



Panaitopol-Bandila-Lascu Type Inequalities for Generalized Hyperbolic Functions

Kwara Nantomah, Osman Kasimu

► To cite this version:

Kwara Nantomah, Osman Kasimu. Panaitopol-Bandila-Lascu Type Inequalities for Generalized Hyperbolic Functions. 2020. <hal-02537130>

HAL Id: hal-02537130

<https://hal.science/hal-02537130v1>

Preprint submitted on 8 Apr 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

PANAITOPOL-BANDILA-LASCU TYPE INEQUALITIES FOR GENERALIZED HYPERBOLIC FUNCTIONS

KWARA NANTOMAH AND OSMAN KASIMU

ABSTRACT. In this paper, we establish Panaitopol-Bandila-Lascu type inequalities for some generalized hyperbolic functions. The established results extend and generalize some earlier results due Barbu and Piscoran.

1. INTRODUCTION AND PRELIMINARY DEFINITIONS

In 2008, Panaitopol, Bandila and Lascu [7] proposed the following problems.

Problem 1.1. If $0 < z < \pi/2$, $u > 0$ and $v > 0$, then

$$\left(1 + \sqrt{2uv}\right)^2 \leq \left(1 + \frac{u}{\sin z}\right) \left(1 + \frac{v}{\cos z}\right). \quad (1)$$

Problem 1.2. If $0 < z < \pi/2$ and $n \in \mathbb{N}$, then

$$\left(1 + 2^{\frac{n}{2}}\right)^2 \leq \left(1 + \frac{1}{\sin^n z}\right) \left(1 + \frac{1}{\cos^n z}\right). \quad (2)$$

Then in 2012, Barbu and Piscoran [1] extended these results to the hyperbolic functions. Specifically, they established that the inequality

$$\left(1 + \sqrt{\frac{2uv}{\sinh 2z}}\right)^2 \leq \left(1 + \frac{u}{\sinh z}\right) \left(1 + \frac{v}{\cosh z}\right), \quad (3)$$

holds for $z > 0$, $u > 0$ and $v > 0$. By using the principle of mathematical induction, they generalize (3) by proving that

$$\left(1 + \sqrt{\frac{2uv}{\sinh 2^{r+1}z}}\right)^2 \leq \left(1 + \frac{u}{\sinh 2^r z}\right) \left(1 + \frac{v}{\cosh 2^r z}\right), \quad (4)$$

holds for $z > 0$, $u > 0$, $v > 0$ and $r \in \mathbb{N}$. They further established that

$$\begin{aligned} \left(1 + \sqrt{\frac{uv}{\sinh^m z \cosh^n z}}\right)^2 &\leq \left(1 + \frac{u}{\sinh^m z}\right) \left(1 + \frac{v}{\cosh^n z}\right) \\ &\leq 1 + \sqrt{(u^2 + v^2) \left(\frac{1}{\sinh^{2m} z} + \frac{1}{\cosh^{2n} z}\right)} + \frac{uv}{\sinh^m z \cosh^n z}, \end{aligned} \quad (5)$$

holds for $z > 0$, $u > 0$, $v > 0$ and $m, n \in \mathbb{N}$.

2010 *Mathematics Subject Classification.* 33B10, 33Bxx, 26D05.

Key words and phrases. Generalized hyperbolic functions, Panaitopol-Bandila-Lascu inequality, arithmetic-geometric mean inequality, Cauchy-Schwarz inequality .

Also, in a bid to generalize a previous work [3], the authors of [5] gave the following generalizations of the hyperbolic functions. See also [4] and [6].

Definition 1.3. The generalized hyperbolic cosine, hyperbolic sine and hyperbolic tangent functions are respectively defined as

$$\cosh_a(z) = \frac{a^z + a^{-z}}{2}, \quad (6)$$

$$\sinh_a(z) = \frac{a^z - a^{-z}}{2}, \quad (7)$$

$$\tanh_a(z) = \frac{\sinh_a(z)}{\cosh_a(z)} = \frac{a^z - a^{-z}}{a^z + a^{-z}} = 1 - \frac{2}{1 + a^{2z}}, \quad (8)$$

where $a > 1$ and $z \in \mathbb{R}$.

