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PANAITOPOL-BANDILA-LASCU TYPE INEQUALITIES FOR GENERALIZED HYPERBOLIC FUNCTIONS

KWARA NANTOMAH AND OSMAN KASIMU

ABSTRACT. In this paper, we establish Panaitopol-Bandila-Lascu type inequalities for some generalized hyperbolic functions. The established results extend and generalize some earlier results due Barbu and Piscoran.

1. INTRODUCTION AND PRELIMINARY DEFINITIONS

In 2008, Panaitopol, Bandila and Lascu [7] proposed the following problems.

Problem 1.1. If $0 < z < \pi/2$, $u > 0$ and $v > 0$, then

$$\left(1 + \sqrt{2uv}\right)^2 \leq \left(1 + \frac{u}{\sin z}\right) \left(1 + \frac{v}{\cos z}\right). \quad (1)$$

Problem 1.2. If $0 < z < \pi/2$ and $n \in \mathbb{N}$, then

$$\left(1 + 2^{\frac{n}{2}}\right)^2 \leq \left(1 + \frac{1}{\sin^n z}\right) \left(1 + \frac{1}{\cos^n z}\right). \quad (2)$$

Then in 2012, Barbu and Piscoran [1] extended these results to the hyperbolic functions. Specifically, they established that the inequality

$$\left(1 + \sqrt{\frac{2uv}{\sinh 2z}}\right)^2 \leq \left(1 + \frac{u}{\sinh z}\right) \left(1 + \frac{v}{\cosh z}\right), \quad (3)$$

holds for $z > 0$, $u > 0$ and $v > 0$. By using the principle of mathematical induction, they generalize (3) by proving that

$$\left(1 + \sqrt{\frac{2uv}{\sinh 2^{r+1}z}}\right)^2 \leq \left(1 + \frac{u}{\sinh 2^r z}\right) \left(1 + \frac{v}{\cosh 2^r z}\right), \quad (4)$$

holds for $z > 0$, $u > 0$, $v > 0$ and $r \in \mathbb{N}$. They further established that

$$\begin{aligned} \left(1 + \sqrt{\frac{uv}{\sinh^m z \cosh^n z}}\right)^2 &\leq \left(1 + \frac{u}{\sinh^m z}\right) \left(1 + \frac{v}{\cosh^n z}\right) \\ &\leq 1 + \sqrt{(u^2 + v^2) \left(\frac{1}{\sinh^{2m} z} + \frac{1}{\cosh^{2n} z}\right)} + \frac{uv}{\sinh^m z \cosh^n z}, \end{aligned} \quad (5)$$

holds for $z > 0$, $u > 0$, $v > 0$ and $m, n \in \mathbb{N}$.

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Also, in a bid to generalize a previous work [3], the authors of [5] gave the following generalizations of the hyperbolic functions. See also [4] and [6].

Definition 1.3. The generalized hyperbolic cosine, hyperbolic sine and hyperbolic tangent functions are respectively defined as

$$\cosh_a(z) = \frac{a^z + a^{-z}}{2}, \quad (6)$$

$$\sinh_a(z) = \frac{a^z - a^{-z}}{2}, \quad (7)$$

$$\tanh_a(z) = \frac{\sinh_a(z)}{\cosh_a(z)} = \frac{a^z - a^{-z}}{a^z + a^{-z}} = 1 - \frac{2}{1 + a^{2z}}, \quad (8)$$

where $a > 1$ and $z \in \mathbb{R}$.

These generalized functions satisfy the following identities.

$$\cosh_a(z) + \sinh_a(z) = a^z, \quad (9)$$

$$\cosh_a(z) - \sinh_a(z) = a^{-z}, \quad (10)$$

$$(\cosh_a(z))' = (\ln a) \sinh_a(z), \quad (11)$$

$$(\sinh_a(z))' = (\ln a) \cosh_a(z), \quad (12)$$

$$(\tanh_a(z))' = \frac{\ln a}{\cosh_a^2(z)}, \quad (13)$$

$$(\cosh_a(z))'' + (\sinh_a(z))'' = (\ln a)^2 a^z, \quad (14)$$

$$(\cosh_a(z))'' - (\sinh_a(z))'' = (\ln a)^2 a^{-z}, \quad (15)$$

$$\cosh_a^2(z) + \sinh_a^2(z) = \cosh_a(2z), \quad (16)$$

$$\cosh_a^2(z) - \sinh_a^2(z) = 1, \quad (17)$$

$$2 \sinh_a(z) \cosh_a(z) = \sinh_a(2z), \quad (18)$$

$$\cosh_a^2(z) = \frac{\cosh_a(2z) + 1}{2}, \quad (19)$$

$$\sinh_a^2(z) = \frac{\cosh_a(2z) - 1}{2}. \quad (20)$$

The generalized hyperbolic secant, hyperbolic cosecant and hyperbolic cotangent functions are respectively defined as

