different types of uncertain models are presented in [15, 10, 2]) and some linear relationships between system signals are specified in the feedforward block. The stability is then tested with the conditions stated in the following theorem.

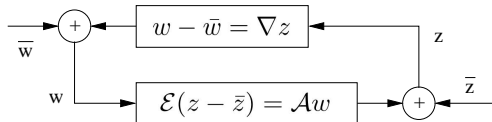


Figure 2: Feedback system.

**Theorem 1 ([15])** *Given the interconnection defined by Figure 2 where  $\mathcal{E}$  and  $\mathcal{A}$  are two real valued matrices and  $\nabla$  is a linear operator which represents the system uncertainties. This latter is assumed to belong to an uncertain set  $\mathbb{W}$ . For simplicity, we assume that  $\mathcal{E}$  is full column rank. The uncertain feedback system of Figure 2 is well-posed and stable if and only if there exists a Hermitian matrix  $\Theta = \Theta^*$  satisfying both conditions*

$$\begin{bmatrix} 1 \\ \nabla \end{bmatrix}^* \Theta \begin{bmatrix} 1 \\ \nabla \end{bmatrix} \leq 0 \quad , \quad \forall \nabla \in \mathbb{W} \quad , \quad (3)$$

$$\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} > 0. \quad (4)$$

Most of the work includes the modeling part (describing matrices  $\mathcal{E}$ ,  $\mathcal{A}$  and  $\nabla$ ) and finding a separator  $\Theta$  (based on some inequality constraints w.r.t. the uncertainty  $\nabla$ ). This latter part is built such that the inequality (3) is satisfied for all possible uncertainties belonging to  $\mathbb{W}$ . Then, the second condition (4) provides the stability test, usually formulated as an LMI condition.

Let us consider, for instance, the stability criterion given below, extracted from [2, Theorem3], and established with the quadratic separation approach. It is a pointwise delay stability condition<sup>2</sup> for systems of the form of (1). It makes use of the Bessel inequality and combines a set of transfer functions that describes the dynamical behaviour of (1). All these transfer functions are embedded in the  $\nabla$  matrix

$$\nabla = [s^{-1}\mathbf{1}_{(N+1)n}, \quad e^{-hs}\mathbf{1}_n, \quad \tilde{\delta}_N(s)]. \quad (5)$$

Transfer functions  $s^{-1}$  and  $e^{-hs}$  are, respectively, the standard integrator and delay related transfer functions. The additional ones  $\tilde{\delta}_N(s) = [\delta_0\mathbf{1}_n \quad \delta_1\mathbf{1}_n \quad \dots \quad \delta_N\mathbf{1}_n]^T$ , with  $\delta_k(s) = \int_{-h}^0 L_k(\theta) e^{s\theta} d\theta$  ( $L_k$  are Legendre polynomials defined over the interval  $[-h, 0]$ ), are delay-related transfer functions introduced to reduce the conservatism of the approach. Matrix  $\nabla$  being considered as an uncertain feedback, each transfer function has to be bounded. The following quadratic inequalities have been proven in [15, 2] for all  $s \in \mathbb{C}$  such that<sup>3</sup>  $Re(s) > 0$ .

$$\begin{bmatrix} \mathbf{1}_n \\ s^{-1}\mathbf{1}_n \end{bmatrix}^* \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{bmatrix} \mathbf{1}_n \\ s^{-1}\mathbf{1}_n \end{bmatrix} \leq 0,$$

$$\begin{bmatrix} \mathbf{1}_n \\ e^{-hs}\mathbf{1}_n \end{bmatrix}^* \begin{bmatrix} -Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} \mathbf{1}_n \\ e^{-hs}\mathbf{1}_n \end{bmatrix} \leq 0,$$

$$\sum_{k=0}^N (2k+1) \delta_k^* R \delta_k \leq h^2 R,$$

<sup>2</sup>Note that any other sophisticated stability conditions (delay-range, uncertain delay, robust w.r.t system matrices...) could have been considered. The pointwise delay case having a less cumbersome LMI test is preferred for sake of simplicity.

