



HAL
open science

HOLOMORPHIC $GL_2(\mathbb{C})$ -GEOMETRY ON COMPACT COMPLEX MANIFOLDS

Indranil Biswas, Sorin Dumitrescu

► **To cite this version:**

Indranil Biswas, Sorin Dumitrescu. HOLOMORPHIC $GL_2(\mathbb{C})$ -GEOMETRY ON COMPACT COMPLEX MANIFOLDS. *Manuscripta mathematica*, 2021, 166 (1-2), pp.251-269. hal-02533912

HAL Id: hal-02533912

<https://hal.science/hal-02533912>

Submitted on 6 Apr 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

HOLOMORPHIC $\mathrm{GL}_2(\mathbb{C})$ -GEOMETRY ON COMPACT COMPLEX MANIFOLDS

INDRANIL BISWAS AND SORIN DUMITRESCU

ABSTRACT. We study holomorphic $\mathrm{GL}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{C})$ geometries on compact complex manifolds.

1. INTRODUCTION

A holomorphic $\mathrm{GL}_2(\mathbb{C})$ geometric structure on a complex manifold X of complex dimension n is a holomorphic point-wise identification between the holomorphic tangent space TX and homogeneous polynomials in two variables of degree $(n - 1)$. More precisely, a $\mathrm{GL}_2(\mathbb{C})$ geometric structure on X is a pair (E, φ) , where E is a rank two holomorphic vector bundle on X and φ is a holomorphic vector bundle isomorphism of TX with the $(n - 1)$ -fold symmetric product $S^{n-1}(E)$ (see Definition 2.1). If E has trivial determinant (i.e. the holomorphic line bundle $\bigwedge^2 E$ is trivial), then (E, φ) is called a holomorphic $\mathrm{SL}_2(\mathbb{C})$ geometric structure.

The above definitions are the holomorphic analogues of the concepts of $\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{R})$ geometries in the real smooth category (for the study of those geometries in the real smooth category we refer to [DG, FK, Kr] and references therein).

This article deals with the classification of compact complex manifolds admitting holomorphic $\mathrm{GL}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{C})$ geometries.

Holomorphic $\mathrm{GL}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{C})$ geometries on X are examples of holomorphic G -structures (see [Kob2] and Section 2). They correspond to the reduction of the structural group of the $\mathrm{GL}_n(\mathbb{C})$ -frame bundle of X to $\mathrm{GL}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{C})$ respectively.

When the dimension n of X is odd, then a $\mathrm{GL}_2(\mathbb{C})$ geometry produces a *holomorphic conformal structure* on X (see, for example, [DG], Proposition 3.2 or Section 2 here). Recall that a holomorphic conformal structure is defined by a holomorphic line bundle L over X and a holomorphic section of $S^2(T^*X) \otimes L$, which is a L -valued fiberwise nondegenerate holomorphic quadratic form on X . Moreover, if $n = 3$, then a $\mathrm{GL}_2(\mathbb{C})$ -geometry on X is exactly a holomorphic conformal structure. The standard flat example is the smooth quadric Q_3 in $\mathbb{C}\mathbb{P}^4$ defined by the equation $Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 = 0$.

A $\mathrm{SL}_2(\mathbb{C})$ -geometry on X defines a *holomorphic Riemannian metric* (i.e. a holomorphic section of $S^2(T^*X)$ which is point-wise non degenerate [Du]) when n is odd. When $n = 3$, a $\mathrm{SL}_2(\mathbb{C})$ -geometry is the same data as a holomorphic Riemannian metric.

1991 *Mathematics Subject Classification.* 53C07, 53C10, 32Q57.

Key words and phrases. $\mathrm{GL}(2)$ -structure, holomorphic conformal structure, twisted holomorphic symplectic form, Kähler manifold, Fujiki class \mathcal{C} manifold.

When the dimension n of X is even, a holomorphic $\mathrm{SL}_2(\mathbb{C})$ -geometry on X produces a nondegenerate holomorphic two form ω on X . Moreover, if X is a compact Kähler manifold, then ω is automatically closed, and hence ω is a *holomorphic symplectic form*, and hence X is a hyper-Kähler manifold (see [Be]).

When n is even, a $\mathrm{GL}_2(\mathbb{C})$ -geometry on X produces a twisted nondegenerate two form ω on X , which means that there is a holomorphic line bundle L over X such that ω is a fiberwise nondegenerate holomorphic global section of $\Omega_X^2 \otimes L$. We use the terminology of [Is1, Is2] and call ω a *twisted holomorphic symplectic form*.

Let us describe the results of this article. In Section 3 we prove Theorem 3.1 asserting that *a compact Kähler manifold of even complex dimension $n \geq 4$ admitting a holomorphic $\mathrm{GL}_2(\mathbb{C})$ -geometry is covered by a compact torus*. A key ingredient of the proof is a result of Istrati [Is1, Is2] proving that Kähler manifolds bearing a twisted holomorphic symplectic form have vanishing first Chern class. By Yau's proof of Calabi's conjecture [Ya] these manifolds admit a Ricci flat metric and the canonical line bundle K_X is trivial (up to a finite cover) [Bo1, Be]. Hence the $\mathrm{GL}_2(\mathbb{C})$ -geometry is induced by an underlying $\mathrm{SL}_2(\mathbb{C})$ -geometry. The proof of Theorem 3.1 involves showing that *any compact Kähler manifold of complex dimension $n \geq 3$ bearing a $\mathrm{SL}_2(\mathbb{C})$ -geometry is covered by a compact complex torus*. Recall that compact Kähler manifolds of odd complex dimension bearing a $\mathrm{SL}_2(\mathbb{C})$ -geometry also admit a holomorphic Riemannian metric and hence has the associated holomorphic (Levi-Civita) affine connections; such manifolds are known to have vanishing Chern classes, [At2], and hence they admit covering by some compact complex torus [IKO].

Theorem 3.4 deals with manifolds X in Fujiki class \mathcal{C} (i.e. holomorphic images of compact Kähler manifolds [Fu]) bearing a $\mathrm{GL}_2(\mathbb{C})$ -geometry. Under the technical assumption that there exists a cohomology class $[\alpha] \in H^{1,1}(X, \mathbb{R})$ which is numerically effective (nef) and has positive self-intersection (meaning $\int_X \alpha^{2m} > 0$, where $2m = \dim_{\mathbb{C}} X$), we prove that *there exists a non-empty Zariski open subset $\Omega \subset X$ admitting a flat Kähler metric*. We recall that simply connected non-Kähler manifolds in Fujiki class \mathcal{C} admitting a holomorphic symplectic form were constructed in [Bo2, Gu1, Gu2].

In Section 4 we obtain the classification of compact Kähler-Einstein manifolds (Theorem 4.1) and of Fano manifolds (Theorem 4.2) bearing a $\mathrm{GL}_2(\mathbb{C})$ -geometry. Theorem 4.1 states that *any compact Kähler-Einstein manifold, of complex dimension at least three, bearing a $\mathrm{GL}_2(\mathbb{C})$ -geometry is*

- *either covered by a torus,*
- *or biholomorphic to the three dimensional quadric Q_3 ,*
- *or covered by the three dimensional Lie ball D_3 (the noncompact dual of Q_3 , as Hermitian symmetric space).*

In all these three situations the $\mathrm{GL}_2(\mathbb{C})$ -geometry is the (flat) standard one.

Theorem 4.2 asserts that *a Fano manifold bearing a holomorphic $\mathrm{GL}_2(\mathbb{C})$ -geometry is isomorphic to the quadric Q_3 endowed with its standard $\mathrm{GL}_2(\mathbb{C})$ -structure.*

Theorem 4.1 and Theorem 4.2 belong to the same circle of ideas as the following known results. Kobayashi and Ochiai proved in [KO3] that compact Kähler-Einstein manifolds

bearing a holomorphic conformal structures are the standard ones: quotients of tori, the smooth n -dimensional quadric Q_n and the quotients of the non compact dual D_n of Q_n . Moreover the same authors proved in [KO2] that holomorphic G -structures, modeled on an irreducible Hermitian symmetric space of rank ≥ 2 (in particular, a holomorphic conformal structure), on compact Kähler-Einstein manifolds are always flat. Let us also mention the main result in [HM] which says that holomorphic irreducible reductive G -structures on uniruled projective manifolds are always flat. Consequently, a uniruled projective manifold bearing a holomorphic conformal structure is biholomorphic to the quadric Q_n (see also [Ye]). The following generalization was proved in [BM]: all holomorphic Cartan geometries (see [Sh]) on manifolds admitting a rational curve are flat.

