Regular and semiregular (uniform) tilings and generating functions

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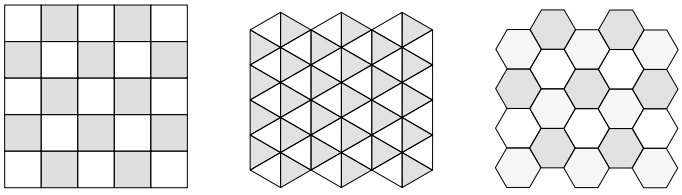
Contents

1 Introduction

In this article we consider the regular and semiregular tilings of the Eucledian plane by convex regular polygons. There are three regular tilings: made of polyominoes with vertex configuration (4^4), polyiamonds (3^6), polyhexes (6^3), see Figure 1.



(a) polyominoes (b) polyiamonds (c) polyhexes

Figure 1: Three regular tilings.

Our task will be to count horizontally convex (or row-convex) regular and semiregular tilings with a given number of tiles. It is known that the number of horizontally convex polynominoes composed by n squares follows the sequence $s_1 = 1$, $s_2 = 2$, $s_3 = 6$, $s_4 = 19$, $s_5 = 61$, $s_6 = 196$, etc. [14] and the recurrent relationship is

$$s_n = 5s_{n-1} - 7s_{n-2} + 4s_{n-3}, \text{ for } n \geq 5. \tag{1}$$

The related generating function is

$$S(x) = \sum_{n=1}^{\infty} s_n x^n = \frac{x(1-x)^3}{1 - 5x + 7x^2 - 4x^3}\,. \tag{2}$$

The recurrent relationship for the number of horizontally convex polyhexes is

$$h_n = 6h_{n-1} - 10h_{n-2} + 7h_{n-3} - h_{n-4}, \text{ for } n \geq 5\,, \tag{3}$$

with the generating function

$$H(x) = \sum_{n=1}^{\infty} h_n x^n = \frac{x(1-x)^3}{1 - 6x + 10x^2 - 7x^3 + x^4}\,, \tag{4}$$

and the sequence looks like $1, 3, 11, 42, 162, 626, 2419, \ldots$ [15].

The recurrent relationship and the generating function is known for the number of horizontally convex polyiamonds as well:

$$t_n = 3t_{n-1} - 4t_{n-3} + t_{n-4} + t_{n-5} + 3t_{n-6} - t_{n-7}, \text{ for } n \geq 8 \tag{5}$$

$$T(x) = \sum_{n=1}^{\infty} t_n x^n = \frac{x(1-x)(2 - x - 4x^2 + 2x^4 + 3x^5)}{1 - 3x + 4x^3 - x^4 - x^5 - 3x^6 + x^7}\,. \tag{6}$$

with the sequence $2, 3, 6, 14, 34, 84, 208, 515, 1272, \ldots$ [16].

2 Semiregular tilings

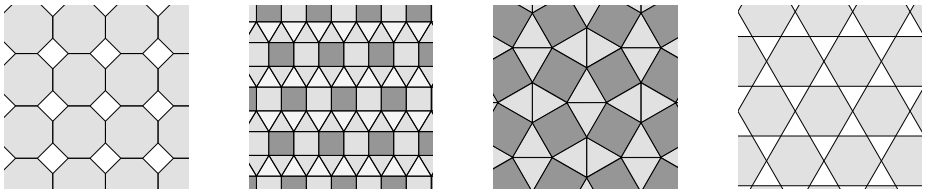
There are eight semiregular tilings with vertex configurations $(4, 8^2)$, $(3^3, 4^2)$, $(3^2, 4, 3, 4)$, $(3, 6, 3, 6)$, $(3^4, 6)$, $(3, 4, 6, 4)$, $(3, 12^2)$, $(4, 6, 12)$, see Figure 2.

3 Horizontally convex semiregular tilings

For counting the number of horizontally convex semiregular tilings we shall use the technique similar to [3]. It is essential to define the layered structure of tilings. There are might be different possibilities to do so, therefore the shape and the number of tilings may be different.

Without going deep into the details we assume that the definition of a horizontally convex semiregular tilings is intuitively clear and very well corresponds to the definition of horizontally convex regular tilings. That is each layer/row is well connected.

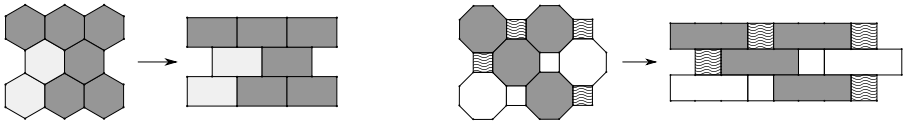
To simplify the reasoning we may use a natural map between some complex tilings and more simple brickwork. For example, there is the bijection between hexagonal tilings and a very standard plain brickwork, see Figure 3a and [13].



(a) $(4, 8^2)$ (b) $(3^3, 4^2)$ (c) $(3^2, 4, 3, 4)$ (d) $(3, 6, 3, 6)$

(e) $(3^4, 6)$ (f) $(3, 4, 6, 4)$ (g) $(3, 12^2)$ (h) $(4, 6, 12)$

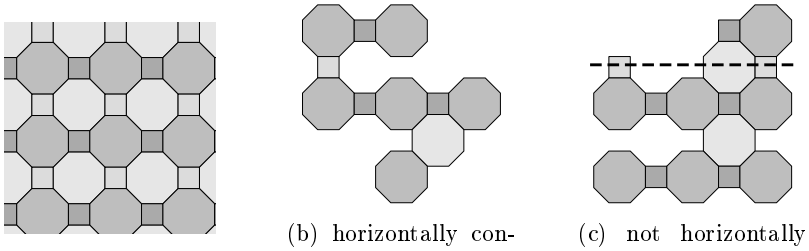
Figure 2: Semiregular tilings.



(a) (b)

Figure 3: Map between tilings and rectangular plain brickwork.

Consider the tiling with the vertex configuration $(4, 8^2)$. It can be naturally transformed into a plain brickwork composed by rectangles and squares, Figure 3b. Therefore, we can use the natural layered structure and related notion of row-convexity, Figure 4.



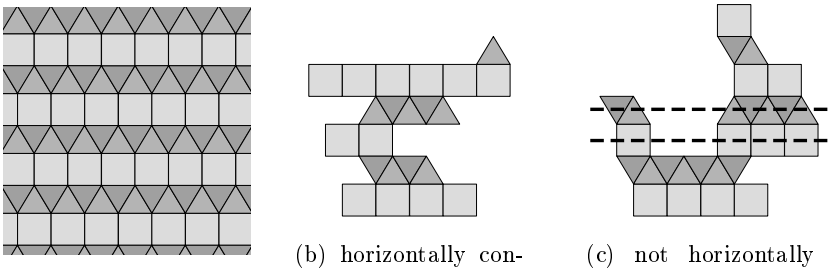
(a) (b) horizontally convex (c) not horizontally convex

Figure 4: Tiling of type $(4, 8^2)$.

Exercise 1. *Find a different layered structure for the tiling of type* $(4, 8^2)$.

Consider the tiling with the vertex configuration $(3^3, 4^2)$. There exists a natural layered structure composed by a layer of squares and another layer of triangles, Figure 5.

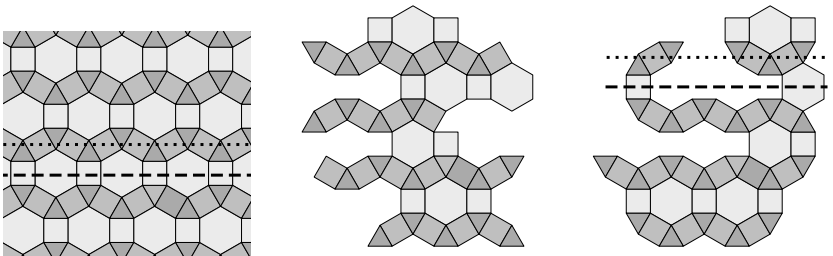
Exercise 2. *Find other layered structures for the tiling of type* $(3^3, 4^2)$.



(a) (b) horizontally convex (c) not horizontally convex

Figure 5: Tiling of type $(3^3, 4^2)$.

Consider the tiling with the vertex configuration $(3, 4, 6, 4)$. There exists a natural layered structure composed by two types of layers: one layer is composed by a sequence of squares and hexagons, another layer is composed by a sequence of squares and triangles, Figure 6.



(a) (b) horizontally convex (c) not horizontally convex

Figure 6: Tiling of type $(3, 4, 6, 4)$.

