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A NXFEM approximation for a nonlinear system of PDEs

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Abstract. Under the motivation of modeling a biofilm model, we introduce a system of semilinear elliptic interface problem. We also propose a technique of decoupling a semilinear problem and apply the Nitsche-Extended Finite Element method to prove the existence and uniqueness of solutions and their convergent properties.

1 Introduction

Biofilms are, generally, observed in aqueous media or in a media exposed to moisture. They can grow on any type of natural or artificial surface. Biofilm structures give bacteria some environment to stay and grow. They have both good and bad impacts for the human life and very important. Biofilm modeling in cooperation with laboratory experience, therefore, has rising in recent years in order to study more about it. Further about biofilm can be found in the works also refer to the works of Chopp and Duddu et al. [13, 7], Cogan et al. [9, 8, 11, 10] and the work of Rittmann [27] for instance.

Under the motivation of modeling a biofilm model, we consider a system of semilinear elliptic equations of two variables in two different regions separated by an interface. This model will be mentioned in more details in section 2. Since it is difficult to work directly on a semilinear system we introduce an intermediary variable to decouple the system into a system of a linear equation and a semilinear one.

As a motivation of biofilm, we always have in mind more general and complicated problems where the biofilms can change their size because of other impacts. It means that the interface of the problem can move with time. We are thus led to study on an unfitted finite element method in which the interface can cut the mesh's element and doesn't make the mesh change with it. This interesting idea has been studied in deep for years with many recent methods.

There are many *unfitted mesh* based methods which have been proposed for recent years, one of them is *Immersed Interface Method* (IIM) which is particularly designed for interface problems. The IIM is a sharp interface method based on Cartesian grids [20]. The work of Li [21] gave an overview of this method and some of its applications. However, when we work with discontinuous coefficients and singular sources, especially with large ones, the accuracy obtained by using IIM is not so good. Other interesting approaches are using *Extended Finite Element Method* (XFEM) [22, 3] and *Nitsche based Finite Element Method* (NFEM) [17]. XFEM is an extension of the standard finite element method in which arbitrary discontinuous functions and derivatives are added to the standard finite element approximation. A. Reusken and his coworkers had many contributions when using XFEM for two-phase incompressible flows problems [16, 19, 25, 26, 23]. NFEM uses the idea of Nitsche [24] to enforce weakly the interface condition in the weak forms. Note that, there is also relation between XFEM and NFEM which is commented in [2]. In this paper we will use NFEM and its innovated versions to work with the purpose of unfitted mesh. We use the name *NXFEM* to allude the relation of XFEM and NFEM. NXFEM is also called “unfitted FEM” or “CutFEM” in some literatures.

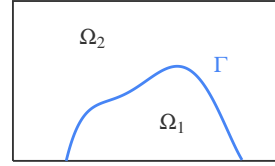
In this article, we are using the Nitsche based one (NXFEM) which first introduced by Hansbo in [17]. The key idea of NXFEM is imposing weakly the interface conditions in the weak formulation by doubling all basis functions whose support cut by the interface into two new ones. Hansbo borrowed this idea from the

method of Nistche [24]. Our purpose in this work is to use NXFEM for the semilinear problem in coupling with the decoupling idea mentioned above. One drawback of NXFEM is that the conditioning of the stiffness matrix is sensitive to the way interface cut the element. It's because there may be some ways in that the ratio between two pieces of cut is very large, the stiffness matrix thus becomes very ill-conditioned. Thanks to the Ghost penalty method proposed by Burman [5] and technique of removing small cut elements by Reusken [25], we can handle this problem. However, we don't present this problem in this work, we just implicitly use it in the implementation for the numerical section 6. In order to prove the convergence of NXFEM discrete solutions to the continuous ones, we apply the idea of proofs in Discontinuous Galerkin Method proposed by Ern and Di Pietro [12]. Their work actually relied on techniques inspired by the Finite Volume literature given in the work of Eymard et al. [15]. Noting that, Ern and Di Pietro worked on the discontinuity on each side of element mesh whereas we only work on the discontinuity of functions on the interface. The article is organized as follows. Section 2 introduces the main model with some necessary assumptions on the coefficients and functions. We also show the technique of decoupling our system, give the details in weak formulations and the proof of existence and uniqueness of continuous solutions. In Section 3 we introduce the NXFEM settings and we give assumptions on the meshes and the discrete formulations. The main part of this work is on Section 4 and Section 5 which give in detail about the existence and uniqueness of solutions of discrete problems and their convergence to the continuous solutions. Finally, in Section 6 we present a numerical test case to validate our approach.

2 The model setting

Let us consider a convex polygonal, Lipschitz and bounded domain Ω in \mathbb{R}^2 such that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$. These two regions are separated by a sufficiently smooth interface Γ . We consider the following stationary problem (2.1) which is a system of semilinear equations.

$$\begin{cases} -\nabla \cdot (\alpha \nabla u) + vg(u) = f_u & \text{in } \Omega, \\ -\nabla \cdot (\beta \nabla v) - \lambda vg(u) = f_v & \text{in } \Omega, \\ \llbracket u \rrbracket = \llbracket \alpha \nabla_{\mathbf{n}} u \rrbracket = 0 & \text{on } \Gamma, \\ v = \nabla_{\mathbf{n}} v = 0 & \text{on } \Gamma, \\ u = \bar{u} & \text{on } \partial\Omega, \\ v = \bar{v} & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$



Here, \mathbf{n} denotes the unit normal at a given point on Γ pointing from Ω_1 to Ω_2 , and the restriction of u on:

- each subdomain Ω_i of Ω for $i = 1, 2$ at a point x on Γ as in (2.2) where \mathbf{n} pointing from Ω_1 to Ω_2 .

is defined by

$$\begin{aligned} u_1(x) &:= \lim_{\epsilon \rightarrow 0} u(x - \epsilon \mathbf{n}), \\ u_2(x) &:= \lim_{\epsilon \rightarrow 0} u(x + \epsilon \mathbf{n}), \end{aligned} \quad (2.2)$$

Then,

- the Jump is defined as

$$\llbracket u \rrbracket := u_1|_{\Gamma} - u_2|_{\Gamma}. \quad (2.3)$$

We also take the following special form of g, f_u, f_v, \bar{v} which are agreed to a biofilm's model, i.e. there is no bacteria outside the biomass region Ω_1 .

$$g(u) = \begin{cases} g_1(u) & \text{in } \Omega_1, \\ 0 & \text{in } \Omega_2. \end{cases}, \quad f_u = \begin{cases} f_{u_1} & \text{in } \Omega_1, \\ f_{u_2} & \text{in } \Omega_2. \end{cases}, \quad f_v = \begin{cases} f_{v_1} & \text{in } \Omega_1, \\ 0 & \text{in } \Omega_2. \end{cases}, \quad \bar{v} = \begin{cases} \bar{v}_1 & \text{on } \partial\Omega_1 \setminus \Gamma, \\ 0 & \text{on } \partial\Omega_2 \setminus \Gamma. \end{cases}$$

Assumption (A) We suppose that $\bar{u}, \lambda > 0, \bar{v}_1 \geq 0$ and two diffusion coefficients α, β are assumed to be piecewise constants $\alpha = \alpha_i > 0, \beta = \beta_i > 0$ in Ω_i for $i = 1, 2$. In general, we have $\alpha_1 \neq \alpha_2, \beta_1 \neq \beta_2$. We also assume that functions g, f_u, f_v are such that

$$\begin{cases} f_u, f_v \in L^2(\Omega), \\ g \text{ measurable with respect to } x \in \Omega \text{ and } 0 \leq \frac{\partial g(x, u)}{\partial u} \leq \xi(x) \in L^1(\Omega). \end{cases}$$

2.1 A weak formulation

We introduce the space $H^k(\Omega_{12})$ defined as,

$$H^k(\Omega_1 \cup \Omega_2) = \{v \in L^2(\Omega) : v_i \in H^k(\Omega_i) \text{ for } i = 1, 2\},$$

for $k = 1, 2$ where $v_i = v|_{\Omega_i}$, which is endowed with the norm:

$$\|u\|_{H^k(\Omega_{12})}^2 := \|u\|_{H^k(\Omega_1)}^2 + \|u\|_{H^k(\Omega_2)}^2.$$

From system (2.1) we derive three uncouple systems for which the weak form is easier to deal with. On letting $w = u + \frac{\beta}{\alpha\lambda}v$, we are able to decouple the system (2.1) into three separated problems (2.4), (2.5) and (2.6) as follows:

$$\begin{cases} -\nabla \cdot (\alpha \nabla w) = f_w := f_u + \frac{1}{\lambda} f_v & \text{in } \Omega_i, i = 1, 2, \\ \llbracket w \rrbracket = \llbracket \alpha \nabla_{\mathbf{n}} w \rrbracket = 0 & \text{on } \Gamma, \\ w = \bar{w} := \bar{u} + \frac{\beta}{\alpha\lambda} \bar{v} & \text{on } \partial\Omega. \end{cases} \quad (2.4)$$

$$\begin{cases} -\nabla \cdot (\beta \nabla v) - \lambda v g(w - \frac{\beta}{\alpha\lambda} v) = f_v & \text{in } \Omega_i, i = 1, 2, \\ v = \nabla_{\mathbf{n}} v = 0 & \text{on } \Gamma, \\ v = \bar{v} & \text{on } \partial\Omega. \end{cases} \quad (2.5)$$

$$u = w - \frac{\beta}{\alpha\lambda} v. \quad (2.6)$$

From the definition of w the following result is easy to check.

