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Nonparametric local linear estimation of the relative error regression function for censorship model

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In this paper, we built a new nonparametric regression estimator with the local linear method by using the mean squared relative error as a loss function when the data are subject to random right censoring. We establish the uniform almost sure consistency with rate over a compact set of the proposed estimator. Some simulations are given to show the asymptotic behavior of the estimate in different cases.

Keywords: Censored data, local Linear fit, mean squared relative error, regression function, survival analysis, uniform almost sure convergence.

AMS Subject Classification: MSC: 62G05, 62G08, 62G20, 62N01, 62N02.

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1. Introduction

When it comes to analyzing the dependence between two random variables (r.v.), regression models have appeared as a common and flexible tool in various disciplines, such as biology, medicine, economics, insurance. Consider a random vector (T, X) taking values in $\mathbb{R}_*^+ \times \mathbb{R}$ where T is the interest r.v. with unknown distribution function (d.f.) F and X is the covariate considered having a density function $f(\cdot)$. In practice, it is well-known that we have to study the association between covariates and responses according to the following relation:

$$T = \mu(X) + \varepsilon$$

where $\mu(X) = \mathbb{E}[T|X]$ denotes the regression function which appears as a quantity that contains all the information about the dependence structure and ε is the unobservable error term

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independent of X . Generally $\mu(X)$ is obtained by the minimization of $\mathbb{E}[(T - \mu(X))^2|X]$. The resulting estimator enjoys some important optimality, such as simplicity, flexibility, and consistency. However, this last loss function is inefficient to the presence of outliers in data, which is a common case in practical situations.

The aim of the present paper is to propose a new approach which reduce these drawbacks. Relative error estimation has been recently used in regression analysis as an alternative to the restrictions imposed by the classical regression approach, which consist by considering the estimation of the regression function μ by minimizing the following mean squared relative error loss function, that is, for $T > 0$

$$\mathbb{E} \left[\left(\frac{T - \mu(X)}{T} \right)^2 \middle| X \right]. \tag{1.1}$$

This criterium has been widely studied for parametric models, we refer to [Chen et al. \(2010\)](#) for a discussion about the previous works and [Hirose and Masuda \(2018\)](#) for a real example on the electricity consumption. When the first two conditional inverse moments of T given X are finite, [Park and Stefanski \(1998\)](#) showed that the solution of (1.1), for any fixed x , is given by the following ratio

$$\mu(x) = \frac{\mathbb{E}[T^{-1}|X = x]}{\mathbb{E}[T^{-2}|X = x]} =: \frac{\mu_1(x)}{\mu_2(x)} \tag{1.2}$$

where $\mu_\ell(x) = r_\ell(x)/f(x)$ and $r_\ell(x) = \int_{\mathbb{R}} t^{-\ell} f_{T,X}(t, x) dt$ for $\ell = 1, 2$ with $f_{T,X}(\cdot, \cdot)$ and $f_X(\cdot)$ are the joint and marginal density of the couple (T, X) and X respectively. For recent works, there have been some literature devoted to the relative error regression (RER) methods for complete data. [Chahad et al. \(2017\)](#) considered the estimation of the regression function for a functional explanatory variable while [Attouch et al. \(2017\)](#) have looked to the case where the data are from a strictly stationary spacial process. [Thiam \(2018\)](#) constructed an estimator based in a deconvolution problem. [Hu \(2019\)](#) established the consistency and the asymptotic normality of the regression function based on a least product relative error.

It is well-known that the local linear method has several advantages over the classical kernel smoothing. In particular, it allows to reduce the bias term and avoid the boundary effects. The local linear smoother is not only superior to the popular kernel regression estimator, but also it is the best among all linear smothers, including those produced by orthogonal series and spline methods. A detailed introduction on the importance of the local linear approach can be found in [Fan \(1992\)](#), [Fan and Gijbels \(1996\)](#) for the univariate case and [Fan and Yao \(2003\)](#) for the multivariate case. For recent works on local linear method, we refer to [Jones et al. \(2008\)](#) for independent data and [El Ghouh and Van Keilegom \(2008, 2009\)](#) for regression and quantile regression respectively in the dependent framework.

All these works concern the complete data except the last two articles. In many situations, the data can not be observed completely. Important examples are the survival time of patients or the unemployment time and many others in different fields. A frequent problem in survival analysis is right-censoring, which may be due to different causes: the loss of some subjects under study, the end of the follow up period. Examples of situations where this kind of data occur can be found in [Klein and Moeschberger \(2006\)](#).