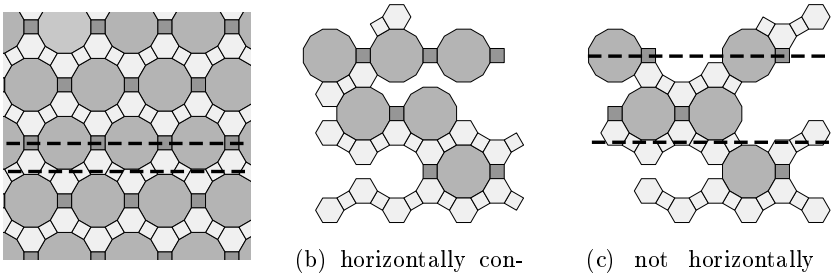
Consider the tiling with the vertex configuration $(4, 6, 12)$. There exists a natural layered structure composed by two types of layers: one layer is composed by a sequence of squares and hexagons, another layer is composed by a sequence of squares and octagons, Figure 7.

Consider the tiling with the vertex configuration $(3^2, 4, 3, 4)$. To see a natural layered structure of this tiling we shall turn this tiling, compare Figure 2c and Figure 8.

We shall turn the tiling with the vertex configuration $(3^4, 6)$ on Figure 2e in order to be able to see a natural layered structure of it, Figure 9.

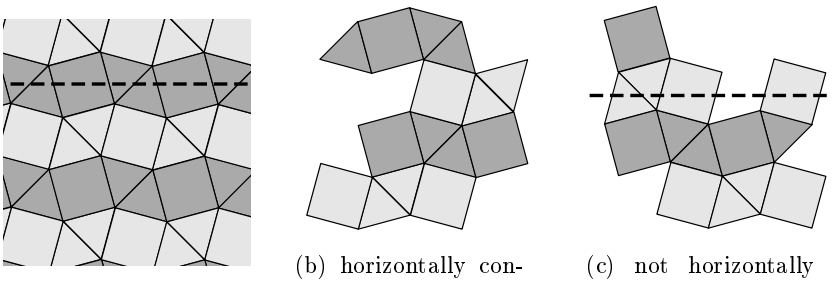
Consider the tiling with the vertex configuration $(3, 6, 3, 6)$. It can be decomposed into equally shaped row layers, see Figure 10a. Nevertheless, the notion of a layer continuity requires some additional considerations which we will not be discussing here leaving a reader to think about it.

The last tiling in the list is of type $(3, 12^2)$, Figure 2g. It looks like it is impossible to determine the layered structure for it. However there exists a bijection between tiling of type $(3, 12^2)$ and the one of type $(3, 6, 3, 6)$, Figure 10b. Therefore, the problem of counting the number of possible tilings of $(3, 12^2)$ is somehow equivalent to the problem of counting the number of tilings



(a) (b) horizontally convex (c) not horizontally convex

Figure 7: Tiling of type $(4, 6, 12)$.



(a) (b) horizontally convex (c) not horizontally convex

Figure 8: Tiling of type $(3^2, 4, 3, 4)$.

for $(3, 6, 3, 6)$.

4 Recurrent relationship for the tiling of type $(4, 8^2)$ with combinatorial proof

Let G denotes the set of all possible horizontally convex tilings of type $(4, 8^2)$. Let G_n be a subset of G with the number of tiles n, let $g_n = |G_n|$.

Theorem 1. *For $n \geq 6$ the following equality holds*

$$g_n = 5g_{n-1} - 3g_{n-2} - 5g_{n-3} + 7g_{n-4} - g_{n-5}. \tag{7}$$

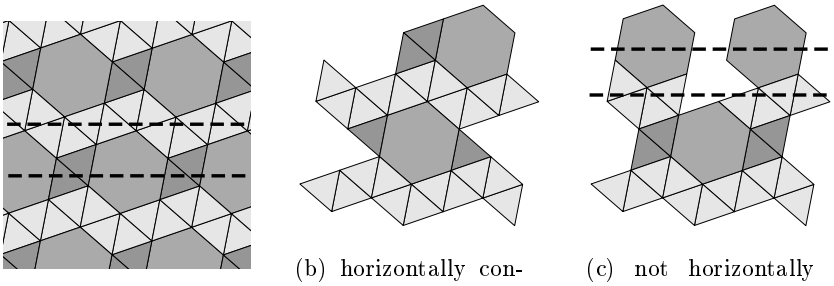
We shall establish the proof by a series of Lemmas. First of all, let us define several subsets of G based on the shape of the top layer, Figure 11:

a) A: the top layer consists of a single square;

b) B: the top layer the number of octagons is larger than the number of squares by 1;

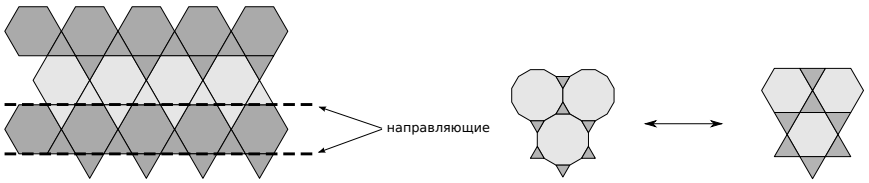
c) C: the top layer contains at least one octagon and the number of octagons is less than the number of squares by 1;

d) D or D^*: the top layer contains the equal number of octagons and squares, with the rightmost figure either square or octagon respectively.



(a) (b) horizontally convex (c) not horizontally convex

Figure 9: Tiling of type $(3^4, 6)$.



(a) $(3, 6, 3, 6)$ (b) mapping $(3, 12^2)$ to $(3, 6, 3, 6)$

Figure 10: Tilings $(3, 6, 3, 6)$ and $(3, 12^2)$.

Consider also the set $H = B \cup C \cup D \cup D^*$, which contains at least one octagon on its top layer. Note that $G = A \cup H$.

Denote by A_n, B_n, C_n, D_n, D_n^* and H_n the subsets of A, B, C, D, D^* и H, containing he number of tiles n. Also $a_n = |A_n|$, $b_n = |B_n|$, $c_n = |C_n|$, $d_n = |D_n|$, $d_n^* = |D_n^*|$ and $h_n = |H_n|$. Due to the symmetry we note that $d_n = d_n^*$.

From the above definitions follows

$$h_n = b_n + c_n + d_n + d_n^* = b_n + c_n + 2d_n, \tag{8}$$

$$g_n = a_n + b_n + c_n + d_n + d_n^* = a_n + h_n. \tag{9}$$

Lemma 1. *For $n \geq 2$ the following equalities hold*

$$\text{a) } d_n = b_{n-1}\,, \qquad \text{b) } c_n = d_{n-1}\,.$$

Proof. a) If we add to a tiling from the set B at the top layer at the right one square then the resulting tiling will be from the set D. This process can be reversed, Figure 12. Therefore $d_n = b_{n-1}$ for $n \geq 2$.

b) Similar reasoning holds for adding the square to the left. □

Taking into account the Lemma 1, for $n \geq 3$ the equality (8) can be written as

$$h_n = b_n + 2b_{n-1} + b_{n-2}. \tag{10}$$

In order to simplify the following reasoning we shall take advantage of the bijection between tiling of the type $(4, 8^2)$ and the brickwork from Figure 3b. The notion of octagons will be replaced by bricks of size 1×3.

We shall introduce subsets of A, Figure 13.

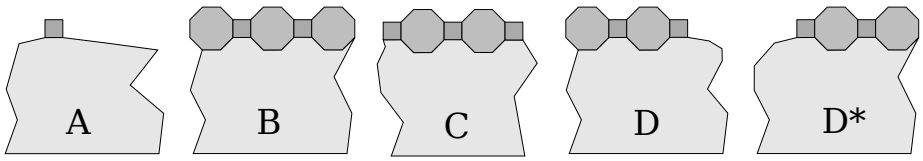


Figure 11: The subsets of the set G.

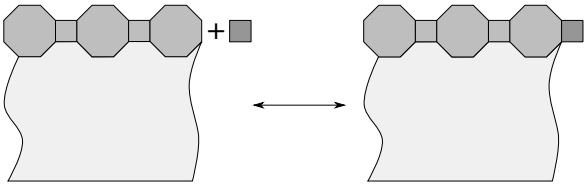


Figure 12: Relationship between sets B and D.

a) I: at the sub-row of the top layer the number of bricks (octagons) is larger than the number of squares by one;

b) J: the sub-row of the top layer contains at least one brick and the number of bricks is less than the number of squares by one;

c) E and E^*: at the sub-row of the top layer the number of bricks (octagons) is equal to the number of squares, either starting or ending by a brick.

Similarly to the above we denote $i_n = |I_n|$, $j_n = |J_n|$, $e_n = |E_n|$, $e_n^* = |E_n^*|$.