Let us clarify that the classification obtained in Theorem 4.1 and Theorem 4.2 does not use results coming from [BM, HM, KO2, KO3, Ye]: the methods are specific to the case of $GL_2(\mathbb{C})$ -geometry and unify the twisted holomorphic symplectic case (even dimension) and the holomorphic conformal case (odd dimension).

The last section discuss some related open problems. Let us emphasis one of those questions dealing with $SL_2(\mathbb{C})$ -geometries on non-Kähler manifolds.

Recall that Ghys constructed in [Gh] exotic deformations of quotients of $SL_2(\mathbb{C})$ by normal lattices. Those Ghys manifolds are non-Kähler and non-parallelizable, but they admit a non-flat holomorphic $SL_2(\mathbb{C})$ -geometric structure (i.e. a holomorphic Riemannian metric). Nevertheless, all Ghys holomorphic Riemannian manifolds are locally homogeneous. Moreover, it was proved in [Du] that holomorphic Riemannian metrics on compact complex threefolds are always locally homogeneous.

For higher complex odd dimensions we conjecture that $SL_2(\mathbb{C})$ -geometries on compact complex manifolds are always locally homogeneous. Some evidence for it was recently provided by the main result in [BD] showing that compact complex simply connected manifolds do not admit any holomorphic Riemannian metric (in particular, in odd dimension, they do not admit a holomorphic $SL_2(\mathbb{C})$ -geometry).

2. HOLOMORPHIC $GL_2(\mathbb{C})$ AND $SL_2(\mathbb{C})$ GEOMETRIES

In this section we introduce the framework of $GL_2(\mathbb{C})$ and $SL_2(\mathbb{C})$ geometries on complex manifolds and describe the geometry of the quadric (model of the flat holomorphic conformal geometry) and that of its noncompact dual. We focus on the complex dimension three, the only case admitting a $GL_2(\mathbb{C})$ -geometry.

The holomorphic tangent bundle of a complex manifold Z will be denoted by TZ . The frame bundle for TZ will be denoted by $R(Z)$. We recall that $R(Z)$ consists of all linear isomorphisms from \mathbb{C}^d to the fibers of TZ , where $d = \dim_{\mathbb{C}} Z$. The $R(Z)$ is a holomorphic principal $GL(d, \mathbb{C})$ -bundle over Z .

Definition 2.1. A holomorphic $GL_2(\mathbb{C})$ -structure (or $GL_2(\mathbb{C})$ -geometry) on a complex manifold X of complex dimension $n \geq 2$ is a holomorphic bundle isomorphism $TX \simeq S^{n-1}(E)$, where $S^{n-1}(E)$ is the $(n - 1)$ -th symmetric power of a rank two holomorphic vector bundle

E over X . If E has trivial determinant, meaning the line bundle $\bigwedge^2 E$ is holomorphically trivial, it is called a holomorphic $\mathrm{SL}_2(\mathbb{C})$ -structure (or $\mathrm{SL}_2(\mathbb{C})$ -geometry) on X .

Holomorphic $\mathrm{GL}_2(\mathbb{C})$ -structures and $\mathrm{SL}_2(\mathbb{C})$ -structures are particular cases of holomorphic irreducible reductive G -structures [Kob2, HM]. They correspond to the holomorphic reduction of the structure group of the frame bundle $R(X)$ of X from $\mathrm{GL}_n(\mathbb{C})$ to $\mathrm{GL}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{C})$ respectively. It should be clarified that for a $\mathrm{GL}_2(\mathbb{C})$ -structure, the corresponding group homomorphism $\mathrm{GL}_2(\mathbb{C}) \rightarrow \mathrm{GL}_n(\mathbb{C})$ is given by the $(n-1)$ -th symmetric product of the standard representation of $\mathrm{GL}_2(\mathbb{C})$. This n -dimensional irreducible linear representation of $\mathrm{GL}_2(\mathbb{C})$ is also given by the induced action on the homogeneous polynomials of degree $(n-1)$ in two variables. For an $\mathrm{SL}_2(\mathbb{C})$ -geometry, the corresponding homomorphism $\mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{GL}_n(\mathbb{C})$ is the restriction of the above homomorphism to $\mathrm{SL}_2(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{C})$.

The standard symplectic form on \mathbb{C}^2 produces a nondegenerate quadratic form (respectively, nondegenerate alternating form) on the symmetric product $S^{2i}(\mathbb{C}^2)$ (respectively, $S^{2i-1}(\mathbb{C}^2)$) for all $i \geq 1$. Consequently, the above linear representation $\mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{GL}_n(\mathbb{C})$ preserves a nondegenerate complex quadratic form on \mathbb{C}^n if n is odd and a nondegenerate two (alternating) form on \mathbb{C}^n if n is even (see, for example, Proposition 3.2 and Sections 2 and 3 in [DG] or [Kr]). The above linear representation $\mathrm{GL}_2(\mathbb{C}) \rightarrow \mathrm{GL}_n(\mathbb{C})$ preserves the line in $(\mathbb{C}^n)^* \otimes (\mathbb{C}^n)^*$ spanned by the above tensor; the action of $\mathrm{GL}_2(\mathbb{C})$ on this line is nontrivial. Consequently, the $\mathrm{GL}_2(\mathbb{C})$ -geometry (respectively, $\mathrm{SL}_2(\mathbb{C})$ -geometry) on X induces a *holomorphic conformal structure* (respectively, *holomorphic Riemannian metric*) if the complex dimension of X is odd, and it induces a *twisted holomorphic symplectic structure* (respectively, a *holomorphic nondegenerate two form*) when the complex dimension of X is even.

A holomorphic $\mathrm{SL}_2(\mathbb{C})$ -structure on a complex surface X is a holomorphic trivialization of the canonical bundle $K_X = \bigwedge^2(TX)^*$.

The simplest nontrivial examples of $\mathrm{GL}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{C})$ structures are provided by the complex threefolds. In this case a $\mathrm{GL}_2(\mathbb{C})$ -structure is a holomorphic conformal structure on X , and an $\mathrm{SL}_2(\mathbb{C})$ -structure is a holomorphic Riemannian metric on X . Indeed, this is deduced from the fact that the $\mathrm{SL}_2(\mathbb{C})$ -representation on the three-dimensional vector space of homogeneous quadratic polynomials in two variables

$$\{aX^2 + bXY + cY^2 \mid a, b, c \in \mathbb{C}\} \quad (2.1)$$

preserves the discriminant $\Delta = b^2 - 4ac$. Consequently, a holomorphic isomorphism between $\mathrm{PSL}_2(\mathbb{C})$ and the complex orthogonal group $\mathrm{SO}(3, \mathbb{C})$ is obtained. In other words, the discriminant, being nondegenerate, induces a holomorphic Riemannian metric on the threefold X . Notice that a holomorphic Riemannian metric coincides with a reduction of the structural group of the frame bundle $R(X)$ to the orthogonal group $\mathrm{O}(3, \mathbb{C})$. On a double unramified cover of a holomorphic Riemannian threefold there is a reduction of the structural group of the frame bundle to the connected component of the identity $\mathrm{SO}(3, \mathbb{C})$. Hence in complex dimension three a $\mathrm{SL}_2(\mathbb{C})$ -geometry is the same data as a holomorphic Riemannian metric, while a reduction of the frame bundle to $\mathrm{PSL}_2(\mathbb{C})$ is the same data as a holomorphic Riemannian metric with an orientation.

Moreover, the $\mathrm{GL}_2(\mathbb{C})$ -representation on the vector space in (2.1) preserves the line generated by the discriminant $\Delta = b^2 - 4ac$. This gives an isomorphism between $\mathrm{GL}_2(\mathbb{C})/(\mathbb{Z}/2\mathbb{Z})$ and the conformal group $\mathrm{CO}(3, \mathbb{C}) = (\mathrm{O}(3, \mathbb{C}) \times \mathbb{C}^*)/(\mathbb{Z}/2\mathbb{Z}) = \mathrm{SO}(3, \mathbb{C}) \times \mathbb{C}^*$. Consequently, the $\mathrm{GL}_2(\mathbb{C})$ -structure coincides with a holomorphic reduction of the structure group of the frame bundle $R(X)$ to $\mathrm{CO}(3, \mathbb{C})$. This holomorphic reduction of the structure group defines a holomorphic conformal structure on X . Notice that $\mathrm{CO}(3, \mathbb{C})$ being connected, the two different orientations of a three dimensional holomorphic Riemannian manifold are conformally equivalent.

Recall that flat conformal structures in complex dimension $n \geq 3$ are locally modeled on the quadric

$$Q_n := \{[Z_0 : Z_1 : \cdots : Z_{n+1}] \mid Z_0^2 + Z_1^2 + \cdots + Z_{n+1}^2 = 0\} \subset \mathbb{C}\mathbb{P}^{n+1}.$$

The holomorphic automorphism group of Q_n is $\mathrm{PSO}(n+2, \mathbb{C})$.