<sup>3</sup>This condition ensures that transfer functions are defined for any  $s$  in the right half plane, and thus no eigenvalues lies in this zone.

where  $P$ ,  $Q$  and  $R$  are  $n \times n$  positive definite matrices. From these inequalities, the separator  $\Theta$  (8) is derived.

This particular choice of matrix  $\nabla$  connects internal signals  $w(t)$  and  $z(t)$  via  $w(t) = \nabla z(t)$  with

$$w(t) = \begin{bmatrix} x(t) \\ \tilde{\delta}_{N-1}[x(t)] \\ x(t-h) \\ \tilde{\delta}_N[\dot{x}(t)] \end{bmatrix}, z(t) = \begin{bmatrix} \dot{x}(t) \\ \tilde{\delta}_{N-1}[\dot{x}(t)] \\ x(t) \\ \dot{x}(t) \end{bmatrix}. \quad (6)$$

Then, the feedforward block is described by :

$$\underbrace{\begin{bmatrix} \mathbf{1}_{(N+3)n} \\ \mathbf{0}_{(N+1)n \times (N+3)n} \end{bmatrix}}_{\mathcal{E}} z(t) = \mathcal{A} w(t) \quad (7)$$

with

$$\mathcal{A} = \left[ \begin{array}{cccc|cccc} A & 0 & \dots & 0 & A_d & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & & 1 & 0 \\ \hline 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ A & 0 & \dots & 0 & A_d & 0 & \dots & 0 & 0 \\ \hline 1 & & & & -1 & & & & \\ 1 & & & & 1 & & & & \\ 0 & & & & 0 & & & & \\ \vdots & & & & \vdots & & & & \\ 0 & & & & 0 & & & & \end{array} \right],$$

and

$$U = \begin{bmatrix} 0 & 0 & \dots & 0 \\ -\frac{2}{h}\mathbf{1} & 0 & \dots & 0 \\ 0 & -\frac{6}{h}\mathbf{1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -\frac{2(2N+1)}{h}\mathbf{1} \end{bmatrix}, V = \begin{bmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & 1 & 0 & -1 \end{bmatrix}.$$

Applying Theorem 1 with this modeling strategy has led to the stability criterion below, which has shown very interesting results in terms of reduction of conservatism.

**Theorem 2 ([2])** *Assume that  $A + A_d$  is a non singular matrix. For a given constant delay  $h$  and for a given  $N \geq 1$ , if there exist positive definite matrices  $P \in \mathbb{R}^{(N+1)n \times (N+1)n}$ ,  $Q, R \in \mathbb{R}^{n \times n}$  such that the following LMI is satisfied:*

$$[\mathcal{E} \quad -\mathcal{A}]^{\perp*} \Theta [\mathcal{E} \quad -\mathcal{A}]^{\perp} > 0,$$

where matrices  $\mathcal{E}$ ,  $\mathcal{A}$  are defined in (7) and  $\Theta$  by (8):

$$\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2^* & \Theta_3 \end{bmatrix} \quad (8)$$

with

$$\begin{aligned}\Theta_1 &= \text{diag}(0_{(N+1)n}, -Q, -h^2R), \\ \Theta_2 &= \text{diag}(-P, 0_n, 0_{n \times (N+1)n}), \\ \Theta_3 &= \text{diag}(0_{(N+1)n}, Q, R, 3R, \dots, (2N+1)R),\end{aligned}$$

then system (1) is asymptotically stable for the given  $h$ .

**Proof 1** See [2].

In this theorem, as in other results [10, 15], the uncertain set  $\mathbb{W}$  is chosen such that

$$\mathbb{W} = \{\nabla(s), \text{Re}(s) > 0\}.$$

We work on that set in the next paragraph to design a performance analysis criterion. The quadratic separation appears to be a convenient framework for this objective because all the structure of the uncertain feedback modeling (5)-(6)-(7) remains unchanged, only the separator matrix  $\Theta$  needs to be recalculated if the domain of definition of  $s$  is modified.