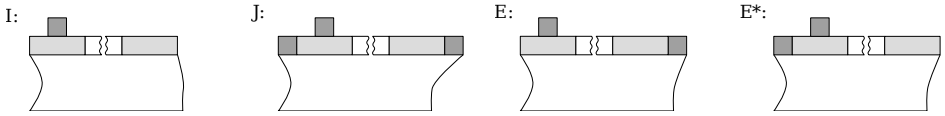


Figure 13: Subsets of A.

Due to the symmetry $e_n = e_n^*$. It is clear that for $n \geq 2$

$$a_n = i_n + j_n + e_n + e_n^* = i_n + j_n + 2e_n. \tag{11}$$

Lemma 2. *For $n \geq 2$ the following equalities hold*

$$\text{a) } e_n = i_{n-1}\,, \qquad \text{b) } j_n = e_{n-1}\,.$$

Proof. a) See Figure 14).

b) Similar reasoning as for the case a). □

Taking into account the above result we can rewrite (11) for $n \geq 3$ as

$$a_n = i_n + 2i_{n-1} + i_{n-2}. \tag{12}$$

We shall define more subsets, see Figure 15. Let P denotes the subset of B, such as the rightmost brick of the top layer is placed on top of the leftmost element (brick or square) of the sub-row.

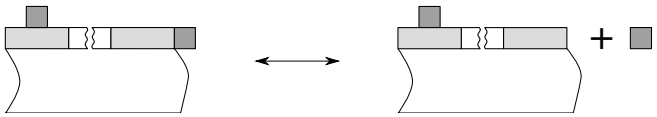


Figure 14: Relationship between E_n and I_{n-1}.

Let Q denote the subset of I, such that none of the squares of the top layer are placed on top of the rightmost brick of the sub-row.

Let R denotes the subset of Q, such that the rightmost brick of the sub-row is placed on the leftmost element of the sub-sub-row, tilings from the set R contain at least three layers.

On the Figure 15 we mark by "xxx" the position where it prohibited to place a square. As usual, $p_n = |P_n|$, $q_n = |Q_n|$, $r_n = |R_n|$.

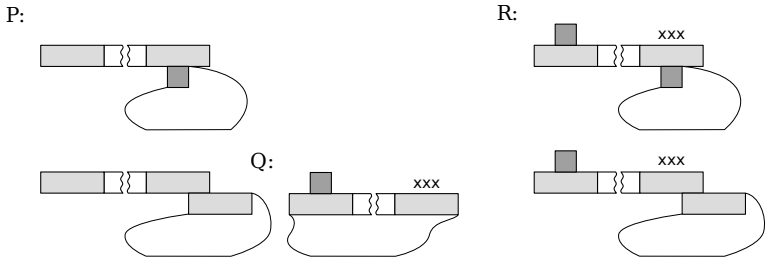


Figure 15: Additional subsets.

Lemma 3. *For $n \geq 3$ holds $p_n = p_{n-2} + g_{n-1}$.*

Proof. If the top layer of tiling from P_n contains at least 3 elements, then taking away a brick and a square we obtain a tiling from P_{n-2}, see Figure 16. If there is only one brick at the top layer of P_n, then this brick should be located exactly on top of the leftmost element of the sub-row. Taking away this brick we obtain a tiling from G_{n-1}. This procedure can be inverted. □

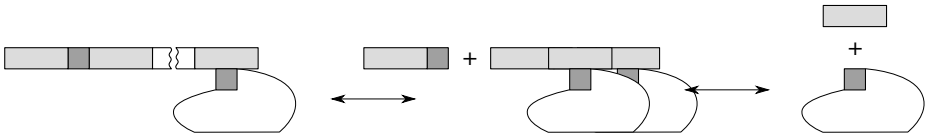


Figure 16: See Lemma 3.

Lemma 4. *For $n \geq 3$ holds $i_n = q_n + b_{n-1}$.*

Proof. The set $I_n \backslash Q_n$ contains tilings such that the only square from the top layer is placed on top of the rightmost brick of the sub-row. If we take away this square, we obtain a tiling from the set B_{n-1}, see Figure 17. This procedure can be inverted. □

Lemma 5. *For $n \geq 3$ holds $q_n = r_n + i_{n-2}$.*

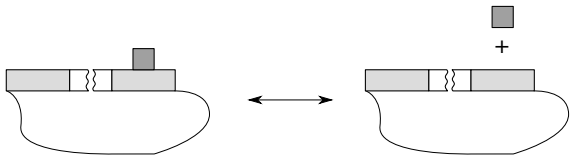


Figure 17: See Lemma 4.

Proof. The set $Q_n \backslash R_n$ contains tilings such that it is possible to take away a square and a brick at the right side of the sub-row and to obtain a tiling from the set I_{n-2}, see Figure 18. This procedure can be inverted. □

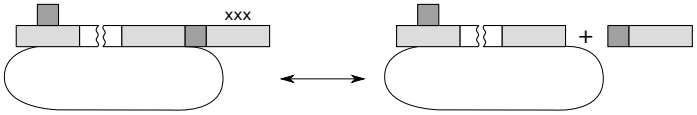


Figure 18: See Lemma 5.

Lemma 6. *For $n \geq 4$ holds $r_n = r_{n-2} + p_{n-3}$.*

Proof. If the square from the top layer of tiling from the set R_n is placed exactly on top of the leftmost brick then it is possible to take away that top square and the leftmost brick with square from the sub-row in order to obtain a tiling from the set P_{n-3}. If the square from the top layer is not placed on top of the leftmost brick of the sub-row, then it is possible to take away the leftmost brick and square in order to obtain a tiling from the set R_{n-2}, see Figure 19. This procedure can be inverted. □

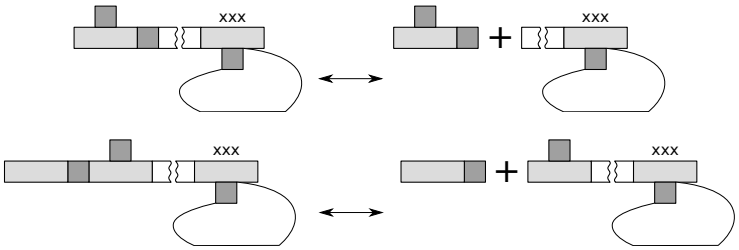


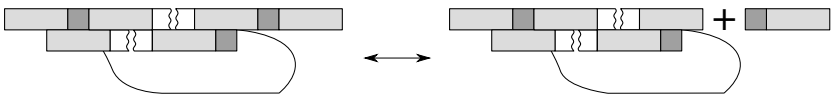
Figure 19: See Lemma 6.

Lemma 7. *For $n \geq 4$ holds $b_n = p_n + b_{n-2} + a_n$.*

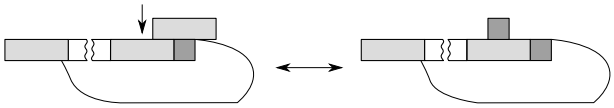
Proof. The top layer of a tiling from the set B contains odd number of elements. Consider a tiling from the set $B_n \backslash P_n$. If the top layer contains tree or more elements, then it is possible to take away the rightmost square and brick in order to obtain a tiling from the set B_{n-2}, see Figure 20a. This procedure can be inverted.

Consider a tiling from the set $B_n \backslash P_n$ such that the top layer contains a single brick. This tiling does not belong to the set P_n, therefore the square just below this brick is not the leftmost element of the sub-row. There should be a

brick at the sub-row and an empty space just on top of it where we can place a square instead of the brick, see Figure 20b. The resulting tiling is from the set A_n. Therefore there is a bijection between A_n and $B_n \backslash P_n$. Which means that for $n \geq 4$ holds $b_n - p_n = b_{n-2} + a_n$. □



(a) top layer contains three or more elements



(b) top layer contains a single brick

Figure 20: See Lemma 7.

