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Comparison of Viability Kernels via Conic Orders

Michel De Lara,^{*} Pedro Gajardo,[†] Diego Vicencio[†]

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Abstract

In natural resource management, decision-makers often aim at maintaining the state of the system within a desirable set for all times: spawning stock biomass over a critical threshold in fishery management, number of infected individuals below a critical threshold in epidemics control, etc. This is possible when the initial state belongs to the so-called viability kernel. However, an exact characterization of such a kernel is generally out of reach. We show how we can compare two viability kernels corresponding to two control systems, using quasimonotonicity under conic preorder. We illustrate the method in the field of epidemics control.

Keywords: convex cone, conic preorder, control theory, viability theory, comparison of flows.

1 Introduction

In natural resource management, one often aims at maintaining the state of the system within a desirable set for all times, for instance the spawning stock biomass over a critical threshold in fishery management [4, 10, 14, 19, 24] or the number of infected individuals below a critical threshold in epidemics control [13]. This is possible when the initial state belongs to the so-called *viability kernel* [2, 3]. The viability approach — consisting in characterizing, computing or estimating the corresponding viability kernel — has notably been applied to the analysis of these topics, as recently reviewed in [22].

In the literature of the last decades, one can find several methods for the challenging task of computing the viability kernel (see for instance, [3, 9, 11, 12, 15, 19, 21, 25]). In general, numerical methods for such computation can be implemented only for systems with a few number of state variables. This is because of the so-called *curse of dimensionality*,

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as these algorithms are based on the dynamic programming principle or, more generally, as they have to explore all possible states (or a large part of them) susceptible to being reached for different choices of controls. Therefore, general methods for computing the viability kernel may only be applied to systems that do not have too many state variables. This is an important drawback in the study of some natural resource management or epidemics control problems that have models composed of many state variables, such as age-structured fish-stock population models which often display more than ten state variables (see [9, 10, 23]).

To overcome the curse of dimensionality, some approaches make use of linearity [20] or monotonicity properties induced by the positive orthant in the state space [11, 12, 13]. In this work, we aim at obtaining a characterization of the viability kernel under monotonicity properties, but in a broad sense, that is, induced by a so-called conic preorder. For this purpose, we consider a convex cone K in the state space and the induced conic preorder \preceq_K . Our key assumption is that the dynamics defining the system under study is K -quasimonotone, a generalization of the cooperativeness property for dynamical systems. Under this assumption, we have at our disposal a comparison theorem for the solutions of the underlying differential equation [17, 18]. Our contribution relies on a second assumption: the existence of a *reduction* of the controls (to be explained later) associated to the convex cone K . The idea of the reduction is that, given a control trajectory and the associated state trajectory, one can find another control path (ideally in a reduced control space) whose associated state trajectory is preordered with respect to the first one. We prove that the problem of computing the viability kernel can be carried out by exploring a smaller set of trajectories, hence reducing the complexity of the problem.

The paper is organized as follows. In Sect. 2, we present the main definitions regarding controlled dynamical systems and viability kernels. Then, in Sect. 3, we introduce conic preorders and prove our main result, that is, a comparison theorem for viability kernels. Finally, Sect. 4 is devoted to an illustration in epidemics control.

2 Controlled dynamical systems and viability kernels

In §2.1, we present controlled dynamical systems and, in §2.2, the viability kernel associated with a controlled dynamical system and a desirable set.

2.1 Controlled dynamical systems

We give a formal definition of controlled dynamics and controlled dynamical systems, including technical assumptions that will be useful in the paper. We consider \mathbb{R}^n for *state space* and \mathbb{R}^m for *control space*, where n and m are positive integers.

Definition 2.1. *A controlled dynamics is a mapping $f : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}^n$, where $\mathbb{X} \subset \mathbb{R}^n$ is a closed subset of \mathbb{R}^n , and $\mathbb{U} \subset \mathbb{R}^m$ is a (Borel) measurable subset of \mathbb{R}^m , with the following two properties: f is jointly measurable in the state and control variables; f is locally Lipschitz in the state variable uniformly in the control variable, that is, for every $x_0 \in \mathbb{X}$ there exists*

$L > 0$ and $\delta > 0$ such that, for any $x, x' \in \mathbb{X}$,

$$\|x - x_0\| \leq \delta \text{ and } \|x' - x_0\| \leq \delta \implies \|f(x, u) - f(x', u)\| \leq L\|x - x'\|, \quad \forall u \in \mathbb{U},$$

where $\|\cdot\|$ is any norm on \mathbb{R}^n .

We define the *set of (admissible) control paths* by

$$\mathcal{U} = \{u(\cdot) : [0, +\infty) \rightarrow \mathbb{U} \mid u(\cdot) \text{ is measurable}\}. \quad (1)$$

Given a controlled dynamics $f : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}^n$, as in Definition 2.1, and a control path $u(\cdot) : [0, +\infty) \rightarrow \mathbb{U}$ as in (1), it can be shown¹ that the differential equation

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \in \mathbb{X} \text{ given} \quad (2)$$

has a unique solution defined on an open time interval $[0, T) \subset [0, +\infty)$. When this unique solution is defined for all $t \in [0, +\infty)$, we denote it by $x(t) = \Psi_f^{u(\cdot)}(t, x_0)$, that is,

$$x(t) = \Psi_f^{u(\cdot)}(t, x_0) \iff \dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0, \quad (3)$$

and we call the mapping Ψ the *flow* of the *controlled dynamical system* (2). We also say that the controlled dynamics f generates a *global flow*.

