# On dual unit balls of Thurston norms

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**Abstract**

Thurston norms are invariants of 3-manifolds defined on their second homology and understanding the shape of their dual unit balls is a widely open problem. In this article, we provide a large family of polytopes in $\mathbb{R}^{2g}$ that appear like dual unit balls of Thurston norms, generalizing Thurston's construction for polygons in $\mathbb{R}^2$.

*mes enseignants premier,*

## 1 Introduction

Let $M$ be a compact orientable 3-manifold with possibly non-empty boundary. In [9], W. Thurston defined a semi-norm on the second homology of $M$. Let $a \in H_2(M, \partial M; \mathbb{Z})$ be an integer class, then $a$ admits representatives that are disjoint unions of properly embedded oriented surfaces $S_i$ in $M$. The Thurston norm of $a$ is given by:

$$x(a) := \min_{[\cup_i S_i]=a} \left\{ \sum_i \max\{0, -\chi(S_i)\} \right\};$$

where $\chi(S_i)$ is the Euler characteristic of $S_i$. When a representative $S$ of $a$ is such that $x(a) = -\chi(S)$, we say that $S$ is ***minimizing***.

If $M$ is prime, i.e., every embedded sphere bounds a ball and atoroidal, i.e., every embedded torus bounds a solid torus, then $x$ extends to a norm on $H_2(M, \partial M; \mathbb{R})$. By construction $x$ takes integer values on $H_2(M, \partial M; \mathbb{Z})$. It is an ***integer norm***: a norm on a vector space that takes integer values on a top dimensional lattice. W. Thurston showed that the dual unit ball of an integer norm on a vector space $E$ relative to a lattice $\Lambda$ is an ***integer polytope*** namely the convex hull of finitely many 1-forms $u_i \in E^*$ that take integer values on $\Lambda$. Moreover, $x$ is completely determined by the vectors $u_i \in E^*$:

$$x(a) = \max_{u_i}\{\langle u_i, a\rangle\}.$$

For a 3-manifold with toral boundary components, Thurston showed ([9]-Page 106) that the vectors defining the dual unit ball $B^1_{x^*}$ of $x$ satisfy the ***parity condition***. More precisely, $B^1_{x^*}$ is the convex hull of finitely many vectors $u_i \in H^2(M, \partial M; \mathbb{R})$ and $u_i = u_j \mod 2$ for all $i; j = 1, .., n$.

Just after defining his norm, Thurston started to compute it on a few examples of 3-manifolds. Even better, he showed the following:

**Theorem 1.** *[W. Thurston, [9]-Theorem 6] Every symmetric integer polygon in $\mathbb{Z}^2$ with vertices satisfying the parity condition is the dual unit ball of the Thurston norm on a 3-manifold.*

Theorem 1 is not stated in the same way like in Thurston article but when we analyse closely the equality established in [9]-Theorem 6, it implies implicitly an equality between the Thurston norm on the 3-manifold constructed by Thurston and a norm —associated to a collection of closed geodesics on the torus— on the first homology of the torus. Now, these norms are known as ***intersection norms***.

Our main result is a generalization of Thurston's theorem to symmetric polytopes in dimension $2g$. We achieve that extension by establishing a bridge between Thurston norms and intersection norms on surfaces (see Section 2 for the definition of intersection norms). Intersection norms are also integer norms on the first homology of a surface and there is a class of polytopes realized by intersection norms called ***homologically non-trivial polytopes*** (see Definition 1). We show:

**Main Theorem.** *Every homologically non-trivial polytope is the dual unit ball of a Thurston norm on a 3-manifold.*

Unlike intersection norms, computing the dual unit ball of a Thurston norm is difficult. There is an algorithm ([1], [3]) that determines whether a given surface $S$ is minimizing or not. This algorithm uses the theory of sutured manifold hierarchies introduced by D. Gabai in [4]. D. Gabai used hierarchies to construct taut foliations in the complement of many knots and as a consequence to determine their genus. M. Scharlemann [8] then showed that hierarchies determine the Thurston norm of a homology class in general. The difficulty in that algorithm is to find a sutured manifold hierarchy for checking that an embedded surface is minimizing or not. The Thurston norm minimizing problem is NP [3].