Let us mention that Q_n is identified with the real Grassmannian of oriented 2-planes in \mathbb{R}^{n+2} . Consequently, $Q_n = \mathrm{SO}(n+2, \mathbb{R})/(\mathrm{SO}(2, \mathbb{R}) \times \mathrm{SO}(n, \mathbb{R}))$ (see, for instance, Section 1 in [JR2]). The action of $\mathrm{SO}(n+2, \mathbb{R})$ on Q_n is via holomorphic automorphisms. To see this, first note that the real tangent space at a point of $Q_n = \mathrm{SO}(n+2, \mathbb{R})/(\mathrm{SO}(2, \mathbb{R}) \times \mathrm{SO}(n, \mathbb{R}))$ is identified with the corresponding quotient of the real Lie algebras $\mathfrak{so}(n+2, \mathbb{R})/(\mathfrak{so}(2, \mathbb{R}) \oplus \mathfrak{so}(n, \mathbb{R}))$, and the stabilizer $\mathrm{SO}(2, \mathbb{R}) \times \mathrm{SO}(n, \mathbb{R})$ acts on this quotient vector space through the adjoint representation. The complex structure of the tangent space (induced by the complex structure of Q_3) is given by the operator J which satisfies the condition that $\{\exp(tJ)\}_{t \in \mathbb{R}}$ is the adjoint action of the factor $\mathrm{SO}(2, \mathbb{R})$. Since $\mathrm{SO}(2, \mathbb{R})$ lies in the center of the stabilizer, the almost complex structure J is preserved by the action of the stabilizer. Consequently, the complex structure is $\mathrm{SO}(n+2, \mathbb{R})$ -invariant.

The quadric Q_n is a Fano manifold, and it is an irreducible Hermitian symmetric space of type **BD I** [Bes, p. 312]. For more about its geometry and that of its noncompact dual D_n (as a Hermitian symmetric space) the reader is referred to [JR2, Section 1].

Theorem 4.2 implies that among the quadrics Q_n , $n \geq 3$, only Q_3 admits a holomorphic $\mathrm{GL}_2(\mathbb{C})$ -structure. Moreover, Theorem 4.1 states that the only compact non-flat Kähler-Einstein manifolds bearing a holomorphic $\mathrm{GL}_2(\mathbb{C})$ -structure are Q_3 and those covered by its noncompact dual D_3 .

Recall that a general result of Borel on Hermitian irreducible symmetric spaces shows that the non compact dual is always realized as an open subset of its compact dual.

We will give below a geometric description of the noncompact dual D_3 of Q_3 as an open subset in Q_3 which seems to be less known.

Consider the complex quadric form $q_{3,2} := Z_0^2 + Z_1^2 + Z_2^2 - Z_3^2 - Z_4^2$ of five variables, and let

$$Q \subset \mathbb{C}\mathbb{P}^4$$

be the quadric Q defined by the equation $q_{3,2} = 0$. Then Q is biholomorphic to Q_3 . Let $\mathrm{O}(3, 2) \subset \mathrm{GL}(5, \mathbb{R})$ be the real orthogonal group of $q_{3,2}$, and denote by $\mathrm{SO}_0(3, 2)$ the connected component of $\mathrm{O}(3, 2)$ containing the identity element. The quadric Q admits a natural holomorphic action of the real Lie group $\mathrm{SO}_0(3, 2)$, which is not transitive, in contrast

to the action of $\mathrm{SO}(5, \mathbb{R})$ on Q_3 . The orbits of the $\mathrm{SO}_0(3, 2)$ -action on Q_3 coincide with the connected components of the complement $Q_3 \setminus S$, where S is the real hypersurface of Q_3 defined by the equation

$$|Z_0|^2 + |Z_1|^2 + |Z_2|^2 - |Z_3|^3 - |Z_4|^2 = 0.$$

Notice that the above real hypersurface S contains all real points of Q . In fact, it can be shown that $S \cap Q$ coincides with the set of point $m \in Q$ such that the complex line (m, \bar{m}) is isotropic (i.e. it also lies in Q). Indeed, since $q_{3,2}(m) = 0$, then the line generated by (m, \bar{m}) lies in Q if and only if m and \bar{m} are perpendicular with respect to the bilinear symmetric form associated to $q_{3,2}$, or equivalently $m \in S$.

For any point $m \in Q \setminus S$, the form $q_{3,2}$ is nondegenerate on the line (m, \bar{m}) . To see this first notice that the complex line generated by (m, \bar{m}) , being real, may be considered as a plane in the real projective space $\mathbb{R}P^4$. The restriction of the (real) quadratic form $q_{3,2}$ to this real plane (m, \bar{m}) vanishes at the points m and \bar{m} which are distinct (because all real points of Q lie in S). It follows that the quadratic form cannot have signature $(0, 1)$ or $(1, 0)$ when restricted to the real plane (m, \bar{m}) . Consequently, the signature of the restriction of $q_{3,2}$ to this plane is either $(2, 0)$ or $(1, 1)$ or $(0, 2)$. Each of these three signature types corresponds to an $\mathrm{SO}_0(3, 2)$ orbit in Q .

Take the point $m_0 = [0 : 0 : 0 : 1 : i] \in Q$. The noncompact dual D_3 of Q_3 is the $\mathrm{SO}_0(3, 2)$ -orbit of m_0 in Q . It is an open subset of Q biholomorphic to a bounded domain in \mathbb{C}^3 ; it is the three dimension Lie ball (the bounded domain IV_3 in Cartan's classification).

The signature of $q_{3,2}$ on the above line (m_0, \bar{m}_0) is $(0, 2)$. The signature of $q_{3,2}$ on the orthogonal part of (m_0, \bar{m}_0) , canonically isomorphic to $T_{m_0}Q$, is $(3, 0)$. Then the $\mathrm{SO}_0(3, 2)$ -orbit of m_0 in Q inherits an $\mathrm{SO}_0(3, 2)$ -invariant Riemannian metric. The stabilizer of m_0 is $\mathrm{SO}(2, \mathbb{R}) \times \mathrm{SO}(3, \mathbb{R})$. Here $\mathrm{SO}(2, \mathbb{R})$ acts on \mathbb{C}^3 through the one parameter group $\exp(tJ)$, with J being the complex structure, while the $\mathrm{SO}(3, \mathbb{R})$ action on \mathbb{C}^3 is given by the complexification of the canonical action of $\mathrm{SO}(3, \mathbb{R})$ on \mathbb{R}^3 . Consequently, the action of $\mathrm{U}(2) = \mathrm{SO}(2, \mathbb{R}) \times \mathrm{SO}(3, \mathbb{R})$ is the unique irreducible action on the symmetric product $S^2(\mathbb{C}^2) = \mathbb{C}^3$. This action coincides with the holonomy representation of this Hermitian symmetric space D_3 ; as mentioned before, D_3 is the noncompact dual of Q_3 . The holonomy representation for Q_3 is the same.

Recall that the automorphism group of the noncompact dual D_3 is $\mathrm{PSO}_0(3, 2)$; it is the subgroup of the automorphism group of Q that preserves D_3 (which lies in Q as the $\mathrm{SO}_0(3, 2)$ -orbit of $m_0 \in Q$). Consequently, any quotient of D_3 by a lattice in $\mathrm{PSO}_0(3, 2)$ admits a flat holomorphic conformal structure induced by that of the quadric Q .

Notice that compact projective threefolds admitting holomorphic conformal structures ($\mathrm{GL}_2(\mathbb{C})$ -structures) were classified in [JR1]. There are in fact only the standard examples: finite quotients of three dimensional abelian varieties, the smooth quadric Q_3 and quotients of its non compact dual D_3 . In [JR2], the same authors classified also the higher dimensional compact projective manifolds admitting a flat holomorphic conformal structure; they showed that the only examples are the standard ones.

3. $\mathrm{GL}_2(\mathbb{C})$ -STRUCTURES ON KÄHLER AND FUJIKI CLASS \mathcal{C} MANIFOLDS

Every compact complex surface of course admits a holomorphic $\mathrm{GL}_2(\mathbb{C})$ -structure by taking E in Definition 2.1 to be the tangent bundle itself. The situation is much more stringent in higher dimensions. The following result shows that a compact Kähler manifold of even dimension $n \geq 4$ bearing a holomorphic $\mathrm{GL}_2(\mathbb{C})$ -structure has trivial holomorphic tangent bundle (up to finite étale cover).