These generalized functions satisfy the following identities.

$$\cosh_a(z) + \sinh_a(z) = a^z, \quad (9)$$

$$\cosh_a(z) - \sinh_a(z) = a^{-z}, \quad (10)$$

$$(\cosh_a(z))' = (\ln a) \sinh_a(z), \quad (11)$$

$$(\sinh_a(z))' = (\ln a) \cosh_a(z), \quad (12)$$

$$(\tanh_a(z))' = \frac{\ln a}{\cosh_a^2(z)}, \quad (13)$$

$$(\cosh_a(z))'' + (\sinh_a(z))'' = (\ln a)^2 a^z, \quad (14)$$

$$(\cosh_a(z))'' - (\sinh_a(z))'' = (\ln a)^2 a^{-z}, \quad (15)$$

$$\cosh_a^2(z) + \sinh_a^2(z) = \cosh_a(2z), \quad (16)$$

$$\cosh_a^2(z) - \sinh_a^2(z) = 1, \quad (17)$$

$$2 \sinh_a(z) \cosh_a(z) = \sinh_a(2z), \quad (18)$$

$$\cosh_a^2(z) = \frac{\cosh_a(2z) + 1}{2}, \quad (19)$$

$$\sinh_a^2(z) = \frac{\cosh_a(2z) - 1}{2}. \quad (20)$$

The generalized hyperbolic secant, hyperbolic cosecant and hyperbolic cotangent functions are respectively defined as

$$\operatorname{sech}_a(z) = \frac{1}{\cosh_a(z)}, \quad \operatorname{cosech}_a(z) = \frac{1}{\sinh_a(z)}, \quad \operatorname{coth}_a(z) = \frac{1}{\tanh_a(z)}. \quad (21)$$

As pointed out in [5], several other identities can be derived from (6), (7) and (8). When $a = e$, where $e = 2.71828\dots$ is the Euler's number, then the above definitions and identities reduce to their ordinary counterparts. For other generalizations of these function, the reader may refer to [2] and [8].

Motivated by the work of Barbu and Piscoran [1], the objective of this work is to extend inequalities (3), (4) and (5) to the generalized hyperbolic functions (6), (7) and (8). Specifically, we provide lower and upper bounds for the functions $(1 + u/\sinh_a^m(z))(1 + v/\cosh_a^n(z))$, $(1 + u/\cosh_a^m(z))(1 + v/\tanh_a^n(z))$ and $(1 + u/\sinh_a^m(z))(1 + v/\tanh_a^n(z))$ where $z > 0$, $u > 0$, $v > 0$ and $m, n \in \mathbb{N}$. We present our results in the following section.

2. RESULTS AND DISCUSSION

Theorem 2.1. *Let $z > 0$, $u > 0$, $v > 0$ and $m, n \in \mathbb{N}$. Then the inequality*

$$\begin{aligned} \left(1 + \sqrt{\frac{uv}{\sinh_a^m(z) \cosh_a^n(z)}}\right)^2 &\leq \left(1 + \frac{u}{\sinh_a^m(z)}\right) \left(1 + \frac{v}{\cosh_a^n(z)}\right) \\ &\leq \left(1 + \frac{u}{2 \sinh_a^m(z)} + \frac{v}{2 \cosh_a^n(z)}\right)^2, \end{aligned} \quad (22)$$

holds.