Since the matter is less complicated for intersection norms, our main theorem provides many polytopes which are dual unit balls of Thurston norms. Nonetheless, we showed in [7] that there are symmetric polytopes

in $\mathbb{Z}^4$ satisfying the parity condition that are not dual unit balls of intersection norms. This result makes the characterization of polytopes (in even dimensions) that appear for those two norms widely open. We wonder those polytopes that are not dual unit balls of intersection norms are also not dual unit balls of Thurston norms.
The proof of our main theorem goes through a detailed analysis of incompressible surfaces in the complement $M_K$ of an oriented knot $K$ (with $[K] \neq 0$) in a circle bundle $M$ with Euler number equal to 1 over a surface. We porve:

**Theorem 2.** *An incompressible surface in $M_K$ is isotopic to an almost vertical surface.*

From Theorem 2 (see Theorem 4 in Section 3 for its elaborate version), we obtain a total description of minimizing surfaces. This approach avoids foliation techniques for checking whether a given surface is minimizing, and also the use of sutured manifold algorithm.

**Outline of this article:** Section 2 starts with the definition of intersection norms and it ends with Thurston's construction of polygons as dual unit balls of Thurston norm on 3-manifolds. Section 3 is about incompressible surfaces and we prove Theorem 2. The proof of Main Theorem is given in Section 4.

## 2 Intersection norms and Thurston's construction of 3-manifolds realizing polygons

In this section, we first recall some basic facts about intersection norms on closed oriented surfaces; see [2] for more details. We finish by explaining the idea in Thurston's proof of Theorem 1 from which one can see our generalization.

**Intersection norms:** They are integer norms defined on the first homology of a closed oriented surface $\Sigma_g$. Introduced by V. Turaev in [10] intersection norms received a new interpretation in the article of M. Cossarini and P. Dehornoy [2]: they used intersection norms to classified Birkhoff sections of the geodesic flow on the unit tangent bundle of a closed oriented surface.

Let $\Gamma = \{\gamma_1, ..., \gamma_n\}$ be a finite collection of closed curves on $\Sigma_g$ with only transverse intersection points. Assume that $\Gamma$ is a ***filling collection***, i.e., its complement in $\Sigma_g$ is a union of topological disks. The function

$$\begin{aligned} N_\Gamma : H_1(\Sigma_g, \mathbb{Z}) &\longrightarrow \mathbb{N} \\ a &\longmapsto \inf\{\mathrm{card}\{\alpha \cap \Gamma\}; [\alpha] = a\}, \end{aligned}$$

where $\alpha$ is an oriented collection of closed curves representing $a$ with each of its components transverse to $\Gamma$, satisfies the following properties:

— **seperation:** $N_\Gamma = 0$ if and only if $a = 0$ —since $\Gamma$ is filling ;

— **linearity on rays:** $N_\Gamma(n.a) = n.N_\Gamma(a)$ for $a \in H_1(\Sigma_g, \mathbb{Z})$ and $n \in \mathbb{N}$ ;

— **convexity**: $N_\Gamma(a + b) \leq N_\Gamma(a) + N_\Gamma(b)$ for $a; b \in H_1(\Sigma_g, \mathbb{Z})$.

In the definition of $N_\Gamma$, $\Gamma$ is fixed in its homotopy class and $N_\Gamma$ computes the minimal intersection number with $\Gamma$ among all the representatives of a homology class.

For $n \in \mathbb{N}^*$ and $a \in H_1(\Sigma_g, \mathbb{Z})$, we set $N_\Gamma(\frac{1}{n}.a) := \frac{1}{n}N_\Gamma(a)$ and by linearity on rays, $N_\Gamma$ extends to a well-defined function on $H_1(\Sigma_g, \mathbb{Q})$. In fact, $N_\Gamma(\frac{n}{n}.a) = \frac{1}{n}N_\Gamma(n.a) = \frac{1}{n}(n.N_\Gamma(a)) = N_\Gamma(a)$. By density, $N_\Gamma$ extends to a norm on $H_1(\Sigma_g, \mathbb{R})$ called the ***intersection norm***.

By definition, $N_\Gamma$ is also an integer norm. Therefore, its dual unit ball is the convex hull of finitely many vectors $v_i \in H^1(\Sigma_g, \mathbb{Z})$. Like dual unit balls of Thurston norms on 3-manifolds with toral boundary components, the vectors $v_i$ also satisfy the parity property.

We recall that the norm is completely determined by the vectors $v_i$:

$$N_\Gamma(a) = \max_{v_i}\{\langle v_i, a\rangle\}.$$

M. Cossarini and P. Dehornoy provided a fast algorithm that computes all the vectors of the dual unit ball of an intersection norm. It is also known that symmetric integer polygons satisfying the parity condition are dual unit balls of intersection norms, but there are examples of such polytopes in dimension 4 that cannot be realized by intersection norms (see [7]). Here is the class of polytopes that interest us.