Theorem 3.1. *Let X be a compact Kähler manifold of even complex dimension $n \geq 4$ admitting a holomorphic $\mathrm{GL}_2(\mathbb{C})$ -structure. Then X admits a finite unramified covering by a compact complex torus.*

Proof. Let E be a holomorphic vector bundle on X and

$$TX \simeq S^{n-1}(E)$$

an isomorphism with the symmetric product, giving a $\mathrm{GL}_2(\mathbb{C})$ -structure on X . Then

$$TX = S^{n-1}(E) = S^{n-1}(E)^* \otimes \left(\bigwedge^2 E \right)^{\otimes(n-1)} = (TX)^* \otimes L,$$

where $L = \left(\bigwedge^2 E \right)^{\otimes(n-1)}$. The above isomorphism between TX and $(TX)^* \otimes L$ produces, when n is even, a holomorphic section

$$\omega \in H^0(X, \Omega_X^2 \otimes L)$$

which is a fiberwise nondegenerate 2-form with values in L . Writing $n = 2m$, the exterior product

$$\omega^m \in H^0(X, K_X \otimes L^m)$$

is a nowhere vanishing section, where $K_X = \Omega_X^n$ is the canonical line bundle on X . Consequently, we have $K_X \simeq (L^*)^m$, in particular, $c_1(X) = mc_1(L)$. Any Hermitian metric on TX induces an associated Hermitian metric on L^m , and hence produces an Hermitian metric on L .

We now use a result of Istrati, [Is1, p. 747, Theorem 2.5], which says that $c_1(X) = c_1(L) = 0$. Hence, by Yau's proof of Calabi's conjecture, [Ya], there exists a Ricci flat metric Kähler metric g on X .

Using de Rham decomposition theorem and Berger's classification of the irreducible holonomy groups of nonsymmetric Riemannian manifolds (see [Jo], Section 3.2 and Theorem 3.4.1 in Section 3.4) we deduce that the universal cover (\tilde{X}, \tilde{g}) of (X, g) is a Riemannian product

$$(\tilde{X}, \tilde{g}) = (\mathbb{C}^l, g_0) \times (X_1, g_1) \times \cdots \times (X_p, g_p), \quad (3.1)$$

where (\mathbb{C}^l, g_0) is the standard flat complete Kähler manifold and (X_i, g_i) is an irreducible Ricci flat Kähler manifold of complex dimension $r_i \geq 2$, for every $1 \leq i \leq p$. The holonomy of each (X_i, g_i) is either $\mathrm{SU}(r_i)$ or the symplectic group $\mathrm{Sp}(\frac{r_i}{2})$, where $r_i = \dim_{\mathbb{C}} X_i$ (in the second case r_i is even). Notice, in particular, that symmetric irreducible Riemannian manifolds of (real) dimension at least two are never Ricci flat. For more details, the reader can refer to [Be, Theorem 1] and [Jo, p. 124, Proposition 6.2.3].

Using Cheeger–Gromoll theorem it is possible to deduce Beauville–Bogomolov decomposition theorem (see [Be, Theorem 1] or [Bo1]) which says that there is a finite étale Galois covering

$$\varphi : \widehat{X} \longrightarrow X,$$

such that

$$(\widehat{X}, \varphi^*g) = (T_l, g_0) \times (X_1, g_1) \times \cdots \times (X_p, g_p), \quad (3.2)$$

where (X_i, g_i) are as in (3.1) and (T_l, g_0) is a flat compact complex torus of dimension l . Note that the Kähler metric φ^*g is Ricci-flat because g is so.

The holomorphic $\mathrm{GL}_2(\mathbb{C})$ -structure on X produces a holomorphic $\mathrm{SL}_2(\mathbb{C})$ -structure on a finite unramified cover of \widehat{X} in (3.2). More precisely, $T\widehat{X} = S^{n-1}(\varphi^*E)$ and since the canonical bundle $K_{\widehat{X}}$ is trivial, the holomorphic line bundle $\bigwedge^2 E$ is torsion. This implies that on a finite unramified cover of \widehat{X} (still denoted by \widehat{X} for notations convenience) we can (and will) assume that $\bigwedge^2 E$ is holomorphically trivial.

The above $\mathrm{SL}_2(\mathbb{C})$ -structure on \widehat{X} gives a holomorphic reduction $R'(\widehat{X}) \subset R(\widehat{X})$ of the structure group of the frame bundle $R(\widehat{X})$ of \widehat{X} from $\mathrm{GL}_n(\mathbb{C})$ to $\mathrm{SL}_2(\mathbb{C})$; recall that the homomorphism $\mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_n(\mathbb{C})$ is given by the $(n-1)$ -th symmetric product of the standard representation, and this homomorphism is injective because n is even. There is a holomorphic tensor θ on \widehat{X} satisfying the condition that the $\mathrm{SL}_2(\mathbb{C})$ -subbundle $R'(\widehat{X}) \subset R(\widehat{X})$ consists of those frames that pointwise preserve θ . Indeed, this is a consequence of Chevalley's theorem asserting that there exists a finite dimensional linear representation W of $\mathrm{GL}_n(\mathbb{C})$, and an element

$$\theta_0 \in W,$$

such that the stabilizer of θ_0 is the image of the above homomorphism $\mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_n(\mathbb{C})$ (see [Hu, p. 80, Theorem 11.2], [DMOS, p. 40, Proposition 3.1(b)]; since $\mathrm{SL}_2(\mathbb{C})$ does not have a nontrivial character, the line in [Hu, p. 80, Theorem 11.2] (or [DMOS, p. 40, Proposition 3.1(b)]) is fixed pointwise in our case). Now, since any irreducible representation W of the reductive group $\mathrm{GL}_n(\mathbb{C})$ is a factor of a representation $(\mathbb{C}^n)^{\otimes p} \otimes ((\mathbb{C}^n)^*)^{\otimes q}$, for some integers $p, q \geq 0$ [DMOS, p. 40, Proposition 3.1(a)], the above element θ_0 gives rise to a holomorphic tensor

$$\theta \in H^0(\widehat{X}, (T\widehat{X})^{\otimes p} \otimes (T^*\widehat{X})^{\otimes q}) \quad (3.3)$$

with $p, q \geq 0$. By construction, θ is stabilized exactly by the frames lying in $R'(\widehat{X})$.

Consider the Levi–Civita connection on \widehat{X} associated to the Ricci-flat Kähler metric φ^*g in (3.2). It is known that the parallel transport for it preserves any holomorphic tensor on \widehat{X} [LT, p. 50, Theorem 2.2.1]. In particular, θ in (3.3) is parallel with respect to this connection. Consequently, the subbundle $R'(\widehat{X}) \subset R(\widehat{X})$ defining the $\mathrm{SL}_2(\mathbb{C})$ -structure (considered as a G -structure) is invariant under the parallel transport by the Levi–Civita connection for φ^*g . This implies that the holonomy group of φ^*g lies in the maximal compact subgroup of $\mathrm{SL}_2(\mathbb{C})$. Hence the holonomy group of φ^*g lies in $\mathrm{SU}(2)$.

From (3.2) it follows that the holonomy of φ^*g is

$$\mathrm{Hol}(\varphi^*g) = \prod_{i=1}^p \mathrm{Hol}(g_i), \quad (3.4)$$

where $\text{Hol}(g_i)$ is the holonomy of g_i . As noted earlier,

- either $\text{Hol}(g_i) = \text{SU}(r_i)$, with $\dim_{\mathbb{C}} X_i = r_i \geq 2$, or
- $\text{Hol}(g_i) = \text{Sp}(\frac{r_i}{2})$, where $r_i = \dim_{\mathbb{C}} X_i$ is even.

Therefore, the above observation, that $\text{Hol}(\varphi^*g)$ is contained in $\text{SU}(2)$, and (3.4) together imply that

- (1) either $(\widehat{X}, \varphi^*g) = (T_l, g_0)$, or
- (2) $(\widehat{X}, \varphi^*g) = (T_l, g_0) \times (X_1, g_1)$, where X_1 is a K3 surface equipped with a Ricci–flat Kähler metric g_1 .

If $(\widehat{X}, \varphi^*g) = (T_l, g_0)$, then the proof of the theorem evidently is complete.

Therefore, we assume that

$$(\widehat{X}, \varphi^*g) = (T_l, g_0) \times (X_1, g_1), \quad (3.5)$$

where X_1 is a K3 surface equipped with a Ricci–flat Kähler metric g_1 . Note that $l \geq 2$ (because $l + 2 = n \geq 4$) and l is even (because n is so).