In this paper, we suggest a new estimator based on the local linear method of the nonparametric relative error regression (LLRER) estimator when the data are censored. We extend the work of [Jones et al. \(2008\)](#) to the censoring framework by stating a strong result. We point out that in the last paper, only a pointwise of the bias and variance terms have been investigated. We

establish that the new estimator is uniformly almost sure consistent with rate over a compact set under appropriate conditions. Simulation experiments emphasize that the LLRER, is highly competitive to the existing estimators for regression function. To the best of our knowledge, this problem is open up to now and there is no analogous result.

This paper is organized as follows. The general idea of the local linear fit of the mean squared relative error regression function in the censoring framework is described in [Section 2](#). Assumptions and theoretical results are given in [Section 3](#) and some simulation results that illustrates the performance of the proposed procedure are given in [Section 4](#). Finally, [Section 5](#) is devoted to auxiliary results and technical details.

2. The model

According to the right-censoring model, instead of observing T we only observe (Y, δ) where $Y = \min(T, C)$ and $\delta = \mathbb{1}_{\{T \leq C\}}$, here $\mathbb{1}(\cdot)$ is the indicator function. The r.v. C represent the censoring time which is independent of T and with d.f. G . The observed data becomes (Y, δ, X) . From now on, we will always make the following assumption:

$$(T, X) \text{ and } C \text{ are independent.} \tag{2.1}$$

This assumption is required to make the estimation of the censoring distribution easier; However, it is reasonable only when the censoring is not associated to the characteristic of the individuals under study. Let $\{(Y_i, \delta_i, X_i), i = 1, \dots, n\}$ be n independent and identically distributed vectors as (Y, δ, X) . Our main aim is to estimate the RER function defined in [\(1.2\)](#) using the local linear fit. The extension of nonparametric local linear procedures to the censored framework requires to replace the unavailable data by a suitable construction of the observed data given by

$$T_i^{*, -\ell} = \frac{\delta_i Y_i^{-\ell}}{\overline{G}(Y_i)} \quad \text{for } 1 \leq i \leq n \tag{2.2}$$

where $\overline{G}(\cdot) = 1 - G(\cdot)$ denotes the survival function of the r.v. C . The later are called "synthetic data" and permits to consider the effect of censoring in the distribution (for more details, we refer to [Carbonez et al. \(1995\)](#) and [Kohler et al. \(2002\)](#)). In this spirit, based on this construction of the data, using the conditional expectation property and under the Assumption [\(2.1\)](#), for $\ell = 1, 2$ we have

$$\begin{aligned} \mathbb{E}[T_1^{*, -\ell} | X_1] &= \mathbb{E} \left[\frac{\delta_1 Y_1^{-\ell}}{\overline{G}(Y_1)} \middle| X_1 \right] \\ &= \mathbb{E} \left[\frac{T_1^{-\ell}}{\overline{G}(T_1)} \mathbb{E} [\mathbb{1}_{\{T_1 \leq C_1\}} | T_1] \middle| X_1 \right] \\ &= \mathbb{E}[T_1^{-\ell} | X_1]. \end{aligned}$$

Modeling by the local linear method (see [Fan \(1992\)](#)), assumes that the twice derivative of $\mu(x)$ at the point x exists and is continuous, so that $\mu(X)$ can be approximated by a linear function that is, $\mu(X) \approx \mu(x) + \mu'(x)(X - x) =: \beta_1 + \beta_2(X - x)$. Then, the RER function [\(1.2\)](#) is estimated as

the solution of the following optimization problem :

$$\arg \min_{\beta_1, \beta_2} \left\{ \sum_{i=1}^n T_i^{*-2} (T_i^* - \beta_1 - \beta_2(X_i - x))^2 K_h(X_i - x) \right\} \quad (2.3)$$

where $K_h(\cdot) := K\left(\frac{\cdot}{h}\right)$ is a kernel function appropriately chosen (Epanechnikov, Gaussian, ...) and $h := h_n$ is a sequence of positive real numbers which converges to 0 when n goes to infinity. By elementary calculus, the solution of the least squares problem (2.3) yields to

$$\tilde{\mu}(x) = \frac{\sum_{i,j=1}^n w_{i,j}(x) T_j^*}{\sum_{i,j=1}^n w_{i,j}(x)} =: \frac{\tilde{\mu}_1(x)}{\tilde{\mu}_2(x)} \quad (2.4)$$

where

$$w_{i,j}(x) = (X_i - x) \left((X_i - x) - (X_j - x) \right) K_h(X_i - x) K_h(X_j - x) T_i^{*-2} T_j^{*-2}. \quad (2.5)$$

