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ON DUAL UNIT BALLS OF THURSTON NORMS

Abdoul Karim SANE

Abstract

Thurston norms are invariants of 3-manifolds defined on their second homology groups, and understanding the shape of their dual unit ball is a (widely) open problem. W. Thurston showed that every symmetric polygon in \mathbb{Z}^2 whose vertices have the same parity is the dual unit ball of a Thurston norm on a 3-manifold. However, it is not known if the parity condition on the vertices of polyhedra is a sufficient condition in higher dimension or if their are polyhedra with mod 2 congruent vertices which are not dual unit ball of any Thurston norm. In this article, we provide a family of polytopes in \mathbb{Z}^{2g} which can be realized as dual unit balls of Thurston norms on 3-manifolds. Those polytopes come from intersection norms on oriented closed surfaces and this article widens the bridge between those two norms.

1 Introduction

Given an oriented 3-manifold M , W. Thurston [9] defined a semi-norm on the second homology group of M . If $a \in H_2(M, \mathbb{Z})$ is an integer class, then a admits representatives which are disjoint unions of properly embedded surfaces S_i in M . The Thurston norm of a is given by :

$$x(a) = \min_{[\cup_i S_i] = a} \left\{ \sum_i \max\{0, -\chi(S_i)\} \right\}.$$

If M is prime —any embedded sphere in M bounds a ball— and atoroidal —has no essential torus—, then x is a norm. By construction x takes integer values on $H_2(M, \mathbb{Z})$; it is an *integer norm* : a norm on a vector space which takes integer values on a top dimensional lattice. W. Thurston showed that the dual unit ball of an integer norm on a vector space E (relatively to a lattice Λ) is the convex hull of finitely many vectors $u_i \in E^*$ which take integer values on Λ :

$$x(a) = \max_{u_i} \{ \langle u_i, a \rangle \}.$$

For a given 3-manifold, the vectors defining the dual unit ball of the Thurston norm satisfy the *parity condition* : $u_i \equiv u_j \pmod{2}$.

When a surface S realizes the norm in its homology class, namely when $x([S]) = -\chi(S)$, S is said to be *minimizing*. Thurston gave some conditions under which a given surface is minimizing. For instance, closed leaves of transversally oriented foliations without Reeb component are minimizing. D. Gabai [4] showed the converse, namely that minimizing surfaces are leaves of foliations without Reeb component on M . Thurston also showed that if an embedded surface S in M is a fiber of a surface bundle over the circle, then there is a unique vector $u_i \in H^2(M, \mathbb{Z})$ such that $x([S]) = \langle u_i, S \rangle$. Moreover, if S' is another surface such that $x([S']) = \langle u_i, S' \rangle$, then S' is the fiber of a surface bundle. Therefore homology classes of fiber of bundles over the circle belong to an open cone on finitely many faces, called *fibred faces*, of the unit ball of the Thurston norm on M . In this article, we are interested in the shape of those polytopes that appear as dual unit balls of Thurston norms.

Just after defining his norm, Thurston started to compute it on a few examples of 3-manifolds. Even better, he shows the following :

Theorem 1 ([9], Thm 6). *Every symmetric polygon in \mathbb{Z}^2 with mod 2 congruent vertices is the dual unit ball of the Thurston norm of some 3-manifold.*

Our main result is an extension of Thurston's theorem to symmetric polytopes in dimension $2g$. To do so, let us introduce intersection norms on oriented closed surfaces Σ_g . Intersection norms were quickly introduced by V. Turaev [10], and they got a new flavor with the article of M. Cossarini and P. Dehornoy [2].

Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a finite collection of closed curves on Σ_g , with only double points. If Γ is a filling collection, that is, its complement on Σ_g is a union of disks, the intersection norm N_Γ associated to Γ restricted to homology with integer coefficient is the function defined by :

$$N_\Gamma : H_1(\Sigma_g, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$a \longmapsto \inf\{\text{card}\{\alpha \cap \Gamma\}; [\alpha] = a\}.$$

Intersection norms are also integer norms. Therefore, the dual unit ball of N_Γ is the convex hull of finitely many vectors $v_i \in H^1(\Sigma, \mathbb{Z})$. As for Thurston norms, the vectors v_i satisfy the parity condition, because geometric intersection and algebraic intersection between curves have the same parity.