**Definition 1.** A filling collection $\Gamma$ on $\Sigma_g$ is ***homologically non-trivial*** if there exists an orientation $\overrightarrow{\Gamma}$ of $\Gamma$ such that $[\overrightarrow{\Gamma}]$ is a non-trivial homology class. A ***homologically non-trivial polytope*** in $\mathbb{Z}^{2g}$ is a symmetric polytope, satisfying the parity condition, that appears like the dual unit ball of an intersection norm on $\Sigma_g$ associated to a homologically non-trivial collection.

A filling collection $\Gamma$ is homologically trivial if and only if its components are all separating curves. So, many filling collections on $\Sigma_g$ are homologically non-trivial and therefore, most of dual unit balls of intersection norms are homologically non-trivial.

**Thurston's construction:** Let $\Gamma = \{\gamma_1, ..., \gamma_n\}$ be a filling collection of closed geodesics on the flat torus $\mathbb{T}$. Since every component of $\Gamma$ is simple and non-separating, there is an orientation of each component of $\Gamma$ making the oriented collection $\overrightarrow{\Gamma}$ non-trivial in homology: every collection of geodesics on the torus is homologically non-trivial.

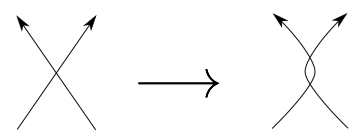

Figure 1 – Attaching two curves at an intersection point.

By applying the operation on Figure 1 at finitely many well-chosen double points, we obtain a filling closed curve $\overrightarrow{\gamma}$ in $\mathbb{T}$ which is no longer a geodesic.

Now, let $\pi : M \longrightarrow \mathbb{T}$ be the circle bundle over $\mathbb{T}$ with Euler number 1. Then, $H_2(M;\mathbb{Z})$ is isomorphic to $H_1(\mathbb{T};\mathbb{Z})$.

Let $K$ be a lift of $\overrightarrow{\gamma}$ in $M$ and $M_K$ the complement in $M$ of a tubular neighborhood $T(K)$ of $K$. The morphism

$$\begin{aligned} r : H_2(M;\mathbb{Z}) &\longrightarrow H_2(M_K, \partial M_K;\mathbb{Z}) \\ [S] &\longmapsto [S \cap M_K] \end{aligned}$$

is an isomorphism. In fact, we have the following exact sequence:

$$0 \to H_2(M;\mathbb{Z}) \to H_2(M, T(K);\mathbb{Z}) \to H_1(T(K);\mathbb{Z}) \to H_1(M;\mathbb{Z})$$

Since $[\overrightarrow{\gamma}] = \pi_*(K)$ is nonzero, the inclusion $H_1(T(K);\mathbb{Z}) \to H_1(M;\mathbb{Z})$ is injective. It follows that the map $H_2(M;\mathbb{Z}) \to H_2(M, T(K);\mathbb{Z})$ is an isomorphism. By excision, we obtain the isomorphism $r$.

Thus, $H_2(M_K, \partial M_K;\mathbb{Z})$ is isomorphic to $H_1(\mathbb{T};\mathbb{Z})$ and canonical representatives of $H_2(M_K, \partial M_K;\mathbb{Z})$ are of the form $\pi^{-1}(\alpha) \cap M_K$, where $\alpha$ is an oriented simple curve in $\mathbb{T}$. Since $\pi^{-1}(\alpha)$ is a torus, the Euler characteristic of $\pi^{-1}(\alpha) \cap M_K$ is given by its number of boundary components:

$$-\chi(\pi^{-1}(\alpha) \cap M_K) = \text{card}\{\pi^{-1}(\alpha) \cap K\} = \text{card}\{\alpha \cap \Gamma\}.$$

Thurston showed that if $\alpha$ minimally intersects $\Gamma$, then $\pi^{-1}(\alpha) \cap M_K$ is minimizing:

$$x([\pi^{-1}(\alpha) \cap M_K]) = \sum_{m=1}^{n} i(\alpha, \gamma_m). \tag{1}$$

The technical part in Thurston's proof is the construction of a foliation on $M_K$ without Reeb component and having $\pi^{-1}(\alpha)\cap M_K$ as a leaf —which by Thurston's characterization of minimizing surface implies that $\pi^{-1}(\alpha)\cap M_K$ realizes the norm in its homology class.

Equation (1) describes exactly an equality between Thurston norm on $M_K$ and the intersection norm on the torus associated to $\Gamma$ and this remark is from us. It can be rewritten as follows:

$$x(a) = N_\Gamma(\pi_*(a)). \tag{2}$$

Polygons satisfying the parity conditions can be realized as dual unit balls of intersection norms on the torus (see [7]-Proposition 9 for the proof of this fact ). Equation 2 implies that they can also be realized as dual unit ball of Thurston norms.