Of course in data analysis, the survival function $\bar{G}(\cdot)$ is unknown and needs to be estimated. This can be done via Kaplan-Meier (KM) as an estimator of $\bar{G}(\cdot)$ (see: [Kaplan and Meier \(1958\)](#))

$$\bar{G}_n(t) = \begin{cases} \prod_{i=1}^n \left(1 - \frac{1 - \delta_i}{n - i + 1} \right)^{\mathbb{1}_{\{Y_i \leq t\}}} & \text{if } t < Y_{(n)}, \\ 0 & \text{otherwise} \end{cases} \quad (2.6)$$

where $Y_{(1)} \leq Y_{(2)} \leq \dots \leq Y_{(n)}$ are the order statistics of the Y_i and δ_i is the indicator of non-censoring. The properties of $\bar{G}_n(t)$ have been studied by many authors. So, (2.2) becomes, for $1 \leq i \leq n$,

$$\widehat{T}_i^{*, -\ell} = \frac{\delta_i Y_i^{-\ell}}{\bar{G}_n(Y_i)}. \quad (2.7)$$

Replacing (2.7) in (2.4) and (2.5) we get a feasible local linear estimator of the relative error regression function (LLRER) expressed as

$$\widehat{\mu}(x) = \frac{\sum_{i,j=1}^n w_{i,j}(x) \widehat{T}_j^*}{\sum_{i,j=1}^n w_{i,j}(x)} =: \frac{\widehat{\mu}_1(x)}{\widehat{\mu}_2(x)} \quad (2.8)$$

where

$$w_{i,j}(x) = (X_i - x) \left((X_i - x) - (X_j - x) \right) K_h(X_i - x) K_h(X_j - x) \widehat{T}_i^{*, -2} \widehat{T}_j^{*, -2}. \quad (2.9)$$

Remark 1 In what follows, we will adopt the convention $0/0 = 0$ in such a case that if, for example, $\widehat{\mu}_1(\cdot) = 0$ and $\widehat{\mu}_2(\cdot) = 0$, the ratio $\widehat{\mu}_1(\cdot)/\widehat{\mu}_2(\cdot)$ in (2.8) will be interpreted as zero.

Throughout this paper, we denote by $\tau_F := \sup\{x : \overline{F}(x) > 0\}$ and $\tau_G := \sup\{x : \overline{G}(x) > 0\}$ be the right support endpoints of \overline{F} and \overline{G} , respectively. We assume that $\tau_F < \infty$, $\overline{G}(\tau_F) > 0$ that implies $0 < \tau_F \leq \tau_G$, which were also assumed in [Guessoum and Ould Saïd \(2008\)](#).

Remark 2 In the simulation part, we will compare our estimator with the classical regression estimator using the local linear method (LLCR). The later is the solution of the following minimization problem:

$$\arg \min_{\alpha, \beta} \left\{ \sum_{i=1}^n (\widehat{T}_i^* - \alpha - \beta(X_i - x))^2 K_h(X_i - x) \right\}$$

for \widehat{T}^* in (2.7), which gives

$$m_n(x) = \frac{\sum_{i,j=1}^n v_{i,j}(x) \widehat{T}_j^*}{\sum_{i,j=1}^n v_{i,j}(x)} \tag{2.10}$$

where

$$v_{i,j}(x) = (X_i - x) \left((X_i - x) - (X_j - x) \right) K_h(X_i - x) K_h(X_j - x).$$

Remark 3 1) We point out that for complete data, i.e. we replace \widehat{T}^* by T in (2.8) and (2.9), we obtain the estimator defined in [Jones et al. \(2008\)](#).

2) Likewise, if we replace \widehat{T}^* by T in (2.10), we obtain the estimator defined in [Nadaraya \(1964\)](#) and [Watson \(1964\)](#).