$$\operatorname{sech}_a(z) = \frac{1}{\cosh_a(z)}, \quad \operatorname{cosech}_a(z) = \frac{1}{\sinh_a(z)}, \quad \operatorname{coth}_a(z) = \frac{1}{\tanh_a(z)}. \quad (21)$$

As pointed out in [5], several other identities can be derived from (6), (7) and (8). When $a = e$, where $e = 2.71828\dots$ is the Euler's number, then the above definitions and identities reduce to their ordinary counterparts. For other generalizations of these function, the reader may refer to [2] and [8].

Motivated by the work of Barbu and Piscoran [1], the objective of this work is to extend inequalities (3), (4) and (5) to the generalized hyperbolic functions (6), (7) and (8). Specifically, we provide lower and upper bounds for the functions $(1 + u/\sinh_a^m(z))(1 + v/\cosh_a^n(z))$, $(1 + u/\cosh_a^m(z))(1 + v/\tanh_a^n(z))$ and $(1 + u/\sinh_a^m(z))(1 + v/\tanh_a^n(z))$ where $z > 0$, $u > 0$, $v > 0$ and $m, n \in \mathbb{N}$. We present our results in the following section.

2. RESULTS AND DISCUSSION

Theorem 2.1. *Let $z > 0$, $u > 0$, $v > 0$ and $m, n \in \mathbb{N}$. Then the inequality*

$$\begin{aligned} \left(1 + \sqrt{\frac{uv}{\sinh_a^m(z) \cosh_a^n(z)}}\right)^2 &\leq \left(1 + \frac{u}{\sinh_a^m(z)}\right) \left(1 + \frac{v}{\cosh_a^n(z)}\right) \\ &\leq \left(1 + \frac{u}{2 \sinh_a^m(z)} + \frac{v}{2 \cosh_a^n(z)}\right)^2, \end{aligned} \quad (22)$$

holds.

Proof. By applying the well-known arithmetic-geometric mean inequality, we obtain

$$\begin{aligned} &\left(1 + \frac{u}{\sinh_a^m(z)}\right) \left(1 + \frac{v}{\cosh_a^n(z)}\right) \\ &= 1 + \frac{u}{\sinh_a^m(z)} + \frac{v}{\cosh_a^n(z)} + \frac{uv}{\sinh_a^m(z) \cosh_a^n(z)} \\ &\geq 1 + 2\sqrt{\frac{uv}{\sinh_a^m(z) \cosh_a^n(z)}} + \frac{uv}{\sinh_a^m(z) \cosh_a^n(z)} \\ &= \left(1 + \sqrt{\frac{uv}{\sinh_a^m(z) \cosh_a^n(z)}}\right)^2, \end{aligned}$$

which gives the left-hand-side of (22). Next, by applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &\left(1 + \frac{u}{\sinh_a^m(z)}\right) \left(1 + \frac{v}{\cosh_a^n(z)}\right) \\ &\leq \frac{1}{2} \left[\left(1 + \frac{u}{\sinh_a^m(z)}\right)^2 + \left(1 + \frac{v}{\cosh_a^n(z)}\right)^2 \right] \\ &= \frac{1}{2} \left[\left(2 + \frac{u}{\sinh_a^m(z)} + \frac{v}{\cosh_a^n(z)}\right)^2 - 2 \left(1 + \frac{u}{\sinh_a^m(z)}\right) \left(1 + \frac{v}{\cosh_a^n(z)}\right) \right] \\ &= 2 \left(1 + \frac{u}{2 \sinh_a^m(z)} + \frac{v}{2 \cosh_a^n(z)}\right)^2 - \left(1 + \frac{u}{\sinh_a^m(z)}\right) \left(1 + \frac{v}{\cosh_a^n(z)}\right), \end{aligned}$$