Proof of the theorem 1. Let us put together the obtained recurrent relations:

$$\begin{cases} g_n &= a_n + h_n, & (9) \\ h_n &= b_n + 2b_{n-1} + b_{n-2}, & (10) \\ a_n &= i_n + 2i_{n-1} + i_{n-2}, & (12) \\ p_n &= p_{n-2} + g_{n-1}, & \text{Lemma 3} \\ i_n &= q_n + b_{n-1}, & \text{Lemma 4} \\ q_n &= r_n + i_{n-2}, & \text{Lemma 5} \\ r_n &= r_{n-2} + p_{n-3}, & \text{Lemma 6} \\ b_n &= p_n + b_{n-2} + a_n, & \text{Lemma 7} \end{cases} \Longrightarrow \begin{cases} g_n &= a_n + h_n, \\ h_n &= b_n + 2b_{n-1} + b_{n-2}, \\ a_n &= i_n + 2i_{n-1} + i_{n-2}, \\ p_n - p_{n-2} &= g_{n-1}, \\ i_n - i_{n-2} &= r_n + b_{n-1}, \\ r_n - r_{n-2} &= p_{n-3}, \\ b_n - b_{n-2} &= p_n + a_n. \end{cases}$$

For $n \geq 4$

$$h_n - h_{n-1} = p_n + a_n + p_{n-1} + a_{n-1}.$$

Therefore

$$g_n - g_{n-1} = 2a_n + p_n + p_{n-1}.$$

Then

$$g_n - g_{n-1} - (g_{n-1} - g_{n-2}) = 2a_n - 2a_{n-1} + g_{n-1}.$$

Finally, for $n \geq 4$

$$g_n - 3g_{n-1} + g_{n-2} = 2a_n - 2a_{n-1}. \tag{13}$$

On the other hand

$$a_n - a_{n-1} = r_n + b_{n-1} + r_{n-1} + b_{n-2}.$$

Therefore, for $n \geq 5$

$$a_n - a_{n-1} - (a_{n-2} - a_{n-3}) = p_{n-3} + p_{n-4} + p_{n-1} + a_{n-1} + p_{n-2} + a_{n-2}.$$

Then

$$a_n - 2a_{n-1} - 2a_{n-2} + a_{n-3} = p_{n-1} + p_{n-2} + p_{n-3} + p_{n-4}.$$

Following

$$a_n - 2a_{n-1} - 2a_{n-2} + a_{n-3} - (a_{n-1} - 2a_{n-2} - 2a_{n-3} + a_{n-4}) = p_{n-1} - p_{n-3} + p_{n-3} - p_{n-5} = g_{n-2} + g_{n-4}.$$

Obtaining

$$g_{n-2} + g_{n-4} = a_n - 3a_{n-1} + 3a_{n-3} - a_{n-4}.$$

The right hand side can be manipulated using (13). Then for $n \geq 6$

$$\begin{aligned} g_{n-2} + g_{n-4} &= a_n - a_{n-1} - 2(a_{n-1} - a_{n-2}) - 2(a_{n-2} - a_{n-3}) + a_{n-3} - a_{n-4} \\ &= \frac{g_n - 3g_{n-1} + g_{n-2}}{2} - (g_{n-1} - 3g_{n-2} + g_{n-3}) - (g_{n-2} - 3g_{n-3} + g_{n-4}) + \frac{g_{n-3} - 3g_{n-4} + g_{n-5}}{2} \\ &= \frac{g_n - 5g_{n-1} + 5g_{n-2} + 5g_{n-3} - 5g_{n-4} + g_{n-5}}{2}. \end{aligned}$$

Which means

$$2g_{n-2} + 2g_{n-4} = g_n - 5g_{n-1} + 5g_{n-2} + 5g_{n-3} - 5g_{n-4} + g_{n-5}.$$

Finally

$$g_n = 5g_{n-1} - 3g_{n-2} - 5g_{n-3} + 7g_{n-4} - g_{n-5}.$$

This completes the proof of the theorem 1. □

The Table 1 contains first few values for the sequences used in the Theorem 1.

n	a	b	c	d	i	j	e	p	q	r	h	g
1	1	1	0	0	0	0	0	0	0	0	1	2
2	1	3	0	1	1	0	0	2	0	0	5	6
3	5	12	1	3	3	0	1	6	0	0	19	24
4	20	49	3	12	13	1	3	26	1	0	76	96
5	83	197	12	49	54	3	13	102	5	2	307	390
6	337	802	49	197	216	13	54	416	19	6	1245	1582
7	1370	3251	197	802	884	54	216	1684	82	28	5052	6422
8	5559	13199	802	3251	3575	216	884	6838	324	108	20503	26062
9	22561	53558	3251	13199	14527	884	3575	27746	1328	444	83207	105768

Table 1: First few elements from the sequences of the Theorem 1.

Exercise 3. *Using the methodology of Theorem 1, prove the formula* (3) *for the number of horizontally convex polyhexes.*

5 Some auxiliary calculations using power series

Exercise 4. *Using definitions, prove the formulas for the following infinite geometric progressions*

$$\text{a)} \sum_{k=1}^{\infty} v^k = \frac{v}{1-v}, \qquad \text{b)} \ (2+2u)\sum_{k=1}^{\infty} u^{2k} = \frac{2u^2}{1-u}.$$

Exercise 5. *Using integration and differentiation of formal power series prove*

$$\text{a)} \sum_{k=1}^{\infty} (k-1)v^k = \frac{v^2}{(1-v)^2}, \qquad \text{b)} \sum_{k=1}^{\infty} kv^k = \frac{v}{(1-v)^2}, \qquad \text{c)} \sum_{k=1}^{\infty} k(k-1)v^k = \frac{2v^2}{(1-v)^3}.$$

Exercise 6. *Using results from exercises 4 and 5 prove*

a) $\displaystyle\sum_{k=1}^{\infty} k u^k v^{k-1} = \frac{u}{(1-uv)^2}\,,$ b) $\displaystyle\sum_{k=1}^{\infty} (kv+1) u^k v^{k-1} = \frac{u+uv-u^2v}{(1-uv)^2}\,,$

c) $\displaystyle\sum_{k=1}^{\infty} k^2 u^k v^{k-1} = \frac{u+u^2v}{(1-uv)^3}\,,$ d) $\displaystyle\sum_{k=1}^{\infty} k(kv+1) u^k v^{k-1} = \frac{u(1+v-uv+uv^2)}{(1-uv)^3}\,.$

Exercise 7. *Prove the following formulas*

a) $\displaystyle\sum_{k=1}^{\infty} \left(ku + (2+u)(k-1)\right) u^{2k} = \frac{u^3}{(1-u)^2}\,,$ b) $\displaystyle\sum_{k=1}^{\infty} (u+2k+2uk) u^{2k} = \frac{u^2(2-u)}{(1-u)^2}\,,$

c) $\displaystyle\sum_{k=1}^{\infty} \left(k^2u + (k-1)(2k+u(k+1))\right) u^{2k} = \frac{u^3(1+2u-u^2)}{(1+u)(1-u)^3}\,.$

Let us define a generating function of three variables

$$H(u,v,y) = \sum_{p,q,m} h(p,q,m) u^p v^q y^m\,.$$

The value of the partial derivative with respect of variable y at point $y=1$ we denote by

$$\chi(u,v) = \frac{\partial}{\partial y} H(u,v,y)\bigg|_{y=1}\,.$$

The formal term by term differentiation of the following series results in

$$\begin{aligned}
\frac{\partial}{\partial y}\left(y^k H(u,v,y)\right)\bigg|_{y=1} &= \frac{\partial}{\partial y}\left(\sum_{p,q,m} h(p,q,m) u^p v^q y^{m+k}\right)\bigg|_{y=1} \\
&= \sum_{p,q,m} (m+k) h(p,q,m) u^p v^q \\
&= \sum_{p,q}\left(\sum_{m} (m+k) h(p,q,m)\right) u^p v^q\,,
\end{aligned}$$

for all integer $k \geq 0$.

On the other hand using the rule of product differentiation

$$\frac{\partial}{\partial y}\left(y^k H(u,v,y)\right)\bigg|_{y=1} = kH(u,v,1) + \chi(u,v).$$

Therefore for all integer $k \geq 0$

$$kH(u,v,1) + \chi(u,v) = \sum_{p,q}\left(\sum_{m} (m+k) h(p,q,m)\right) u^p v^q. \tag{14}$$

In particular for $k=0$

$$\chi(u,v) = \sum_{p,q}\left(\sum_{m} m h(p,q,m)\right) u^p v^q. \tag{15}$$

6 Generating functions for the number of horizontally convex tilings of type $(4, 8^2)$

Theorem 2. *The number of horizontally convex tilings of type*$(4, 8^2)$ *can be computed using the following generating functions*

$$G(x) = \sum_n g_n\, x^n = \frac{2x(1-x)^3(1+x)}{1-5x+3x^2+5x^3-7x^4+x^5},$$

and

$$G(u,v,1) = \frac{(1-uv)^3(u+v+2uv)}{1-2u-9uv+4u^2v-4uv^2-u^3v+14u^2v^2-uv^3-6u^3v^2+4u^2v^3-13u^3v^3-4u^3v^4+u^4v^4},$$

where x is the total number of tiles (squares and octagons), u – the number of octagons, v – the number of squares.

In order to prove this theorem we shall use the subsets A, B, C, D, D^*, H and G defined in Section 4. In addition we denote $a(p,q,0)$, $b(p,q,m)$, $c(p,q,m)$, $d(p,q,m)$ and $d^*(p,q,m)$ — the number of tilings in each subset A, B, C, D и D^* correspondingly, which contain p octagons and q squares, and the top layer contains m octagons.