2.2 Desirable set and viability kernel

In *viability theory*, one aims to determine a set of initial conditions which allow to keep the trajectories of a dynamical system inside a so-called desirable set by means of suitable control paths [2, 3].

Definition 2.2. *Let be given a controlled dynamics $f : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}^n$ as in Definition 2.1, and suppose that it generates a global flow Ψ . Given a subset*

$$\mathbb{D} \subset \mathbb{X} \times \mathbb{U},$$

called the desirable set, we define the viability kernel, associated with the controlled dynamics f and with the desirable set \mathbb{D} , by

$$\mathbb{V}(f, \mathbb{D}) = \left\{ x_0 \in \mathbb{X} \mid \exists u(\cdot) \in \mathcal{U}, \left(\Psi_f^{u(\cdot)}(t, x_0), u(t) \right) \in \mathbb{D}, \quad \forall t \in [0, +\infty) \right\}. \quad (4)$$

Thus, the viability kernel represents the set of initial conditions $x_0 \in \mathbb{X}$ such that there exists a control path $u(\cdot) \in \mathcal{U}$ for which the associated state and control paths, generated by (2), remain in the desirable set \mathbb{D} for all times.

For a decision maker, knowing the viability kernel has practical interest since it describes the set of states from which controls can be found that maintain the system in a desirable configuration forever. Nevertheless, computing this kernel is not an easy task in general because of the so-called *curse of dimensionality*, as most of algorithms are based on the dynamic programming principle. However, under additional assumptions on the dynamics and on the desirable set, it is possible to simplify the computation of the viability kernel in presence of many state variables, as we show in the following section.

¹See [8, Theorem 7.4.1, p. 263] or [7, Theorem 1.1, p. 178].

3 Comparison of viability kernels via conic preorders

Now, we present our main result, which is a comparison theorem for viability kernels by means of so-called conic preorders. In §3.1, we recall the notions of conic preorder and of conic quasimonotonicity. Then, we propose the new definition of *conic quasimonotonicity reducibility for controlled dynamical systems* in §3.2. Thus equipped, we state our main result in §3.3.

3.1 Conic preorders and conic quasimonotonicity

Let $K \subset \mathbb{R}^n$ be a convex cone, that is, $\alpha K \subset K$ for all $\alpha \in \mathbb{R}_+$ (hence $0 \in K$), and $K + K \subset K$. It is well-known [1, 17, 18, 26] that such a convex cone induces a preorder (that is, a transitive and reflexive relation) on \mathbb{R}^n , denoted by \preceq_K and given by

$$(\forall x, x' \in \mathbb{R}^n) \quad x \preceq_K x' \iff x' - x \in K. \quad (5)$$

Let $\langle \cdot, \cdot \rangle$ stands for the usual inner product in \mathbb{R}^n . The *positive polar cone* associated with the cone K is [18]

$$K^+ = \{y \in \mathbb{R}^n \mid \langle y, x \rangle \geq 0, \forall x \in K\}. \quad (6)$$

The following definition is introduced in [17, 18]. We reframe it with our own notations.

Definition 3.1 ([17, 18]). *We say that a mapping (dynamics) $h : \mathbb{X} \times [0, +\infty) \rightarrow \mathbb{R}^n$, where $\mathbb{X} \subset \mathbb{R}^n$ is a closed subset of \mathbb{R}^n , is K -quasimonotone if the following condition holds*

$$(\forall x, x' \in \mathbb{X}, \forall y \in K^+) \quad x \preceq_{K \cap \{y\}^\perp} x' \implies h(x, t) \preceq_{\{y\}^+} h(x', t), \quad \forall t \in [0, +\infty). \quad (7)$$

In this definition, we use the preorders given by the convex cones $K \cap \{y\}^\perp$ and $\{y\}^+$, where $\{y\}^\perp$ denotes the orthogonal space to y . From the definition (5) of the preorder induced by a convex cone, we obtain that

$$x \preceq_{K \cap \{y\}^\perp} x' \iff x' - x \in K \text{ and } \langle x' - x, y \rangle = 0, \quad (8a)$$

$$x \preceq_{\{y\}^+} x' \iff \langle x' - x, y \rangle \geq 0. \quad (8b)$$

When the mapping h displays additional regularity properties, there exists a more amenable characterization of K -quasimonotonicity, as presented in the next proposition [26].

Proposition 3.1. ([26, Proposition 5.1]) *If the mapping $h : \mathbb{X} \times [0, +\infty) \rightarrow \mathbb{R}^n$ in Definition 3.1 is differentiable with respect to the first variable, where $\mathbb{X} \subset \mathbb{R}^n$ is the closure of an open subset of \mathbb{R}^n , and if K is one of the orthants of \mathbb{R}^n , that is*

$$K = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid (-1)^{m_j} x_j \geq 0, \quad j = 1, \dots, n\},$$

where $(m_1, \dots, m_n) \in \{0, 1\}^n$, then the mapping h is K -quasimonotone if and only if

$$(-1)^{m_i+m_j} \frac{\partial h_i}{\partial x_j}(x, t) \geq 0, \quad \forall i \neq j, \quad \forall (x, t) \in \mathbb{X} \times [0, +\infty). \quad (9)$$

An example of such cones and characterization will be used in Sect. 4. When the convex cone is the positive orthant $K = \mathbb{R}_+^n$, condition (9) is called *cooperativeness* in [26], as it reads $\frac{\partial h_i}{\partial x_j}(x, t) \geq 0, \quad \forall i \neq j, \quad \forall (x, t) \in \mathbb{X} \times [0, +\infty)$.

3.2 Conic quasimonotonicity reducibility for controlled dynamical systems

The following definition is new.