which gives the right-hand-side of (22). \square

Corollary 2.2. *Let $z > 0$, $u > 0$, $v > 0$ and $r \in \mathbb{N}_0$. Then the inequality*

$$\begin{aligned} \left(1 + \sqrt{\frac{2uv}{\sinh_a(2^{r+1}z)}}\right)^2 &\leq \left(1 + \frac{u}{\sinh_a(2^r z)}\right) \left(1 + \frac{v}{\cosh_a(2^r z)}\right) \\ &\leq \left(1 + \frac{u \cosh_a(2^r z) + v \sinh_a(2^r z)}{\sinh_a(2^{r+1}z)}\right)^2, \end{aligned} \quad (23)$$

holds.

Proof. This follows from Theorem 2.1 by letting $m = n = 1$ and replacing z by $2^r z$. \square

Corollary 2.3. *Let $z > 0$, $u > 0$, $v > 0$ and $n \in \mathbb{N}$. Then the inequality*

$$\begin{aligned} \left(1 + \sqrt{\frac{2^n uv}{\sinh_a^n(2z)}}\right)^2 &\leq \left(1 + \frac{u}{\sinh_a^n(z)}\right) \left(1 + \frac{v}{\cosh_a^n(z)}\right) \\ &\leq \left(1 + \frac{2^{n-1}[u \cosh_a^n(z) + v \sinh_a^n(z)]}{\sinh_a^n(2z)}\right)^2, \end{aligned} \quad (24)$$

holds.

Proof. This follows from Theorem 2.1 by letting $m = n$. \square

Corollary 2.4. *For $z > 0$, then the inequality*

$$\begin{aligned} \left(1 + \sqrt{\frac{2}{\sinh_a(2z)}}\right)^2 &\leq \left(1 + \frac{1}{\sinh_a(z)}\right) \left(1 + \frac{1}{\cosh_a(z)}\right) \\ &\leq \left(1 + \frac{a^z}{\sinh_a(2z)}\right)^2, \end{aligned} \quad (25)$$

holds.

Proof. This follows from Theorem 2.1 by letting $m = n = 1$ and $u = v = 1$. \square

Corollary 2.5. *For $z > 0$, then the inequality*

$$\begin{aligned} \left(1 + \frac{2}{\sinh_a(2z)}\right)^2 &\leq \left(1 + \frac{1}{\sinh_a^2(z)}\right) \left(1 + \frac{1}{\cosh_a^2(z)}\right) \\ &\leq \left(1 + \frac{2}{\sinh_a(2z) \tanh_a(2z)}\right)^2, \end{aligned} \quad (26)$$

holds.

Proof. This follows from Theorem 2.1 by letting $m = n = 2$ and $u = v = 1$. \square

Theorem 2.6. *Let $z > 0$, $u > 0$, $v > 0$ and $m, n \in \mathbb{N}$. Then the inequality*

$$\begin{aligned} \left(1 + \sqrt{\frac{uv}{\cosh_a^m(z) \tanh_a^n(z)}}\right)^2 &\leq \left(1 + \frac{u}{\cosh_a^m(z)}\right) \left(1 + \frac{v}{\tanh_a^n(z)}\right) \\ &\leq \left(1 + \frac{u \tanh_a^n(z) + v \cosh_a^m(z)}{2 \cosh_a^m(z) \tanh_a^n(z)}\right)^2, \end{aligned} \quad (27)$$

holds.

Proof. This follows the same procedure as that for Theorem 2.1. Hence we omit it. \square

Corollary 2.7. *Let $z > 0$, $u > 0$, $v > 0$ and $n \in \mathbb{N}$. Then the inequality*

$$\begin{aligned} \left(1 + \sqrt{\frac{uv}{\sinh_a^n(z)}}\right)^2 &\leq \left(1 + \frac{u}{\cosh_a^n(z)}\right) \left(1 + \frac{v}{\tanh_a^n(z)}\right) \\ &\leq \left(1 + \frac{u \tanh_a^n(z) + v \cosh_a^n(z)}{2 \sinh_a^n(z)}\right)^2, \end{aligned} \quad (28)$$

holds.