Proposition 1. *If w, u and v are solution of (2.4)-(2.6) then u, v and is solution of (2.1). Conversely, if u, v is a solution of (2.1) then w, v and u are solutions of (2.4)-(2.6).*

A weak formulation of (2.4) reads as

$$\begin{cases} \text{Seek } w \in \{s \in H^1(\Omega)\} \text{ such that } s = \bar{w} \text{ on } \partial\Omega \text{ and} \\ \langle \alpha \nabla w, \nabla \varphi \rangle_{\Omega} = \langle f_w, \varphi \rangle_{\Omega}, \quad \forall \varphi \in H_0^1(\Omega). \end{cases} \quad (2.7)$$

On enforcing weakly the interface conditions owing to the Nitsche technique, a weak formulation of (2.5) reads as:

$$\begin{cases} \text{Seek } v \in V \text{ such that } v = \bar{v} \text{ on } \partial\Omega \text{ and} \\ \langle \beta \nabla v, \nabla \varphi \rangle_{\Omega} - \langle q(v), \varphi \rangle_{\Omega} = \langle f_v, \varphi \rangle_{\Omega}, \quad \forall \varphi \in V_0. \end{cases} \quad (2.8)$$

Next, we need to verify the equivalence between the weak problems (2.4), (2.5) and the decoupled system (2.7), (2.8). Indeed,

Proposition 2. *If (w, v) is a solution of (2.4)-(2.5) then it is also a solution of (2.7)-(2.8). Conversely, if the weak solution (w, v) of (2.7)-(2.8) belongs to $H^2(\Omega)$, then it solves (2.4)-(2.5).*

Proof. The first statement can be obtained easily from the derivation of weak formulations. We now prove that if (w, v) solves (2.7, 2.8) and $w, v \in H^2(\Omega)$ then it also solves (2.4, 2.5).

First, consider problem (2.7) and $w \in H^2(\Omega)$ is a solution of it, we have $w = \bar{w}$ on $\partial\Omega$ and

$$\langle \alpha \nabla w, \nabla \varphi \rangle_{\Omega} = \langle f_w, \varphi \rangle_{\Omega}, \quad \forall \varphi \in H_0^1(\Omega).$$

Performing integration by parts on the $\langle \alpha \nabla w, \nabla \varphi \rangle_{\Omega_{12}}$ backwards on each subdomain Ω_i , we have

$$-\langle \nabla \cdot (\alpha \nabla w), \varphi \rangle_{\Omega_{12}} + \langle \llbracket \alpha \nabla_{\mathbf{n}} w \rrbracket, \{\!\!\{ \varphi \}\!\!\} \rangle_{\Gamma} = \langle f_w, \varphi \rangle_{\Omega}, \quad \forall \varphi \in H_0^1(\Omega). \quad (2.9)$$

We choose $\varphi = 0$ on Γ , (2.9) becomes

$$-\langle \nabla \cdot (\alpha \nabla w), \varphi \rangle_{\Omega_{12}} = \langle f_w, \varphi \rangle_{\Omega}, \quad \forall \varphi \in H_0^1(\Omega) \cap \{\varphi = 0 \text{ on } \Gamma\}.$$

We could argue in each subdomain Ω_i ,

$$-\nabla \cdot (\alpha \nabla w) = f_w \text{ a.e. on } \Omega_i.$$

Now back to (2.9) we have

$$\langle \llbracket \alpha \nabla_{\mathbf{n}} w \rrbracket, \{\!\!\{ \varphi \}\!\!\} \rangle_{\Gamma} = 0, \quad \forall \varphi \in H_0^1(\Omega) \cap \{\varphi \neq 0 \text{ on } \Gamma\}.$$

This implies, $\llbracket \alpha \nabla_{\mathbf{n}} w \rrbracket = 0$ on Γ . To sum up, we have shown that w solves (2.7) and w also satisfies all conditions of problem (2.4).

With the same technique, we can easily obtain the same result for problem (2.5) and (2.6). Indeed, from (2.8) and for any $\varphi \in H_0^1(\Omega)$, we have

$$-\langle \nabla \cdot (\beta \nabla v), \varphi \rangle_{\Omega_{12}} + \langle \llbracket \beta \nabla_{\mathbf{n}} v \rrbracket, \{\!\!\{ \varphi \}\!\!\} \rangle_{\Gamma} + \langle \llbracket \varphi \rrbracket, \{\!\!\{ \beta \nabla_{\mathbf{n}} v \}\!\!\} \rangle_{\Gamma} - \langle q(v), \varphi \rangle_{\Omega} = \langle f_v, \varphi \rangle_{\Omega}.$$

Choose $\varphi = 0$ on Γ then

$$-\langle \nabla \cdot (\beta \nabla v), \varphi \rangle_{\Omega_{12}} - \langle q(v), \varphi \rangle_{\Omega} = \langle f_v, \varphi \rangle_{\Omega}, \quad \forall \varphi \in \{\psi \in H_0^1(\Omega), \psi = 0 \text{ on } \Gamma\}.$$

This implies $-\nabla \cdot (\beta \nabla v) - q(v) = f_v$. Thus,

$$\langle \llbracket \beta \nabla_{\mathbf{n}} v \rrbracket, \{\!\!\{ \varphi \}\!\!\} \rangle_{\Gamma} + \langle \llbracket \varphi \rrbracket, \{\!\!\{ \beta \nabla_{\mathbf{n}} v \}\!\!\} \rangle_{\Gamma} = 0, \quad \forall \varphi \in \{\psi \in H_0^1(\Omega), \psi \neq 0 \text{ on } \Gamma\}. \quad (2.10)$$

Don't forget that $\varphi \in H_0^1(\Omega)$ or we have $\llbracket \varphi \rrbracket = 0$, this leads to

$$\langle \llbracket \beta \nabla_{\mathbf{n}} v \rrbracket, \{\!\!\{ \varphi \}\!\!\} \rangle_{\Gamma} = 0, \quad \forall \varphi \in \{\psi \in H_0^1(\Omega), \psi \neq 0 \text{ on } \Gamma\},$$

or we have

$$\llbracket \beta \nabla_{\mathbf{n}} v \rrbracket = 0, \text{ on } \Gamma. \quad (2.11)$$

Replace (2.11) in (2.10), we get

$$\langle \llbracket \varphi \rrbracket, \{\{\beta \nabla_{\mathbf{n}} v\}\} \rangle_{\Gamma} = 0, \quad \forall \varphi \in \{\psi \in H_0^1(\Omega), \psi \neq 0 \text{ on } \Gamma, \llbracket \psi \rrbracket \neq 0 \text{ on } \Gamma\}.$$

or,

$$\{\{\beta \nabla_{\mathbf{n}} v\}\} = 0, \quad \text{on } \Gamma. \quad (2.12)$$

Coupling (2.11) and (2.12), we have $\nabla_{\mathbf{n}} v = 0$ on Γ . To sum up, we have shown that v solves (2.8) and v also satisfies all conditions of problem (2.5). ■

Proposition 3. *Under assumption (A), the problem (2.7)-(2.8) has a unique solution.*

Proof. If $f_w \in L^2(\Omega)$ and thanks to [6], problem (2.7) has unique solution in H^2 on each subdomain and further,

$$\|w\|_{H^1(\Omega_{12})} + \|w\|_{H^2(\Omega_{12})} \leq C \|f_w\|_{L^2(\Omega_{12})}.$$

It's also known in [18, Theorem 2.1] that the semilinear problem (2.8) has unique solution in V if $f_v \in L^2(\Omega)$ and $g(x, u(x))$ satisfies

$$g \text{ measurable w.r.t } x \in \Omega \text{ and } 0 \leq \frac{\partial g(x, u)}{\partial u} \leq \xi(x) \in L^1(\Omega). \quad (2.13)$$

■

3 A NXFEM approximation

3.1 The discret setting

Triangulation. Let h_K be the diameter of each element K in \mathcal{T}_h and $h := \max_{K \in \mathcal{T}_h} h_K$. Based on the location of interface Γ , we also define the set of triangles covering each subdomain Ω_i by $\mathcal{T}_h^i := \{K \in \mathcal{T}_h : K \cap \Omega_i \neq \emptyset\}$ and $G_h := \{K \in \mathcal{T}_h : K \cap \Gamma \neq \emptyset\}$ is the set of all cells that are intersected by Γ . For each $K \in G_h$, let $\Gamma_K := \Gamma \cap K$ be the part of Γ in K . For each $K \in \mathcal{T}_h$, let $K_i := K \cap \Omega_i$ be the part of K in Ω_i . For any part, e of Ω , we denote by $|e|$ the measure of e .

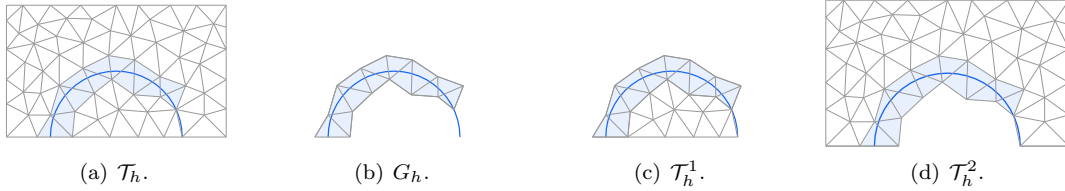


Figure 1: Illustration of the triangulation.

Assumption 1. *The triangulation is non-degenerate, i.e. $\frac{h_K}{\rho_K} \leq C, \forall K \in \mathcal{T}_h$ for some $C > 0$ where h_K and ρ_K are the diameter of K and the diameter of the largest ball in K respectively.*

Assumption 2. *For $K \in G_h$, Γ cuts each element boundary ∂K exactly twice and each open edge at most once.*

Assumption 3. Let $\Gamma_{K,h}$ be the segment connecting the intersection points between Γ and ∂K , we assume that $\Gamma_{K,h}$ is the function of length on Γ_K in the local coordinates (3.1).

$$\begin{aligned}\Gamma_{K,h} &= \{(x, y) : 0 < x < |\Gamma_{K,h}|, y = 0\}, \\ \Gamma_K &= \{(x, y) : 0 < x < |\Gamma_{K,h}|, y = \delta(x)\}.\end{aligned}\tag{3.1}$$

Definition 1. Let v be a scalar-valued function defined on Ω and assume that v is smooth enough to admit on all part of the interface a possibly two-valued trace. The average of v is defined as

$$\{\!\{u\}\!\} := \kappa_1 u_1|_{\Gamma} + \kappa_2 u_2|_{\Gamma},\tag{3.2}$$

where weight parameters κ_1, κ_2 satisfies $\sum \kappa_i = 1, \kappa_i > 0$.