Definition 3.2. Let $K \subset \mathbb{R}^n$ be a convex cone. Let $\phi : \mathbb{U} \rightarrow \mathbb{U}$ be a measurable mapping. We say that a controlled dynamics $f : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}^n$, as in Definition 2.1, is (K, ϕ) -quasimonotone reducible if the two following conditions hold.

(H1) For all control path $u(\cdot) \in \mathcal{U}$, the mapping $h_{u(\cdot)} : \mathbb{X} \times [0, +\infty) \rightarrow \mathbb{R}^n$ defined by $h_{u(\cdot)}(x, t) = f(x, u(t))$ is K -quasimonotone (as in Definition 3.1).

(H2) The measurable mapping $\phi : \mathbb{U} \rightarrow \mathbb{U}$ has the property that

$$f(x, u) \preceq_K f(x, \phi(u)), \quad \forall (x, u) \in \mathbb{X} \times \mathbb{U}. \quad (10)$$

The measurable mapping ϕ is called a K -reduction for the controlled dynamics f .

The notion of K -reduction is interesting in practice if the mapping $\phi : \mathbb{U} \rightarrow \mathbb{U}$ is not surjective (that is², $\phi(\mathbb{U}) \subsetneq \mathbb{U}$), and more precisely if its image $\phi(\mathbb{U})$ is “small” because, in some way, we are reducing the control space \mathbb{U} . The following result provides a sufficient condition to compare flows of controlled dynamics, based on (K, ϕ) -quasimonotone reducibility.

Proposition 3.2. Let $K \subsetneq \mathbb{R}^n$ be a closed convex cone with nonempty interior. Let $\phi : \mathbb{U} \rightarrow \mathbb{U}$ be a measurable mapping. Let be given a controlled dynamics $f : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}^n$ as in Definition 2.1, and suppose that it generates a global flow Ψ , and that it is (K, ϕ) -quasimonotone reducible, as in Definition 3.2.

Then, for any control path $u(\cdot) \in \mathcal{U}$, we have that

$$x_0, x'_0 \in \mathbb{X} \text{ and } x_0 \preceq_K x'_0 \implies \Psi_f^{u(\cdot)}(t, x_0) \preceq_K \Psi_f^{u_\phi(\cdot)}(t, x'_0), \quad \forall t \in [0, +\infty), \quad (11)$$

where the reduced control path $u_\phi(\cdot)$ is defined by

$$u_\phi(t) = \phi(u(t)), \quad \forall t \in [0, +\infty). \quad (12)$$

² $J \subsetneq L$ stands for $J \subset L$ and $J \neq L$.

Proof. The control path $u_\phi(\cdot)$, defined by $u_\phi(t) = \phi(u(t))$ for all $t \in [0, +\infty)$, is measurable as both $u(\cdot)$ and ϕ are measurable mappings. For a control path $u(\cdot) \in \mathcal{U}$, we define the dynamics mappings $h_{u(\cdot)} : \mathbb{X} \times [0, +\infty) \rightarrow \mathbb{R}^n$ and $h_{u_\phi(\cdot)} : \mathbb{X} \times [0, +\infty) \rightarrow \mathbb{R}^n$ by

$$h_{u(\cdot)}(x, t) = f(x, u(t)) \text{ and } h_{u_\phi(\cdot)}(x, t) = f(x, u_\phi(t)) , \quad \forall (x, t) \in \mathbb{X} \times [0, +\infty) .$$

By assumption (H1) in Definition 3.2, the dynamics mapping $h_{u(\cdot)}$ is K -quasimonotone. By assumption (H2) in Definition 3.2, Equation (10) gives that

$$h_{u(\cdot)}(x, t) \preceq_K h_{u_\phi(\cdot)}(x, t) , \quad \forall (x, t) \in \mathbb{X} \times [0, +\infty) .$$

Looking at the assumptions of Lemma A.1 in Appendix A, we can check that they are all satisfied. The result (11) follows directly. \square

Let us contrast Proposition 3.2 with the conditions to ensure monotonicity given in [1]:

- ($\hat{H}1$) The control set \mathbb{U} is a convex subset of \mathbb{R}^m and there exists a preorder $\preceq_{K_{\mathbb{U}}}$ given by a closed convex cone $K_{\mathbb{U}} \subset \mathbb{R}^m$;
- ($\hat{H}2$) For all $u(\cdot) \in \mathcal{U}$, the mapping $(x, t) \rightarrow f(x, u(t))$ is K -quasimonotone;
- ($\hat{H}3$) For any control paths $u(\cdot), u'(\cdot) \in \mathcal{U}$, as in (1), one has that, if $u(t) \preceq_{K_{\mathbb{U}}} u'(t)$ for all $t \in [0, +\infty)$, then $f(x, u(t)) \preceq_K f(x, u'(t))$ for all $x \in \mathbb{X}$ and $t \in [0, +\infty)$.

The result in [1] is a particular case of our result, because the assumptions ($\hat{H}1$), ($\hat{H}2$) and ($\hat{H}3$) imply our assumptions (H1) and (H2) in Definition 3.2. Indeed, first, condition (H1) is the same as ($\hat{H}2$). Second, by taking for K -reduction mapping any measurable mapping $\phi : \mathbb{U} \rightarrow \mathbb{U}$ such that $\phi(u) \in (u + K_{\mathbb{U}}) \cap \mathbb{U}$, we see that conditions ($\hat{H}1$) and ($\hat{H}3$) imply (H1). Therefore, to obtain the monotonicity property (11), it is not necessary to have a preorder defined on the control space \mathbb{R}^m as in condition ($\hat{H}1$) in [1], but it is enough to find a K -reduction as in condition (H2).