We recall that the norm is completely determined by the vectors v_i :

$$N_\Gamma(a) = \max_{v_i} \{\langle v_i, a \rangle\}.$$

M. Cossarini and P. Dehornoy gave an algorithm for the computation of all the vectors defining an intersection norm.

Definition 1. A filling collection Γ is *homologically non trivial* if there exists an orientation $\vec{\Gamma}$ of Γ such $[\vec{\Gamma}]$ is a non-trivial homology class.

A *homologically non-trivial polytope* in \mathbb{Z}^{2g} is a symmetric polytope with mod 2 congruent vectors which appears as the dual unit ball of an intersection on Σ_g associated to homologically non-trivial collection.

Our main theorem brings Thurston norms closer to intersection norms.

Main Theorem. *If P is a homologically non-trivial polytope, then it is the dual unit ball of a Thurston norm on a 3-manifold.*

Unlike intersection norms, computing the dual unit ball of a Thurston norm is difficult. There is an algorithm [1]-[3] which asserts whether a given surface S is Thurston norm minimizing or not. This algorithm uses the theory of sutured manifold hierarchies introduced by D. Gabai in [4]. He used hierarchies to construct taut foliations on the exterior of many knots and as a consequence to determine their genus; and M. Scharlemann [8] showed that hierarchies determine the Thurston norm of a homology class in general. The difficulty in this algorithm is to find a sutured manifold hierarchy for checking that an embedded surface is minimizing or not : the Thurston norm minimizing problem is NP-complete [3].

Since the matter is less complicated for intersection norms, our Main Theorem provides many polytopes which are dual unit balls of Thurston norms. Nonetheless, we have showed in [7] that there are symmetric polytopes in \mathbb{Z}^4 which are not dual unit ball of intersection norms. This result makes the characterization of polytopes in even dimension which appear for those two norms widely open; and one wonders if those polytopes which are not dual unit ball of intersection norms [7] are also not dual unit ball of Thurston norms.

The proof of our Main Theorem goes through a detailed analysis of incompressible surfaces in the exterior of a knot on a circle bundle over a surface. It is an alternative to the difficulty of constructing foliations without Reeb component to check if a given surface is minimizing.

Definition 2. Let M be a 3-manifold. An embedded surface S in M is incompressible if any simple curve on S which bounds a disk in M also bounds a disk in S . It is equivalent to say that the inclusion $i_* : \pi_1(S) \longrightarrow \pi_1(M)$ is injective.

If S is not incompressible in M , then one can cut S along a compressing disk. This cutting operation reduces the complexity of the surface. Therefore, a minimizing surface for the Thurston norm is incompressible.

Theorem 2 (F. Waldhausen [11]). *Let $\pi : M \rightarrow \Sigma_g$ be circle bundle. Then, up to isotopy, an incompressible surface S in M is either vertical, that is, $\pi^{-1}(\pi(S)) = S$, or horizontal, that is, $\pi|_S : S \rightarrow \Sigma_g$ is a finite covering.*

For $\pi : M \rightarrow \Sigma_g$ a circle bundle with Euler class 1 and K an oriented knot in M such that $\pi(K)$ is a non-trivial homology class, denote by M_K the complement of a tubular neighborhood of K .

Definition 3. Let S be a surface embedded in M_K . The closure of S in M , denoted by \bar{S} , is the surface obtained by gluing finitely many disk on boundary component of S

The closure \bar{S} is considered as embedded in M and it is easy to see that $S = \bar{S} \cap M_K$.

For proving our Main Theorem, we show the following :

Proposition 1. Let S be an incompressible surface in M_K and \bar{S} its closure. There is sequence of incompressible surfaces $S_0 \rightarrow S_1 \rightarrow \dots \rightarrow S_n = \bar{S}$ such that :

- S_0 is a disjoint union of vertical surfaces ;
- S_{i+1} is obtained by attaching a handle on S_i .

Outline of this article : In Section 2 we extend Waldhausen classification on a knot complement in a circle bundle with non zero Euler class. Section 3 recalls Thurston's construction of polygons as dual unit ball of a Thurston norm. Section 4 is devoted to the proof of the Main Theorem.

2 Thurston's construction of 3-manifolds realizing polygons

In this section, we review Thurston's proof [9] of the fact symmetric polygons on \mathbb{Z}^2 with congruent vertices are dual unit balls of Thurston norms. It will be helpful for understanding our generalization.

Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a filling collection of closed geodesic on a flat torus \mathbb{T}^2 . Since any component of Γ is simple and non-separating, then there is an orientation of each component of Γ such that the oriented collection $\vec{\Gamma}$

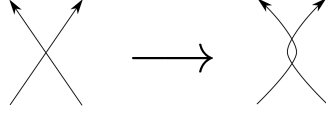


FIGURE 1 – Attaching two curves on a double point.

is non-trivial in homology (every collection of geodesics on the torus is homologically non trivial).

By applying the operation depicted on Figure 1 at finitely many double points, we obtain a filling closed curve $\vec{\gamma}$ in \mathbb{T} (which is not geodesic anymore).

Now, let $\pi : N \rightarrow \mathbb{T}$ be the circle bundle over \mathbb{T} with Euler class 1. Then, $H_2(N)$ is isomorphic to $H_1(\Sigma_g)$.

Let K be a lift of $\vec{\gamma}$ and N_K be the complement in N of a tubular neighborhood $T(K)$ of K . The morphism

$$\begin{aligned} r : H_2(N) &\longrightarrow H_2(N_K, \partial N_K) \\ [S] &\longmapsto [S \cap N_K] \end{aligned}$$

is an isomorphism. In fact, we have the following exact sequence :

$$\dots \rightarrow H_2(T(K)) = 0 \rightarrow H_2(N) \rightarrow H_2(N, T(K)) \rightarrow H_1(T(K)) \rightarrow H_1(N) \rightarrow \dots$$

Since $[\vec{\gamma}] = \pi_*(K)$ is non zero, the inclusion $H_1(T(K)) \rightarrow H_1(N)$ is injective. It follows that the map $H_2(N) \rightarrow H_2(N, T(K))$ is an isomorphism. By excision, we obtain the isomorphism $r : H_2(N) \rightarrow H_2(N_K, \partial N_K)$.

Canonical representatives of $H_2(N_K, \partial N_K)$ are of the form $\pi^{-1}(\alpha) \cap N_K$, where α is an oriented simple curve in \mathbb{T} . Since $\pi^{-1}(\alpha)$ is a torus, then

$$-\chi(\pi^{-1}(\alpha) \cap N_K) = |(\pi^{-1}(\alpha) \cap K)| = |\alpha \cap \Gamma|.$$

Thurston showed that if α minimally intersects Γ , so that $\pi^{-1}(\alpha) \cap N_K$ is minimizing :

$$x([\pi^{-1}(\alpha) \cap N_K]) = \sum_i i(\alpha, \gamma_i). \quad (1)$$

The technical part of Thurston's proof is the construction of a foliation on N_K without Reeb component having $\pi^{-1}(\alpha) \cap N_K$ as a leaf.

Equation 1 describes exactly an equality between Thurston norm on N_K and the intersection norm on the torus associated to Γ . It can be rewritten as follows :

$$x(a) = N_\Gamma(\pi_*(a)).$$

Now, we extend Thurston's construction to higher genus surfaces using intersection norms, namely on any circle bundle $\pi : N \rightarrow \Sigma_g$ of Euler class 1. For the general case, there are essentially two differences :

- there exists filling collections which are not homologically non-trivial since the collection can be made of only separating closed simple curves,
- there are examples of filling collections Γ and N_Γ -minimizing oriented curves α for which $\pi^{-1}(\alpha) \cap N_K$ is not minimizing for the Thurston norm (see Figure 2).

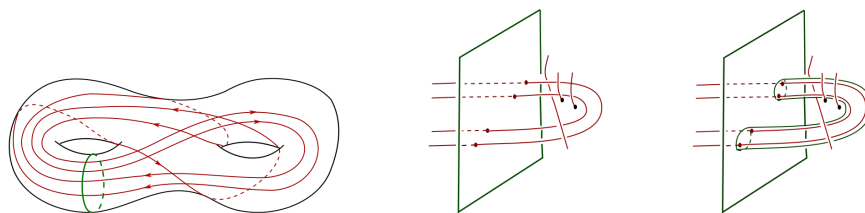


FIGURE 2 – The vertical surface (middle) over the green curve intersects four times the lift K . We reduce its complexity by attaching a handle (right picture).

3 Incompressible surfaces in non-trivial knot complements in circle bundles.

Given a 3-manifold, a natural question is to classify incompressible surfaces up to isotopy. For the case of a circle bundle, a complete answer has been given by Waldhaussen [11]. Let us recall it in its general form.