Fix a point $t \in T_l$. The holomorphic vector bundle on X_1 obtained by restricting φ^*E to $\{t\} \times X_1 \subset T_l \times X_1$ will be denoted by F . Restricting the isomorphism

$$T\widehat{X} \xrightarrow{\sim} S^{l+1}(\varphi^*E)$$

to $\{t\} \times X_1 \subset T_l \times X_1$, from (3.5) we conclude that

$$S^{l+1}(F) = TX_1 \oplus \mathcal{O}_{X_1}^{\oplus l}; \quad (3.6)$$

note that $(T\widehat{X})|_{\{t\} \times X_1} = TX_1 \oplus \mathcal{O}_{X_1}^{\oplus l}$. The vector bundle TX_1 is polystable of degree zero (we may use the Kähler structure g_1 on X_1 to define the degree), because X_1 admits a Kähler–Einstein metric. Hence $TX_1 \oplus \mathcal{O}_{X_1}^{\oplus(l+1)}$ is polystable of degree zero. This implies that $\text{End}(TX_1 \oplus \mathcal{O}_{X_1}^{\oplus(l+1)})$ is polystable of degree zero, because for any Hermitian–Einstein structure on $TX_1 \oplus \mathcal{O}_{X_1}^{\oplus(l+1)}$ (which exists by [UY]), the Hermitian structure on $\text{End}(TX_1 \oplus \mathcal{O}_{X_1}^{\oplus(l+1)})$ induced by it is also Hermitian–Einstein. On the other hand, the holomorphic vector bundle $\text{End}(F)$ is a direct summand of $\text{End}(S^{l+1}(F))$. Since $\text{End}(S^{l+1}(F)) = \text{End}(TX_1 \oplus \mathcal{O}_{X_1}^{\oplus(l+1)})$ is polystable of degree zero, this implies that the holomorphic vector bundle $\text{End}(F)$ is also polystable of degree zero.

Since $\text{End}(F)$ is polystable, it follows that F is polystable [AB, p. 224, Corollary 3.8].

Let H be a Hermitian–Einstein metric on F . The holonomy of the Chern connection on F associated to H is contained in $\text{SU}(2)$, because $\bigwedge^2 F = \mathcal{O}_{X_1}$ (since $\bigwedge^2 E$ is trivial). Let $\text{Hol}(H)$ denote the holonomy of the Chern connection on F associated to H . We note that H is connected because X_1 is simply connected. Therefore, either $\text{Hol}(H) = \text{SU}(2)$ or $\text{Hol}(H) = \text{U}(1)$ (since TX_1 is not trivial, from (3.6) it follows that F is not trivial and hence the holonomy of H cannot be trivial).

First assume that $\text{Hol}(H) = \text{SU}(2)$. Let \mathcal{H} denote the Hermitian–Einstein metric on $S^{n-1}(F)$ induced by H . Since $\text{Hol}(H) = \text{SU}(2)$, the holonomy $\text{Hol}(\mathcal{H})$ of the Chern connection on $S^{n-1}(F)$ associated to \mathcal{H} is the image of the homomorphism

$$h_0 : \text{SU}(2) \longrightarrow \text{SU}(n) \quad (3.7)$$

given by the $(n-1)$ -th symmetric power of the standard representation. The action of $h_0(\text{SU}(2))$ on \mathbb{C}^n , obtained by restricting the standard action of $\text{SU}(n)$, is irreducible. From this it follows that

$$H^0(X_1, S^{n-1}(F)) = 0$$

because any holomorphic section of $S^{n-1}(F)$ is flat with respect to the Chern connection on $S^{n-1}(F)$ associated to \mathcal{H} [LT, p. 50, Theorem 2.2.1]. On the other hand, from (3.6) we have

$$H^0(X_1, S^{n-1}(F)) = \mathbb{C}^l. \quad (3.8)$$

In view of this contradiction we conclude that $\text{Hol}(H) \neq \text{SU}(2)$.

So assume that $\text{Hol}(H) = \text{U}(1)$. Then

$$F = \mathcal{L} \oplus \mathcal{L}^*, \quad (3.9)$$

where \mathcal{L} is a holomorphic line bundle on X_1 . Note that $\text{degree}(\mathcal{L}) = 0$, because F is polystable. From (3.9) it follows that

$$S^{n-1}(F) = \bigoplus_{j=0}^{n-1} \mathcal{L}^{1-n+2j}.$$

Now using (3.8) we conclude that $H^0(X_1, \mathcal{L}^k) \neq 0$ for some nonzero integer k . Since $\text{degree}(\mathcal{L}) = 0$, this implies that the holomorphic line bundle \mathcal{L} is trivial; note that the Picard group $\text{Pic}(X_1)$ is torsionfree. Hence $S^{n-1}(F)$ is trivial. Since TX_1 is not trivial, this contradicts (3.6) (see [At1, p. 315, Theorem 2]).

Therefore, (3.5) can't occur. This completes the proof of the theorem. \square

Recall that a compact Kähler manifold of odd complex dimension bearing a holomorphic $\text{SL}_2(\mathbb{C})$ -structure also admits a holomorphic Riemannian metric and inherits of the associated holomorphic (Levi-Civita) affine connection. Those manifolds are known to have vanishing Chern classes [At2] and, consequently, all of them are covered by compact complex tori [IKO].

Therefore Theorem 3.1 has the following corollary.

Corollary 3.2. *Let X be a compact Kähler manifold of complex dimension $n \geq 3$ bearing a holomorphic $\text{SL}_2(\mathbb{C})$ -structure. Then X admits a finite unramified covering by a compact complex torus.*

We will prove an analog of Theorem 3.1 for Fujiki class \mathcal{C} manifolds. Recall that a compact complex manifold Y is in the class \mathcal{C} of Fujiki if Y is the image of a Kähler manifold through a holomorphic map. By a result of Varouchas in [Va] we know that Fujiki class \mathcal{C} manifolds are precisely those that are bimeromorphic to Kähler manifolds.

In order to use a result in [BDG] (Theorem A) we will need to make a technical assumption. Let us recall some standard terminology:

Definition 3.3. Let X be a compact complex manifold of dimension n , and let

$$[\alpha] \in H^{1,1}(X, \mathbb{R})$$

be a cohomology class represented by a smooth, closed $(1, 1)$ -form α .

- (1) $[\alpha]$ is *numerically effective (nef)* if for any $\epsilon > 0$, there exists a smooth representative $\omega_\epsilon \in [\alpha]$ such that $\omega_\epsilon \geq -\epsilon\omega_X$, where ω_X is some fixed (independent of ϵ) hermitian metric on X .
- (2) $[\alpha]$ has *positive self-intersection* if $\int_X \alpha^n > 0$.

The reader is referred to Section 2 of [BDG] for a discussion on Definition 3.3.

Theorem 3.4. *Let X be a compact complex manifold in Fujiki class \mathcal{C} bearing a holomorphic $GL_2(\mathbb{C})$ -structure. Assume that there exists a cohomology class $[\alpha] \in H^{1,1}(X, \mathbb{R})$ which is nef and has positive self-intersection. Then the following two statements hold.*

- (i) *If the complex dimension of X is even and at least 4, then there exists a non-empty Zariski open subset $\Omega \subset X$ admitting a flat Kähler metric.*
- (ii) *If the complex dimension of X is odd and the first Chern class of X vanishes, then X admits a finite unramified cover which is a torus.*

Proof. (i) Since the complex dimension of X is even, X inherits a twisted holomorphic symplectic form $\omega \in H^0(X, \Omega_X^2 \otimes L)$, defined by a holomorphic line bundle L and a non degenerate L -valued holomorphic two form ω .

The proof of Theorem 2.5 in [Is1] shows that L inherits a holomorphic connection (see Remark 2.7 in [Is1]) and hence its curvature, representing the first Chern class of L , is a holomorphic two form. In particular, $c_1(L)$ admits a representative which is a two form on X of type $(2, 0)$. On the other hand, starting with a Hermitian metric on L , the classical computation of the first Chern class using the associated Chern connection gives a representative of $c_1(L)$ which is a two form on X of type $(1, 1)$. Since on Fujiki class \mathcal{C} manifolds (just as for Kähler manifolds) forms of different types are cohomologous only in the trivial class this implies that $c_1(L) = 0$ and, consequently, as in the proof of Theorem 3.1, $c_1(X) = 0$.

Then Theorem A of [BDG] constructs a closed, positive $(1, 1)$ -current on X , lying in the class $[\alpha]$, which induces a genuine Ricci-flat Kähler metric g on a non-empty Zariski open subset $\Omega \subset X$. Furthermore, given any global holomorphic tensor $\theta \in H^0(X, (TX)^{\otimes p} \otimes T^*X)^{\otimes q}$, with $p, q \geq 0$, the restriction of θ to Ω is parallel with respect to the Levi-Civita connection on Ω associated to g .