3.3 Comparison of viability kernels

Now, we are ready to provide a comparison result for viability kernels, the main purpose of this work.

Proposition 3.3. *Let $K \subsetneq \mathbb{R}^n$ be a closed convex cone with nonempty interior. Let $\phi : \mathbb{U} \rightarrow \mathbb{U}$ be a measurable mapping. Let be given a controlled dynamics $f : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}^n$ as in Definition 2.1, and suppose that it generates a global flow Ψ , and that it is (K, ϕ) -quasimonotone reducible, as in Definition 3.2. Let $\mathbb{D} \subset \mathbb{X} \times \mathbb{U}$ be a desirable set. We introduce*

1. the reduced controlled dynamics $f_\phi : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}^n$ defined by

$$f_\phi(x, u) = f(x, \phi(u)) , \quad \forall (x, u) \in \mathbb{X} \times \mathbb{U} , \tag{13}$$

2. the extended desirable set $\mathbb{D}_K \subset \mathbb{X} \times \mathbb{U}$ defined by

$$\mathbb{D}_K = \mathbb{D} + (K \times \{0\}) . \quad (14)$$

Then, we have the following inclusion of viability kernels:

$$\mathbb{V}(f, \mathbb{D}) \subset \mathbb{V}(f_\phi, \mathbb{D}_K) . \quad (15)$$

Furthermore, if

$$\bigcup_{(x,u) \in \mathbb{D}} (x + K) \times \phi(u) \subset \mathbb{D} , \quad (16)$$

then we have the following equality between viability kernels:

$$\mathbb{V}(f, \mathbb{D}) = \mathbb{V}(f_\phi, \mathbb{D}_K) . \quad (17)$$

Proof. It is easily checked that the mapping f_ϕ in (13) indeed is a controlled dynamics as in Definition 2.1. Moreover, it is immediate, from definition (12) of the reduced control path $u_\phi(\cdot)$ and from definition (3) of the flow, that

$$\Psi_f^{u_\phi(\cdot)}(t, x_0) = \Psi_{f_\phi}^{u(\cdot)}(t, x_0) , \quad \forall (t, x_0) \in [0, +\infty) \times \mathbb{X} . \quad (18)$$

• First, we prove the inclusion (15), that is, $\mathbb{V}(f, \mathbb{D}) \subset \mathbb{V}(f_\phi, \mathbb{D}_K)$. For this purpose, we consider $x_0 \in \mathbb{V}(f, \mathbb{D})$, and we show that $x_0 \in \mathbb{V}(f_\phi, \mathbb{D}_K)$.

By definition (4) of the viability kernel $\mathbb{V}(f, \mathbb{D})$, there exists a control path $u(\cdot) \in \mathcal{U}$ such that

$$(\Psi_f^{u(\cdot)}(t, x_0), u(t)) \in \mathbb{D} , \quad \forall t \in [0, +\infty) .$$

As the assumptions of Proposition 3.2 are satisfied, Equation (11) gives $\Psi_f^{u(\cdot)}(t, x_0) \preceq_K \Psi_{f_\phi}^{u(\cdot)}(t, x_0)$, $\forall t \in [0, +\infty)$. Thus, from (18), we deduce that

$$\Psi_f^{u(\cdot)}(t, x_0) \preceq_K \Psi_{f_\phi}^{u(\cdot)}(t, x_0) , \quad \forall t \in [0, +\infty) . \quad (19)$$

Then, we write

$$\Psi_{f_\phi}^{u(\cdot)}(t, x_0) = \Psi_f^{u(\cdot)}(t, x_0) + \overbrace{(\Psi_{f_\phi}^{u(\cdot)}(t, x_0) - \Psi_f^{u(\cdot)}(t, x_0))}^{\in K} ,$$

where the second term $(\Psi_{f_\phi}^{u(\cdot)}(t, x_0) - \Psi_f^{u(\cdot)}(t, x_0))$ belongs to K by (19) and by the definition (5) of the preorder \preceq_K . Therefore, from definition (14) of $\mathbb{D}_K = \mathbb{D} + (K \times \{0\})$, we deduce that, for all $t \in [0, +\infty)$,

$$(\Psi_{f_\phi}^{u(\cdot)}(t, x_0), u(t)) = \underbrace{(\Psi_f^{u(\cdot)}(t, x_0), u(t))}_{\in \mathbb{D}} + \underbrace{(\Psi_{f_\phi}^{u(\cdot)}(t, x_0) - \Psi_f^{u(\cdot)}(t, x_0), 0)}_{\in K \times \{0\}} \in \mathbb{D}_K .$$

This implies that $x_0 \in \mathbb{V}(f_\phi, \mathbb{D}_K)$, hence the first part of the proof is complete.

• Second, we suppose that (16) holds true and we prove the equality (17), that is, $\mathbb{V}(f, \mathbb{D}) = \mathbb{V}(f_\phi, \mathbb{D})$. By the just proven inclusion (15), it suffices to show the reverse inclusion, that is, $\mathbb{V}(f_\phi, \mathbb{D}_K) \subset \mathbb{V}(f, \mathbb{D})$. For this purpose, we consider $x_0 \in \mathbb{V}(f_\phi, \mathbb{D}_K)$ and we show that $x_0 \in \mathbb{V}(f, \mathbb{D})$.