We aim to extend Thurston's construction to higher genus surfaces using intersection norms, namely for every circle bundle $\pi : M \longrightarrow \Sigma_g$ with Euler number equal to 1. For the general case, there are essentially two differences.

— There exists filling collections that are not homologically non-trivial. A consequence of this fact is that the dimension of $H_2(M_K)$ increases by one with one homology class corresponding to $K$. For instance, filling collections made with separating simple closed curves are homologically trivial.

— There are examples of filling collections $\Gamma$ and $N_\Gamma$-minimizing oriented curves $\alpha$ for which $\pi^{-1}(\alpha)\cap N_K$ is not minimizing for the Thurston norm which contrasts with the case of the torus (see Figure 2).

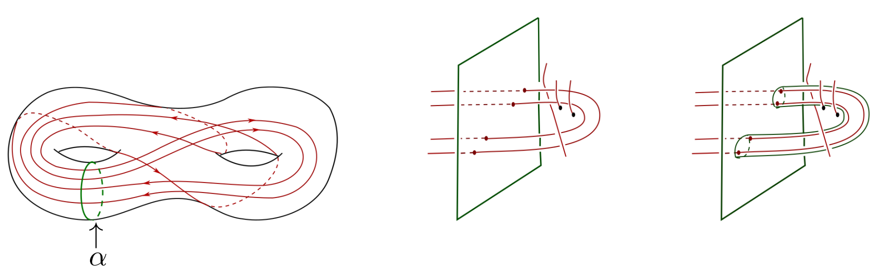


Figure 2 – The vertical surface $S := \pi^{-1}(\alpha) \subset M_K$ (in the middle) is a torus with four boundary components. By replacing these four boundary components by a handle, we obtain a genus 2 closed surface $S'$ (on the right-picture) and $|\chi(S')| < |\chi(S)|$.

## 3 Incompressible surfaces in non-trivial knot complements in circle bundles.

This section is devoted to the study of incompressible surfaces in the complement of a knot in a circle bundle over a closed surface.

**Definition 2.** Let $M$ be a 3-manifold. An embedded surface $S$ in $M$ is ***incompressible*** if every simple curve on $S$ which bounds an embedded disk in $M$ also bounds a disk in $S$. Otherwise we say that $S$ is ***compressible*** and the disk in $M$ bounded by $\alpha$ is called a ***compression disk***.

If $S$ is compressible in $M$, then one can cut $S$ along a compression disk (see Figure 3). This cutting operation reduces the complexity of the surface. Therefore, a minimizing surface with nonzero Euler characteristic is incompressible.

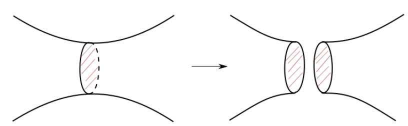

FIGURE 3 – Cutting a surface $S$ along a compression disk. This cutting operation reduces the genus by one.

Classification (up to isotopy) of incompressible surfaces of 3-manifolds is an interesting question in topology. For the case of circle bundles over closed surfaces, a complete answer is given in [11].
A circle bundle $M$ over a closed surface is obtained as follows. Let $\Sigma_{g,1}$ be a closed surface with one boundary component and $M' := \Sigma_{g,1} \times \mathbb{S}^1$ be the trivial circle bundle. The bundle structure on $M'$ induces a foliation by vertical closed curves $F$ on its boundary component which is a torus, and let $\alpha$ be the trace of a section $\Sigma_{g,1} \times \{*\}$ of $M'$ on its boundary. Let $\mathbb{D}^2 \times \mathbb{S}^1$ be a solid torus, $l := \{*\} \times \mathbb{S}^1 \subset \partial(\mathbb{D}^2 \times \mathbb{S}^1)$ be its longitude and $m := \partial\mathbb{D}^2 \times \{*\}$ be its meridian. We obtain a closed 3-manifold $M$ by Dehn filling $M'$ with $\mathbb{D}^2 \times \mathbb{S}^1$ and the bundle structure of $M'$ extend to $M$ if and only if the vertical the meridian $m$ is mapped to a curve $\beta \in \partial M'$ which intersects $F$ exactly one time. The geometric intersection between $\alpha$ and $\beta$ is called the ***Euler number*** of the circle bundle $M$. All circle bundles over a closed surface are obtained in this way and the Euler number classified them (see [6]).