Following the definitions we note that the top layer of a tiling from the set B contains m octagons and $m-1$ squares for $m \geq 1$. The top layer of a tiling from the set C contains m octagons and $m+1$ squares for $m \geq 1$. The top layer of a tiling from the set D or D^* contains m octagons and m squares for $m \geq 1$, also $d(p,q,m) = d^*(p,q,m)$ due to symmetry, which also means that generating functions related to the sets D and D^* are the same.

Let us consider the following generating functions of three variables with u counting the number of octagons, v counting the number of squares and y counting the number of octagons at the top layer.

$$A(u,v,y) = \sum_{p,q} a(p,q,0)u^p v^q,$$
$$B(u,v,y) = \sum_{p,q,m} b(p,q,m)u^p v^q y^m,$$
$$C(u,v,y) = \sum_{p,q,m} c(p,q,m)u^p v^q y^m,$$
$$D(u,v,y) = D^*(u,v,y) = \sum_{p,q,m} d(p,q,m)u^p v^q y^m.$$

Since $A(u,v,y)$ does not depend on y, then $A(u,v,y) = A(u,v,1)$ and

$$\frac{\partial}{\partial y} A(u,v,y) = 0.$$

Lemma 8. *For all $p \geq 1$, $q \geq 0$, $m \geq 1$ holds*

$$b(p,q,m) = d(p,q+1,m) = c(p,q+2,m).$$

Proof. See the proof of Lemma 1, Figure 12. □

From Lemma 8 follows

$$D(u,v,y) = vB(u,v,y), \qquad C(u,v,y) = v^2 B(u,v,y).$$

Then

$$H(u,v,y) = B(u,v,y)+C(u,v,y)+D(u,v,y)+D^*(u,v,y) = (1+2v+v^2)B(u,v,y), \tag{16}$$
$$G(u,v,y) = A(u,v,y) + H(u,v,y) = A(u,v,y) + (1+2v+v^2)B(u,v,y).$$

Note that in our reasoning we do not need to find the exact expression for three-variables function $G(u,v,y)$. It is sufficient to find the value of this function at $y=1$. As before, denote

$$\chi(u,v) = \left.\frac{\partial}{\partial y} H(u,v,y)\right|_{y=1}.$$

Proof of Theorem 2. First, we shall obtain the recurrent relations for $a(p,q,m)$, $b(p,q,m)$, $h(p,q,m)$ and then we shall compose the generating functions.

Let us start with the set A. Take a square and place it on top to some initial tiling of type G. This type of operation is impossible if the initial tiling was from the set A. If the initial tiling was from the set H and its top layer contains already m octagons, then this type of operation can produce m different tilings from A. Therefore, for any p and q

$$a(p,q+1,0) = \sum_m m\, h(p,q,m).$$

For a single square we have $a(0,1,0) = 1$. And finally,

$$A(u,v,y) = v + v\, \left.\left(\frac{\partial H(u,v,y)}{\partial y}\right)\right|_{y=1} = v + v\,\chi(u,v). \tag{17}$$

Taking into account (15), obtain

$$\begin{aligned} v + v\chi(u,v) &= v + \sum_{p,q}\left(\sum_m m h(p,q,m)\right) u^p v^{q+1} \\ &= v + \sum_{p,q} a(p,q+1,0) u^p v^{q+1} \\ &= A(u,v,y). \end{aligned}$$

Let us work now with the set B. Consider a tiling from the set B which consists of a single row with k octagons and $k-1$ squares. This row can be placed on top of some initial tiling of type G. If the initial tiling was from the set A, then this operation can be produce k different tilings. If the initial tiling was from the set H and its top layer contain already m octagons, then we are able to produce $m+k$ different tilings. This way it is possible to produce any tiling from the set B but the ones that are made of a single row. Therefore, for any p and q holds

$$b(p+k, q+k-1, k) = k\, a(p,q,0) + \sum_m (m+k)\, h(p,q,m).$$

And for the tilings from B made of a single row we have $b(k,k-1,k)=1$.

Using (14), obtain

$$\begin{aligned}
B(u,v,y) &= \sum_{p,q,m} b(p,q,m)u^p v^q y^m \\
&= \sum_{k} b(k,k-1,k)u^k v^{k-1} y^k + \sum_{p,q,k} b(p+k,q+k-1,k)u^{p+k} v^{q+k-1} y^k \\
&= \sum_{k=1}^{\infty} u^k v^{k-1} y^k + \sum_{k=1}^{\infty} u^k v^{k-1} y^k \left(\sum_{p,q} \left(ka(p,q,0) + \sum_{m} (m+k)h(p,q,m) \right) u^p v^q \right) \\
&= \sum_{k=1}^{\infty} u^k v^{k-1} y^k + \sum_{k=1}^{\infty} u^k v^{k-1} y^k (kA(u,v,1) + kH(u,v,1) + \chi(u,v)) \,.
\end{aligned}$$

From (17) follows

$$\begin{aligned}
B(u,v,y) &= \sum_{k=1}^{\infty} u^k v^{k-1} y^k + \sum_{k=1}^{\infty} u^k v^{k-1} y^k (kv + kv\chi(u,v) + kH(u,v,1) + \chi(u,v)) \\
&= H(u,v,1) \sum_{k=1}^{\infty} k u^k v^{k-1} y^k + (1+\chi(u,v)) \sum_{k=1}^{\infty} (kv+1) u^k v^{k-1} y^k \,.
\end{aligned}$$

Taking into account (16) we obtain some relations for generating function $H(u,v,y)$

$$\frac{H(u,v,y)}{1+2v+v^2} = H(u,v,1) \sum_{k=1}^{\infty} k u^k v^{k-1} y^k + (1+\chi(u,v)) \sum_{k=1}^{\infty} (kv+1) u^k v^{k-1} y^k. \quad (*)$$

From here we can compose a system of equations for $H(u,v,1)$ and $\chi(u,v)$. The first equation can be obtained by substituting $y=1$ in $(*)$. The second equation can be obtaned by differentiation of $(*)$ with relation to y and substituting $y=1$.

$$\begin{cases}
\dfrac{H(u,v,1)}{(1+v)^2} &= H(u,v,1) \displaystyle\sum_{k=1}^{\infty} k u^k v^{k-1} + (1+\chi(u,v)) \sum_{k=1}^{\infty} (kv+1) u^k v^{k-1}, \\
\dfrac{\chi(u,v)}{(1+v)^2} &= H(u,v,1) \displaystyle\sum_{k=1}^{\infty} k^2 u^k v^{k-1} + (1+\chi(u,v)) \sum_{k=1}^{\infty} k(kv+1) u^k v^{k-1}.
\end{cases}$$

Now we shall use the computations made in Exercise 6. The system of equations can be rewritten as

$$\begin{cases}
H(u,v,1) \left(\dfrac{1}{(1+v)^2} - \dfrac{u}{(1-uv)^2} \right) &= (1+\chi(u,v)) \dfrac{u+uv-u^2v}{(1-uv)^2}, \\
(1+\chi(u,v)) \left(\dfrac{1}{(1+v)^2} - \dfrac{u(1+v-uv+uv^2)}{(1-uv)^3} \right) &= H(u,v,1) \dfrac{u+u^2v}{(1-uv)^3} + \dfrac{1}{(1+v)^2}.
\end{cases}$$

From the first equation obtain

$$1+\chi(u,v) = \frac{H(u,v,1)\left((1-uv)^2 - u(1+v)^2\right)}{(1+v)^2(u+uv-u^2v)}. \quad (**)$$

Substituting this relation into the second equation and performing some technical calculations we obtain

$$H(u,v,1) = \frac{(1-uv)^2(1+v)^2(u+uv-u^2v)}{1-2u-9uv+4u^2v-4uv^2-u^3v+14u^2v^2-uv^3-6u^3v^2+4u^2v^3-13u^3v^3-4u^3v^4+u^4v^4}.$$

From (17) and $(**)$ follows

$$A(u,v,y) = \frac{v(1-uv)^2(1-u-4uv-uv^2+u^2v^2)}{1-2u-9uv+4u^2v-4uv^2-u^3v+14u^2v^2-uv^3-6u^3v^2+4u^2v^3-13u^3v^3-4u^3v^4+u^4v^4}.$$

Recall that $G(u,v,1) = A(u,v,1) + H(u,v,1)$, then

$$G(u,v,1) = \frac{(1-uv)^3(u+v+2uv)}{1-2u-9uv+4u^2v-4uv^2-u^3v+14u^2v^2-uv^3-6u^3v^2+4u^2v^3-13u^3v^3-4u^3v^4+u^4v^4}.$$