Let $\pi : M \rightarrow \Sigma_{g,k}$ be a circle bundle over a genus g orbifold with k cone points. We denote by $e(M)$ the Euler class of M .

Theorem 3 (F. Waldhaussen [11]). *Let S be an incompressible surface in M . Then, up to isotopy, S is vertical, namely $\pi^{-1}(\pi(S)) = S$, or horizontal, that is, $\pi|_S : S \rightarrow \Sigma_{g,k}$ is a branched cover.*

One can also see the proof of Theorem 3 in Hatcher's notes [5].

Vertical surfaces correspond to pre-images of (essential) simple closed curves in $\Sigma_{g,k}$, and a horizontal surface S satisfies :

$$\chi(\Sigma_{g,k}) - \frac{\chi(S)}{n} = \sum_i \left(1 - \frac{1}{q_i}\right);$$

where n is the degree of the branched cover and the q_i are the multiplicities of the fibers. When the base is a regular surface Σ_g , then $\chi(S) = n \cdot \chi(\Sigma_g)$. Moreover, S can be made transverse to the fibers and this implies that the algebraic intersection between S and a given fiber F (fibers are canonically oriented) is $\pm n$ according to the orientation of S . It follows that horizontal surfaces are torsion free.

Since $H_2(M)$ is isomorphic to $H_1(\Sigma_g) \times \mathbb{Z}[F]/e(M)F$, any homology class in $H_2(M)$ decomposes in to a vertical part and another part represented by an element in $\mathbb{Z}[F]/e(M)F$. If $e(M) \neq 0$, then M does not admit horizontal surfaces since torsion free part of $H_2(M)$ is represented by vertical surfaces.

Now, we push a little bit further Waldhausen's classification. Let K be an oriented knot in M which is non-trivial in homology, and $M_K := M - \overset{\circ}{T}(K)$ be the exterior of K in M .

Proposition 3.1. Let S be a surface in M_K not isotopic to ∂M_K and \bar{S} its closure in M . If S is incompressible, then there is sequence $S_0 \rightarrow S_1 \rightarrow \dots \rightarrow S_n = \bar{S}$ of closed surfaces in M such that :

1. S_0 is a disjoint union of vertical surfaces,
2. S_{i+1} is obtained from S_i by attaching a handle.

Proof. Since the closure \bar{S} of S is a priori not incompressible in M , one obtains a sequence $\bar{S} \rightarrow S_1 \rightarrow \dots \rightarrow S_n$ where each step consists in cutting S_i along an essential simple curve which bounds a disk in M , and taking the closure of the obtained surface. This process ends to a (possibly non connected) incompressible surface S_n which is a disjoint union of vertical surfaces. The reverse sequence achieves the proof. \square

Definition 3.1. Let S be an incompressible surface in M_K and α an essential simple curve on \bar{S} which bounds a disk \mathbb{D}_α in M . The *weight* of α is the integer $w(\alpha)$ defined by :

$$w(\alpha) = \min\{\text{card}\{\mathbb{D}_{\alpha'} \cap K\}, \alpha' \text{ isotopic to } \alpha\}$$

The *verticality defect* of S is the integer $vd(S)$ defined by :

$$vd(S) = \max_{\alpha} \{w(\alpha), \alpha \text{ essential}\}.$$

It is easy to see that if S has verticality defect equal to zero, then S is a vertical surface :

$$S = \pi^{-1}(\alpha) \cap M_K,$$

where α is a simple closed curve on Σ_g . Moreover, if $vd(\bar{S}) = 1$, then S is homologous to a vertical surface with the same Euler characteristic. In fact, if α is a simple curve in \bar{S} such that $w(\alpha) = 1$, we can cut \bar{S} along α to obtain a surface \bar{S}_1 . The surface $S_1 := \bar{S}_1 \cap M_K$ has two more boundary components than S and one handle less and is homologous to S . It follows that $\chi(S) = \chi(S_1)$. Repeating this process, we obtain a vertical surface S_n with the same Euler characteristic than S .

We end this section with some definitions.

Let A and B be two sub-arcs of K such that $a := \pi(A)$ and $b := \pi(B)$ are simple arcs with extremities $\partial a = \{t, x\}$ and $\partial b = \{y, z\}$. Let λ_1 and λ_2 be two arcs from t to x and y to z , respectively, such that λ_1, a, λ_2 and b bound a disk.