As in the proof of Theorem 3.1, this implies that the holonomy group of Levi-Civita connection for g lies inside $SU(2)$.

Take any $u \in \Omega$. Using de Rham's local splitting theorem, there exists a local decomposition of an open neighborhood $U^u \subset \Omega$ of u such that (U^u, g) is a Riemannian product

$$(U^u, g) = (U_0, g_0) \times \cdots \times (U_p, g_p), \quad (3.10)$$

where (U_0, g_0) is a flat Kähler manifold and (U_i, g_i) is an irreducible Kähler manifold of complex dimension $r_i \geq 2$ for every $1 \leq i \leq p$ (the reader is referred to [GGK, Proposition 2.9] for more details on this local Kähler decomposition). Since g is Ricci flat, each (U_i, g_i) is also Ricci flat. For every $1 \leq i \leq p$, the holonomy $\text{Hol}(g_i)$ of the Levi–Civita connection for g_i satisfies the following:

- either $\text{Hol}(g_i) = \text{SU}(r_i)$, or
- $\text{Hol}(g_i) = \text{Sp}(\frac{r_i}{2})$ (in this case r_i is even);

see [GGK, Proposition 5.3]. The holonomy $\text{Hol}(g)$ of the Levi–Civita connection for g is

$$\text{Hol}(g) = \prod_{i=1}^p \text{Hol}(g_i). \quad (3.11)$$

As observed above, $\text{Hol}(g)$ lies inside $\text{SU}(2)$. First assume that

$$\text{Hol}(g) \subsetneq \text{SU}(2).$$

Therefore, $\dim \text{Hol}(g) < 3$. Now from (3.11) it follows that $\text{Hol}(g) = 1$ and there is no factor (U_i, g_i) with $1 \leq i \leq p$ in the decomposition (3.10) (because $\dim \text{Hol}(g_i) \geq 3$ for every $1 \leq i \leq p$). It follows that $(U^u, g) = (U_0, g_0)$ and hence Ω admits the flat Kähler metric g .

Therefore, now assume that

$$\text{Hol}(g) = \text{SU}(2).$$

From (3.11) we conclude the following:

- $p = 1$, meaning $(U^u, g) = (U_0, g_0) \times (U_1, g_1)$,
- $\dim_{\mathbb{C}} U_1 = 2$, and
- $\text{Hol}(g_1) = \text{SU}(2)$.

Consider the homomorphism in (3.7). Consider \mathbb{C}^n as a $\text{SU}(2)$ –representation using h_0 and the standard representation of $\text{SU}(n)$. This $\text{SU}(2)$ –representation is irreducible, in particular, there are no nonzero $\text{SU}(2)$ –invariant in \mathbb{C}^n . From this it can be deduced that $\dim U_0 = 0$, where U_0 is the factor in (3.10). Indeed,

- the direct summand of TU^u given by the tangent bundle TU_0 is preserved by the Levi–Civita connection on TU^n corresponding to g , and
- this direct summand of TU^u given by TU_0 is generated by flat sections of TU^u .

Since TU^u does not have any flat section, we conclude that $\dim_{\mathbb{C}} U_0 = 0$.

Therefore, $(U^u, g) = (U_1, g_1)$, and $\dim_{\mathbb{C}} U^n = \dim_{\mathbb{C}} U_1 = 2$. This contradicts the assumption that $\dim_{\mathbb{C}} X \geq 4$. Hence the proof of (i) is complete.

(ii) Fix a $\text{GL}_2(\mathbb{C})$ –structure on X . As the dimension of X is odd, the $\text{GL}_2(\mathbb{C})$ –structure on X produces a holomorphic conformal structure on X . Since the first Chern class of X vanishes, by Theorem 1.5 of [To], there exists a finite unramified covering of X with trivial canonical bundle. Replacing X by this finite unramified covering we shall assume that K_X is trivial.

As in the proof of Theorem 3.1, this implies that, up to a finite unramified cover, the $GL_2(\mathbb{C})$ -structure of X is induced by a $SL_2(\mathbb{C})$ -structure on X . In particular, X admits a holomorphic Riemannian metric. Now Theorem C of [BDG] says that X admits a finite unramified cover which is a compact complex torus. \square

If X is Kähler, the open subset $\Omega \subset X$ in Theorem 3.4 (point (i)) is the entire manifold. The proof of point (i) in Theorem 3.4 encapsulates an alternative proof to Theorem 3.1.

We note that there are simply connected non-Kähler manifolds in Fujiki class \mathcal{C} that admit a holomorphic symplectic form. Such examples were constructed in [Bo2, Gu1, Gu2].

A compact complex manifold is called Moishezon if it is bimeromorphic to a projective manifold [Mo]. It is known that Moishezon manifolds bearing a holomorphic Cartan geometry (in particular, a holomorphic conformal structure [Sh]) are projective (see Corollary 2 in [BM]). Therefore a Moishezon manifold admitting a $SL_2(\mathbb{C})$ -structure is covered by an abelian variety.

We conjecture that the statement of Theorem 3.4 holds for all Fujiki class \mathcal{C} manifolds. In particular, we conjecture that a Fujiki class \mathcal{C} manifold bearing a $SL_2(\mathbb{C})$ -structure is covered by a compact complex torus.

4. $GL_2(\mathbb{C})$ -STRUCTURES ON KÄHLER-EINSTEIN AND FANO MANIFOLDS

Theorem 4.1. *Let X be a compact Kähler–Einstein manifold, of complex dimension at least three, bearing a holomorphic $GL_2(\mathbb{C})$ -structure. Then one of the following three holds:*

- (1) X admits an unramified covering by a compact complex torus;
- (2) X is the three dimensional quadric Q_3 equipped with its standard $GL_2(\mathbb{C})$ -structure;
- (3) X is covered by the three-dimensional Lie ball D_3 (the noncompact dual of the Hermitian symmetric space Q_3) endowed with the standard $GL_2(\mathbb{C})$ -structure.

Proof. Let n be the complex dimension of X . Let g be a Kähler–Einstein metric on X with Einstein factor e_g .

If $e_g = 0$, then X is Calabi–Yau. Up to a finite unramified covering we can assume that the canonical bundle K_X is trivial [Bo1, Be], and hence X admits a holomorphic $SL_2(\mathbb{C})$ -structure. Now Corollary 3.2 implies that X is covered by a compact complex torus.

Assume now $e_g \neq 0$.

For any integers m, l , the Hermitian structure H on $(TX)^{\otimes m} \otimes ((TX)^*)^{\otimes l}$ induced by g also satisfies the Hermitian–Einstein condition. The Einstein factor e_H for the Hermitian–Einstein metric H is $(m - l)e_g$. In particular, $e_H = 0$ when $m = l$. Consequently, any holomorphic section ψ of $(TX)^{\otimes m} \otimes ((TX)^*)^{\otimes m}$ is flat with respect to the Chern connection on $(TX)^{\otimes m} \otimes ((TX)^*)^{\otimes m}$ corresponding to the Hermitian–Einstein structure H [LT, p. 50, Theorem 2.2.1].

Fix a $GL_2(\mathbb{C})$ -structure on X . It produces a holomorphic reduction of the structure group of the frame bundle $R(X)$ defined by a $GL_2(\mathbb{C})$ principal subbundle denoted by $R'(X)$; recall that the homomorphism $GL_2(\mathbb{C}) \rightarrow GL_n(\mathbb{C})$ is given by the $(n - 1)$ -th symmetric product

of the standard representation of $\mathrm{GL}_2(\mathbb{C})$. This holomorphic reduction $R'(X)$ induces a holomorphic reduction, to $\mathrm{GL}_2(\mathbb{C})$, of the structure group of $(TX)^*$, and hence we get a holomorphic reduction, to $\mathrm{GL}_2(\mathbb{C})$, of the structure group of $TX \otimes (TX)^*$. This reduction of structure group of $TX \otimes (TX)^*$ is given by a holomorphic tensor ψ (as in the proof of Theorem 3.1 this is a consequence of Chevalley's theorem). As noted above, ψ is parallel with respect to the Chern connection on $TX \otimes (TX)^*$ associated to the Hermitian–Einstein metric H .