Lemma 3.1. With the jump and average operators defined in (??) and (3.2) and u, v two discontinuous functions across Γ , we easily have,

$$\begin{aligned}[[uv]] &= [[u]]\{\!\{v\}\!\} + \{\!\{u\}\!\}[[v]] + (\kappa_2 - \kappa_1)[[u]][[v]] \text{ and} \\ \{\!\{uv\}\!\} &= \{\!\{u\}\!\}\{\!\{v\}\!\} + \kappa_1 \kappa_2 [[u]][[v]].\end{aligned}$$

We introduce the finite element spaces $V_h^\Gamma = V_h^1 \times V_h^2$ and V_h^0 defined as:

$$V_h^i := \{v_h \in L^2(\Omega) : v_h|_{\Omega_i} \in H^1(\Omega_i) \text{ and } v_h|_K \in \mathbb{P}^1(K), \forall K \in \cup_i \mathcal{T}_h^i\}, \quad i = 1, 2.\tag{3.3}$$

$$V_h^0 := \{v_h \in V_h^\Gamma : v_h = 0 \text{ on } \partial\Omega\}.\tag{3.4}$$

Remark 3.1. Note that, $V_h^\Gamma \subset H^1(\Omega_{12})$ and the functions in V_h^Γ are no need to be continuous across the interface.

We recall below the norm used in [17] and some results established in [17].

$$\|v\|_{1/2}^2 := \sum_{K \in \mathcal{G}_h} h_K^{-1} \|v\|_{L^2(\Gamma_K)}^2, \quad \|v\|_{-1/2}^2 := \sum_{K \in \mathcal{G}_h} h_K \|v\|_{L^2(\Gamma_K)}^2.\tag{3.5}$$

Proposition 4. For $u \in V_h^\Gamma$, $\|\cdot\|_{-1/2}$ defined in (3.5) and $\kappa_i = \frac{|K_i|}{|K|}$, the following inverse inequality holds,

$$\|\nabla_{\mathbf{n}} u\|_{-1/2}^2 \leq C \|\nabla u\|_{L^2(\Omega_{12})}^2.\tag{3.6}$$

Definition 2 (Interpolant). Consider an operator $E_i : H^2(\Omega_i) \rightarrow H^2(\Omega)$ such that $(E_i w)|_{\Omega_i} = w$ and

$$\|E_i w\|_{s,\Omega} \leq C \|w\|_{s,\Omega_i}, \quad \forall w \in H^s(\Omega_i), \quad s = 0, 1, 2.$$

Let I_h be the standard P^1 finite element interpolation operator and define

$$I_h^* v := (I_{h,1}^* v_1, I_{h,2}^* v_2) \text{ where } I_{h,i}^* v_i := (I_h E_i v_i)|_{\Omega_i}.\tag{3.7}$$

3.2 Discrete formulations

The discrete form of problem (2.4) we introduce is the following:

$$\begin{cases} \text{Seek } w_h \in V_h^\Gamma \text{ such that } w_h = \bar{w} \text{ on } \partial\Omega \\ a_{wh}(w_h, \varphi_h) = K_{wh}(\varphi_h), \quad \forall \varphi_h \in V_h^0, \end{cases}\tag{3.8}$$

where

$$\begin{aligned} a_{wh}(w_h, \varphi_h) &= \langle \alpha \nabla w_h, \nabla \varphi_h \rangle_{\Omega_{12}} - \langle \llbracket w_h \rrbracket, \{\{\alpha \nabla_{\mathbf{n}} \varphi_h\}\} \rangle_{\Gamma} - \langle \{\{\alpha \nabla_{\mathbf{n}} w_h\}\}, \llbracket \varphi_h \rrbracket \rangle_{\Gamma} + \zeta \langle \llbracket w_h \rrbracket, \llbracket \varphi_h \rrbracket \rangle_{\Gamma}, \\ K_{wh}(\varphi_h) &= \langle f_w, \varphi_h \rangle_{\Omega}. \end{aligned}$$

The discrete form of problem (2.5) we introduce is the following:

$$\begin{cases} \text{Seek } v_h \in V_h^{\Gamma} \text{ such that } v_h = \bar{v} \text{ on } \partial\Omega \\ a_{vh}(v_h, \varphi_h) - \langle q(v_h), \varphi_h \rangle_{\Omega} = K_{vh}(\varphi_h), \quad \forall \varphi_h \in V_h^0, \end{cases} \quad (3.9)$$

where,

$$\begin{aligned} a_{vh}(v_h, \varphi_h) &:= \langle \beta \nabla v_h, \nabla \varphi_h \rangle_{\Omega_{12}} - \langle \llbracket v_h \rrbracket, \{\{\beta \nabla_{\mathbf{n}} \varphi_h\}\} \rangle_{\Gamma} - \langle \{\{\beta \nabla_{\mathbf{n}} v_h\}\}, \llbracket \varphi_h \rrbracket \rangle_{\Gamma} \\ &\quad + \theta \langle \{\{v_h\}\}, \{\{\varphi_h\}\} \rangle_{\Gamma} + \theta \kappa_1 \kappa_2 \langle \llbracket v_h \rrbracket, \llbracket \varphi_h \rrbracket \rangle_{\Gamma}, \\ K_{vh}(\varphi_h) &:= \langle f_v, \varphi_h \rangle_{\Omega}. \end{aligned} \quad (3.10)$$

Proposition 5. For $u, v \in V_h^{\Gamma}$ and the norms $\|\cdot\|_{\frac{1}{2}}, \|\cdot\|_{-\frac{1}{2}}$ defined as in (3.5), we have

$$\langle u, v \rangle_{\Gamma} \leq \|u\|_{\frac{1}{2}} \|v\|_{-\frac{1}{2}}. \quad (3.11)$$

Proof. Thanks to Hölder's inequality and Schwarz's inequality, we have

$$\begin{aligned} \langle u, v \rangle_{\Gamma} &= \sum_{T \in G_h} \langle h_T^{-\frac{1}{2}} u, h_T^{\frac{1}{2}} v \rangle_{\Gamma_T} \leq \sum_{T \in G_h} \langle \|h_T^{-\frac{1}{2}} u\|_{L^2(\Gamma_T)}, \|h_T^{\frac{1}{2}} v\|_{L^2(\Gamma_T)} \rangle_{\Gamma_T} \\ &\leq \left(\sum_{T \in G_h} h_T^{-1} \|u\|_{L^2(\Gamma_T)}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in G_h} h_T \|v\|_{L^2(\Gamma_T)}^2 \right)^{\frac{1}{2}} = \|u\|_{\frac{1}{2}} \|v\|_{-\frac{1}{2}}. \end{aligned}$$

■

Proposition 6 (Consistency). If w, v solve the continuous problems (2.4), (2.5) respectively then w, v also solve the discrete problems (3.8), (3.9) respectively.

Proof. For w , using the same technique given in [17, Lemma 1], we are able to get

$$a_{wh}(w, \varphi_h) = K_{wh}(\varphi_h), \quad \forall \varphi_h \in V_{\Gamma}^0.$$

For v solving (3.9), it's even easier because of the condition of v on Γ ($v = \nabla_{\mathbf{n}} v = 0$). After put v into a_{vh} , all terms on the interface will disappear and we get instantly the result. ■

Proposition 7. The following estimates hold

- (i) $a_{wh}(w_h, \varphi_h) \leq C \|w_h\|_1 \|\varphi_h\|_1$, for all $w_h \in V_h^{\Gamma}$ and $\varphi_h \in V_h^0$,
- (ii) $a_{wh}(w_h, w_h) \geq C \|w_h\|_1^2$, $\forall w_h \in V_h^{\Gamma}$ for all $w_h \in V_h^{\Gamma}$,

where $\|\cdot\|_1$ is defined by

$$\|v_h\|_1^2 := \|\nabla v_h\|_{L^2(\Omega_{12})}^2 + \|\{\{\alpha \nabla_{\mathbf{n}} v_h\}\}\|_{-1/2}^2 + \|\llbracket v_h \rrbracket\|_{1/2}^2. \quad (3.12)$$

Proof. The estimates (i) and (ii) can be obtained from [17, Lemma 5]. ■

Theorem 1. The discrete problem (3.8) has unique solution in V_h^{Γ} .

Proof. Thanks to the Lax-Milgram theorem. ■

Theorem 2. *The discrete problem (3.9) has a solution $v_h \in V_h^\Gamma$.*

Proof. Given $\tilde{v}_h \in V_h^\Gamma$, consider the problem : Find $v_h \in V_h^\Gamma$ such that

$$a_{vh}(v_h, \varphi_h) - \langle q(\tilde{v}_h), \varphi_h \rangle_\Omega = \langle f_v, \varphi_h \rangle_\Omega, \quad \forall \varphi_h \in V_h^0, \quad (3.13)$$