Remark 4 A crucial point in censored regression is to extend the identifiability assumption on the independence of T and C defined in (2.1) to the case where the explanatory variables are present. In this spirit of KM estimator, one may impose that T and C are independent conditionally to X . Then, (2.7) becomes

$$\widehat{T}_i^* = \frac{\delta_i Y_i}{\overline{G}_n(Y_i|X_i)} \tag{2.11}$$

where $\overline{G}_n(Y_i|X_i)$ is Beran's estimator of the survival conditional function of the r.v. C given X , for more details see [Beran \(1981\)](#). The property of this estimator has been studied by [Dabrowska \(1987\)](#) and [Dabrowska \(1989\)](#). Replacing (2.11) in (2.8) and (2.9) we obtain a feasible estimator of the LLRER function $\mu(\cdot)$.

Remark 5 A frequently used bandwidth selection technique is the cross-validation method, which choose h to minimize

$$\sum_{i=1}^n (\widehat{T}_i^* - \widehat{\mu}_{-i}(X_i))^2 \tag{2.12}$$

where $\widehat{\mu}_{-i}(\cdot)$ is the LLRER estimator defined in (2.8) without using the i^{th} observation (X_i, T_i) .

3. Hypotheses and main results

We will use the following notation C to refer to a compact set of C_0 where $C_0 = \{x \in \mathbb{R}^+ / f(x) > 0\}$ is an open set. Furthermore, when no confusion is possible, we will denote by C any generic positive constant and we assume that

$$\forall T > 0, \exists C, \text{ such that } |T|^{-\ell} \leq C. \tag{3.1}$$

- H1** The bandwidth h satisfies $\lim_{n \rightarrow \infty} h = 0, \lim_{n \rightarrow \infty} nh = +\infty, \lim_{n \rightarrow \infty} \frac{\log n}{nh} = 0.$
- H2** The kernel $K(\cdot)$ is bounded, symmetric non-negative function on C .
 - i.** $\int t^j K(t) dt < \infty,$ for $j = 2, 3.$
 - ii.** $\int t^j K^2(t) dt < \infty$ for $j = 2, 3.$
- H3** The density function $f(\cdot)$ is continuously differentiable and $\sup_{x \in C} |f'(x)| < +\infty.$
- H4** The function $r_\varrho(x) = \int t^{-\varrho} f_{T,X}(t, x) dt$ for $\varrho = 1, 2, 3, 4$ is continuously, differentiable and $\sup_{x \in C} |r'_\varrho(x)| < +\infty.$
- H5** The function

$$v_{\ell,k}(x) = \int t^{-\ell k} f_{T,X}(t, x) dt, \ell = 1, 2 \text{ and } 0 \leq k \leq \nu$$

is continuously differentiable and $\sup_{x \in C} |v'_{\ell,k}(x)| < +\infty.$

3.1. Comments on the Hypotheses:

The hypothesis **H1** concern the bandwidth and is very common in nonparametric estimation. The hypothesis **H2** regards the Kernel K and are needed for the convergence of the estimator. Analogous hypotheses on the kernel has been also made by [Fan \(1992\)](#). The hypothesis **H3** deals with the density function $f(\cdot)$. The hypothesis **H4** and **H5** are regularity conditions for $r_\varrho(\cdot)$ and $v_{\ell,k}(\cdot)$ respectively for different value of ℓ, ϱ and k .

THEOREM 3.1 *Under Hypotheses **H1-H5**, for n large enough, we have*

$$\sup_{x \in C} |\widehat{\mu}(x) - \mu(x)| = O(h^3) + O_{a.s.} \left(\sqrt{\frac{\log n}{nh}} \right).$$

The proof of the **Theorem 1** is made up on the following decomposition:

$$\begin{aligned} \widehat{\mu}(x) - \mu(x) &= \frac{1}{\widehat{\mu}_2(x)} \left\{ \widehat{\mu}_1(x) - \widetilde{\mu}_1(x) + \widetilde{\mu}_1(x) - \mathbb{E}[\widetilde{\mu}_1(x)] + \mathbb{E}[\widetilde{\mu}_1(x)] - r_1(x)r_2(x) \right. \\ &\quad \left. + \mu(x) \left\{ r_2^2(x) - \mathbb{E}[\widetilde{\mu}_2(x)] + \mathbb{E}[\widetilde{\mu}_2(x)] - \widetilde{\mu}_2(x) + \widetilde{\mu}_2(x) - \widehat{\mu}_2(x) \right\} \right\}. \end{aligned}$$

Remark that by Hypothesis **H4** and condition (3.1), there exists $\eta > 0$ such that $\sup_{x \in C} |r_2(x)| \leq \eta.$