When the Euler number of $M$ is equal to 1, then $\beta = F + \alpha$. It implies that $[F + \alpha] = 0$ in $M$. Since $\alpha = \partial(\Sigma_{g,1} \times \{*\})$, we obtained that $[F] = 0$.

So, $\pi_* : H_1(M;\mathbb{Z}) \longrightarrow H_1(\Sigma_g;\mathbb{Z})$ is an isomorphim.

**Theorem 3.** *[F. Waldhausen [11]] Let $\pi : M \longrightarrow \Sigma_g$ be a circle bundle. Then, an incompressible surface in $M$ is either isotopic to a* ***vertical surface*** *$S$, that is $\pi^{-1}(\pi(S)) = S$, or a* ***horizontal surface****, that is $\pi|_S : S \longrightarrow \Sigma_g$ is a finite covering.*

One can check the proof of Waldhausen's theorem in Hatcher's notes [5]-Proposition 1-11.

The existence of horizontal surfaces in $M$ depends on its Euler number. More precisely, a circle bundle admits a horizontal surface if and only if its Euler number is zero ([5], Proposition 2.2).

We push Waldhausen's classification a bit further. Let $\pi : M \longrightarrow \Sigma_g$ be a circle bundle with Euler number 1 and $K$ an oriented knot in $M$ such that $\pi(K)$ is a non-trivial homology class. We denote by $M_K$ the complement in $M$ of a tubular neighborhood of $K$. Let $S$ be a surface embedded in $M_K$.

**Definition 3.** The ***closure*** of $S$ in $M$ denoted $\bar{S}$, is the surface embedded in $M$ obtained by forgetting $K$ and gluing disks along all the boundary components of $S$.

The closure $\bar{S}$ is embedded in $M$ and $S = \bar{S} \cap M_K$. So, to classify incompressible surfaces in $M_K$, all we need is to understand their closure in $M$.

For the proof of Main Theorem, we show the following elaborate version of Theorem 2 in the introduction:

**Theorem 4.** *Let $S$ be an incompressible surface in $M_K$ and $\bar{S}$ its closure in $M$. There is a sequence $S_0 \longrightarrow S_1 \longrightarrow ... \longrightarrow S_n = \bar{S}$ of embedded surfaces in $M$ such that:*

- *$S_0$ is a disjoint union of vertical surfaces ;*
- *$S_{i+1}$ is obtained by attaching a handle to $S_i$ ;*
- *$S_i \cap M_K$ is incompressible in $M_K$.*

*Proof.* Let $\bar{S}$ be the closure of $S$ in $M$. Since $[K] \neq 0$, then $\partial S \neq K$. If $\bar{S}$ is incompressible in $M$, then $\bar{S}$ is vertical and $S_0 = S_n = \bar{S}$.

If $\bar{S}$ is not incompressible, we obtain a sequence $\bar{S} \longrightarrow S_1 \longrightarrow ... \longrightarrow S_n$, where each step consists in cutting $S_i$ along an essential simple curve which bounds a disk in $M$, and gluing disks on boundary components of the surface obtained. This process ends with a possibly non-connected incompressible surface $S_n$ in $M$ which is a disjoint union of vertical surfaces. The reverse sequence achieves the proof. □

Theorem 4 shows that the only obstruction for an incompressible surface to be vertical comes from attaching handles like in Figure 2.

**Definition 3.1.** Let $S$ be an incompressible surface in $M_K$ and $\alpha$ an essential simple curve on $\bar{S}$ which bounds a disk $\mathbb{D}_\alpha$ in $M$. The ***weight*** of $\alpha$ is the integer $w(\alpha)$ defined by:

$$w(\alpha) = \min\{\text{card}\{\mathbb{D}_{\alpha'} \cap \text{K}\}, \alpha' \text{ isotopic to } \alpha\}$$

The ***verticality defect*** of $S$ is the integer $vd(S)$ defined by:

$$vd(S) = \max_{\alpha}\{w(\alpha)\}.$$

One can see that $vd(S)$ is equal to zero if and only if $S$ is a vertical surface up to isotopy namely

$$S = \pi^{-1}(\alpha) \cap M_K,$$

where $\alpha$ is a simple closed curve on $\Sigma_g$. Moreover, if $vd(S) = 1$, then $S$ is homologous to a vertical surface with the same Euler characteristic. In fact, if $\alpha$ is a simple curve on $\bar{S}$ such that $w(\alpha) = 1$, we can cut $\bar{S}$ along $\alpha$ to obtain a surface $\bar{S}_1$. The surface $S_1 := \bar{S}_1 \cap M_K$ has two more boundary components than $S$ and one handle less and is homologous to $S$. It follows that $\chi(S) = \chi(S_1)$. Repeating this process, we obtain a vertical surface $S_n$ in the same homology class and with the same Euler characteristic like $S$.