### 3.2 Performance analysis

The performance of a time delay system is directly linked to the placement of its eigenvalues in the complex plane. Constraining the infinitely many eigenvalues to a desired region is impossible, except for vertical half planes. The principle of the proposed approach consists in specifying that the Laplace variable  $s$ , used in quadratic constraints for each transfer functions defined in  $\nabla$ , belongs to the desired region  $\mathcal{D}$ . As mentioned earlier, starting from a given stability criterion, the key idea is to redefine condition (3) with a new separator  $\Theta$  depending on  $\mathcal{D}$ , while the other matrices remain unchanged. The resulting condition of Theorem 1 will then imply that there is no eigenvalue in  $\mathcal{D}$ , what can be translated into some performance indexes (responsiveness, damping). This methodology requires additional constraints on the definition of the uncertain matrix  $\nabla$ . The following lemmas show the new separator for each transfer functions in  $\nabla$ .

**Lemma 1** A quadratic constraint for  $s^{-1}$  is given by the following inequality for any positive definite matrix  $P$  in  $\mathbb{R}^{n \times n}$ ,

$$\begin{bmatrix} \mathbf{1}_n \\ s^{-1}\mathbf{1}_n \end{bmatrix}^* \begin{bmatrix} d_1P & d_2P \\ d_2^*P & d_3P \end{bmatrix} \begin{bmatrix} \mathbf{1}_n \\ s^{-1}\mathbf{1}_n \end{bmatrix} \leq 0, \\ \forall s \in \mathcal{D}, s \neq 0.$$

**Proof 2** By assumption,  $s \neq 0$ . Let us divide the inequality (2) by the strictly positive number  $|s|^2$  to make appear the integrator transfer function:

$$d_1 + d_2 \frac{1}{s} + d_2^* \frac{1}{s^*} + d_3 \frac{1}{ss^*} \leq 0.$$

Then, factorizing by  $[1 \ s^{-1}]^*$  on the left and by  $[1 \ s^{-1}]^T$  on the right, and inserting the positive definite matrix  $P$ , the inequality of the lemma is obtained.



**Lemma 2** A quadratic constraint for  $e^{-hs}$  is given by the following inequality for any positive definite matrix  $Q$  in  $\mathbb{R}^{n \times n}$ ,

$$\begin{bmatrix} \mathbf{1}_n \\ e^{-hs}\mathbf{1}_n \end{bmatrix}^* \begin{bmatrix} -e^{2ah}Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} \mathbf{1}_n \\ e^{-hs}\mathbf{1}_n \end{bmatrix} \leq 0, \quad \forall s \in \mathcal{D},$$

where  $a$  is the real part of the leftmost point of the region  $\mathcal{D}$  in the complex plan.

**Proof 3** Let us express the modulus the delay transfer function:

$$e^{-hs}e^{-hs^*} = e^{-2\operatorname{Re}(s)h} \leq e^{-2ah},$$

the scalar  $a$  being the leftmost point of the admissible region  $\mathcal{D}$  for  $s$  so that  $\operatorname{Re}(s) > a$ . Usually,  $a$  is negative since the simple stability property is included in the condition. Again, factorizing by  $[1 \ e^{-hs^*}]$  on the left and by  $[1 \ e^{-hs}]^T$  on the right, and inserting the positive definite matrix  $Q$ , the inequality of the lemma is obtained.

**Lemma 3** A quadratic constraint for  $\tilde{\delta}_N(s) = [\delta_0 \ \delta_1 \ \dots \ \delta_N]^T$  is given by the following inequality for any positive definite matrix  $R \in \mathbb{R}^{n \times n}$ ,

$$\begin{bmatrix} \mathbf{1}_n \\ \delta_0\mathbf{1}_n \\ \delta_1\mathbf{1}_n \\ \vdots \\ \delta_N\mathbf{1}_n \end{bmatrix}^* Z(\alpha, h) \begin{bmatrix} \mathbf{1}_n \\ \delta_0\mathbf{1}_n \\ \delta_1\mathbf{1}_n \\ \vdots \\ \delta_N\mathbf{1}_n \end{bmatrix} \leq 0, \quad \forall s \in \mathcal{D}.$$

with

$$Z(\alpha, h) \begin{bmatrix} \mu R & 0 & 0 & \dots & 0 \\ 0 & R & 0 & \dots & 0 \\ 0 & 0 & 3R & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & (2N+1)R \end{bmatrix}$$

and  $\mu = -\frac{h}{2a}(1 - e^{-2ah})$ .