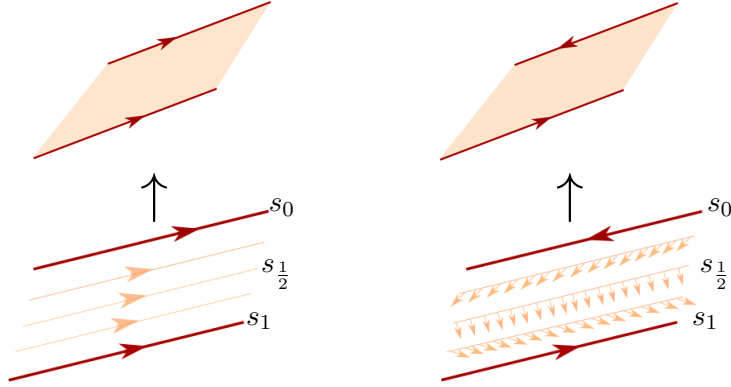


FIGURE 3 – Rectangle between two arcs obtained by lifting a homotopy of section. On the left, the case where the orientations of the arcs agree and on the right the case where the orientations are opposite.

The arcs a and b can be seen as sections of the unit tangent bundle of their supports, and there is a homotopy (see Figure 3) of sections s_t such that :

- $\text{pr}_1(s_t)$ is an isotopy between the support of a and b , with extremities gliding in λ_1 and λ_2 ;
- $s_0 = a$ and $s_1 = b$.

The lift of s_t in M gives a rectangle R from A to B and when we blow-up R , we obtain a handle enclosing A and B .

Lemma 3.1. *If H is a handle in M enclosing two sub-arcs A and B whose projections are simple arcs, then H is isotopic to the blowing up of a rectangle*

between A and B .

Proof. Since H is a compressible handle enclosing A and B , then there is an isotopy between A and B inside H . This isotopy gives a rectangle R between A and B and the blowing up of that rectangle is inside H . Therefore, H is isotopic to the blowing up of R . \square

The construction described above works for more than two sub-arcs and in what follows, we will consider handles as blowing-ups of rectangles between sub-arcs.

4 Proof of the main theorem

Let us start this section with the following statement : if two filling collections Γ and Γ' differ by an "attachment" (see Figure 1), then the intersection norm associated to Γ is equal to the one associated to Γ' (see [7]). Therefore any intersection norm is realized by one filling curve γ , not necessarily in minimal position.

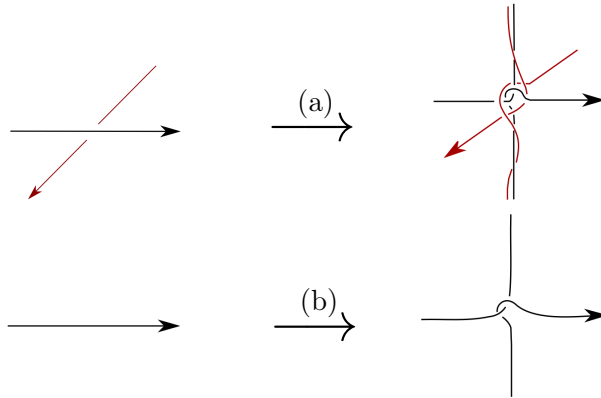


FIGURE 4 – (a) Modification on K around a fiber of a double point of γ . Each of the two arcs is modified as depicted in (b) and they wind around each other (twice) along the fiber.

Let $\vec{\gamma}$ be an oriented filling curve with non vanishing class in homology. Let $\pi : N \rightarrow \Sigma_g$ be a circle bundle with Euler class one. As we have seen, $H_2(N)$ is isomorphic to $H_1(\Sigma_g)$. Instead of only taking a lift of $\vec{\gamma}$ in N , we add the modification depicted on Figure 4 on the neighborhood of the fiber of double point of γ . Let \hat{K} be the obtained knot and $N_{\hat{K}}$ be

the exterior of \hat{K} . Since $\pi(\hat{K})$ is still homologous to $\vec{\gamma}$, then $H_2(M_{\hat{K}})$ is isomorphic to $H_1(\Sigma_g)$ with vertical surfaces as canonical represents.

As we have seen, Thurston's construction does not trivially extends to higher genus surfaces since a minimizing surface S could have verticality defect greater than two. Our modification, which consists in braiding the knot K along fibers (see Figure 4), increases the complexity of incompressible surfaces with vertical defect greater than two.