Proof. This follows from Theorem 2.6 by letting $m = n$. \square

Remark 2.8. Inequality (28) implies that

$$4uv \sinh_a^n(z) - [u \tanh_a^n(z) + v \cosh_a^n(z)]^2 \leq 0. \quad (29)$$

Theorem 2.9. *Let $z > 0$, $u > 0$, $v > 0$ and $m, n \in \mathbb{N}$. Then the inequality*

$$\begin{aligned} \left(1 + \sqrt{\frac{uv}{\sinh_a^{m+n}(z)}}\right)^2 &< \left(1 + \frac{u}{\sinh_a^m(z)}\right) \left(1 + \frac{v}{\tanh_a^n(z)}\right) \\ &\leq \left(1 + \frac{u \tanh_a^n(z) + v \sinh_a^m(z)}{2 \sinh_a^m(z) \tanh_a^n(z)}\right)^2, \end{aligned} \quad (30)$$

holds.

Proof. By adopting the technique of the proof of Theorem 2.1 and using the fact that $\cosh_a(z) > 1$ for all $z \neq 0$, we obtain

$$\begin{aligned} \left(1 + \frac{u}{\sinh_a^m(z)}\right) \left(1 + \frac{v}{\tanh_a^n(z)}\right) &\geq \left(1 + \sqrt{\frac{uv \cosh_a^n(z)}{\sinh_a^{m+n}(z)}}\right)^2 \\ &> \left(1 + \sqrt{\frac{uv}{\sinh_a^{m+n}(z)}}\right)^2, \end{aligned}$$

which gives the left-hand side of (30). Likewise, the right-hand side of (30) also obtained by applying the Cauchy-Schwarz inequality. \square

Corollary 2.10. *Let $z > 0$, $u > 0$ and $v > 0$. Then the inequality*

$$\begin{aligned} \left(1 + \sqrt{\frac{2uv}{\cosh_a(2z) - 1}}\right)^2 &< \left(1 + \frac{u}{\sinh_a(z)}\right) \left(1 + \frac{v}{\tanh_a(z)}\right) \\ &< \left(1 + \frac{u + v}{2 \tanh_a(z)}\right)^2, \end{aligned} \quad (31)$$

holds.

Proof. By letting $m = n = 1$ in the left-hand side of Theorem 2.9, and applying identity (20), we obtain the left-hand side of (31). Next, by letting $m = n = 1$

in the right-hand side of Theorem 2.9, and using the fact that $\cosh_a(z) > 1$, we obtain

$$\begin{aligned} \left(1 + \frac{u}{\sinh_a(z)}\right) \left(1 + \frac{v}{\tanh_a(z)}\right) &\leq \left(1 + \frac{u \tanh_a(z) + v \sinh_a(z)}{2 \sinh_a(z) \tanh_a(z)}\right)^2 \\ &= \left(1 + \frac{\sinh_a(z) \left[\frac{u}{\cosh_a(z)} + v\right]}{2 \sinh_a(z) \tanh_a(z)}\right)^2 \\ &< \left(1 + \frac{u + v}{2 \tanh_a(z)}\right)^2, \end{aligned}$$

which gives the right-hand side of (31). This completes the proof. \square

3. CONCLUDING REMARKS

In this work, we have provided lower and upper bounds for the functions

$$\begin{aligned} f(z) &= \left(1 + \frac{u}{\sinh_a^m(z)}\right) \left(1 + \frac{v}{\cosh_a^n(z)}\right), \\ g(z) &= \left(1 + \frac{u}{\cosh_a^m(z)}\right) \left(1 + \frac{v}{\tanh_a^n(z)}\right), \\ h(z) &= \left(1 + \frac{u}{\sinh_a^m(z)}\right) \left(1 + \frac{v}{\tanh_a^n(z)}\right), \end{aligned}$$

where $z > 0$, $u > 0$, $v > 0$ and $m, n \in \mathbb{N}$. These extend and generalize some earlier results due Barbu and Piscoran. By applying some mean value inequalities, other bounds could be established for these functions. The largest possible range of $f(z)$, $g(z)$ and $h(z)$ are respectively $(1, \infty)$, $(1+v, \infty)$ and $(1+v, \infty)$. This follows from the fact that $f(z)$, $g(z)$ and $h(z)$ are decreasing with $\lim_{z \rightarrow \infty} f(z) = 1$, $\lim_{z \rightarrow \infty} g(z) = 1 + v$ and $\lim_{z \rightarrow \infty} h(z) = 1 + v$. When $a = e$, the results of this paper reduce to those involving the ordinary hyperbolic functions.

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