By definition (4) of the viability kernel $\mathbb{V}(f_\phi, \mathbb{D}_K)$, there exists a control path $u(\cdot) \in \mathcal{U}$ such that $(\Psi_{f_\phi}^{u(\cdot)}(t, x_0), u(t)) \in \mathbb{D}_K, \forall t \in [0, +\infty)$. From (18), we deduce that

$$(\Psi_f^{u_\phi(\cdot)}(t, x_0), u(t)) \in \mathbb{D}_K, \quad \forall t \in [0, +\infty).$$

Now, by definition (14) of \mathbb{D}_K , for all $t \in [0, +\infty)$ there exist $v_t \in \mathbb{R}^n$ and $w_t \in \mathbb{R}^n$ such that

$$\Psi_f^{u_\phi(\cdot)}(t, x_0) = v_t + w_t, \quad (v_t, u(t)) \in \mathbb{D}, \quad w_t \in K. \quad (20)$$

We now show that the control path $u_\phi(\cdot)$ in (12) maintains the state and control trajectory $(\Psi_f^{u_\phi(\cdot)}(t, x_0), u_\phi(t))$ in \mathbb{D} . Indeed, we have

$$\begin{aligned} (\Psi_f^{u_\phi(\cdot)}(t, x_0), u_\phi(t)) &= (\Psi_f^{u_\phi(\cdot)}(t, x_0), \phi(u(t))) && \text{(by definition (12) of } u_\phi(\cdot)\text{)} \\ &= (v_t + w_t, \phi(u(t))) && \text{(by definition (20) of } v_t \text{ and } w_t\text{)} \\ &\in \bigcup_{(x', u') \in \mathbb{D}} (x' + K) \times \phi(u') && \text{(as } (v_t, u(t)) \in \mathbb{D} \text{ and } w_t \in K \text{ by (20))} \\ &\in \mathbb{D}. && \text{(by (16))} \end{aligned}$$

By definition (4) of the viability kernel $\mathbb{V}(f, \mathbb{D})$, we conclude that $x_0 \in \mathbb{V}(f, \mathbb{D})$.

This ends the proof. □

4 Application to viable control of the Wolbachia bacterium

In this Section, we apply the result established in Proposition 3.3 to a problem related to epidemics control.

The mosquito species *Aedes aegypti* is the main transmitter of dengue. When these mosquitoes are infected with *Wolbachia* bacterium, they become less capable of transmitting the dengue virus to human hosts. Thanks to this discovery, *Wolbachia*-based biocontrol is accepted as an ecologically friendly and potentially cost-effective method for prevention and control of dengue and other arboviral infections. We introduce now a model borrowed from [5] representing the dynamics of a mosquito population infected with *Wolbachia*. This model is described by four state variables

$$x = (L_U, A_U, L_W, A_W) \in \mathbb{R}^4,$$

where L_U and A_U represent the uninfected mosquitoes abundances (larva and adults respectively), whereas L_W and A_W are the infected (with Wolbachia) mosquitoes abundances (larva and adults respectively). The population dynamics model is described by the following system of differential equations

$$\dot{L}_U = \alpha_U A_U \frac{A_U}{A_U + A_W} - \nu L_U - \mu (1 + k (L_U + L_W)) L_U , \quad (21a)$$

$$\dot{A}_U = \nu L_U - \mu_U A_U , \quad (21b)$$

$$\dot{L}_W = \alpha_W A_W - \nu L_W - \mu (1 + k (L_U + L_W)) L_W , \quad (21c)$$

$$\dot{A}_W = \nu L_W - \mu_W A_W , \quad (21d)$$

where all parameters are assumed to be positive [5].

In biocontrol, one can choose the quantity of mosquitoes infected with Wolbachia larvae to be introduced [6]. This is why, in the context of the model (21), we consider the control variable

$$u \in \mathbb{U} = [0, u^\sharp] \subset \mathbb{R} ,$$

where $u^\sharp > 0$ is the maximal quantity of mosquitoes infected with Wolbachia larvae that can be introduced. Then, by (21), we obtain a controlled dynamics which reads as (2) with

$$f(x, u) = (F_L(x), F_A(x), G_L(x) + u, G_A(x)) , \quad \forall x \in \mathbb{X} = \mathbb{R}_+^4 , \quad \forall u \in \mathbb{U} , \quad (22)$$

where

$$F_L(L_U, A_U, L_W, A_W) = \alpha_U A_U \frac{A_U}{A_U + A_W} - \nu L_U - \mu (1 + k (L_U + L_W)) L_U , \quad (23a)$$

$$F_A(L_U, A_U, L_W, A_W) = \nu L_U - \mu_U A_U , \quad (23b)$$

$$G_L(L_U, A_U, L_W, A_W) = \alpha_W A_W - \nu L_W - \mu (1 + k (L_U + L_W)) L_W , \quad (23c)$$

$$G_A(L_U, A_U, L_W, A_W) = \nu L_W - \mu_W A_W . \quad (23d)$$

By (22) and (23), the mapping f is well defined on $\mathbb{X} = \mathbb{R}_+^4$, except for points where $A_U = A_W = 0$. But, from the expression (23a) of the first component F_L of $f(\cdot, u)$, the mapping f can be defined in such points by continuity.