**Proof 4** In the former result for simple stability, presented in [2], it has been shown using the Bessel inequality that

$$\sum_{k=0}^N (2k+1)\delta_k\delta_k^* \leq \langle e^{s\theta}, e^{s\theta} \rangle. \quad (9)$$

In the case of performance analysis, the upperbound on the inner product<sup>4</sup>  $\langle e^{s\theta}, e^{s\theta} \rangle$  need to be re-evaluated:

$$\begin{aligned} \langle e^{s\theta}, e^{s\theta} \rangle &= \int_{-h}^0 e^{s\theta} e^{s^*\theta} d\theta \leq \int_{-h}^0 e^{2a\theta} d\theta \\ &= \frac{1}{2a}(1 - e^{-2ah}), \end{aligned}$$

<sup>4</sup>In this framework, considering time delay systems, the inner product was defined as  $\langle f, g \rangle = \int_{-h}^0 f(\theta)g^*(\theta) d\theta$ .

where  $a$  is the real part of the leftmost point of the region  $\mathcal{D}$  in the complex plan. At last, factorizing by  $[1 \ \delta_0^* \ \dots \ \delta_N^*]$  on the left and by  $[1 \ \delta_0 \ \dots \ \delta_N]^T$  on the right, and inserting the positive definite matrix  $R$ , the inequality of the lemma is obtained.

We can now state a theorem for the performance analysis for time-delay systems (1).

**Theorem 3** *Assume that  $A + A_d$  is a non singular matrix. For some given complex scalar parameters  $\{d_1, d_2, d_3\}$  that define a region  $\mathcal{D}$  in the complex plan (2). For a given constant delay  $h$  and for a given  $N \geq 1$ , if there exist positive definite matrices  $P \in \mathbb{R}^{(N+1)n \times (N+1)n}$ ,  $Q, R \in \mathbb{R}^{n \times n}$  such that the following LMI is satisfied:*

$$[\mathcal{E} \quad -\mathcal{A}]^{\perp*} \Theta [\mathcal{E} \quad -\mathcal{A}]^{\perp} > 0,$$

where matrices  $\mathcal{E}, \mathcal{A}$  are defined in (7) and  $\Theta$  by:

$$\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2^* & \Theta_3 \end{bmatrix}$$

with

$$\begin{aligned} \Theta_1 &= \text{diag}(d_1 P, -e^{2ah} Q, -\frac{h}{2a}(1 - e^{-2ah})R), \\ \Theta_2 &= \text{diag}(d_2 P, 0_n, 0_{n \times (N+1)n}), \\ \Theta_3 &= \text{diag}(d_3 P, Q, R, 3R, \dots, (2N+1)R), \end{aligned}$$

then system (1) has no eigenvalues in  $\mathcal{D}$  for the given  $h$ .

The parameter  $N$  corresponds to the degree of the Legendre polynomial that is used to approximate the delay transfer function in the uncertain feedback modeling [2]. It has been shown that increasing  $N$  reduces the conservatism of the criterion, at the expense of the numerical burden.

**Remark 1** *Theorem 3 provides a sufficient condition ensuring that system (1) has no eigenvalues in a region  $\mathcal{D}$  defined by the triplet  $\{d_1, d_2, d_3\}$ . To check if the system has no eigenvalues belonging to multiple regions, it is sufficient to merely repeat the condition of the theorem as many as the number of regions with the corresponding triplet  $\{d_{1_i}, d_{2_i}, d_{3_i}\}$ . This convenience is due to the fact that the structure and the matrices of the system (5)-(6)-(7) remain unchanged and each condition assesses if there is any eigenvalue in the region to be tested.*

**Remark 2** *Notice that the infinite dimensional nature of the system implies an infinite numbers of roots for the characteristic equation. Then, contrary to the  $D$ -stability, where some LMI criteria are developed to ensure that the poles are belonging to some prescribed sets, here, in the case of time delay systems, it is impossible to impose such constraints. That being said, one ensures that the eigenvalues do not belong to some given sets (generally bounded). The proposed Theorem 3 gives therefore an optimistic method to analyse the performances of the time delay system.*