We can obtain the continuity of a_{vh} thanks to Hölder's inequality and (3.11),

$$\begin{aligned} |a_{vh}(v_h, \varphi_h)| &\leq |\langle \beta \nabla v_h, \nabla \varphi_h \rangle_{\Omega_{12}}| + |\langle \llbracket v_h \rrbracket, \{\{\beta \nabla_{\mathbf{n}} \varphi_h\}\}_\Gamma \rangle| + |\langle \{\{\beta \nabla_{\mathbf{n}} v_h\}\}, \llbracket \varphi_h \rrbracket \rangle_\Gamma| \\ &\quad + |\theta \langle \{\{v_h\}\}, \{\{\varphi_h\}\}_\Gamma \rangle| + |\theta \kappa_1 \kappa_2 \langle \llbracket v_h \rrbracket, \llbracket \varphi_h \rrbracket \rangle_\Gamma| \\ &\leq C \|\nabla v_h\|_{L^2(\Omega_{12})} \|\nabla \varphi_h\|_{L^2(\Omega_{12})} + C \|\llbracket v_h \rrbracket\|_{\frac{1}{2}} \|\{\{\beta \nabla_{\mathbf{n}} \varphi_h\}\}\|_{-\frac{1}{2}} \\ &\quad + C \|\{\{\beta \nabla_{\mathbf{n}} v_h\}\}\|_{-\frac{1}{2}} \|\llbracket \varphi_h \rrbracket\|_{\frac{1}{2}} + C \|\{\{v_h\}\}\|_{L^2(\Gamma)} \|\{\{\varphi_h\}\}\|_{L^2(\Gamma)} \\ &\quad + C \|\llbracket v_h \rrbracket\|_{\frac{1}{2}} \|\llbracket \varphi_h \rrbracket\|_{\frac{1}{2}} \\ &\leq C \|\llbracket v_h \rrbracket\|_2 \|\llbracket \varphi_h \rrbracket\|_2, \end{aligned}$$

where $\|\cdot\|_2$ is defined by

$$\|\llbracket v_h \rrbracket\|_2^2 := \|\nabla v_h\|_{L^2(\Omega_{12})}^2 + \|\{\{\beta \nabla_{\mathbf{n}} v_h\}\}\|_{-1/2}^2 + \|\llbracket v_h \rrbracket\|_{1/2}^2 + \|\{\{v_h\}\}\|_{L^2(\Gamma)}^2 \quad (3.14)$$

Using the same technique as in the proof of [17, Lemma 5] with any $\xi > 0$, we have

$$\begin{aligned} a_{vh}(v_h, v_h) &\geq \frac{1}{2} \|\beta^{1/2} \nabla v_h\|_{L^2(\Omega_{12})}^2 + C \|v_h\|_{L^2(\partial\Omega_1 \setminus \Gamma)}^2 \\ &\quad + \left(\frac{1}{2} - \frac{2C_I \beta_{\max}}{\xi} \right) \|\beta^{1/2} \nabla v_h\|_{L^2(\Omega_{12})}^2 + \frac{1}{\xi} \|\{\{\beta \nabla_{\mathbf{n}} v_h\}\}\|_{-\frac{1}{2}}^2 \\ &\quad + \sum_{T \in G_h} \left(\theta \kappa_1 \kappa_2 - \frac{\xi}{h_T} \right) \|\llbracket v_h \rrbracket\|_{L^2(\Gamma_T)}^2 + \theta \|\{\{v_h\}\}\|_{L^2(\Gamma)}^2. \end{aligned}$$

By choosing $\xi = 4C_I \beta_{\max}$ (C_I is the coefficient in the inverse inequality (3.6)) and $\theta > \frac{\xi}{h_T \kappa_1 \kappa_2}$, we have

$$a_{vh}(v_h, v_h) \geq C \|\llbracket v_h \rrbracket\|_2^2. \quad (3.15)$$

Thanks to Lax-Milgram theorem, it is easily to check that (3.13) has unique solution in V_h^Γ . From this, we define the operator T by

$$T : \begin{array}{ccc} V_h^\Gamma & \longrightarrow & V_h^\Gamma \\ \tilde{v}_h & \longmapsto & T(\tilde{v}_h) = v_h \text{ solving (3.13)} \end{array}$$

Owing to (3.15),

$$a_{vh}(v_h, v_h) \geq C \|\llbracket v_h \rrbracket\|_2^2 \geq C \|\nabla v_h\|_{L^2(\Omega_{12})}^2.$$

Besides that,

$$\begin{aligned} a_{vh}(v_h, v_h) &= \langle q(\tilde{v}_h), v_h \rangle_{\Omega_{12}} + \langle f_v, v_h \rangle_\Omega \leq |\langle q(\tilde{v}_h), v_h \rangle_{\Omega_{12}}| + \langle f_v, v_h \rangle_\Omega \\ &\leq \|q\|_{L^2(\Omega)} \|v_h\|_{L^2(\Omega)} + \|f_v\|_{L^2(\Omega)} \|v_h\|_{L^2(\Omega)} \leq C \|\nabla v_h\|_{L^2(\Omega_{12})}. \end{aligned}$$

Therefore $\|\nabla v_h\|_{L^2(\Omega_{12})} \leq C$ (thanks to Poincaré's inequality), then T is bounded in V_h^Γ . Take R "large enough", T maps $B_R = \{v_h \in V_h^\Gamma : \|v_h\|_{L^2(\Omega_{12})} \leq R\}$ to B_R . It's enough to prove that T is

continuous to conclude by the Brouwer fixed point theorem (cf. [4]). Indeed, let $\{\tilde{v}_h^n\}_n$ be a sequence in V_h^Γ such that $\tilde{v}_h^n \rightarrow \tilde{v}_h$. Putting $v_h^n = T(\tilde{v}_h^n)$. Because T is bounded in V_h^Γ , thanks to Bolzano-Weierstrass theorem, there exists a subsequence, also denoted v_h^n , converges to a quantity so-called $z_h \in V_h^\Gamma$. What we need to do now is to prove that z_h is a solution of (3.13). Take $\varphi_h \in V_h^0$, we have

$$a_{vh}(v_h^n, \varphi_h) - \langle q(\tilde{v}_h^n), \varphi_h \rangle_{\Omega_{12}} = \langle f_v, \varphi_h \rangle_\Omega.$$

With assumptions like in 3, $\langle q(\tilde{v}_h^n), \varphi_h \rangle_{\Omega_{12}} \rightarrow \langle q(\tilde{v}_h), \varphi_h \rangle_{\Omega_{12}}$. Moreover, $v_h^n \rightarrow z_h$ implies

$$\begin{aligned} \langle \beta \nabla v_h^n, \nabla \varphi_h \rangle_{\Omega_{12}} &\rightarrow \langle \beta \nabla z_h, \nabla \varphi_h \rangle_{\Omega_{12}}, \\ \|\llbracket v_h^n \rrbracket - \llbracket z_h \rrbracket\|_{L^2(\Gamma)} &= \|\llbracket v_h^n - z_h \rrbracket\|_{L^2(\Gamma)} \sum_{i=1}^2 \|(v_h^n - z_h)|_{\Omega_i}\|_{L^2(\Gamma)} \\ &\leq C \sum_{i=1}^2 \|(v_h^n - z_h)|_{\Omega_i}\|_{L^2(\Omega_i)} \rightarrow 0, \\ \|\{\{v_h^n\}\} - \{\{z_h\}\}\|_{L^2(\Gamma)} &= \|\sum_{i=1}^2 \kappa_i (v_h^n - z_h)|_{\Omega_i}\|_{L^2(\Gamma)} \leq C \sum_{i=1}^2 \|(v_h^n - z_h)|_{\Omega_i}\|_{L^2(\Omega_i)} \rightarrow 0. \end{aligned}$$

$$\begin{aligned} &\|\{\{\beta \nabla_{\mathbf{n}} v_h^n\}\} - \{\{\beta \nabla_{\mathbf{n}} z_h\}\}\|_{L^2(\Gamma)}^2 \\ &\leq C \sum_{i=1}^2 \|\kappa_i \nabla_{\mathbf{n}}(v_h^n - z_h)|_{\Omega_i}\|_{L^2(\Gamma)}^2 \leq C \sum_{i=1}^2 \sum_K \|\kappa_i \nabla_{\mathbf{n}}(v_h^n - z_h)|_{\Omega_i}\|_{L^2(\Gamma_K)}^2 \\ &\leq C \sum_{i=1}^2 \sum_K \kappa_i^2 |\Gamma_K| |\nabla_{\mathbf{n}}(v_h^n - z_h)|_{\Omega_i}|^2 = C \sum_{i=1}^2 \sum_K \frac{|K_i| |\Gamma_K|}{|K|^2} \|\nabla_{\mathbf{n}}(v_h^n - z_h)|_{\Omega_i}\|_{L^2(K_i)}^2 \\ &\leq C \sum_{i=1}^2 \sum_K \|\nabla_{\mathbf{n}}(v_h^n - z_h)|_{\Omega_i}\|_{L^2(K_i)}^2 = C \sum_{i=1}^2 \|\nabla_{\mathbf{n}}(v_h^n - z_h)|_{\Omega_i}\|_{L^2(\Omega_i)}^2 \rightarrow 0. \end{aligned}$$

In the estimate $\|\{\{\beta \nabla_{\mathbf{n}} v_h^n\}\} - \{\{\beta \nabla_{\mathbf{n}} z_h\}\}\|_{L^2(\Gamma)}$, we have used the fact that $|\Gamma_K| \leq h_K$, $|K_i| \leq h_K^2$, $|K| \geq Ch_K^2$. And therefore, $a_{vh}(v_h^n, \varphi_h) \rightarrow a_{vh}(z_h, \varphi_h)$. Finally, we get,

$$a_{vh}(z_h, \varphi_h) - \langle q(\tilde{v}_h), \varphi_h \rangle_{\Omega_{12}} = \langle f_v, \varphi_h \rangle_\Omega,$$

or z_h solves (3.13). Sum up, we conclude that T admits a fixed point $z_h = T(z_h)$ which solves the problem (3.9). \blacksquare

4 A convergence result

In this section, we are going to show that the solution of discrete problems (3.8) and (3.9) converges to the solution of the weak problems (2.7) and (2.8) respectively. In order to do that, we have to define some operators defined on the interface and also their convergence result in V_h^Γ .

Definition 3 (Lifting operator). For $v_h \in V_h^\Gamma$, let $\mathcal{L}_h : L^2(\Gamma) \rightarrow [V_h^\Gamma]^2$ such that

$$\forall \varphi \in [V_h^0]^2, \quad \langle \mathcal{L}_h(\llbracket v_h \rrbracket), \varphi \rangle_{\Omega_{12}} := \langle \{\{\varphi\}\} \cdot \mathbf{n}, \llbracket v_h \rrbracket \rangle_\Gamma. \quad (4.1)$$

We observe that the support of \mathcal{L}_h consists of two subdomains of which Γ is part of the boundary $\partial\Omega_i$, or

$$\text{supp}(\mathcal{L}_h) = \bar{\Omega}_1 \cup \bar{\Omega}_2 = \bar{\Omega}.$$

The following discrete gradient operators will play an important role in the analysis.