Then, by triangle inequality, we have

$$\begin{aligned} \sup_{x \in C} |\widehat{\mu}(x) - \mu(x)| &\leq \frac{1}{\eta^2 - \sup_{x \in C} |\widehat{\mu}_2(x) - r_2^2(x)|} \left\{ \sup_{x \in C} |\widehat{\mu}_1(x) - \widetilde{\mu}_1(x)| + \sup_{x \in C} |\widetilde{\mu}_1(x) - \mathbb{E}[\widetilde{\mu}_1(x)]| \right. \\ &\quad + \sup_{x \in C} |\mathbb{E}[\widetilde{\mu}_1(x)] - r_1(x)r_2(x)| + \sup_{x \in C} |\mu(x)| \left\{ \sup_{x \in C} |\mathbb{E}[\widetilde{\mu}_2(x)] - r_2^2(x)| \right. \\ &\quad \left. \left. + \sup_{x \in C} |\widetilde{\mu}_2(x) - \mathbb{E}[\widetilde{\mu}_2(x)]| + \sup_{x \in C} |\widetilde{\mu}_2(x) - \mu_2(x)| \right\} \right\}. \end{aligned}$$

The proof will be achieved with the following propositions.

PROPOSITION 3.2 Under Hypotheses **H1**, **H2 i)**, **H3** and **H4**, for $\ell = 1, 2$, for n large enough, we have

$$\sup_{x \in C} |\widehat{\mu}_\ell(x) - \widetilde{\mu}_\ell(x)| = O_{a.s.} \left(\sqrt{\frac{\log \log n}{n}} \right).$$

PROPOSITION 3.3 Under Hypotheses **H1**, **H2 i)**, **H3**, **H4** and **H5**, for $\ell = 1, 2$, for n large enough, we have

$$\sup_{x \in C} |\widetilde{\mu}_\ell(x) - \mathbb{E}[\widetilde{\mu}_\ell(x)]| = O_{a.s.} \left(\sqrt{\frac{\log n}{nh}} \right).$$

PROPOSITION 3.4 Under Hypotheses **H1**, **H2** and **H4**, for $\ell = 1, 2$, for n large enough, we have

$$\sup_{x \in C} |\mathbb{E}[\widetilde{\mu}_\ell(x)] - r_\ell(x)r_2(x)| = O(h^3).$$

4. Numerical study

To evaluate the quality of this method, we perform several simulations of the proposed estimator $\widehat{\mu}(\cdot)$ with different level of censoring. For that, we generate the data as follows:

Inputs: Generate n i.i.d. $\{X_i \rightsquigarrow \mathcal{N}(0, 1), C_i \rightsquigarrow \mathcal{N}(3 + c, 1) \text{ and } \epsilon_i \rightsquigarrow \mathcal{N}(0, 1)\}$ for $1 \leq i \leq n$ where c is a constant that adjusts the percentage of censoring (C.P).

Step 1 : Calculate the interest variable $T_i = 2X_i + 1 + 0.2 \epsilon_i$ where X_i and ϵ_i are independent.

Step 2 : Compute the observed data $\{T_i^*, 1 \leq i \leq n\}$ from (2.7) with the KM estimator from (2.6).

Step 3 : We employ the Gaussian Kernel. Furthermore, we apply the cross-validation method (see : **Remark 2.5**) to choose the bandwidth. For a predetermined sequence of h 's from a wide range (0.01 to 2) with an increment 0.01, we choose the optimal bandwidth (h_{opt}) that minimize the cross-validation criterium (2.12).

Ouputs: Compute the LLRER estimator from (2.8) for $x \in [1, 4]$ and h_{opt} .

In all the simulation study, we use the following proposition of **Port (1994)** which permit to calculate the theoretical RER function (see formula (4.1) below).

PROPOSITION 4.1 Let $q_1(X)$ and $q_2(X)$ be two random variable with means: μ_1 and μ_2 and vari-

ances: v_1 and v_2 respectively, and covariance v_{12} . Let $(X_i)_{1 \leq i \leq n}$ be an i.i.d. sequence of r.v. and defined by

$$\widehat{\Sigma}_1 = \frac{1}{n} \sum_{i=1}^n q_1(X_i) \quad \text{and} \quad \widehat{\Sigma}_2 = \frac{1}{n} \sum_{i=1}^n q_2(X_i)$$

and $\widehat{R} = \frac{\widehat{\Sigma}_1}{\widehat{\Sigma}_2}$ then the second order approximation of $\mathbb{E}[\widehat{R}]$ is

$$\mathbb{E}[\widehat{R}] \approx \frac{\mu_1}{\mu_2} + \frac{1}{n} \left(\frac{\mu_1 v_2}{\mu_2^3} - \frac{v_{12}}{\mu_2^2} \right).$$