Proof. By applying the well-known arithmetic-geometric mean inequality, we obtain

$$\begin{aligned} &\left(1 + \frac{u}{\sinh_a^m(z)}\right) \left(1 + \frac{v}{\cosh_a^n(z)}\right) \\ &= 1 + \frac{u}{\sinh_a^m(z)} + \frac{v}{\cosh_a^n(z)} + \frac{uv}{\sinh_a^m(z) \cosh_a^n(z)} \\ &\geq 1 + 2\sqrt{\frac{uv}{\sinh_a^m(z) \cosh_a^n(z)}} + \frac{uv}{\sinh_a^m(z) \cosh_a^n(z)} \\ &= \left(1 + \sqrt{\frac{uv}{\sinh_a^m(z) \cosh_a^n(z)}}\right)^2, \end{aligned}$$

which gives the left-hand-side of (22). Next, by applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &\left(1 + \frac{u}{\sinh_a^m(z)}\right) \left(1 + \frac{v}{\cosh_a^n(z)}\right) \\ &\leq \frac{1}{2} \left[\left(1 + \frac{u}{\sinh_a^m(z)}\right)^2 + \left(1 + \frac{v}{\cosh_a^n(z)}\right)^2 \right] \\ &= \frac{1}{2} \left[\left(2 + \frac{u}{\sinh_a^m(z)} + \frac{v}{\cosh_a^n(z)}\right)^2 - 2 \left(1 + \frac{u}{\sinh_a^m(z)}\right) \left(1 + \frac{v}{\cosh_a^n(z)}\right) \right] \\ &= 2 \left(1 + \frac{u}{2 \sinh_a^m(z)} + \frac{v}{2 \cosh_a^n(z)}\right)^2 - \left(1 + \frac{u}{\sinh_a^m(z)}\right) \left(1 + \frac{v}{\cosh_a^n(z)}\right), \end{aligned}$$

which gives the right-hand-side of (22). \square

Corollary 2.2. *Let $z > 0$, $u > 0$, $v > 0$ and $r \in \mathbb{N}_0$. Then the inequality*

$$\begin{aligned} \left(1 + \sqrt{\frac{2uv}{\sinh_a(2^{r+1}z)}}\right)^2 &\leq \left(1 + \frac{u}{\sinh_a(2^r z)}\right) \left(1 + \frac{v}{\cosh_a(2^r z)}\right) \\ &\leq \left(1 + \frac{u \cosh_a(2^r z) + v \sinh_a(2^r z)}{\sinh_a(2^{r+1}z)}\right)^2, \end{aligned} \quad (23)$$

holds.

Proof. This follows from Theorem 2.1 by letting $m = n = 1$ and replacing z by $2^r z$. \square

Corollary 2.3. *Let $z > 0$, $u > 0$, $v > 0$ and $n \in \mathbb{N}$. Then the inequality*

$$\begin{aligned} \left(1 + \sqrt{\frac{2^n uv}{\sinh_a^n(2z)}}\right)^2 &\leq \left(1 + \frac{u}{\sinh_a^n(z)}\right) \left(1 + \frac{v}{\cosh_a^n(z)}\right) \\ &\leq \left(1 + \frac{2^{n-1}[u \cosh_a^n(z) + v \sinh_a^n(z)]}{\sinh_a^n(2z)}\right)^2, \end{aligned} \quad (24)$$

holds.

Proof. This follows from Theorem 2.1 by letting $m = n$. \square

Corollary 2.4. *For $z > 0$, then the inequality*

$$\begin{aligned} \left(1 + \sqrt{\frac{2}{\sinh_a(2z)}}\right)^2 &\leq \left(1 + \frac{1}{\sinh_a(z)}\right) \left(1 + \frac{1}{\cosh_a(z)}\right) \\ &\leq \left(1 + \frac{a^z}{\sinh_a(2z)}\right)^2, \end{aligned} \quad (25)$$

holds.

Proof. This follows from Theorem 2.1 by letting $m = n = 1$ and $u = v = 1$. \square

Corollary 2.5. *For $z > 0$, then the inequality*

$$\begin{aligned} \left(1 + \frac{2}{\sinh_a(2z)}\right)^2 &\leq \left(1 + \frac{1}{\sinh_a^2(z)}\right) \left(1 + \frac{1}{\cosh_a^2(z)}\right) \\ &\leq \left(1 + \frac{2}{\sinh_a(2z) \tanh_a(2z)}\right)^2, \end{aligned} \quad (26)$$

holds.