Definition 4 (Discrete gradient operators). For all $v_h \in V_h^\Gamma$, let $\mathcal{G}_h : V_h^\Gamma \rightarrow [V_h^\Gamma]^2$ such that,

$$\mathcal{G}_h(v_h) := \nabla v_h - \mathcal{L}_h(\llbracket v_h \rrbracket). \quad (4.2)$$

Symbolic notations. Let $V_\sigma = \begin{cases} H^1(\Omega) & \text{if } \sigma = \alpha \\ V & \text{if } \sigma = \beta \end{cases}$, $V_\sigma^0 = \begin{cases} H_0^1(\Omega) & \text{if } \sigma = \alpha \\ V_0 & \text{if } \sigma = \beta \end{cases}$, and we define a symbolic norm $\|\cdot\|$ which stands for $\|\cdot\|_i, i = 1, 2$ given as below

$$\|z\|^2 = \|\nabla z\|_{L^2(\Omega_{12})}^2 + \|\llbracket z \rrbracket\|_{\frac{1}{2}}^2 + \|\{\{\sigma \nabla_{\mathbf{n}} z\}\}\|_{-\frac{1}{2}}^2 + \mu \|\{\{z\}\}\|_{L^2(\Gamma)}^2, \quad (4.3)$$

where $\sigma = \alpha, \sigma = \beta$ are corresponding to $\|\cdot\|_1, \|\cdot\|_2$ respectively and $\mu = 0, \mu = 1$ are corresponding to $\|\cdot\|_1, \|\cdot\|_2$ respectively.

Theorem 3. Let $I_h^* : V_\sigma \cap H^2(\Omega_{12}) \rightarrow V_h^\Gamma$ be the interpolation operator defined in (3.7), then

$$\|v - I_h^* v\| \leq Ch \|v\|_{L^2(\Omega_{12})}, \quad \forall v \in V_\sigma \cap H^2(\Omega_{12}). \quad (4.4)$$

Proof. Look back to definition of norm $\|\cdot\|$, there are 4 terms. For the first 3 terms, using the result given from the proof of [17, Theorem 2], we have

$$\|\nabla z\|_{L^2(\Omega_{12})}^2 + \|\llbracket z \rrbracket\|_{\frac{1}{2}}^2 + \|\{\{\sigma \nabla_{\mathbf{n}} z\}\}\|_{-\frac{1}{2}}^2 \leq Ch^2 \|v\|_{L^2(\Omega_{12})}^2, \quad \forall v \in V_\sigma \cap H^2(\Omega_{12}),$$

where $z = v - I_h^* v$. For the last term,

$$\begin{aligned} \|\{\{z\}\}\|_{L^2(\Gamma)} &= \left\| \sum_{i=1}^2 \kappa_i z_i \right\|_{L^2(\Gamma)} \leq C \sum_{i=1}^2 \|z_i\|_{L^2(\Gamma)} \leq C \sum_{i=1}^2 \|z_i\|_{L^2(\partial\Omega_i)} \\ &\leq C \sum_{i=1}^2 \|\nabla z_i\|_{L^2(\Omega_i)} = C \|\nabla z\|_{L^2(\Omega_{12})} \leq Ch \|v\|_{L^2(\Omega_{12})}. \end{aligned}$$

■

Proposition 8. Let $\|\cdot\|_{-\frac{1}{2}}$ and \mathcal{L}_h be defined in (3.5) and (4.1) respectively, we have the boundedness for the lifting operator \mathcal{L}_h as following

$$\|\mathcal{L}_h(\llbracket v_h \rrbracket)\|_{[L^2(\Omega_{12})]^2} \leq C \|\llbracket v_h \rrbracket\|_{\frac{1}{2}}, \quad \forall v_h \in V_h^\Gamma. \quad (4.5)$$

Proof. Coming from the left hand side of (4.5), we have

$$\begin{aligned} \|\mathcal{L}_h(\llbracket v_h \rrbracket)\|_{[L^2(\Omega_{12})]^2}^2 &= \langle \{\{\mathcal{L}_h(\llbracket v_h \rrbracket)\}\} \cdot \mathbf{n}, \llbracket v_h \rrbracket \rangle_\Gamma \leq C \|\{\{\mathcal{L}_h(\llbracket v_h \rrbracket)\}\}\|_{[L^2(\Gamma)]^2} \|\llbracket v_h \rrbracket\|_{L^2(\Gamma)} \\ &\leq C \left(\kappa_1 \|\mathcal{L}_h(\llbracket v_h \rrbracket)\|_{\Omega_1} \|_{[L^2(\Gamma)]^2} + \kappa_2 \|\mathcal{L}_h(\llbracket v_h \rrbracket)\|_{\Omega_2} \|_{[L^2(\Gamma)]^2} \right) \|\llbracket v_h \rrbracket\|_{L^2(\Gamma)} \\ &\leq C \left(\kappa_1 \|\mathcal{L}_h(\llbracket v_h \rrbracket)\|_{\Omega_1} \|_{[L^2(\partial\Omega_1)]^2} + \kappa_2 \|\mathcal{L}_h(\llbracket v_h \rrbracket)\|_{\Omega_2} \|_{[L^2(\partial\Omega_2)]^2} \right) \|\llbracket v_h \rrbracket\|_{L^2(\Gamma)} \\ &\stackrel{(A)}{\leq} C \left(\kappa_1 h^{-\frac{1}{2}} \|\mathcal{L}_h(\llbracket v_h \rrbracket)\|_{\Omega_1} \|_{[L^2(\Omega_1)]^2} + \kappa_2 h^{-\frac{1}{2}} \|\mathcal{L}_h(\llbracket v_h \rrbracket)\|_{\Omega_2} \|_{[L^2(\Omega_2)]^2} \right) \|\llbracket v_h \rrbracket\|_{L^2(\Gamma)} \\ &\leq C \|\mathcal{L}_h(\llbracket v_h \rrbracket)\|_{[L^2(\Omega_{12})]^2} h^{-\frac{1}{2}} \|\llbracket v_h \rrbracket\|_{L^2(\Gamma)} \\ &\leq C \|\mathcal{L}_h(\llbracket v_h \rrbracket)\|_{[L^2(\Omega_{12})]^2} \|\llbracket v_h \rrbracket\|_{\frac{1}{2}}. \end{aligned}$$

In above estimate, the reason (A) comes from following estimate owing to the trace theorem and [1, Theorem 1.3],

$$\|v_h\|_{L^2(\partial\Omega_i)} \leq C\|\nabla v_h\|_{L^2(\Omega_i)} \leq Cd_i^{-1}\|v_h\|_{L^2(\Omega_i)} \leq Ch^{-\frac{1}{2}}\|v_h\|_{L^2(\Omega_i)}, \quad (4.6)$$

where $d_i := \sup_{x,y \in \Omega_i} \|x - y\|$ which is the diameter of the domain Ω_i satisfies an assumption on the domain that there exists $C_i > 0$ such that $d_i \geq C_i$. ■

Proposition 9. For the discrete gradient operator \mathcal{G}_h defined in (4.2),

- 1) $\forall v_h \in V_h^\Gamma, \quad \|\mathcal{G}_h(v_h)\|_{[L^2(\Omega_{12})]^2} \leq C\|v_h\|.$
- 2) $\mathcal{G}_h(I_h^*\varphi) \rightarrow \nabla\varphi$ strongly in $[L^2(\Omega_{12})]^2$ for all $\varphi \in V_\sigma^0$

Proof. a) Using the definition (4.2) of \mathcal{G}_h and the triangle inequality coupling with the boundedness (4.5) of \mathcal{L}_h , we have

$$\begin{aligned} \|\mathcal{G}_h(v_h)\|_{[L^2(\Omega_{12})]^2} &= \|\nabla v_h - \mathcal{L}_h(\llbracket v_h \rrbracket)\|_{[L^2(\Omega_{12})]^2} \\ &\leq C\|\nabla v_h\|_{L^2(\Omega_{12})} + C\|\mathcal{L}_h(\llbracket v_h \rrbracket)\|_{[L^2(\Omega_{12})]^2} \\ &\leq C\|\nabla v_h\|_{L^2(\Omega_{12})} + C\|\llbracket v_h \rrbracket\|_{\frac{1}{2}} \leq C\|v_h\|. \end{aligned}$$

b) For all $\varphi \in V_\sigma^0$,

$$\begin{aligned} \|\mathcal{G}_h(I_h^*\varphi) - \nabla\varphi\|_{[L^2(\Omega_{12})]^2} &\leq \|\nabla(I_h^*\varphi) - \mathcal{L}_h(\llbracket I_h^*\varphi \rrbracket) - \nabla\varphi\|_{[L^2(\Omega_{12})]^2} \\ &= \|\nabla(I_h^*\varphi - \varphi) - \mathcal{L}_h(\llbracket (I_h^*\varphi - \varphi) \rrbracket)\|_{[L^2(\Omega_{12})]^2} \\ &\leq C\|\nabla(I_h^*\varphi - \varphi)\|_{L^2(\Omega_{12})} + C\|\mathcal{L}_h(\llbracket (I_h^*\varphi - \varphi) \rrbracket)\|_{[L^2(\Omega)]^2} \\ &\leq C\|I_h^*\varphi - \varphi\| + C\|\llbracket (I_h^*\varphi - \varphi) \rrbracket\|_{\frac{1}{2}} \\ &\leq C\|I_h^*\varphi - \varphi\| \leq Ch\|\varphi\|_{L^2(\Omega_{12})}. \end{aligned}$$

■

Proposition 10. Let $\{v_h\}_h$ be a sequence in V_h^Γ and assume that this sequence is bounded in the $\|\cdot\|$ -norm. Then, the family $\{v_h\}_h$ is relatively compact in $L^2(\Omega)$.