We take the stand that one of the objectives of biocontrol is to keep the population of infected mosquitoes with Wolbachia above some thresholds (see [5, 6] and the references therein). In this context, we consider positive upper population levels $(\overline{L}_W, \overline{A}_W)$ and positive lower population levels $(\underline{L}_U, \underline{A}_U)$. Our aim is to have the infected population of mosquitoes to be above both $\overline{A}_W, \overline{L}_W$, and the uninfected population to be below both $\underline{L}_U, \underline{A}_U$, permanently. Thus, we define the following desirable set

$$\mathbb{D} = \{ (L_U, A_U, L_W, A_W, u) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times [0, u^\sharp] \mid L_U \leq \underline{L}_U, A_U \leq \underline{A}_U, L_W \geq \overline{L}_W, A_W \geq \overline{A}_W \} . \quad (24)$$

Proposition 4.1. *Let the controlled dynamics mapping $f^\# : \mathbb{R}_+^4 \times [0, u^\#] \rightarrow \mathbb{R}^4$ be defined from the controlled dynamics (22) by*

$$f^\#(x, u) = f(x, u^\#), \quad \forall (x, u) \in \mathbb{R}_+^4 \times [0, u^\#]. \quad (25)$$

Then, we have

$$\mathbb{V}(f, \mathbb{D}) = \mathbb{V}(f^\#, \mathbb{D}). \quad (26)$$

The advantage of the equality (26) over the definition (4) of the viability kernel $\mathbb{V}(f, \mathbb{D})$ is that $f^\#$ in (25) is not really a controlled dynamics, as it does not depend on the control u . In other words, an initial condition $x_0 = (L_U, A_U, L_W, L_W)_0$ belongs to the viability kernel $\mathbb{V}(f, \mathbb{D})$ if and only if, using the stationary control $u^\#$ in the differential equation $\dot{x}(t) = f(x(t), u^\#)$, the state and control $(L_U(t), A_U(t), L_W(t), A_W(t), u^\#)$ lies in \mathbb{D} , defined in (24), for all $t \in [0, +\infty)$. Hence, the problem has been reduced to compute the viability kernel for a single constant control policy, instead of a family of controls, which is a far more easier problem to handle than the original problem.

Proof. The proof consists in applying Proposition 3.3. In order to ensure that all assumptions are satisfied, we divide the proof in three parts.

- First, we prove that the mapping f given by (22) is a controlled dynamics as in Definition 2.1. Indeed, on the one hand, by (22) and (23), it is straightforward that $f(\cdot, u)$ is locally Lipschitz on $\mathbb{X} = \mathbb{R}_+^4$, with Lipschitz constant independent of u . On the other hand, it is proved in [5, Theorem 1] that, for all initial condition with nonnegative components $x_0 \in \mathbb{R}_+^4$, the solutions of the controlled system $\dot{x}(t) = f(x(t), u(t))$, where $x(0) = x_0 \in \mathbb{R}_+^4$ and where $u(\cdot) : [0, +\infty) \rightarrow [0, u^\#]$ is a measurable control path, remain in \mathbb{R}_+^4 , and that the solution is defined for all time $t \in [0, +\infty)$.

- Second, we show that the controlled dynamics f is (K, ϕ) -quasimonotone reducible according to Definition 3.2, for suitable cone $K \subset \mathbb{R}^n$ and mapping $\phi : [0, u^\#] \rightarrow [0, u^\#]$.

On the one hand, we define the cone

$$K = \mathbb{R}_- \times \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_+, \quad (27)$$

and the associated preorder given by, for any two vectors $x = (x_1, x_2, x_3, x_4)$, $x' = (x'_1, x'_2, x'_3, x'_4)$, $x \preceq_K x'$ if and only if $x_1 \geq x'_1$, $x_2 \geq x'_2$, $x_3 \leq x'_3$, $x_4 \leq x'_4$. As the cone K in (27) is one of the orthants of \mathbb{R}^4 , we deduce from Proposition 3.1 that, for any measurable control path $u(\cdot) : [0, +\infty) \rightarrow [0, u^\#]$, the mapping $h_{u(\cdot)}(x, t) = f(x, u(t))$ is K -quasimonotone if and only if

- (a) $\frac{\partial F_L}{\partial A_U} \geq 0$, $\frac{\partial F_L}{\partial L_W} \leq 0$, $\frac{\partial F_L}{\partial A_W} \leq 0$,
- (b) $\frac{\partial F_A}{\partial L_U} \geq 0$, $\frac{\partial F_A}{\partial L_W} \leq 0$, $\frac{\partial F_A}{\partial A_W} \leq 0$,
- (c) $\frac{\partial G_L}{\partial A_W} \geq 0$, $\frac{\partial G_L}{\partial L_U} \leq 0$, $\frac{\partial G_L}{\partial A_U} \leq 0$,

$$(d) \quad \frac{\partial G_A}{\partial L_W} \geq 0, \quad \frac{\partial G_A}{\partial L_U} \leq 0, \quad \frac{\partial G_A}{\partial A_U} \leq 0.$$

Now, these inequalities can easily be verified for the functions F_L , F_A , G_L , and G_A defined in (23). Therefore, the controlled dynamics f in (22) satisfies assumption (H1) in Definition 3.2.

On the other hand, we define the mapping $\phi : [0, u^\sharp] \rightarrow [0, u^\sharp]$ by

$$\phi(u) = u^\sharp, \quad \forall u \in [0, u^\sharp]. \quad (28)$$

Then, we observe that, by (22), one has, for all $u \in [0, u^\sharp]$,

$$f(x, \phi(u)) - f(x, u) = f(x, u^\sharp) - f(x, u) = (0, 0, u^\sharp - u, 0) \in \mathbb{R}_- \times \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_+ = K.$$

By definition (5) of the preorder \preceq_K and by expression (27) of the cone K , we get that

$$f(x, u) \preceq_K f(x, \phi(u)) = f(x, u^\sharp), \quad \forall (x, u) \in \mathbb{R}_+^4 \times [0, u^\sharp].$$

Thus, the mapping ϕ in (28) is a K -reduction for the controlled dynamics f , and condition (H2) in Definition 3.2 is satisfied.

• Third, we prove (26).