Proof. This follows from Theorem 2.1 by letting $m = n = 2$ and $u = v = 1$. \square

Theorem 2.6. *Let $z > 0$, $u > 0$, $v > 0$ and $m, n \in \mathbb{N}$. Then the inequality*

$$\begin{aligned} \left(1 + \sqrt{\frac{uv}{\cosh_a^m(z) \tanh_a^n(z)}}\right)^2 &\leq \left(1 + \frac{u}{\cosh_a^m(z)}\right) \left(1 + \frac{v}{\tanh_a^n(z)}\right) \\ &\leq \left(1 + \frac{u \tanh_a^n(z) + v \cosh_a^m(z)}{2 \cosh_a^m(z) \tanh_a^n(z)}\right)^2, \end{aligned} \quad (27)$$

holds.

Proof. This follows the same procedure as that for Theorem 2.1. Hence we omit it. \square

Corollary 2.7. *Let $z > 0$, $u > 0$, $v > 0$ and $n \in \mathbb{N}$. Then the inequality*

$$\begin{aligned} \left(1 + \sqrt{\frac{uv}{\sinh_a^n(z)}}\right)^2 &\leq \left(1 + \frac{u}{\cosh_a^n(z)}\right) \left(1 + \frac{v}{\tanh_a^n(z)}\right) \\ &\leq \left(1 + \frac{u \tanh_a^n(z) + v \cosh_a^n(z)}{2 \sinh_a^n(z)}\right)^2, \end{aligned} \quad (28)$$

holds.

Proof. This follows from Theorem 2.6 by letting $m = n$. \square

Remark 2.8. Inequality (28) implies that

$$4uv \sinh_a^n(z) - [u \tanh_a^n(z) + v \cosh_a^n(z)]^2 \leq 0. \quad (29)$$

Theorem 2.9. *Let $z > 0$, $u > 0$, $v > 0$ and $m, n \in \mathbb{N}$. Then the inequality*

$$\begin{aligned} \left(1 + \sqrt{\frac{uv}{\sinh_a^{m+n}(z)}}\right)^2 &< \left(1 + \frac{u}{\sinh_a^m(z)}\right) \left(1 + \frac{v}{\tanh_a^n(z)}\right) \\ &\leq \left(1 + \frac{u \tanh_a^n(z) + v \sinh_a^m(z)}{2 \sinh_a^m(z) \tanh_a^n(z)}\right)^2, \end{aligned} \quad (30)$$

holds.

Proof. By adopting the technique of the proof of Theorem 2.1 and using the fact that $\cosh_a(z) > 1$ for all $z \neq 0$, we obtain

$$\begin{aligned} \left(1 + \frac{u}{\sinh_a^m(z)}\right) \left(1 + \frac{v}{\tanh_a^n(z)}\right) &\geq \left(1 + \sqrt{\frac{uv \cosh_a^n(z)}{\sinh_a^{m+n}(z)}}\right)^2 \\ &> \left(1 + \sqrt{\frac{uv}{\sinh_a^{m+n}(z)}}\right)^2, \end{aligned}$$

which gives the left-hand side of (30). Likewise, the right-hand side of (30) also obtained by applying the Cauchy-Schwarz inequality. \square

Corollary 2.10. *Let $z > 0$, $u > 0$ and $v > 0$. Then the inequality*

$$\begin{aligned} \left(1 + \sqrt{\frac{2uv}{\cosh_a(2z) - 1}}\right)^2 &< \left(1 + \frac{u}{\sinh_a(z)}\right) \left(1 + \frac{v}{\tanh_a(z)}\right) \\ &< \left(1 + \frac{u + v}{2 \tanh_a(z)}\right)^2, \end{aligned} \quad (31)$$

holds.