Proof. We will borrow the idea of proofs in the work of [12] and [14]. While the authors of [12] work on Discontinuous Galerkin Method in which they consider the discontinuity throughout faces of all elements of the mesh and the authors of [14] work on Finite Volume Method, our work will focus only on the zone around the interface.

For $v \in L^1(\mathbb{R}^2)$, define a space $BV := \{v \in L^1(\mathbb{R}^2) : \|v\|_{BV} < +\infty\}$ where

$$\|v\|_{BV} := \sum_{i=1}^2 \sup\left\{ \int_{\mathbb{R}^2} v \partial_i \varphi \, dx; \varphi \in C_c^\infty(\mathbb{R}^2), \|\varphi\|_{L^\infty(\mathbb{R}^2)} \leq 1 \right\}.$$

Extending the functions v_h by zero outside Ω and for all $\varphi \in C_c^\infty(\mathbb{R}^2)$ with $\|\varphi\|_{L^\infty(\mathbb{R}^2)} \leq 1$, integrating by parts gives us

$$\begin{aligned} \int_{\mathbb{R}^2} (v_h) \partial_i \varphi \, dx &= \int_{\Omega} (v_h) \partial_i \varphi \, dx = \int_{\Omega_1} (v_h) \partial_i \varphi \, dx + \int_{\Omega_2} (v_h) \partial_i \varphi \, dx \\ &= - \int_{\Omega_{12}} (e_i \cdot \nabla(v_h)) \varphi \, dx + \sum_{T \in \mathcal{G}_h} \int_{\Gamma_T} (e_i \cdot \mathbf{n}) \llbracket v_h \rrbracket \varphi \, ds. \end{aligned}$$

Hölder's inequality and the fact that $\|\varphi\|_{L^\infty(\mathbb{R}^2)} \leq 1$ will give us

$$\begin{aligned}
& - \int_{\Omega_{12}} (e_i \cdot \nabla(v_h)) \varphi \, dx \leq \|\nabla(v_h)\|_{L^1(\Omega_{12})} \|\varphi\|_{L^\infty(\Omega_{12})} \leq \|\nabla(v_h)\|_{L^1(\Omega_{12})}, \\
& \sum_{T \in G_h} \int_{\Gamma_T} (e_i \cdot \mathbf{n}) \llbracket v_h \rrbracket \varphi \, ds \leq \sum_{T \in G_h} \|\llbracket v_h \rrbracket\|_{L^1(\Gamma_T)} \|\varphi\|_{L^\infty(\Gamma_T)} \leq \sum_{T \in G_h} \|\llbracket v_h \rrbracket\|_{L^1(\Gamma_T)}.
\end{aligned}$$

Applying Hölder's inequality again,

$$\begin{aligned}
\|\nabla v_h\|_{L^1(\Omega_{12})} & \leq \|1\|_{L^2(\Omega_{12})} \|\nabla v_h\|_{L^2(\Omega_{12})} \leq C \|\nabla v_h\|_{L^2(\Omega_{12})} \\
\sum_{T \in G_h} \|\llbracket v_h \rrbracket\|_{L^1(\Gamma_T)} & = \sum_{T \in G_h} \|h_T^{\frac{1}{2}} h_T^{-\frac{1}{2}} \llbracket v_h \rrbracket\|_{L^1(\Gamma_T)} \\
& \leq \left(\sum_{T \in G_h} h_T \|1\|_{L^2(\Gamma_T)}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in G_h} h_T^{-1} \|\llbracket v_h \rrbracket\|_{L^2(\Gamma_T)}^2 \right)^{\frac{1}{2}} \\
& \leq C \|\llbracket v_h \rrbracket\|_{\frac{1}{2}},
\end{aligned}$$

in which we have used that $|\Gamma_T| \leq h_T$ and the non-degenerate property of the mesh, $h_T^2 \leq C|T|$. Therefore, $\int_{\mathbb{R}^2} (v_h) \partial_i \varphi \, dx \leq C \|\llbracket v_h \rrbracket\|$ or we have

$$\|\sigma v_h\|_{\text{BV}} \leq C \|\llbracket v_h \rrbracket\| \leq C.$$

From [14], for all $y \in \mathbb{R}^2$,

$$C \|v_h(\cdot + y) - v_h\|_{L^1(\mathbb{R}^2)} \leq |y| \|\sigma v_h\|_{\text{BV}} \leq C|y|,$$

where $|y|$ is the Euclidean norm of y . From this and thanks to Kolmogorov's Compactness Criterion, we have that $\{v_h\}_h$ is relatively compact in $L^1(\mathbb{R}^2)$. Besides that, Poincaré's inequality helps us

$$\|v_h\|_{L^2(\mathbb{R}^2)} = \|v_h\|_{L^2(\Omega)} = \|v_h\|_{L^2(\Omega_{12})} \leq C \|\nabla v_h\|_{L^2(\Omega_{12})} \leq C \|\llbracket v_h \rrbracket\| \leq C,$$

or $\{v_h\}_h$ is also bounded in $L^2(\mathbb{R}^2)$, hence it is also relatively compact in $L^2(\mathbb{R}^2)$. Finally, we have $\{v_h\}_h$ is relatively compact in $L^2(\Omega)$ because v_h has been extended by zero outside Ω . \blacksquare

Theorem 4. *Let $\{v_h\}_h$ be a sequence in V_h^Γ . Assume that this sequence is bounded in $\|\cdot\|$ -norm. There exists a function $v \in V_\sigma$ such that as $h \rightarrow 0$, up to a subsequence, $v_h \rightarrow v$ strongly in $L^2(\Omega)$ and $\mathcal{G}_h(v_h) \rightharpoonup \nabla v$ weakly in $[L^2(\Omega)]^2$.*

Proof. Thanks to the Proposition 10 and Rellich's theorem, there exists a function $v \in L^2(\Omega)$ and a subsequence, also denoted by v_h , such that $v_h \rightarrow v$ strongly in $L^2(\Omega)$. Moreover, the Proposition 9 gives us the boundedness of \mathcal{G}_h in $[L^2(\Omega)]^2$, thus there exists a new subsequence, again denoted as v_h , and $\mathbf{w} \in [L^2(\Omega)]^2$ such that $\mathcal{G}_h(v_h) \rightharpoonup \mathbf{w}$ weakly in $[L^2(\Omega)]^2$. What we need to do is to prove that $\mathbf{w} = \nabla v$. Indeed, for all $\varphi \in [C_c^\infty(\Omega)]^2$ (note that, $\llbracket \varphi \rrbracket = 0$ on Γ and $\varphi = 0$ on $\partial\Omega$),

$$\begin{aligned}
\langle \mathcal{G}_h(v_h), \varphi \rangle_{\Omega_{12}} & = \langle \nabla v_h - \mathcal{L}_h(\llbracket v_h \rrbracket), \varphi \rangle_{\Omega_{12}} \\
& = \langle \nabla v_h, \varphi \rangle_\Omega - \langle \mathcal{L}_h(\llbracket v_h \rrbracket), \varphi \rangle_{\Omega_{12}} \\
& = -\langle v_h, \nabla \cdot \varphi \rangle_\Omega + \int_\Gamma \llbracket v_h(\varphi \cdot \mathbf{n}) \rrbracket \, ds - \langle \mathcal{L}_h(\llbracket v_h \rrbracket), \varphi \rangle_{\Omega_{12}}
\end{aligned}$$

$$\begin{aligned}
&= -\langle v_h, \nabla \cdot \varphi \rangle_\Omega + \langle \llbracket v_h \rrbracket, \{\{\varphi\}\} \cdot \mathbf{n} \rangle_\Gamma - \langle \mathcal{L}_h(\llbracket v_h \rrbracket), \varphi \rangle_{\Omega_{12}} \\
&= -\langle v_h, \nabla \cdot \varphi \rangle_\Omega.
\end{aligned}$$

Observe that when $h \rightarrow 0$, $\langle v_h, \nabla \cdot \varphi \rangle_\Omega \rightarrow \langle v, \nabla \cdot \varphi \rangle_\Omega$ because of the strong convergence of v_h in $L^2(\Omega)$. As a result,

$$\langle \mathbf{w}, \varphi \rangle_\Omega = \lim_{h \rightarrow 0} \langle \mathcal{G}_h(v_h), \varphi \rangle_{\Omega_{12}} = -\langle v, \nabla \cdot \varphi \rangle_\Omega, \quad \forall \varphi \in [C_c^\infty(\Omega)]^2.$$

If we can prove that $\llbracket v \rrbracket = 0$, we can obtain $\mathbf{w} = \nabla v$, hence $v \in V_\sigma$. Indeed, considering the relation,

$$\begin{aligned}
| \|\llbracket v_h \rrbracket\|_{L^2(\Gamma)} - \|\llbracket v \rrbracket\|_{L^2(\Gamma)} | &\leq \|\llbracket v_h - v \rrbracket\|_{L^2(\Gamma)} \\
&\leq \|(v_h - v)|_{\Omega_1}\|_{L^2(\Gamma)} + \|(v_h - v)|_{\Omega_2}\|_{L^2(\Gamma)}
\end{aligned}$$

Using the relation (4.6), we have

$$\begin{aligned}
\|(v_h - v)|_{\Omega_i}\|_{L^2(\Gamma)} &\leq \|(v_h - v)|_{\Omega_i}\|_{L^2(\partial\Omega_i)} \\
&\leq C d_i^{-1} \|(v_h - v)|_{\Omega_i}\|_{L^2(\Omega_i)} \leq C \|(v_h - v)|_{\Omega_i}\|_{L^2(\Omega_i)}
\end{aligned}$$

Thus,

$$| \|\llbracket v_h \rrbracket\|_{L^2(\Gamma)} - \|\llbracket v \rrbracket\|_{L^2(\Gamma)} | \leq C \sum_i \|(v_h - v)|_{\Omega_i}\|_{L^2(\Omega_i)},$$

which tends to zero because $v_h \rightarrow v$ in $L^2(\Omega)$. This implies

$$\|\llbracket v_h \rrbracket\|_{L^2(\Gamma)} \rightarrow \|\llbracket v \rrbracket\|_{L^2(\Gamma)}. \quad (4.7)$$

Besides that, v_h is bounded in $\|\cdot\|$ -norm,

$$\begin{aligned}
h^{-1} \|\llbracket v_h \rrbracket\|_{L^2(\Gamma)}^2 &= \sum_{T \in \mathcal{G}_h} h^{-1} \|\llbracket v_h \rrbracket\|_{L^2(\Gamma_T)}^2 \leq \sum_{T \in \mathcal{G}_h} h_T^{-1} \|\llbracket v_h \rrbracket\|_{L^2(\Gamma_T)}^2 \\
&= \|\llbracket v_h \rrbracket\|_{\frac{1}{2}}^2 \leq \|v_h\|^2 \leq C,
\end{aligned}$$

which yields that as $h \rightarrow 0$,

$$\|\llbracket v_h \rrbracket\|_{L^2(\Gamma)} \rightarrow 0. \quad (4.8)$$

From (4.7), (4.8) we have $\|\llbracket v \rrbracket\|_{L^2(\Gamma)} = 0$. This yields $\llbracket v \rrbracket = 0$. ■

Now, we have all needed tools to consider the convergence of the solutions of discrete problems to a solution of the weak problems.