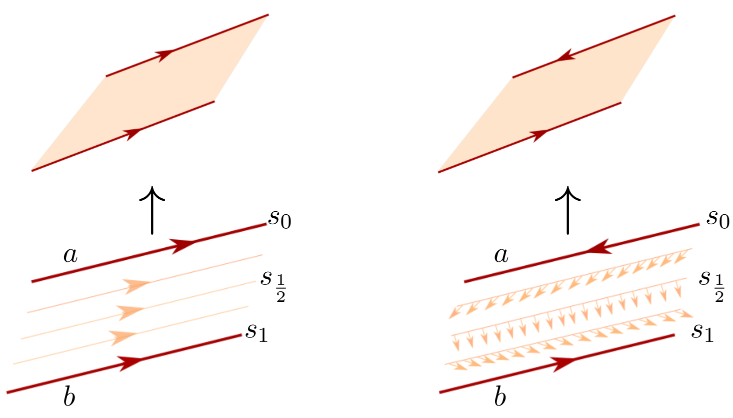


Figure 4 – Rectangle between two arcs obtained by lifting a homotopy between two sections. On the left, we have the case where the orientations of the arcs agree and on the right we have the case where the orientations are opposite.

We end this section with some definitions. Let $A$ and $B$ be two sub-arcs of $K$ such that $a := \pi(A)$ and $b := \pi(B)$ are disjoint simple arcs with extremities $\partial a = \{t, x\}$ and $\partial b = \{y, z\}$. Let $\lambda_1$ and $\lambda_2$ be two arcs from $t$ to $y$ and $x$ to $z$, respectively, such that $\lambda_1$, $a$, $\lambda_2$ and $b$ bound a topological disk.

The oriented arcs $a$ and $b$ can be seen as sections of the unit tangent bundle of their supports, and there is a homotopy (see Figure 4) of sections $s_t$ such that:

— $s_t$ is an isotopy between the support of $a$ and $b$, with extremities gliding in $\lambda_1$ and $\lambda_2$ ;

— $s_0 = a$ and $s_1 = b$.

The isotopy $s_t$ lifts to a rectangle $R$ from $A$ to $B$ and when we blow up $R$ —the blow up of a rectangle $R$ consists in replacing $R$ by the boundary of a tubular neighborhood of $R$ (see Figure 5)— , we obtain a handle (homeomorphic to $\mathbb{S}^1 \times [0, 1]$) enclosing $A$ and $B$.

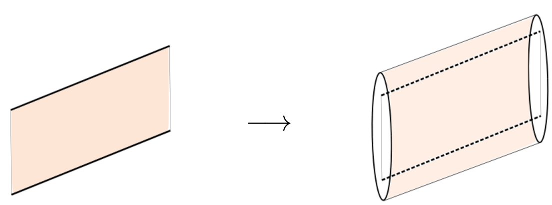

Figure 5 – Blowing up of a rectangle.

**Lemma 3.1.** *If $H$ is a handle in $M$ enclosing two sub-arcs $A$ and $B$ whose projections are disjoint simple arcs, then $H$ is isotopic to a blow up of a rectangle between $A$ and $B$.*

*Proof.* Since $H$ is a compressible handle enclosing $A$ and $B$, then there is an isotopy between $A$ and $B$ inside $H$. This isotopy gives a rectangle $R$ between $A$ and $B$ and the blow up of that rectangle is inside $H$. Therefore, $H$ is isotopic to the blow up of $R$. □

The construction described above works for more than two sub-arcs and in what follows, we will consider handles as blow up of rectangles between sub-arcs.

## 4 Proof of the main theorem

Let us start this section with the following statement: if two filling collections $\Gamma$ and $\Gamma'$ differ by an "attachment" (see Figure 1), then the intersection norm associated to $\Gamma$ is equal to the one associated to $\Gamma'$ (see [7] ; Lemma 11). Therefore any intersection norm is realized by one filling curve $\gamma$, not necessarily in minimal position.