### 3.3 Numerical example

As an illustrative example, let us consider the following system with a delay  $h = 1s$ :

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-1) \quad (10)$$

Several tests have been run to illustrate the ability of Theorem 3 to assess the absence of any eigenvalue of (10) in some specific location. Those simulations are summarized in Table 1 and plotted on Figures 3, 4 and 5. For each test, the forbidden region  $\mathcal{D}$  for eigenvalues (orange zones) is compared to the actual location of (10). The eigenvalues location is determined with a MATLAB function that approximates the characteristic roots of linear delay differential equations (based on the work of [3]). Obviously, when any region of  $\mathcal{D}$  overlaps any pole spot, the LMI condition of Theorem 3 is unfeasible. It can be noticed that in some cases it is necessary to increase  $N$  so that Theorem 3 is able to detect the “ $\mathcal{D}$ -stability”. Reducing the conservatism is especially required when a region gets closer to an eigenvalue.

| test | regions  | Thm. 3 result               |
|------|--|-----------------------------|
| 1    | half-plane: $\text{Re}(s) > -0.3$<br>disc: $r = 2, s_0 = -3 + 20j$ | LMI feasible for $N \geq 2$ |
| 2    | half-plane: $\text{Re}(s) > -0.3$<br>disc: $r = 3, s_0 = -8 + 6j$  | LMI feasible for $N \geq 4$ |
| 3    | half-plane: $\text{Re}(s) > -0.3$<br>disc: $r = 1, s_0 = -4 + 11j$ | LMI feasible for $N \geq 8$ |

Table 1: Configuration parameters for the analysis of (10).

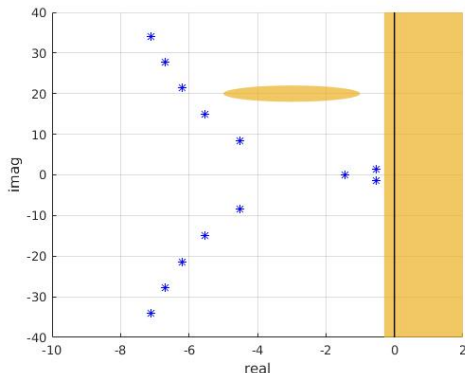


Figure 3: Analysis of (10), test 1.

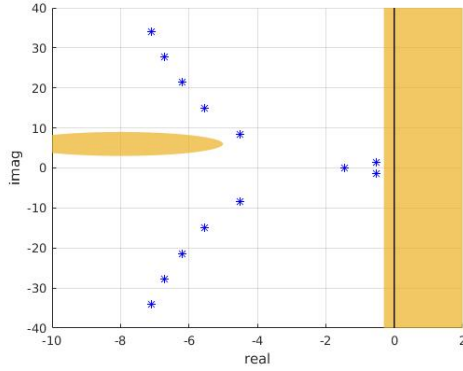


Figure 4: *Analysis of (10), test 2.*

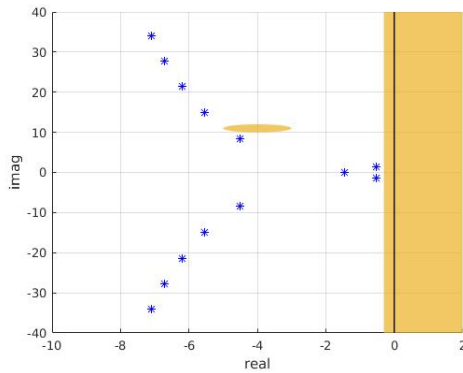


Figure 5: *Analysis of (10), test 3.*

## 4 Application to an active mass damper system

### 4.1 Description of the system

The practical application considered in this work is a bench-scale building like tall structure. The test stand is a Quanser experimental model and is depicted in Figure 1. The aim is to design a control system that dampens actively vibrations with an actuated cart on the top.