Theorem 5. *For $\{w_h\}_h, \{v_h\}_h$ be the sequence of discrete solutions generated by solving discrete problems (3.8),(3.9) respectively, there exist solutions w, v solving (2.7),(2.8) respectively such that as $h \rightarrow 0$, $(w_h, v_h) \rightarrow (w, v)$ strongly in $L^2(\Omega)$.*

Proof. Using 7, (3.15), Hölder's inequality and Poincaré's inequality, it is inferred that,

$$\begin{aligned} \|w_h\|_1^2 &\leq Ca_{wh}(w_h, w_h) = K_{wh}(w_h) \leq C\|w_h\|_{L^2(\Omega)} \leq C\|\nabla w_h\|_{L^2(\Omega)} \leq C\|w_h\|, \\ \|v_h\|_2^2 &\leq Ca_{vh}(v_h, v_h) = C\langle q(v_h), v_h \rangle_\Omega + C\langle f_v, v_h \rangle_\Omega \\ &\leq \|q\|_{L^2(\Omega)}\|v_h\|_{L^2(\Omega)} + \|f_v\|_{L^2(\Omega)}\|v_h\|_{L^2(\Omega)} \\ &\leq C\|k\|_{L^\infty(\Omega)}(\text{mesh}(\Omega))^{1/2}\|\nabla v_h\|_{L^2(\Omega)} + C\|\nabla v_h\|_{L^2(\Omega)} \leq C\|v_h\|_2. \end{aligned}$$

Hence, $\{w_h\}_h, \{v_h\}_h$ are bounded in $\|\cdot\|_1$ -norm and $\|\cdot\|_2$ -norm respectively. Thanks to 4, there exist $w^* \in H^1(\Omega)$ and $v^* \in V$ such that, as $h \rightarrow 0$, up to a subsequence, $w_h \rightarrow w^*, v_h \rightarrow v^*$ strongly in $L^2(\Omega)$ and $\mathcal{G}_h(w_h) \rightharpoonup \nabla w^*, \mathcal{G}_h(v_h) \rightharpoonup \nabla v^*$ weakly in $[L^2(\Omega)]^2$.

We want to prove that w^* and v^* are solutions of problems (2.7),(2.8) respectively. In deed,

(i) For all $\varphi \in H_0^1(\Omega)$,

$$a_{wh}(w_h, I_h^* \varphi) = \langle \alpha^{\frac{1}{2}} \mathcal{G}_h(w_h), \alpha^{\frac{1}{2}} \mathcal{G}_h(I_h^* \varphi) \rangle_{\Omega_{12}} + j_{wh}(w_h, I_h^* \varphi) = F_1 + F_2,$$

where

$$\begin{aligned} F_1 &= \langle \alpha^{\frac{1}{2}} \mathcal{G}_h(w_h), \alpha^{\frac{1}{2}} \mathcal{G}_h(I_h^* \varphi) \rangle_{\Omega_{12}}, \\ F_2 &= j_{wh}(w_h, I_h^* \varphi) = \zeta(\llbracket w_h \rrbracket, \llbracket I_h^* \varphi \rrbracket)_\Gamma - \langle \alpha^{\frac{1}{2}} \mathcal{L}_h(\llbracket w_h \rrbracket), \alpha^{\frac{1}{2}} \mathcal{L}_h(\llbracket I_h^* \varphi \rrbracket) \rangle_{\Omega_{12}}. \end{aligned}$$

From the weak convergence of $\mathcal{G}_h(w_h)$ to ∇w^* and the strong convergence of $\mathcal{G}_h(I_h^* \varphi)$ to $\nabla \varphi$ (cf. Lemma 9), as $h \rightarrow 0$, we have $F_1 \rightarrow \langle \alpha \nabla w^*, \nabla \varphi \rangle_{\Omega_{12}}$. We show that $F_2 \rightarrow 0$ also. Indeed,

$$\begin{aligned} |\zeta(\llbracket w_h \rrbracket, \llbracket I_h^* \varphi \rrbracket)_\Gamma| &\leq C\|\llbracket w_h \rrbracket\|_{\frac{1}{2}}\|\llbracket I_h^* \varphi \rrbracket\|_{\frac{1}{2}} \leq C\|w_h\|_1\|\llbracket I_h^* \varphi - \varphi \rrbracket\|_{\frac{1}{2}} \\ &\leq C\|w_h\|_1\|I_h^* \varphi - \varphi\|_1 \rightarrow 0, \\ |\langle \alpha^{\frac{1}{2}} \mathcal{L}_h(\llbracket w_h \rrbracket), \alpha^{\frac{1}{2}} \mathcal{L}_h(\llbracket I_h^* \varphi \rrbracket) \rangle_\Gamma| &\leq C\|\mathcal{L}_h(\llbracket w_h \rrbracket)\|_{[L^2(\Omega_{12})]^2}\|\mathcal{L}_h(\llbracket I_h^* \varphi \rrbracket)\|_{[L^2(\Omega_{12})]^2} \\ &\leq C\|\llbracket w_h \rrbracket\|_{\frac{1}{2}}\|\llbracket I_h^* \varphi \rrbracket\|_{\frac{1}{2}} \rightarrow 0. \end{aligned}$$

because w_h is bounded in $\|\cdot\|_1$ and owing to 3 and Proposition ??prop:sys-liftOperBound . Besides that, we also have $K_{wh}(I_h^* \varphi) \rightarrow K_{wh}(\varphi)$. It's because

$$\begin{aligned} |K_{wh}(I_h^* \varphi) - K_{wh}(\varphi)| &= |K_{wh}(I_h^* \varphi - \varphi)| \leq \|f_w\|_{1,\infty,\Omega}\|I_h^* \varphi - \varphi\|_{L^2(\Omega)} \\ &\leq C\|\nabla(I_h^* \varphi - \varphi)\|_{L^2(\Omega)} \leq C\|I_h^* \varphi - \varphi\|_1 \rightarrow 0. \end{aligned}$$

In short, for all $\varphi \in H_0^1(\Omega)$,

$$a_w(w^*, \varphi) \leftarrow a_{wh}(w_h, I_h^* \varphi) = K_w(I_h^* \varphi) \rightarrow K_w(\varphi). \quad (4.9)$$

The remaining thing to be verified is the boundary condition $w^* = \bar{w}$ on $\partial\Omega$. It's easy to obtained thanks to the strong convergence of w_h to w in $L^2(\Omega)$ and the trace theorem.

In one word, w^* is a solution of discrete problem (3.8). Since the solution of this problem is unique (cf. 3), we also have that the whole sequence $\{w_h\}_h$ strongly converges to w^* in $L^2(\Omega)$.

(ii) Similarly, for all $\varphi \in V_0$,

$$a_{vh}(v_h, I_h^* \varphi) = \langle \beta^{\frac{1}{2}} \mathcal{G}_h(v_h), \beta^{\frac{1}{2}} \mathcal{G}_h(I_h^* \varphi) \rangle_{\Omega_{12}} + j_{vh}(v_h, I_h^* \varphi),$$

where

$$j_{vh}(v_h, I_h^* \varphi) = \theta \kappa_1 \kappa_2 \langle \llbracket v_h \rrbracket, \llbracket I_h^* \varphi \rrbracket \rangle_\Gamma + \theta \langle \{\!\!\{ v_h \}\!\!\}, \{\!\!\{ I_h^* \varphi \}\!\!\} \rangle_\Gamma - \langle \beta^{\frac{1}{2}} \mathcal{L}_h(\llbracket v_h \rrbracket), \beta^{\frac{1}{2}} \mathcal{L}_h(\llbracket I_h^* \varphi \rrbracket) \rangle_{\Omega_{12}}.$$

With the same technique as in a_{wh} in a notice that,

$$\begin{aligned} |\langle \{\!\!\{ v_h \}\!\!\}, \{\!\!\{ I_h^* \varphi \}\!\!\} \rangle_\Gamma| &= |\langle \{\!\!\{ v_h \}\!\!\}, \{\!\!\{ \varphi - I_h^* \varphi \}\!\!\} \rangle_\Gamma| \quad (\{\!\!\{ \varphi \}\!\!\} = \varphi = 0 \text{ on } \Gamma) \\ &\leq C \|v_h\|_2 \|\varphi - I_h^* \varphi\|_2 \rightarrow 0, \end{aligned}$$

we have

$$a_{vh}(v_h, I_h^* \varphi) \rightarrow \langle \beta \nabla v^*, \nabla \varphi \rangle_{\Omega_{12}}.$$