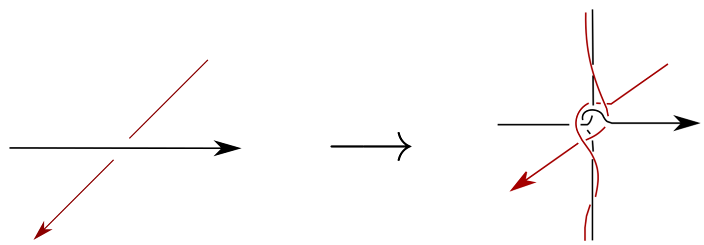

Figure 6 – Modification of $K$ around a fiber of a double point of $\gamma$. The red arc is coming out of the page. Each vertical arc (the dark and the red one) individually follows a fiber and is linked to itself. Along the fiber, the modified arcs form a braid with two strands twisted three times.

Let $\overrightarrow{\gamma}$ be an oriented filling curve with non vanishing class in homology. Let $\pi : M \longrightarrow \Sigma_g$ be a circle bundle with Euler number equal to 1. Then, $H_2(M;\mathbb{Z})$ is isomorphic to $H_1(\Sigma_g;\mathbb{Z})$. In fact, by Poincarality, $H_2(M;\mathbb{Z})$ is isomorphic to $H_1(M;\mathbb{Z})$ which is isomorphic to $H_1(\Sigma_g,\mathbb{Z})$ since the Euler number is equal to 1. Instead of only taking a lift of $\overrightarrow{\gamma}$ in $M$, we add the modification depicted in Figure 6 on the neighborhood of each fiber of a double point of $\gamma$. Let $\hat{K}$ be the knot obtained and $M_{\hat{K}}$ be the complement of $\hat{K}$. Since $\pi(\hat{K})$ is still homologous to $\overrightarrow{\gamma}$, one can use exact sequence and excision theorem in homology, like in Thurston's proof (Section 2), to check that $H_2(M_{\hat{K}};\mathbb{Z})$ is isomorphic to $H_1(\Sigma_g;\mathbb{Z})$ with vertical surfaces as canonical representatives.

Thurston's construction does not extend in a trivial way to higher genus surfaces since a minimizing surface $S$ could have verticality defect greater than two (see Figure 2). Our modification, which consists in braiding the knot $K$ along fibers (see Figure 6), as we will see increases the complexity of incompressible surfaces with verticality defect greater than two. The modification involves a choice (which is not unique) and the goal of the modification along fibers of double points is to avoid handles attaching that reduce the complexity of vertical surfaces like in Figure 2.

**Definition 4.1.** Let $H_\alpha$ be a handle with $\partial H_\alpha = \{\alpha_1, \alpha_2\}$. Let $\lambda$ be a simple arc from $\alpha_1$ to $\alpha_2$. The handle $H_\alpha$ is ***horizontal*** if the homotopy class —with fixed extremities— of $\lambda$ in $M$ has no fibers.

**Lemma 4.1.** *Let $S_1$ and $S_2$ be two vertical surfaces in $M_{\hat{K}}$ on which we attach a handle $H_\alpha$ to obtained a surface $S := S_1 \underset{H_\alpha}{\#} S_2$.*

*If $w(\alpha) \geq 2$, then there is a surface $S'$ homologous to $S$ such that*

$$-\chi(S') < -\chi(S).$$

*Proof.* Their are two alternatives concerning the configuration of a handle depending on wether $\pi(H_\alpha)$ contains a double point or not.

If $\pi(H_\alpha)$ does not contain a double point of $\overrightarrow{\gamma}$, then $S$ is compressible. In fact, the curve $\beta$ (Figure 7-a) which is obtained by summing two fibers in $S_1$ and $S_2$ along $H_\alpha$ is essential in $S$ (since fibers are essential in $S_1$ and $S_2$) and bounds a disk in $M_{\hat{K}}$ (see Figure 7-a). So, we can reduce the complexity of $S$ in this case by cutting $S$ along the disk bounded by $\beta$.

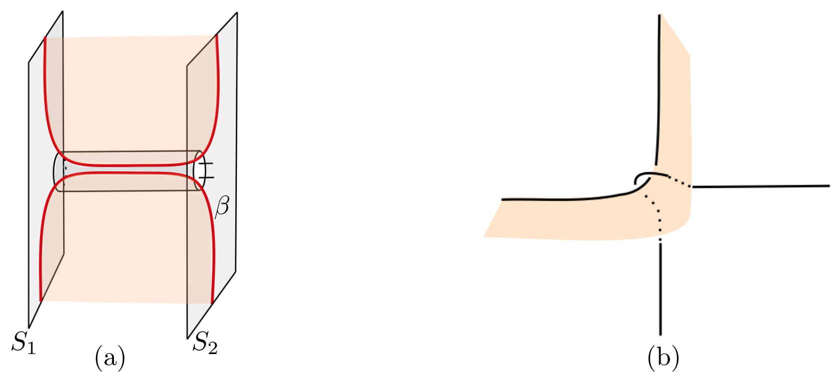


FIGURE 7 – (a) Compression disk in $M_{\hat{K}}$ bounded by an essential curve $\beta$ in $S$. (b) The arc around the fiber of a double point which intersects a rectangle. This shows that the rectangle cannot go completely along the fiber.