Let $\mathrm{ad}(R(X))$ and $\mathrm{ad}(R'(X))$ be the adjoint vector bundles for the principal bundles $R(X)$ and $R'(X)$ respectively. Note that $\mathrm{ad}(R(X)) = \mathrm{End}(TX) = TX \otimes (TX)^*$, and the Lie algebra structure of the fibers of $\mathrm{ad}(R(X))$ is the Lie algebra structure of the fibers of $\mathrm{End}(TX)$ given by the usual Lie bracket $(A, B) \mapsto AB - BA$. We have a holomorphic inclusion of Lie algebra bundles

$$\mathrm{ad}(R'(X)) \hookrightarrow \mathrm{ad}(R(X)) = TX \otimes (TX)^*$$

induced by the above holomorphic reduction of structure group $R'(X) \subset R(X)$. As noted before, the Kähler–Einstein structure g produces a Hermitian–Einstein structure H on $TX \otimes (TX)^*$. In particular, the vector bundle $\mathrm{ad}(R(X))$ is polystable of degree zero.

On the other hand, we have $\mathrm{degree}(\mathrm{ad}(R'(X))) = 0$. Indeed, any $\mathrm{GL}_2(\mathbb{C})$ -invariant nondegenerate symmetric bilinear form on the Lie algebra $M(2, \mathbb{C})$ of $\mathrm{GL}_2(\mathbb{C})$ (for example, $(A, B) \mapsto \mathrm{Tr}(AB)$) produces a fiberwise nondegenerate symmetric bilinear form on $\mathrm{ad}(R'(X))$, which in turn holomorphically identifies $\mathrm{ad}(R'(X))$ with $\mathrm{ad}(R'(X))^*$.

Since $\mathrm{ad}(R'(X))$ is a subbundle of degree zero of the polystable vector bundle $TX \otimes (TX)^*$ of degree zero, the Hermitian–Einstein connection on $TX \otimes (TX)^*$ given by H preserves this subbundle $\mathrm{ad}(R'(X))$. From this it can be deduced that the Levi–Civita connection on TX corresponding to the Kähler–Einstein metric g induces a connection on the principal $\mathrm{GL}_2(\mathbb{C})$ -bundle $R'(X)$. To see this we note the following general fact. Let $A \subset B \subset C$ be Lie groups such that A is normal in C . Let E_B be a principal B -bundle and $E_C = E_B(C)$ the principal C -bundle obtained by extending the structure group of E_B . Since A is normal in C , the quotient E_B/A (respectively, E_C/A) is a principal B/A -bundle (respectively, C/A -bundle). Let $\tilde{\nabla}$ be a connection on E_C such that the connection on the principal C/A -bundle E_C/A induced by $\tilde{\nabla}$ preserves the subbundle E_B/A . Then $\tilde{\nabla}$ preserves the subbundle $E_B \subset E_C$. In our situation, $C = \mathrm{GL}(n, \mathbb{C})$ and A is its center, while B is the image of $\mathrm{GL}(2, \mathbb{C})$.

The above observation, that the Levi–Civita connection for g induces a connection the principal $\mathrm{GL}_2(\mathbb{C})$ -bundle $R'(X)$, implies that the holonomy group of g lies in the subgroup

$$\mathrm{GL}_2(\mathbb{C}) \cap \mathrm{U}(n) = \mathrm{U}(2) \subset \mathrm{U}(n).$$

Using de Rham local Riemannian decomposition (see Proposition 2.9 in [GGK] for a proof adapted to the Kähler case), and Berger's list of holonomy (see Proposition 3.4.1 in [Jo, p. 55]) of irreducible holonomies, we conclude that this holonomy, namely a subgroup of $\mathrm{U}(2)$, appears only for locally symmetric Hermitian spaces. The classification of the holonomies of locally symmetric Hermitian spaces shows that X is

- either biholomorphic to the Hermitian symmetric space $SO(5, \mathbb{R})/(SO(2, \mathbb{R}) \times SO(3, \mathbb{R}))$ [Bes, p. 312] (this is the case **BD I** in the list) (recall that $SO(5, \mathbb{R})/(SO(2, \mathbb{R}) \times SO(3, \mathbb{R}))$ is biholomorphic to the quadric Q_3);
- or X is covered by the bounded domain which is the noncompact dual (recall that the noncompact dual of Q_3 is $D_3 = SO_0(3, 2)/(SO(2, \mathbb{R}) \times SO(3, \mathbb{R}))$).

The holonomy group of the Kähler-Einstein metric of the quadric Q_3 is $SO(2, \mathbb{R}) \times SO(3, \mathbb{R})$ [Bes, p. 312] (case **BD I** in the list). Here $SO(2, \mathbb{R})$ acts on \mathbb{C}^3 by the one parameter group $\exp(tJ)$ with J being the complex structure, while the action of $SO(3, \mathbb{R})$ on \mathbb{C}^3 is given by the complexification of the standard action of $SO(3, \mathbb{R})$ on \mathbb{R}^3 . Consequently, the action of the covering $U(2)$ of $SO(2, \mathbb{R}) \times SO(3, \mathbb{R})$ on \mathbb{C}^3 coincides with the action on the second symmetric power $S^2(\mathbb{C}^2)$ of the standard representation.

The $GL_2(\mathbb{C})$ -structure on the quadric Q_3 must be flat [KO, Kl, HM, Ye]. Since the quadric is simply connected this flat $GL_2(\mathbb{C})$ -structure coincides with the standard one [Oc, HM, Ye].

Also, the only holomorphic conformal structure on any compact manifold covered by the noncompact dual D_3 of the quadric is the standard one [Kl, Ye]. \square

The next result deals with Fano manifolds. Recall that a Fano manifold is a compact complex projective manifold such that the anticanonical line bundle K_X^{-1} is ample. Fano manifolds are known to be rationally connected [Ca2], [KMM], and they are simply connected [Ca1].

A basic invariant of a Fano manifold is its *index*, which is, by definition, the maximal positive integer l such that the canonical line bundle K_X is divisible by l in the Picard group of X .

Theorem 4.2. *Let X be a Fano manifold, of complex dimension $n \geq 3$, that admits a holomorphic $GL_2(\mathbb{C})$ -structures. Then $n = 3$, and X is biholomorphic to the quadric Q_3 (the $GL_2(\mathbb{C})$ -structure being the standard one).*

Proof. Let

$$TX \xrightarrow{\sim} S^{n-1}(E)$$

be a holomorphic $GL_2(\mathbb{C})$ -structure on X , where E is a holomorphic vector bundle of rank two on X . A direct computation shows that

$$K_X = \left(\bigwedge^2 E^* \right)^{\frac{n(n-1)}{2}}.$$

Hence the index of X is at least $\frac{n(n-1)}{2}$. It is a known fact that the index of a Fano manifold Y of complex dimension n is at most $n+1$. Moreover, the index is maximal ($= n+1$) if and only if Y is biholomorphic to the projective space $\mathbb{C}P^n$, and the index equals n if and only if Y is biholomorphic to the quadric [KO1]. These imply that $n = 3$ and X is biholomorphic to the quadric Q_3 . As in the proof of Theorem 4.1, the results in [HM, KO, Oc] imply that the only $GL_2(\mathbb{C})$ -structure on the quadric Q_3 is the standard one. \square

5. RELATED OPEN QUESTIONS

In this section we collect some open questions on compact complex manifolds bearing a holomorphic $GL_2(\mathbb{C})$ -structure or a holomorphic $SL_2(\mathbb{C})$ -structure.

 $SL_2(\mathbb{C})$ -structure on Fujiki class \mathcal{C} manifolds.

We think that the statement of Theorem 3.4 holds without the technical assumption described in Definition 3.3 (the existence of a cohomology class $[\alpha] \in H^{1,1}(X, \mathbb{R})$ which is nef and has positive self-intersection).

In particular, we conjecture that *an odd dimensional compact complex manifold X in Fujiki class \mathcal{C} bearing a holomorphic $SL_2(\mathbb{C})$ -structure is covered by a compact torus*. Notice that in this case X admits the Levi-Civita holomorphic affine connection corresponding to the associated holomorphic Riemannian metric, which implies, as in the Kähler case, that all Chern classes of X vanish [At2].

 $SL_2(\mathbb{C})$ -structure on compact complex manifolds.

Recall that Ghys manifolds, constructed in [Gh] by deformation of quotients of $SL_2(\mathbb{C})$ by normal lattices, admit non-flat locally homogeneous holomorphic Riemannian metrics (or equivalently, $SL_2(\mathbb{C})$ -structures). It was proved in [Du] that all holomorphic Riemannian metrics on compact complex threefolds are locally homogeneous.

We conjecture that *$SL_2(\mathbb{C})$ -structures on compact complex manifolds of odd dimension are always locally homogeneous*.

 $GL_2(\mathbb{C})$ -structures on compact Kähler manifolds of odd dimension.

Recall that these manifolds admit a holomorphic conformal structure. Flat conformal structures on compact projective manifolds were classified in [JR2]: beside some projective surfaces, there are only the standard examples.