A schematic of the plant and notations are illustrated in Figure 6. For small floor deflection, the top of the structure is modeled as a linear spring-mass system.  $x_f$  is the floor horizontal displacement and  $x_c$  is the cart position. This latter is actuated with a DC motor that induces a linear force  $F_c$ . Applying the Lagrange's method and combining with the equation of the motor, a linear

model of the plant is derived:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 278.9 & -18.6 & 0 \\ 0 & -336 & 5.9 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 2.99 \\ -0.96 \end{bmatrix} u(t) \quad (11)$$

where the state space vector consists of  $x = [x_c \ x_f \ \dot{x}_c \ \dot{x}_f]^T$ . The control signal  $u$  is the voltage input of the DC motor driving the cart. A standard pole placement design provides the state feedback gain  $K = [6.9 \ -103 \ -2.8 \ -26.9]$  to have the closed-loop poles  $\{-8, -16, -6 \pm 2i\}$ .

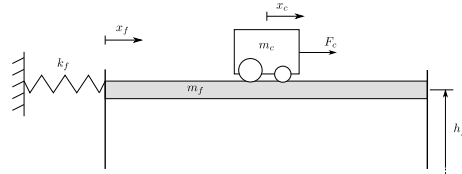


Figure 6: *Schematic of the plant and notations.*

## 4.2 Delay based control

A state feedback control requires the full state  $x$  to be available. This requirement may be a major drawback in practice as engineers usually aim at limiting the use of sensors for several reasons: feasibility, reliability, cost, maintenance... It is proposed to approximate a state feedback control with a time-delay system approach. The key idea is to replace the static state feedback control with a static output feedback combined with a static delayed output feedback. This approach recently updated by [16] assumes basically that,

$$u(t) = -Kx(t)$$

can be approximated by

$$u(t) = -K_1 y(t) - K_2 y(t - h),$$

where  $y = [x_c \ x_f]^T$  is the measured output, namely the positions. Note that the control law remains quite simple compared to a dynamical control system when a state observer is implemented. As often in mechanical system representation, the second part of the state  $x$  is the derivative of the first one. The method is based on the following approximation of the derivative non causal transfer function:

$$\dot{y}(t) \simeq \frac{y(t) - y(t - h)}{h}. \quad (12)$$

Applying a static state feedback control to system (11) with a state structured as  $x = [y \ \dot{y}]^T$ , the following closed-loop formulation is obtained

$$\dot{x}(t) = Ax(t) - BK_\alpha y(t) - BK_\beta \dot{y}(t)$$

where  $K_\alpha$  and  $K_\beta$  are components of the state feedback gain  $K = [K_\alpha \ K_\beta]$ . Approximating the last term with (12), a delay based feedback formulation is obtained:

$$\begin{aligned}\dot{x}(t) &= Ax(t) - B\left(K_\alpha + \frac{1}{h}K_\beta\right)y(t) + \frac{1}{h}BK_\beta y(t-h), \\ &= Ax(t) - BK_1y(t) + BK_2y(t-h).\end{aligned}\tag{13}$$

The analysis of the properties of the above feedback system amounts to the analysis of a standard time-delay system:

$$\dot{x}(t) = A_1x(t) + A_2x(t-h),\tag{14}$$

with  $A_1 = A - B\left(K_\alpha + \frac{1}{h}K_\beta\right)C$  and  $A_2 = \frac{1}{h}BK_\beta C$ . Let us exploit Theorem 3 to analyze the stability and performances of (14). In this application, besides stability, it is required to dampen the structure oscillations. This feature can be addressed with an appropriate setting of eigenvalues location. The objective is to prove that the properties of the theoretical closed-loop system (the linear system (11) with a state feedback) is preserved with the delay based feedback (static output feedback + static delayed output feedback).

### 4.3 Simulation

As mentioned above, a standard state feedback control with gain  $K = [6.9 \ -103 \ -2.8 \ -26.9]$  for the linear system (11) leads to a stable and damped closed-loop system with poles  $\{-8, -16, -6 \pm 2i\}$ . The parameters for the delay based control are calculated according to (13). For instance, setting a delay  $h = 60 \text{ ms}$ , the resulting static gains are

$$K_1 = [-40.5 \ -552.8] \quad \text{and} \quad K_2 = [-47.5 \ -449.5].$$

The closed-loop system is then turned into a time-delay system as (14):