Now, suppose that $\pi(H_\alpha)$ contains double points of $\overrightarrow{\gamma}$. We claim that $H_\alpha$ is horizontal. Since $w(\alpha) \geq 2$, $H_\alpha$ is isotopic to a blow up of a rectangle between sub-arcs of $\hat{K}$. A rectangle stays on one side of an arc. Thus, it cannot follow a sub-arc of $\hat{K}$ along a fiber (see Figure 7-b).

Finally, if $H_\alpha$ is horizontal and $\pi(H_\alpha)$ contains a double point $p$, then the fiber $\pi^{-1}(p)$ intersects $H_\alpha$ twice. Therefore, $H_\alpha$ intersects $\hat{K}$ four times the braiding above $p$ and these intersection points define four boundary components on $S$. By attaching a new handle along the fiber $\pi^{-1}(p)$ which encloses those four boundary components (see Figure 8) we obtain a surface $S'$ with one more handle and four boundary components less. So $-\chi(S') \leq -\chi(S)$. □

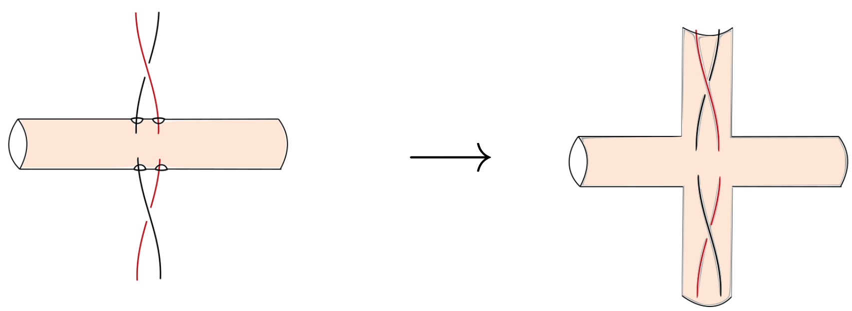

FIGURE 8 – Replacing four boundary components by attaching a handle which enclosed the modification along a fiber.

**Corollary 4.1.** Let $S$ be a surface embedded in $M_{\hat{K}}$. If $S$ is Thurston norm minimizing, then $vd(S) \leq 1$.

*Proof.* Since a minimizing surface $S$ is incompressible, $S$ is obtained by attaching finitely many handle between vertical surfaces embedded in $M_{\hat{K}}$ according to Theorem 4. By Lemma 4.1, the weight of each handle is less or equal to 1. It follows that $vd(S) \leq 1$. □

Now, we are able to prove the main theorem.

*Proof of the main theorem.* Let $S$ be a minimizing surface in $M_{\hat{K}}$. By Corollary 4.1, $vd(S) \leq 1$. If $vd(S) = 0$ then $S = \pi^{-1}(\alpha) \cap M_{\hat{K}}$. So $x(S) = N_\gamma(\alpha)$.

If $vd(S) = 1$, then one can replace each handle of $S$ by two boundary components by cutting along essential simple curves in $S$ which are trivial in $M$. This operation does not increase the Euler characteristic and we obtain at the end an incompressible surface $S'$ in the homology class of $S$ such that $vd(S') = 0$. Again in this case, there is a vertical surface which minimizes the Thurston norm. So $x(a) = N_\Gamma(\pi_*(a))$. □

Homologically non-trivial polytopes realized by our construction do not have fibered faces since a fibration of $M_{\hat{K}}$ by vertical surfaces would give a foliation on $\Sigma_g$ without singularities.

Our main theorem links the realization problems of intersection norms and Thurston norms. In [7], we showed that every polytope $P$ in $\mathcal{P}_8$: the set of non degenerate symmetric sub-polytopes of $[-1,1]^4$ with eight vertices, is not the dual unit ball of an intersection norm.

**Question 1.** *Let $P \in \mathcal{P}_8$. Is $P$ the dual unit ball of a Thurston norm on a 3-manifold ?*

By Gabai's theorem which states that minimizing surfaces are leaves of foliations without Reeb component, this question is somehow related to the studying of the topology of foliated (without Reeb component) 3-manifolds with pairs of pants or one-holed torus as leaves.

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