On the one hand, the new reduced controlled dynamics (13) satisfies $f_\phi = f^\sharp$, because of the expression (25) of f^\sharp and by $\phi(u) = u^\sharp$ in (28). On the other hand, the desirable set \mathbb{D} in (24) has the expression

$$\mathbb{D} = ((\underline{L}_U, \underline{A}_U, \overline{L}_W, \overline{A}_W) + K) \times [0, u^\sharp]. \quad (29)$$

We deduce that the new extended desirable set in (14) satisfies

$$\mathbb{D}_K = \mathbb{D} + (K \times \{0\}) = ((\underline{L}_U, \underline{A}_U, \overline{L}_W, \overline{A}_W) + K + K) \times ([0, u^\sharp] + 0) = \mathbb{D},$$

where we have used the property that $K + K = K$, as the cone K is convex and contains 0. There remains to check that (16) holds true. Now, by (28) and (29), we have

$$\bigcup_{(x,u) \in \mathbb{D}} (x + K) \times \phi(u) = ((\underline{L}_U, \underline{A}_U, \overline{L}_W, \overline{A}_W) + K) \times \{u^\sharp\} \subset \mathbb{D}.$$

Therefore, we apply Proposition 3.3 and we obtain that

$$\mathbb{V}(f, \mathbb{D}) = \mathbb{V}(f_\phi, \mathbb{D}_K) = \mathbb{V}(f_\phi, \mathbb{D}) = \mathbb{V}(f^\sharp, \mathbb{D}).$$

This ends the proof. □

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A Comparison of flows

The purpose of this appendix is to prove a result about the comparison of flows generated by two dynamics.

Lemma A.1. *Let $\mathbb{X} \subset \mathbb{R}^n$ be a closed subset of \mathbb{R}^n , and $g, h : \mathbb{X} \times [0, +\infty) \rightarrow \mathbb{R}^n$ be two mappings that are locally Lipschitz in the first variable and measurable in the second variable, and such that the two differential equations*

$$\dot{x} = g(x, t), \quad \dot{x} = h(x, t), \quad x(0) = x_0$$

have unique solutions, for all time $t \in [0, +\infty)$ and $x_0 \in \mathbb{X}$, denoted by $\Psi_g(t, x_0)$ and $\Psi_h(t, x_0)$. The mappings Ψ_g and Ψ_h are called flows.

Let $K \subsetneq \mathbb{R}^n$ be a closed convex cone with nonempty interior. Suppose that

- one of the two mappings, either g or h , is K -quasimonotone (as in Definition 3.1),
- we have that $g(x, t) \preceq_K h(x, t)$, for all $(x, t) \in \mathbb{X} \times [0, +\infty)$.

Then, the two flows Ψ_g and Ψ_h have the following property:

$$x_0, x'_0 \in \mathbb{X} \text{ and } x_0 \preceq_K x'_0 \implies \Psi_g(t, x_0) \preceq_K \Psi_h(t, x'_0), \quad \forall t \in [0, +\infty). \quad (30)$$

This Lemma is a generalization of Theorem 1.1 in [18], where the implication (30) is established in the particular case when $g = h$. (When $g = h$, it is also proven in [18] that (30) is a sufficient condition for the K -quasimonotonicity of g .)

Proof. Observe that, for any initial conditions x_0 and x'_0 , the solutions $x(t) = \Psi_g(t, x_0)$ and $y(t) = \Psi_h(t, x'_0)$ satisfy

$$x(t) = x_0 + \int_0^t g(x(s), s) ds, \quad y(t) = x'_0 + \int_0^t h(y(s), s) ds, \quad \forall t \in [0, +\infty).$$

As the mappings g and h are locally Lipschitz in the first variable, we obtain that $\Psi_g(\cdot, \cdot)$ and $\Psi_h(\cdot, \cdot)$ are continuous in the couple argument.

As the closed convex cone $K \subsetneq \mathbb{R}^n$ has nonempty interior $\text{int}K$, we introduce the following notation

$$x \prec\prec_K x' \Leftrightarrow x' - x \in \text{int}K. \quad (31)$$

The relation $\prec\prec_K$ is transitive (as $\text{int}K + \text{int}K \subset \text{int}K$), but not necessarily reflexive (as 0 may or may not be in $\text{int}K$). The following result is established in [17, Proposition 3.1]

$$x \in \text{int}K \Leftrightarrow x \in K \text{ and } \langle y, x \rangle > 0, \quad \forall y \in K^+ \setminus \{0\}, \quad (32)$$

where the positive polar cone K^+ has been defined in (6). As a consequence, if $x \in \partial K = K \setminus \text{int}K$, then there exists an element $y \in K^+ \setminus \{0\}$ such that $\langle y, x \rangle = 0$ (indeed, $K^+ \setminus \{0\} \neq \emptyset$ because of the assumption that $K \subsetneq \mathbb{R}^n$, hence $K \neq \mathbb{R}^n$).

We assume that g is K -quasimonotone. In the case of h being K -quasimonotone, the proof is analogous.