Definition 4.1. Let H_α be a handle with $\partial H_\alpha = \{\alpha_1, \alpha_2\}$. Let λ be any simple arc from α_1 to α_2 . The handle H_α is *horizontal* if the homotopy class—with fixed extremities—of α in M has no fibers.

Lemma 4.1. Let S_1 and S_2 be two vertical surfaces in $M_{\hat{K}}$ on which we attach a handle H_α and obtained a surface $S := S_1 \#_{H_\alpha} S_2$.

If $w(\alpha) \geq 2$, then there is a surface S' homologous to S such that

$$-\chi(S') < -\chi(S).$$

Proof. If $\pi(H_\alpha)$ does not contain a double point of $\vec{\gamma}$, then S is compressible; the curve β (Figure 5-a) which is obtained by summing to fibers in S_1 and S_2 along H_α is essential in S and vanishes in $M_{\hat{K}}$. So, we can reduce the complexity of S in this case.

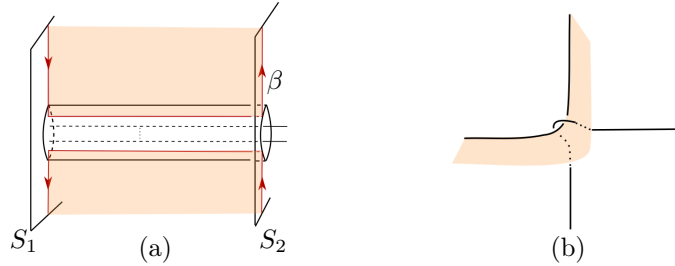


FIGURE 5 – (a) Compression disk in $M_{\hat{K}}$ bounded by an essential curve β in S . (b) The arc around the fiber of a double point which intersects a rectangle. This shows that the rectangle cannot go completely along the fiber.

Now, suppose that $\pi(H_\alpha)$ contains double points of $\vec{\gamma}$. We claim that H_α is horizontal. Let us see H_α as the blowing up of a rectangle between sub-arcs of \hat{K} . Since a rectangle stays on one side of an arc, it follows that it cannot follow a sub-arc of \hat{K} along a fiber (see Figure 5-b).

Finally, if H_α is horizontal and $\pi(H_\alpha)$ contains a double point p , then the fiber $\pi^{-1}(p)$ intersects H_α twice. Therefore H_α intersects \hat{K} four times the modification above p and those four intersection points define four boundary components on S . By attaching a new handle along the fiber $\pi^{-1}(p)$ which encloses those four boundary components we obtain a surface S' with one more handle and four boundary components less. So $-\chi(S') \leq -\chi(S)$. \square

Corollary 4.1. Let S be a surface embedded in $M_{\hat{K}}$. If S is Thurston norm minimizing, then $vd(S) \leq 1$.

Proof. Since a minimizing surface S is incompressible, by Proposition 3.1, S is obtained by attaching finitely many handle between vertical surfaces embedded in $M_{\hat{K}}$. By Lemma 4.1, each handle has height less than or equal to 1. It follows that $vd(S) \leq 1$. \square

Now, we are able to prove the main theorem.

Proof of the main theorem. Let S be a (Thurston norm) minimizing surface in $M_{\hat{K}}$. By Corollary 4.1, $vd(S) \leq 1$. If $vd(S) = 0$ then $S = \pi^{-1}(\alpha) \cap M_{\hat{K}}$. So $x(S) = N_\gamma(\alpha)$.

If $vd(S) = 1$, then one can replace every handle of S by two boundaries by cutting along essential simple curves in S which are trivial in M . This operation does not increase the positive part of the Euler characteristic and we obtain at the end an incompressible surface S' in the same homology class as S and such that $vd(S') = 0$. Again in this case, there is a vertical surface which minimizes the Thurston norm. So $x(a) = N_\Gamma(\pi_*(a))$. \square

Our main theorem links the realization problems of intersection norms and Thurston norms. In [7], we showed that any polytope P in \mathcal{P}_8 (the set of non degenerate symmetric sub-polytopes of $[-1, 1]^4$ with eight vertices) is not the dual unit ball of an intersection norm.

Question 1. Let $P \in \mathcal{P}_8$. Is P the dual unit ball of an Thurston norm on a 3-manifold?

By Gabai theorem on the fact that minimizing surfaces are leaves of foliations without Reeb component, this question is somehow related to the studying of the topology of (Reebless) foliated 3-manifold with pairs of pants or one-holed torus as a leaves.

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