As for the conclusion in Theorem 4.1 and in Theorem 4.2, we think that *Kähler manifolds of odd complex dimension ≥ 5 and bearing a holomorphic $GL_2(\mathbb{C})$ -structure are covered by compact tori*.

ACKNOWLEDGEMENTS

The authors would like to thank Charles Boubel for illuminating discussions on the geometry of Hermitian symmetric spaces.

This work has been supported by the French government through the UCAJEDI Investments in the Future project managed by the National Research Agency (ANR) with the reference number ANR2152IDEX201. The first-named author is partially supported by a J. C. Bose Fellowship, and school of mathematics, TIFR, is supported by 12-R&D-TFR-5.01-0500. The second-named author wishes to thank TIFR Mumbai, ICTS Bangalore and IISc Bangalore for hospitality.

REFERENCES

- [AB] B. Anchoche and I. Biswas, Einstein-Hermitian connections on polystable principal bundles over a compact Kähler manifold, *Amer. Jour. Math.* **123** (2001), 207–228.
- [At1] M. F. Atiyah, On the Krull-Schmidt theorem with application to sheaves, *Bull. Soc. Math. Fr.* **84** (1956), 307–317.
- [At2] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* **85** (1957), 181–207.
- [Be] A. Beauville, Variétés kähleriennes dont la première classe de Chern est nulle, *Jour. Diff. Geom.* **18** (1983), 755–782.
- [Bo1] F. A. Bogomolov, Kähler manifolds with trivial canonical class, *Izv. Akad. Nauk. SSSR* **38** (1974), 11–21; English translation in *Math. USSR Izv.* **8** (1974), 9–20.
- [Bo2] F. A. Bogomolov, On Guan’s examples of simply connected non-Kähler compact complex manifolds, *Amer. Journal of Math.* **118** (1996), 1037–1046.
- [Bes] A. L. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin, Heidelberg (2008).
- [BD] I. Biswas and S. Dumitrescu, Holomorphic Riemannian metric and fundamental group, *Bull. Soc. Math. Fr.* **147** (2019), 455–468.
- [BDG] I. Biswas, S. Dumitrescu and H. Guenancia, A Bochner principle and its applications to Fujiki class \mathcal{C} manifolds with vanishing first Chern class, <https://arxiv.org/abs/1901.02656>, *Comm. Contemp. Math.* (to appear).
- [BM] I. Biswas and B. McKay, Holomorphic Cartan geometries and rational curves, *Complex Manifolds* **3** (2016), 145–168.
- [Ca1] F. Campana, On twistor spaces of the class \mathcal{C} , *Jour. Diff. Geom.* **33** (1991), 541–549.
- [Ca2] F. Campana, Connexité rationnelle des variétés de Fano, *Ann. Sci. École Norm. Sup.* **25** (1992), 539–545.
- [DMOS] P. Deligne, J. S. Milne, A. Ogus and K-y. Shih, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Mathematics, 900, Springer-Verlag, Berlin-New York, 1982.
- [Du] S. Dumitrescu, Homogénéité locale pour les métriques riemanniennes holomorphes en dimension 3, *Ann. Inst. Fourier* **57** (2007), 739–773.
- [DG] M. Dunajski and M. Godlinski, $GL_2(\mathbb{R})$ -structures, G_2 geometry and twistor theory, *Quarterly J. Math.* **63** (2012), 101–132.
- [FK] E. Ferapontov and B. Kruglikov, Dispersionless integrable hierarchies and $GL_2(\mathbb{R})$ geometry, *Math. Proc. of Cambridge Ph. Society* (2019) DOI: <https://doi.org/10.1017/S0305004119000355>.
- [Fu] A. Fujiki, On the structure of compact manifolds in \mathcal{C} , *Advances Studies in Pure Mathematics*, **1**, Algebraic Varieties and Analytic Varieties, (1983), 231–302.
- [Gh] E. Ghys, Déformations des structures complexes sur les espaces homogènes de $SL(2, \mathbb{C})$, *J. Reine Angew. Math.* **468** (1995), 113–138.
- [GGK] D. Greb, H. Guenancia and S. Kebekus, Klt varieties with trivial canonical class: holonomy, differential forms, and fundamental groups, [arxiv:1704.01408](https://arxiv.org/abs/1704.01408), (2017), to appear in *Geom. Topol.*
- [Gu1] D. Guan, Examples of compact holomorphic symplectic manifolds which are not Kählerian II, *Invent. Math.* **121** (1995), 135–145.
- [Gu2] D. Guan, Examples of compact holomorphic symplectic manifolds which are not Kählerian III, *Internat. J. Math.* **6** (1995), 709–718.
- [Hu] J. E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics, No. 21, Springer-Verlag, New York-Heidelberg, 1975.
- [HM] J.-M. Hwang and N. Mok, Uniruled projective manifolds with irreducible reductive G -structures, *J. reine angew. Math.* **490** (1997), 55–64.
- [IKO] M. Inoue, S. Kobayashi and T. Ochiai, Holomorphic affine connections on compact complex surfaces, *J. Fac. Sci. Univ. Tokyo* **27** (1980), 247–264.
- [Is1] N. Istrati, Twisted holomorphic symplectic forms, *Bull. London Math. Soc.* **48** (2016), 745–756.
- [Is2] N. Istrati, Conformal structures on compact complex manifolds, Ph. D. Thesis, Université Sorbonne Paris Cité, (2018).
- [JR1] P. Jahnke and I. Radloff, Projective threefolds with holomorphic conformal structures, *Internat. J. Math.* **16** (2005), 595–607.

- [JR2] P. Jahnke and I. Radloff, Projective manifolds modeled after hyperquadrics, *Internat. J. Math.* **29** (2018).
- [Jo] D. D. Joyce, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000.
- [Kl] B. Klingler, Un théorème de rigidité non métrique pour les variétés localement symétriques hermitiennes, *Comment. Math. Helv.* **76** (2001), 200–217.
- [Kob2] S. Kobayashi, *Transformation groups in differential geometry*, Springer, 1995.
- [KO] S. Kobayashi and T. Ochiai, Holomorphic projective structures on compact complex surfaces, *Math. Ann.* **249** (1980), 75–94.
- [KO1] S. Kobayashi and T. Ochiai, Characterizations of complex projective spaces and hyperquadrics, *J. Math. Kyoto. Univ.* **13** (1973), 31–47.
- [KO2] S. Kobayashi and T. Ochiai, Holomorphic structures modeled after compact hermitian symmetric spaces, in Hano et al. (eds.), *Manifolds and Lie groups*, Papers in honor of Yozo Matsushima, Progr. Math. **14** (1981), 207–221.
- [KO3] S. Kobayashi and T. Ochiai, Holomorphic structures modeled after hyperquadrics, *Tohoku Math. J.* **34** (1982), 587–629.
- [KMM] J. Kollár, Y. Miyaoka and S. Mori, Rational connectedness and boundedness of Fano manifolds, *Jour. Diff. Geom.* **36** (1992), 765–779.
- [Kr] W. Krynski, GL(2)-geometry and complex structures, arXiv:1910.12669. (2019).
- [LT] M. Lübke and A. Teleman, *The Kobayashi-Hitchin correspondence*, World Scientific Publishing Co., Inc., River Edge, NJ, 1995.
- [Mo] B. Moishezon, On n dimensional compact varieties with n independent meromorphic functions, *Amer. Math. Soc. Transl.* **63** (1967), 51–77.
- [Oc] T. Ochiai, Geometry associated with semisimple flat homogeneous spaces, *Trans. Amer. Math. Soc.* **152** (1970), 159–193.
- [Sh] R. Sharpe, *Differential Geometry: Cartan's generalization of Klein's Erlangen program*, Graduate Texts in Mathematics, 166. Springer-Verlag, New York, 1997.
- [To] V. Tosatti, Non-Kähler Calabi-Yau manifolds, *Contemp. Math.* **644** (2015), 261–277.
- [UY] K. Uhlenbeck and S.-T. Yau, On the existence of Hermitian-Yang-Mills connections in stable vector bundles, *Comm. Pure Appl. Math.* **39** (1986), 257–293.
- [Va] J. Varouchas, Kähler spaces and proper open morphisms, *Math. Ann.* **283** (1989), 13–52.
- [Ya] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I, *Comm. Pure Appl. Math.* **31** (1978), 339–411.
- [Ye] Y. Ye, Extremal rays and null geodesics on a complex conformal manifold, *Internat. J. Math.* **5** (1994), 141–168.

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, MUMBAI 400005, INDIA

Email address: `indranil@math.tifr.res.in`

UNIVERSITÉ CÔTE D'AZUR, CNRS, LJAD, FRANCE

Email address: `dumitres@unice.fr`