- First, we prove that, if $g(x, t) \prec\prec_K h(x, t)$, $\forall (x, t) \in \mathbb{X} \times [0, +\infty)$, then

$$x_0 \prec\prec_K x'_0 \implies \Psi_g(t, x_0) \prec\prec_K \Psi_h(t, x'_0), \quad \forall t \in [0, +\infty). \quad (33)$$

Indeed, let us assume that this is not the case. Then, there would exist initial conditions x_0 and x'_0 in \mathbb{X} , and $s \in [0, +\infty)$, $s > 0$, such that

$$\Psi_g(t, x_0) \prec\prec_K \Psi_h(t, x'_0), \quad \forall t \in [0, s) \text{ and } \Psi_h(s, x'_0) \not\prec\prec_K \Psi_g(s, x_0),$$

that is, $\Psi_g(t, x_0) - \Psi_h(t, x'_0) \in \text{int}K$, $\forall t \in [0, s)$, and $\Psi_h(s, x'_0) - \Psi_g(s, x_0) \notin \text{int}K$. Since K is closed and the flows are continuous in their two arguments, we would deduce that $\Psi_h(s, x'_0) - \Psi_g(s, x_0) \in K \setminus \text{int}K = \partial K$. By (32), there would exist $y \in K^+ \setminus \{0\}$ such that both $\langle y, \Psi_h(s, x'_0) - \Psi_g(s, x_0) \rangle = 0$, and $\langle y, \Psi_h(t, x'_0) - \Psi_g(t, x_0) \rangle > 0$, for $0 \leq t < s$, giving thus $\frac{d}{dt} \langle y, \Psi_h(t, x'_0) - \Psi_g(t, x_0) \rangle|_{t=s} \leq 0$. From the definition of the flows, we would finally obtain that

$$\langle y, h(\Psi_h(s, x'_0), s) \rangle \leq \langle y, g(\Psi_g(s, x_0), s) \rangle. \quad (34)$$

As we have seen that $\langle y, \Psi_h(s, x'_0) - \Psi_g(s, x_0) \rangle = 0$, and $\Psi_h(s, x'_0) - \Psi_g(s, x_0) \in K$, we would deduce that $\Psi_h(s, x'_0) - \Psi_g(s, x_0) \in K \cap \{y\}^\perp$, that is, $\Psi_g(s, x_0) \preceq_{K \cap \{y\}^\perp} \Psi_h(s, x'_0)$ by definition (5) of the preorder $\preceq_{K \cap \{y\}^\perp}$. Now, since g is K -quasimonotone, we would deduce from (7) that

$$\langle y, g(\Psi_g(s, x_0), s) \rangle \leq \langle y, g(\Psi_h(s, x'_0), s) \rangle. \quad (35)$$

Combining Inequalities (34) and (35) would give

$$\langle y, h(\Psi_h(s, x'_0), s) \rangle \leq \langle y, g(\Psi_h(s, x'_0), s) \rangle.$$

Now, this would contradict the assumption that $g(x, t) \prec\prec_K h(x, t)$, $\forall (x, t) \in \mathbb{X} \times [0, +\infty)$, which indeed implies that $h(\Psi_h(s, x'_0), s) - g(\Psi_h(s, x'_0), s) \in \text{int}K$, and, by (32), that

$$\langle y, g(\Psi_h(s, x'_0), s) \rangle < \langle y, h(\Psi_h(s, x'_0), s) \rangle.$$

Therefore, the implication (33) holds true.

- Second, we prove (30).

For this purpose, we consider $x_0, x'_0 \in \mathbb{X}$ such that $x_0 \preceq_K x'_0$. Then, we take $v \in \text{int}K \neq \emptyset$ and, for any $\epsilon > 0$, we consider the following differential equation

$$\dot{x} = h_\epsilon(x, t) = h(x, t) + \epsilon v, \quad x(0) = x'_0 + \epsilon v.$$

From the assumptions on the dynamics mapping h , the above differential equation has a unique solution $x_\epsilon(t) = \Psi_{h_\epsilon}(t, x'_0 + \epsilon v)$, defined for all $t \in [0, +\infty)$, and which satisfies

$$x_\epsilon(t) = x'_0 + (1+t)\epsilon v + \int_0^t h(x_\epsilon(s), s) ds, \quad \forall t \in [0, +\infty). \quad (36)$$

By an easy adaptation of the classical proof that solutions of ordinary differential equations continuously depend on a continuous parameter (see for instance [16]), we get the following result: for every $t \geq 0$, we have $x_\epsilon(t) \rightarrow x(t)$ when $\epsilon \downarrow 0$, where $x(\cdot)$ is solution of the differential equation $\dot{x} = h(x, t)$, $x(0) = x'_0$, that is, $x_\epsilon(t) \rightarrow \Psi_h(t, x'_0)$ when $\epsilon \downarrow 0$.

Now, since $x_0 \preceq_K x'_0$ and $g(x, t) \preceq_K h(x, t)$, for all $(x, t) \in \mathbb{R}^n \times [0, +\infty)$, we obtain that $x_0 \prec\prec_K x'_0 + \epsilon v$ and $g(x, t) \prec\prec_K h_\epsilon(x, t)$, $\forall (x, t) \in \mathbb{X} \times [0, +\infty)$, by the definition (31) of the relation $\prec\prec_K$, where we have used that $K + \text{int}K \subset \text{int}K$ and $\epsilon(\text{int}K) \subset \text{int}K$, for all $\epsilon > 0$. Thus, we can apply the implication (33) established in the first part of the proof, and get

$$\Psi_g(t, x_0) \prec\prec_K \Psi_{h_\epsilon}(t, x'_0 + \epsilon v) = x_\epsilon(t), \quad \forall t \in [0, +\infty),$$

where $x_\epsilon(t)$ is given by (36). Since $x_\epsilon(t) \rightarrow \Psi_h(t, x'_0)$ when $\epsilon \downarrow 0$, for all $t \in [0, +\infty)$, and since the cone K is closed, we finally get that

$$\Psi_g(t, x_0) \preceq_K \Psi_h(t, x'_0), \quad \forall t \in [0, +\infty),$$

which is the desired result (30). □

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