Proof. By letting $m = n = 1$ in the left-hand side of Theorem 2.9, and applying identity (20), we obtain the left-hand side of (31). Next, by letting $m = n = 1$

in the right-hand side of Theorem 2.9, and using the fact that $\cosh_a(z) > 1$, we obtain

$$\begin{aligned} \left(1 + \frac{u}{\sinh_a(z)}\right) \left(1 + \frac{v}{\tanh_a(z)}\right) &\leq \left(1 + \frac{u \tanh_a(z) + v \sinh_a(z)}{2 \sinh_a(z) \tanh_a(z)}\right)^2 \\ &= \left(1 + \frac{\sinh_a(z) \left[\frac{u}{\cosh_a(z)} + v\right]}{2 \sinh_a(z) \tanh_a(z)}\right)^2 \\ &< \left(1 + \frac{u + v}{2 \tanh_a(z)}\right)^2, \end{aligned}$$

which gives the right-hand side of (31). This completes the proof. \square

3. CONCLUDING REMARKS

In this work, we have provided lower and upper bounds for the functions

$$\begin{aligned} f(z) &= \left(1 + \frac{u}{\sinh_a^m(z)}\right) \left(1 + \frac{v}{\cosh_a^n(z)}\right), \\ g(z) &= \left(1 + \frac{u}{\cosh_a^m(z)}\right) \left(1 + \frac{v}{\tanh_a^n(z)}\right), \\ h(z) &= \left(1 + \frac{u}{\sinh_a^m(z)}\right) \left(1 + \frac{v}{\tanh_a^n(z)}\right), \end{aligned}$$

where $z > 0$, $u > 0$, $v > 0$ and $m, n \in \mathbb{N}$. These extend and generalize some earlier results due Barbu and Piscoran. By applying some mean value inequalities, other bounds could be established for these functions. The largest possible range of $f(z)$, $g(z)$ and $h(z)$ are respectively $(1, \infty)$, $(1+v, \infty)$ and $(1+v, \infty)$. This follows from the fact that $f(z)$, $g(z)$ and $h(z)$ are decreasing with $\lim_{z \rightarrow \infty} f(z) = 1$, $\lim_{z \rightarrow \infty} g(z) = 1 + v$ and $\lim_{z \rightarrow \infty} h(z) = 1 + v$. When $a = e$, the results of this paper reduce to those involving the ordinary hyperbolic functions.

REFERENCES

- [1] C. Barbu and L-I. Piscoran *On Panaitopol and Jordan type inequalities*, <http://ijgeometry.com/wp-content/uploads/2012/04/Untitled1.pdf>.
- [2] P. Lindqvist *Some remarkable sine and cosine functions*, Ric. Mat., 44(2)(1995), 269-290.
- [3] K. Nantomah *On Some Properties of the Sigmoid Function*, Asia Mathematika, 3(1)(2019), 79-90.
- [4] K. Nantomah *Cusa-Huygens, Wilker and Huygens Type Inequalities for Generalized Hyperbolic Functions*. 2020. hal-02527401f. <https://hal.archives-ouvertes.fr/hal-02527401>
- [5] K. Nantomah, C. A. Okpoti and S. Nasiru *On a Generalized Sigmoid Function and its Properties*, Asian Journal of Mathematics and Applications, 2020(2020), Article ID ama0527, 11 pages
- [6] K. Nantomah and E. Prempeh *Some Inequalities for Generalized Hyperbolic Functions*. 2020. hal-02524105f. <https://hal.archives-ouvertes.fr/hal-02524105>
- [7] L. Panaitopol, V. Bandila and M. Lascu *Inegalitati*, Ed. GIL Zalau, p. 131, 2008.
- [8] S. Takeuchi *Generalized Jacobian elliptic functions and their application to bifurcation problems associated with p-Laplacian*, J. Math. Anal. Appl., 385(1)(2012), 24-35.

DEPARTMENT OF MATHEMATICS, FACULTY OF MATHEMATICAL SCIENCES, UNIVERSITY FOR DEVELOPMENT STUDIES, NAVRONGO CAMPUS, P. O. Box 24, NAVRONGO, UE/R, GHANA.

E-mail address: knantomah@uds.edu.gh

FACULTY OF EDUCATION, COLLEGE OF ARTS, LAW AND EDUCATION, UNIVERSITY OF TASMANIA, LAUNCESTON TAS 7248, AUSTRALIA.

E-mail address: osman.kasimu@utas.edu.au