Let $u = v = x$, then

$$\begin{aligned}
H(x) &= \frac{x(1-x)^2(1+x)(1+x-x^2)}{1-5x+3x^2+5x^3-7x^4+x^5}\,, \\
A(x) &= \frac{x(1-x)^2(1+x)(1-3x+x^2)}{1-5x+3x^2+5x^3-7x^4+x^5}\,, \\
B(x) &= \frac{x(1-x)^2(1+x-x^2)}{(1+x)(1-5x+3x^2+5x^3-7x^4+x^5)}\,, \\
G(x) &= \sum_n g_n\, x^n = \frac{2x(1-x)^3(1+x)}{1-5x+3x^2+5x^3-7x^4+x^5}\,.
\end{aligned}$$

This finishes the proof of the theorem. □

We use a mathematical software, for example [17], to obtain the power series of two variable based on the above generating functions

$$\begin{aligned}
A(u,v,1) &= v + uv + 2u^2v + 3uv^2 + 4u^3v + 13u^2v^2 + 3uv^3 + 8u^4v + 41u^3v^2 + 33u^2v^3 + uv^4 + \ldots \\
G(u,v,1) &= u + v + 2u^2 + 4uv + 4u^3 + 14u^2v + 6uv^2 + 8u^4 + 42u^3v + 42u^2v^2 + 4uv^3 \\
&\quad + 16u^5 + 113u^4v + 192u^3v^2 + 68u^2v^3 + uv^4 + \ldots
\end{aligned}$$

Similarly

$$\begin{aligned}
A(x) &= x + x^2 + 5x^3 + 20x^4 + 83x^5 + 337x^6 + 1370x^7 + 5559x^8 + 22561x^9 + 91554x^{10} + \ldots \\
G(x) &= 2x + 6x^2 + 24x^3 + 96x^4 + 390x^5 + 1582x^6 + 6422x^7 + 26062x^8 + 105768x^9 + 429228x^{10} + \ldots
\end{aligned}$$

These coefficients are obviously the same as in Table 1. It is easy to obtain now another proof of Theorem 1.

Proof of Theorem 1, as Corollary from Theorem 2. Using Theorem 2, for $G(x)$ holds

$$(1-5x+3x^2+5x^3-7x^4+x^5)\sum_n g_n\, x^n = 2x(1-x)^3(1+x).$$

Since the right hand side is a polynomial, then the left hand side should be the same polynomial. Therefore the sequence g_n follows the recurrent relation of the 5th order:

$$g_{n+5} = 5g_{n+4} - 3g_{n+3} - 5g_{n+2} + 7g_{n+1} - g_n.$$ □

The characteristic equation for the above recurrent relation is (see [7])

$$x^5 - 5x^4 + 3x^3 + 5x^2 - 7x + 1 = 0.$$

This equation has real roots and the largest root is $x_{max} \approx 4,058206109671$.

Since this root is not a root of the numerator of the generating function, then we can obtain the following asymptotic boundary $4.0582^n \le g_n \le 4.05821^n$.

Let us point onto the particular substitution $v = 1$,

$$G(u,1,1) = \frac{(1-u)^3(1+3u)}{1-16u+22u^2-24u^3+u^4} = \sum_n w_n\, u^n.$$

The term w_n is a number of horizontally convex tilings of type $(4,8^2)$ containing n octagons. We assume that $w_0 = 1$ corresponds to the case of a single square. The first few values in this sequence are $w_1 = 16$, $w_2 = 228$, $w_3 = 3328$, $w_4 = 48612$, $w_5 = 710032$. This sequence satisfies the recurrent relation of the 4th order:

$$w_{n+4} = 16w_{n+3} - 22w_{n+2} + 24w_{n+1} - w_n\,,$$

with the characteristic equation

$$u^4 - 16u^3 + 22u^2 - 24u + 1 = 0.$$

This equation has real roots and the largest root is $u_{max} \approx 14.6059427255653$. The asymptotic boundary is $14.6059^n \le w_n \le 14.606^n$

7 Generating functions for the number of horizontally convex tilings of type $(3^3, 4^2)$

Denote by F the set of all possible horizontally convex tilings of type $(3^3,4^2)$ and by F_n the subset of F, such that the number of elements (triangles and squares) is n. Also denote $f_n = |F_n|$.

In order to simplify the announce of the theorem let us introduce the following polynomials of two variables

$$\begin{aligned} S(u,v) = {} & 2u + v - 4u^2 - 9uv - 3v^2 + 17u^2v + 17uv^2 + 3v^3 + 4u^4 - 3u^3v - 26u^2v^2 - 15uv^3 - v^4 \\ & - 2u^5 - 17u^4v - 4u^3v^2 + 17u^2v^3 + 5uv^4 + 9u^5v + 35u^4v^2 + 15u^3v^3 - 6u^2v^4 - 5u^5v^2 - 29u^4v^3 \\ & - 4u^3v^4 - 9u^6v^2 - 11u^5v^3 + 9u^4v^4 + u^7v^2 + 10u^6v^3 + 3u^5v^4 + 4u^7v^3 - 3u^6v^4 - u^7v^4, \end{aligned}$$

$$\begin{aligned} R(u,v) = {} & 1 - 3u - 4v + 2u^2 + 12uv + 6v^2 + 2u^3 - 10u^2v - 18uv^2 - 4v^3 - 3u^4 - 6u^3v \\ & + 12u^2v^2 + 12uv^3 + v^4 + u^5 + 12u^4v + 16u^3v^2 - 6u^2v^3 - 3uv^4 - 4u^5v - 16u^4v^2 - 14u^3v^3 \\ & + 2u^2v^4 - 4u^5v^2 + 8u^4v^3 + 2u^3v^4 + 4u^6v^2 + 8u^5v^3 - u^4v^4 - u^6v^3 - u^5v^4 - u^7v^3. \end{aligned}$$

Theorem 3. *Generating functions for the number of horizontally convex tiles of type* $(3^3,4^2)$ *are*

$$F(x) = \sum_n f_n\, x^n = \frac{x(1-x)(3-13x+24x^2-17x^3-18x^4+35x^5-3x^6-14x^7+x^9)}{1-7x+20x^2-30x^3+16x^4+20x^5-32x^6+6x^7+11x^8-2x^9-x^{10}},$$

and

$$F(u,v,1) = \frac{S(u,v)}{R(u,v)},$$

where $S(u,v)$ and $R(u,v)$ are defined just above, the variable u counts the number of triangles in given tiling, v – the number of squares.

Following the methodology we developed in previous sections let us define six subsets of F depending on the shape of the top layer, refer to Figure 21.

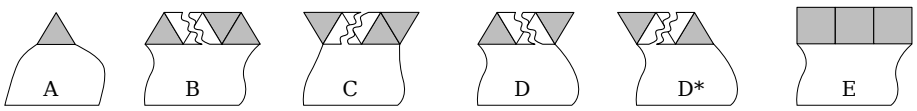


Figure 21: The subsets of F.

Let us define the set H as a union of B, C, D and D^*. Therefore the set F is a union of A, H and E.

Proof of Theorem 3. We shall be using some helpful functions of three variables (u,v,y), where u counts the number of triangles, v – number of squares, y – the number of free slots at the top layer, that is the length of the top side of top layer assuming the side length of an element equals to 1.

These functions $a(p,q,0)$, $b(p,q,m)$, $c(p,q,m)$, $d(p,q,m)$, $d^*(p,q,m)$ and $e(p,q,m)$ counts the number of elements in their corresponding sets. Remark that due to symmetry $d(p,q,m) = d^*(p,q,m)$. Let us look into more details:

$$\begin{aligned}
A(u,v,y) &= \sum_{p,q} a(p,q,0)u^p v^q\,,\\
B(u,v,y) &= \sum_{p,q,m} b(p,q,m)u^p v^q y^m\,,\\
C(u,v,y) &= \sum_{p,q,m} c(p,q,m)u^p v^q y^m\,,\\
D(u,v,y) &= D^*(u,v,y) = \sum_{p,q,m} d(p,q,m)u^p v^q y^m\,,\\
E(u,v,y) &= \sum_{p,q,m} e(p,q,m)u^p v^q y^m\,.
\end{aligned}$$

Obviously, $A(u,v,y)$ does not depend on y (the top side length is zero). Therefore

$$A(u,v,y) = A(u,v,1) \quad \text{and} \quad \frac{\partial}{\partial y}A(u,v,y) = 0.$$

Lemma 9. *For all $p \geq 1$, $q \geq 0$, $m \geq 0$ holds*

$$\begin{aligned}
&\text{a)} \quad d(p+1,q,m+1) = a(p,q,0) + b(p,q,m)\\
&\text{b)} \quad c(p+2,q,m+2) = a(p,q,0) + b(p,q,m)
\end{aligned}$$

Proof. a) If we add a triangle of the type $\bigtriangledown$ to the right of the top layer of a tiling of type A or B the we obtain a tiling of type D. The top side length will increase by 1. This operation can be inverted. Therefore, for all $p \geq 1$, $q \geq 0$, $m \geq 0$ holds $d(p+1,q,m+1) = a(p,q,0) + b(p,q,m)$.