We want also that $\langle q(v_h), I_h^* \varphi \rangle_\Omega \rightarrow \langle q(v^*), \varphi \rangle_\Omega$. Indeed,

$$\begin{aligned} &|\langle q(v_h), I_h^* \varphi \rangle_\Omega - \langle q(v^*), \varphi \rangle_\Omega| \\ &= |\langle q(v_h), I_h^* \varphi \rangle_\Omega - \langle q(v^*), I_h^* \varphi \rangle_\Omega + \langle q(v^*), I_h^* \varphi \rangle_\Omega - \langle q(v^*), \varphi \rangle_\Omega| \\ &\leq |\langle q(v_h) - q(v^*), I_h^* \varphi \rangle_\Omega| + |\langle q(v^*), I_h^* \varphi - \varphi \rangle_\Omega|. \end{aligned}$$

With the same assumptions for q as in 3, $q(x, v_h(x)) \rightarrow q(x, v^*(x))$, a.e. in $L^2(\Omega)$ and we will get $\langle q(x, v_h(x)), I_h^* \varphi(x) \rangle_\Omega \rightarrow \langle q(x, v^*(x)), I_h^* \varphi(x) \rangle_\Omega$ thanks to convergence dominated theorem. Moreover,

$$\begin{aligned} |\langle q(v^*), I_h^* \varphi - \varphi \rangle_\Omega| &\leq \|q\|_{L^2(\Omega)} \|I_h^* \varphi - \varphi\|_{L^2(\Omega)} \leq \|k\|_{L^\infty} (\text{mes}(\Omega))^{1/2} \|I_h^* \varphi - \varphi\|_{L^2(\Omega)} \\ &\leq C \|\nabla(I_h^* \varphi - \varphi)\|_{L^2(\Omega)} \leq C \|I_h^* \varphi - \varphi\|_2 \rightarrow 0. \end{aligned}$$

Again, we do similarly get $K_{vh}(I_h^* \varphi) \rightarrow K_v(\varphi)$ and $v^* = \bar{v}$ on $\partial\Omega$. We get finally that v^* satisfies

$$\langle \beta \nabla v^*, \nabla \varphi \rangle_{\Omega_{12}} - \langle q(v^*), \varphi \rangle_\Omega \leftarrow a_{vh}(v_h, I_h^* \varphi) - \langle q(v_h), I_h^* \varphi \rangle_\Omega = K_{vh}(I_h^* \varphi) \rightarrow K_v(\varphi),$$

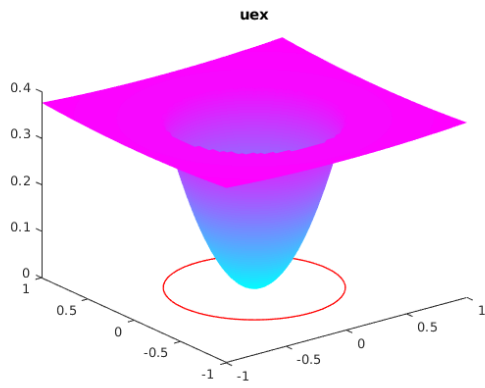
or v^* is a solution of discrete problem (3.9). Since the solution of (3.9) is unique (cf. 3), we also have the strong convergence of v_h to v^* in $L^2(\Omega)$. ■

5 A numerical test

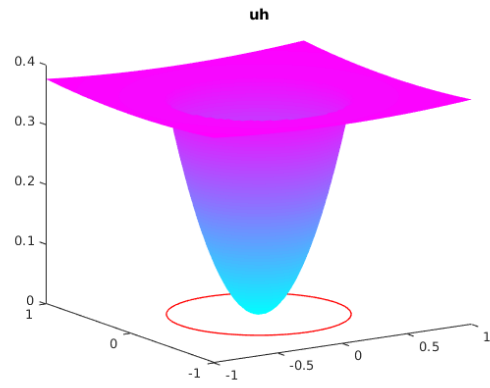
We consider the problem (2.1) in which a domain $\Omega = [0, 1] \times [0, 1]$ with an interface is a circle centered at the origin with a radius r_0 . The boundary condition and the source term f_u, f_v are determined from the exact solutions

$$u(x, y) = \begin{cases} \frac{r^2}{\alpha_1} & \text{if } r \leq r_0, \\ \frac{r^2 - r_0^2}{\alpha_2} + \frac{r_0^2}{\alpha_1} & \text{otherwise,} \end{cases} \quad v(x, y) = \begin{cases} \frac{(r^2 - r_0^2)^2}{\beta_1} & \text{if } r \leq r_0, \\ 0 & \text{otherwise,} \end{cases}$$

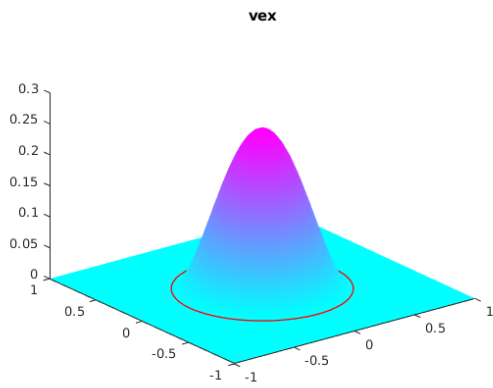
where $r = \sqrt{x^2 + y^2}$, $r_0 = 0.6$. Notice that the exact solutions satisfy interface conditions in equation (2.1). The coefficients of this test are taken as $\alpha_1 = 1, \alpha_2 = 100, \beta_1 = 0.5$.



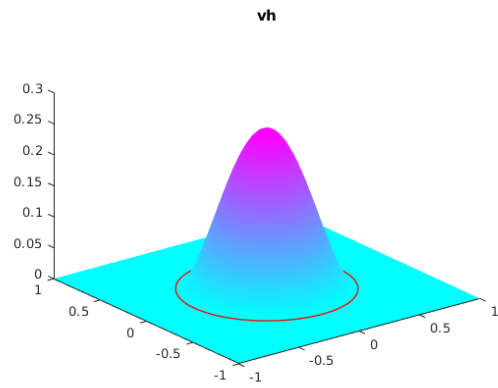
(a) An exact solution u .



(b) A numerical solution u_h .

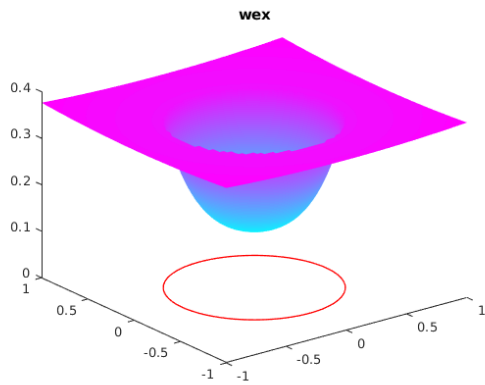


(c) An exact solution v .

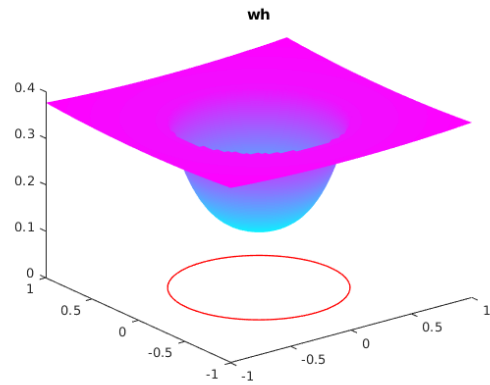


(d) A numerical solution v_h .

Figure 2: An exact solution and a numerical solution of u, v in a fine mesh.



(a) An exact solution w .



(b) A numerical solution w_h .

Figure 3: An exact solution and a numerical solution of w in a fine mesh.

Numerical solutions u_h, v_h in comparison with the exact solutions u, v are given Figure ??fig:chap4-solutions-uv (here we are showing the solutions in 3D view with the z-index is the value of solutions at points on Oxy). You can see, with a smooth mesh, we obtain almost the same results for both exact and numerical solutions.

For a reference, we also plot the two solutions of $w = u + \frac{\beta}{\alpha\lambda}v$ as in Figure ??fig:chap4-solutions-w. The corresponding L^2 norm errors and convergence rates of $w - w_h$ and $v - v_h$ are given in 1 and 4.

h	$\ w - w_h\ _{L^2}$	order	$\ v - v_h\ _{L^2}$	order
1.34×10^{-1}	7×10^{-3}		2.5×10^{-3}	
6.9×10^{-2}	2.1×10^{-3}	1.82	5.94×10^{-4}	2.14
3.49×10^{-2}	5.33×10^{-4}	2.01	1.16×10^{-4}	2.40
1.76×10^{-2}	1.38×10^{-4}	1.93	2.57×10^{-5}	2.18

Table 1: L^2 norm errors of the solutions with different mesh sizes.

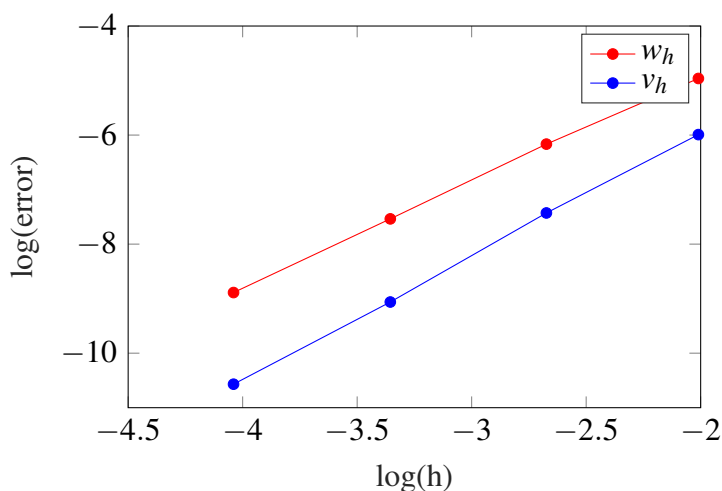


Figure 4: The convergence of numerical solutions to exact solutions of the system.

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