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 121 & 1936 & -18.6 & 0 \\ -39 & -866 & 6 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -142 & 1347 & 0 & 0 \\ 46 & 431 & 0 & 0 \end{bmatrix} x(t-0.06)\tag{15}$$

Theorem 3 can then be applied to perform a performance analysis and to confirm that the eigenvalues of the resulting system (15) are sufficiently damped (see Figure 7). We aim at ensuring that the oscillatory behavior of the response is reduced, similarly to the full state feedback control case. Figure 8 shows simulations of both control laws and an open-loop test. The initial condition for the floor position is  $0.5m$ . The open-loop response shows the highly oscillatory behavior of the flexible structure. The system response with the delay based control is fairly similar to the one with the state feedback control and a significant vibration reduction is observed. Several simulations have been run to find a delay  $h$  for the control (13) that minimizes the amplitude of oscillations. Results with  $h = 60 \text{ ms}$  are satisfactory as shown in Figure 8.

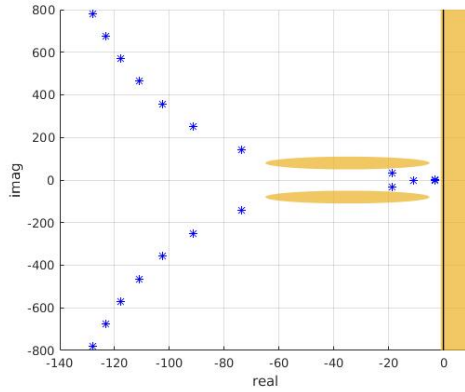


Figure 7: *Performance analysis for system (15).*

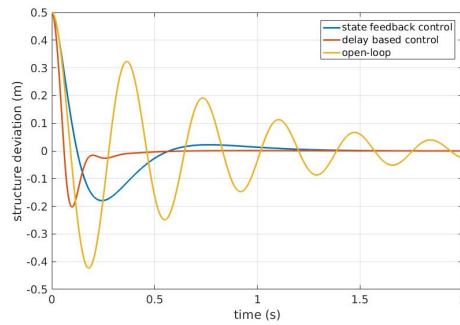


Figure 8: *Damping the floor deflection, simulation results.*

#### 4.4 Experimentation

This paragraph presents the experimental test with the bench-scale building (see Figure 1). The delay based control law is now implemented on a data acquisition device, a *Q2-usb* board with real-time computing capability. The sampling period is  $1\text{ ms}$ . The experimental results confirm the simulations of the previous paragraph. The experiment starts at rest (zero initial conditions), the system is stimulated by a brief push with the hand as an impulsive disturbance. The three configurations have been tested, and the corresponding measurements of the structure deviation  $x_f$  are plotted on Figure 9. Once again, the delay based control response is similar to the one with the state feedback control, and both are able to dampen oscillations. However, the former one requires only position information, that is two measures are used instead of four. The performance requirement in terms of damping was beforehand assessed with a performance analysis for the equivalent time-delay system.

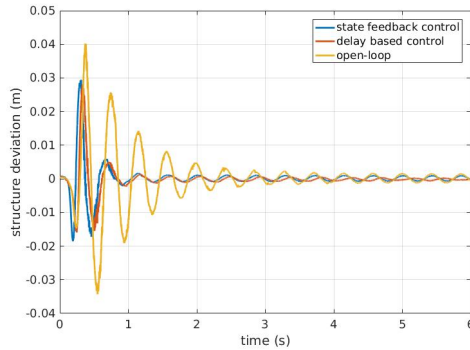


Figure 9: *Damping the floor deflection, experimental results.*

## 5 Conclusion

This paper studies the performance analysis for linear time-delay system. A sufficient condition ensuring that all eigenvalues are absent from a specific region  $\mathcal{D}$  is proposed. The condition is built with the quadratic separation technique and is formulated as an LMI condition. Then, the criterion is used to analyze the performances of an active mass damper system controlled with a delayed output feedback. Indeed, the closed-loop system is expressed as a time-delay system, the delay  $h$  being also a design parameter. Analyzing the eigenvalues location in the complex plane, we are able to assess the damping property of the control law. The validity of the approach is demonstrated with simulations and experimental tests. Future works include the design of more complex delayed control laws and the conservatism reduction of performance analysis.

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