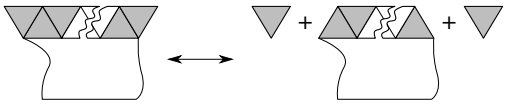


Figure 22: See Lemma 9.

b) If we add two triangles of type $\bigtriangledown$ one to the right and another to the left to the top layer of a tiling of type A or B, then the obtained tiling will be from the set C, Figure 22. The top side length will be increased by 2. This operation can be inverted. Therefore, for all $p \geq 1$, $q \geq 0$, $m \geq 0$ holds $c(p+2, q, m+2) = a(p, q, 0) + b(p, q, m)$. □

From Lemma 9a) follows

$$D(u, v, y) = uy(A(u, v, y) + B(u, v, y)).$$

Taking into account the fact that the triangle $\bigtriangledown$ corresponds to the monomial uy, from Lemma 9b) follows

$$C(u, v, y) = uy + u^2y^2(A(u, v, 1) + B(u, v, y)) = uy + uyD(u, v, y).$$

Since $H(u, v, y) = B(u, v, y) + C(u, v, y) + D(u, v, y) + D^*(u, v, y)$, then

$$H(u, v, y) = uy + B(u, v, y) + (2 + uy)D(u, v, y)\,, \tag{18}$$
$$F(u, v, y) = A(u, v, y) + H(u, v, y) + E(u, v, y)\,. \tag{19}$$

Recall that we do not need to construct the complete expression for $F(u, v, y)$ as a function of three variables, it is sufficient to find this expression for $y = 1$, i.e. $F(u, v, 1)$.

Let us introduce the following notation for the value of partial derivative of $E(u, v, y)$ for $y = 1$

$$\left.\frac{\partial E(u, v, y)}{\partial y}\right|_{y=1} = \epsilon(u, v).$$

In order to construct a tiling that belongs to the set A we take the triangle $\triangle$ and place it to the top of some initial tiling. This is only possible for the initial tiling from the set E. Therefore, for all p, q holds

$$a(p + 1, q, 0) = \sum_m m\, e(p, q, m).$$

For a single triangle $\triangle$ we have $a(1, 0, 0) = 1$. Finally,

$$A(u, v, 1) = A(u, v, y) = u + u\,\left(\frac{\partial E(u, v, y)}{\partial y}\right)\Bigg|_{y=1} = u + u\,\epsilon(u, v). \qquad (\dagger)$$

Recalling (15), obtain

$$\begin{aligned} u + u\,\epsilon(u, v) &= u + \sum_{p,q}\left(\sum_m m\, e(p, q, m)\right) u^{p+1}v^q \\ &= u + \sum_{p,q} a(p + 1, q, 0)u^{p+1}v^q = A(u, v, y)\,. \end{aligned}$$

Let us construct a tiling from the set B. Let us take a single row tiling from B with top side length k (and bottom side length $k+1$). Let us place it on top of some initial tiling from F in order to create a new tiling of F. This is possible only if the initial tiling belongs to the set E. If this initial tiling has the top side length m, then we are able to produce $m+k$ new tilings. This way, we are able to produce any tiling from B but a single-row tiling. Therefore, for all p, q holds

$$b(p+2k+1,q,k) = \sum_m (m+k)e(p,q,m).$$

And for a single-row tiling of B holds $b(2k+1,0,k)=1$. Using (14), obtain

$$\begin{aligned}
B(u,v,y) &= \sum_{p,q,m} b(p,q,m)u^p v^q y^m \\
&= \sum_k b(2k+1,0,k)u^{2k+1}y^k + \sum_{p,q,k} b(p+2k+1,q,k)u^{p+2k+1}v^q y^k \\
&= \sum_{k=1}^{\infty} u^{2k+1}y^k + \sum_{k=1}^{\infty} u^{2k+1}y^k \left(\sum_{p,q} \left(\sum_m (m+k)e(p,q,m) \right) u^p v^q \right) \\
&= \sum_{k=1}^{\infty} u^{2k+1}y^k + \sum_{k=1}^{\infty} u^{2k+1}y^k \left(kE(u,v,1) + \epsilon(u,v) \right).
\end{aligned}$$

Therefore,

$$B(u,v,y) = (1+\epsilon(u,v)) \sum_{k=1}^{\infty} u^{2k+1}y^k + E(u,v,1) \sum_{k=1}^{\infty} k u^{2k+1} y^k. \qquad (\ddagger)$$

Let us construct a tiling from the set D. Let us take a single-row tiling from D with the top side length k and place it on top of some initial tiling in order to construct a new tiling from F. This is only possible for the initial tiling from the set E, and if the initial tiling has the top side length m than we are able to produce $m+k-1$ new tilings. For all p, q holds

$$d(p+2k+1,q,k) = \sum_m (m+k-1)e(p,q,m).$$

For a single-row tiling from D holds $d(2k,0,k)=1$. Using (14), obtain

$$\begin{aligned}
D(u,v,y) &= \sum_{p,q,m} d(p,q,m)u^p v^q y^m \\
&= \sum_k d(2k,0,k)u^{2k}y^k + \sum_{p,q,k} d(p+2k,q,k)u^{p+2k}v^q y^k \\
&= \sum_{k=1}^{\infty} u^{2k}y^k + \sum_{k=1}^{\infty} u^{2k}y^k \left(\sum_{p,q} \left(\sum_m (m+k-1)e(p,q,m) \right) u^p v^q \right) \\
&= \sum_{k=1}^{\infty} u^{2k}y^k + \sum_{k=1}^{\infty} u^{2k}y^k \left((k-1)E(u,v,1) + \epsilon(u,v) \right).
\end{aligned}$$

Therefore,

$$D(u,v,y) = (1+\epsilon(u,v))\sum_{k=1}^{\infty} u^{2k}y^k + E(u,v,1)\sum_{k=1}^{\infty}(k-1)u^{2k}y^k. \qquad (\dagger\dagger)$$

Let us construct a tiling from the set E. Let us take a single-row tiling from E with the top side length k and place it on top of some initial tiling in order to construct a new tiling from F. This is only possible for the initial tiling from the set H, and if the initial tiling has the top side length m than we are able to produce $m+k-1$ new tilings. For all p, q holds

$$e(p,q+k,k) = \sum_m (m+k-1)h(p,q,m).$$

For a single-row tiling of type E holds $e(0,k,k)=1$. Using (14) obtain

$$E(u,v,y) = (1+\chi(u,v))\sum_{k=1}^{\infty} v^k y^k + H(u,v,1)\sum_{k=1}^{\infty}(k-1)v^k y^k. \qquad (\ddagger\ddagger)$$

From (18) taking into account ($\ddagger$) and ($\dagger\dagger$) obtain

$$\begin{aligned} H(u,v,y) &= uy + B(u,v,y) + (2+uy)D(u,v,y) \\ &= uy + (1+\epsilon(u,v))\sum_{k=1}^{\infty} u^{2k+1}y^k + E(u,v,1)\sum_{k=1}^{\infty} ku^{2k+1}y^k \\ &\quad + (2+uy)\left((1+\epsilon(u,v))\sum_{k=1}^{\infty} u^{2k}y^k + E(u,v,1)\sum_{k=1}^{\infty}(k-1)u^{2k}y^k\right). \end{aligned}$$

Therefore

$$H(u,v,y) = uy + (1+\epsilon(u,v))(2+u+uy)\sum_{k=1}^{\infty} u^{2k}y^k + E(u,v,1)\sum_{k=1}^{\infty}\left(ku + 2(k-1) + (k-1)uy\right)u^{2k}y^k. \qquad (\dagger\dagger\dagger)$$

Using ($\ddagger\ddagger$), ($\dagger\dagger\dagger$) obtain the system of equations for $H(u,v,1)$, $\chi(u,v)$, $E(u,v,1)$ and $\epsilon(u,v)$. Two equations come from the substitution $y=1$. Two more are obtained from differentiation by y and substitution $y=1$:

$$\begin{cases} H(u,v,1) &= u + (1+\epsilon(u,v))(2+2u)\sum_{k\geq 1} u^{2k} \\ &\quad + E(u,v,1)\sum_{k\geq 1}\left(ku + (2+u)(k-1)\right)u^{2k}, \\ \chi(u,v) &= u + (1+\epsilon(u,v))\sum_{k\geq 1}(2k+uk+u(k+1))u^{2k} \\ &\quad + E(u,v,1)\sum_{k\geq 1}\left(k^2u + (k-1)(2k+u(k+1))\right)u^{2k}, \\ E(u,v,1) &= (1+\chi(u,v))\sum_{k\geq 1} v^k + H(u,v,1)\sum_{k\geq 1}(k-1)v^k, \\ \epsilon(u,v) &= (1+\chi(u,v))\sum_{k\geq 1} kv^k + H(u,v,1)\sum_{k\geq 1} k(k-1)v^k. \end{cases}$$

Using computations from exercises 4–7 rewrite the system as

$$\begin{cases} H(u,v,1) &= u &+ &(1+\epsilon(u,v))\dfrac{2u^2}{1-u} &+ &E(u,v,1)\dfrac{u^3}{(1-u)^2}, \\ \chi(u,v) &= u &+ &(1+\epsilon(u,v))\dfrac{u^2(2-u)}{(1-u)^2} &+ &E(u,v,1)\dfrac{u^3(1+2u-u^2)}{(1+u)(1-u)^3}, \\ E(u,v,1) &= & &(1+\chi(u,v))\dfrac{v}{1-v} &+ &H(u,v,1)\dfrac{v^2}{(1-v)^2}, \\ \epsilon(u,v) &= & &(1+\chi(u,v))\dfrac{v}{(1-v)^2} &+ &H(u,v,1)\dfrac{2v^2}{(1-v)^3}. \end{cases}$$

From last two equations obtain

$$\begin{aligned} \chi(u,v) &= \frac{E(u,v,1)(1-v)}{v} - \frac{H(u,v,1)v}{1-v} - 1 \\ \epsilon(u,v) &= \frac{E(u,v,1)}{1-v} + \frac{H(u,v,1)v^2}{(1-v)^3} \end{aligned} \qquad (\ddagger\ddagger\ddagger)$$

Substituting these into the first two equations obtain generating functions $E(u,v,1)$ and $H(u,v,1)$.

$$\begin{aligned} E(u,v,1) &= \frac{v(1+u)(1-u)^2 P_1(u,v)}{R(u,v)} \\ H(u,v,1) &= \frac{u(1+u)(1-v)^2 P_2(u,v)}{R(u,v)} \end{aligned}$$

where $R(u,v)$ was defined at the beginning of the section,

$$\begin{aligned} P_1(u,v) &= 1 - u - 3v + u^2 + 4uv + 3v^2 - 3u^2v - 5uv^2 - v^3 - u^3v + u^2v^2 + 2uv^3 + 4u^3v^2 - u^2v^3 \\ &\quad - u^3v^3 + u^4v^2\,, \\ P_2(u,v) &= 1 - 2u - 2v + v^2 + 6uv - 2uv^2 - 5u^2v + 2u^3 - u^2v^2 - u^3v - u^4 + 4u^3v^2 + u^4v - u^5v^2\,. \end{aligned}$$

From (†) and (‡‡‡) obtain

$$\begin{aligned} A(u,v,1) &= u + u\epsilon(u,v) \\ &= u + \frac{uE(u,v,1)}{1-v} + \frac{uH(u,v,1)v^2}{(1-v)^3} \\ &= \frac{u(1-u)P_3(u,v)}{R(u,v)}, \end{aligned}$$

where

$$\begin{aligned} P_3(u,v) &= 1 - 2u - 3v + 7uv + 4v^2 + 2u^3 - 2u^2v - 8uv^2 - 3v^3 - u^4 - 7u^3v + 5uv^3 + v^4 + 3u^4v \\ &\quad + 10u^3v^2 + 2u^2v^3 - 2uv^4 + 2u^4v^2 - 6u^3v^3 - 2u^5v^2 - 5u^4v^3 + 2u^3v^4 - u^5v^3 + u^4v^4. \end{aligned}$$

Finally, from (19) obtain $F(u,v,1)$.

Substituting $u = v = x$, everything becomes very simple. Indeed,

$$R(x,x) = 1 - 7x + 20x^2 - 30x^3 + 16x^4 + 20x^5 - 32x^6 + 6x^7 + 11x^8 - 2x^9 - x^{10}\,,$$
$$P_1(x,x) = 1 - 4x + 8x^2 - 9x^3 + 2x^4 + 3x^5\,,$$
$$P_2(x,x) = (1-x)(1-x-x^2)(1-2x+3x^2-x^4)\,,$$
$$P_3(x,x) = (1-x-x^2)(1-4x+8x^2-7x^3-x^4+5x^5)\,.$$

Then

$$E(x) = \frac{x(1+x)(1-x)^2(1-4x+8x^2-9x^3+2x^4+3x^5)}{1-7x+20x^2-30x^3+16x^4+20x^5-32x^6+6x^7+11x^8-2x^9-x^{10}},$$
$$H(x) = \frac{x(1-x)^3(1+x)(1-x-x^2)(1-2x+3x^2-x^4)}{1-7x+20x^2-30x^3+16x^4+20x^5-32x^6+6x^7+11x^8-2x^9-x^{10}},$$
$$A(x) = \frac{x(1-x)(1-x-x^2)(1-4x+8x^2-7x^3-x^4+5x^5)}{1-7x+20x^2-30x^3+16x^4+20x^5-32x^6+6x^7+11x^8-2x^9-x^{10}},$$
$$F(x) = \frac{x(1-x)(3-13x+24x^2-17x^3-18x^4+35x^5-3x^6-14x^7+x^9)}{1-7x+20x^2-30x^3+16x^4+20x^5-32x^6+6x^7+11x^8-2x^9-x^{10}}.$$

This completes the proof of the Theorem. □

We use online mathematical software to obtain the power series for above generating functions.

$$\begin{aligned}F(u,v,1) = {} & 2u + v + 2u^2 + 2uv + v^2 + 2u^3 + 5u^2v + 4uv^2 + v^3 + 2u^4 + 10u^3v + 14u^2v^2 + 6uv^3 + v^4 \\ & + 2u^5 + 18u^4v + 34u^3v^2 + 29u^2v^3 + 8uv^4 + v^5 + 2u^6 + 30u^5v + 74u^4v^2 + 88u^3v^3 + 52u^2v^4 \\ & + 10uv^5 + v^6 + 2u^7 + 47u^6v + 146u^5v^2 + 228u^4v^3 + 194u^3v^4 + 85u^2v^5 + 12uv^6 + v^7 + \dots\end{aligned}$$

Also

$$E(x) = x + 2x^2 + 5x^3 + 13x^4 + 34x^5 + 94x^6 + 266x^7 + 751x^8 + 2093x^9 + 5793x^{10} + \dots\,,$$
$$H(x) = x + 2x^2 + 4x^3 + 11x^4 + 32x^5 + 92x^6 + 255x^7 + 698x^8 + 1925x^9 + 5362x^{10} + \dots\,,$$
$$A(x) = x + x^2 + 3x^3 + 9x^4 + 26x^5 + 71x^6 + 194x^7 + 539x^8 + 1511x^9 + 4222x^{10} + \dots\,,$$
$$F(x) = 3x + 5x^2 + 12x^3 + 33x^4 + 92x^5 + 257x^6 + 715x^7 + 1988x^8 + 5529x^9 + 15377x^{10} + \dots$$

Corollary 1. *The sequence f_n satisfies the recurrent relation of 10th order:*

$$f_{n+10} = 7f_{n+9} - 20f_{n+8} + 30f_{n+7} - 16f_{n+6} - 20f_{n+5} + 32f_{n+4} - 6f_{n+3} - 11f_{n+2} + 2f_{n+1} + f_n. \tag{20}$$

Proof. For $F(x)$ holds

$$\begin{aligned}(1 - 7x + 20x^2 - 30x^3 + 16x^4 + 20x^5 - 32x^6 + 6x^7 + 11x^8 - 2x^9 - x^{10}) & \sum_n f_n\, x^n \\ & = x(1-x)(3 - 13x + 24x^2 - 17x^3 - 18x^4 + 35x^5 - 3x^6 - 14x^7 + x^9)\,.\end{aligned}$$

These are equal polynomials, therefore the recurrent relation (20) holds. □

The corresponding characteristic equation is

$$x^{10} - 7x^9 + 20x^8 - 30x^7 + 16x^6 + 20x^5 - 32x^4 + 6x^3 + 11x^2 - 2x - 1 = 0.$$

This equation has real roots, and the largest root is $x_{max} \approx 2.77906203737$. Therefore $2.779^n \le f_n \le 2.7791^n$.

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