In the following figures, the solid line represent the theoretical curve (TC) of the RER function which is generated according to the following formula:

$$m(x) = 2x + 1 + 0.04(2x + 1)^{-1} \quad \text{for } x \in [1, 4] \tag{4.1}$$

Furthermore, a comparative study with other existing kernel methods: the classical regression (CR) estimator defined in [Guessoum and Ould Saïd \(2008\)](#) by

$$\widehat{m}(x) = \frac{\sum_{i=1}^n \widehat{T}_i^* K_h(X_i - x)}{\sum_{i=1}^n K_h(X_i - x)}$$

and the local linear classical regression (LLCR) estimators defined in [\(2.10\)](#) was carried out.

4.1. Effect of sample size:

We plot the true RER curve (TC) together with the LLRER estimator in [Figure 1](#). We can see that the quality of fit is better when n rises.

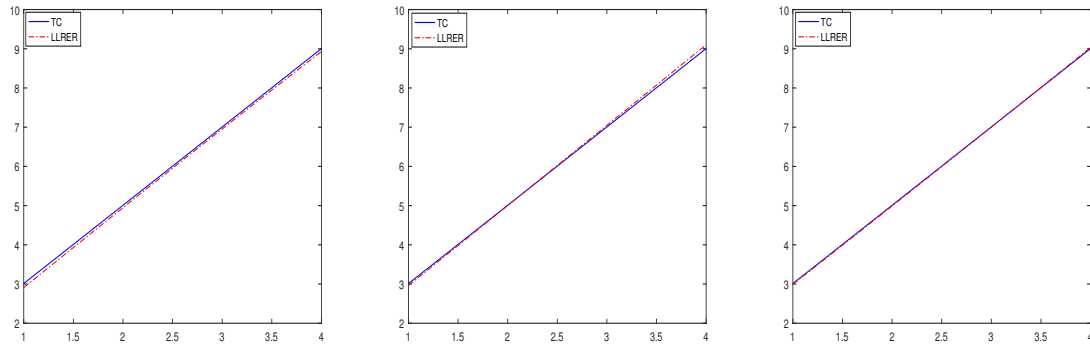


Figure 1. $\mu(\cdot)$, $\widehat{\mu}(\cdot)$ with C.P. $\approx 65\%$ for $n = 100, 300,$ and 500 respectively.

4.2. Effect of C.P.:

From **Figure 2**, it can be seen for a fixed sample size that the LLRER estimator quality is a little bit affected by the percentage of observed data.

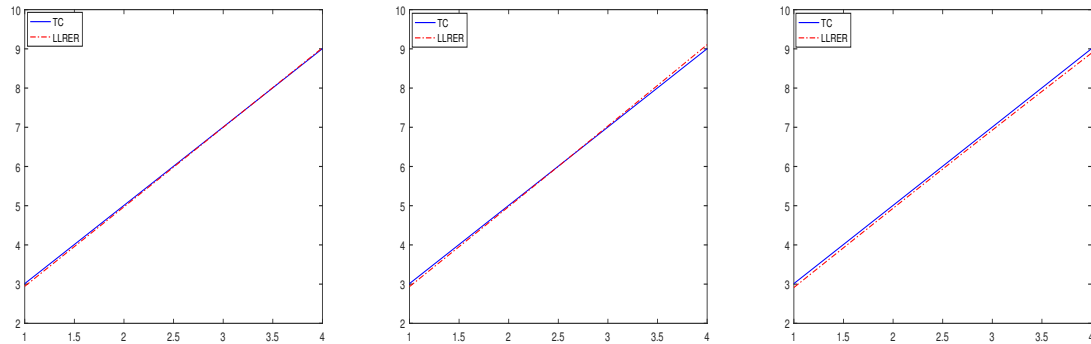


Figure 2. $\mu(\cdot), \widehat{\mu}(\cdot)$ with $n = 300$ for C.P. $\approx 35, 50,$ and 70% respectively.

4.3. Effect of outliers:

In order to assess the robustness to outliers of our new estimator, we generate samples of size $n = 300$ and multiply the values of 15 among them by a multiplying coefficient (M.C.). We can observe that the quality of fit decreases as the value of M.C. increases but remains consistent.

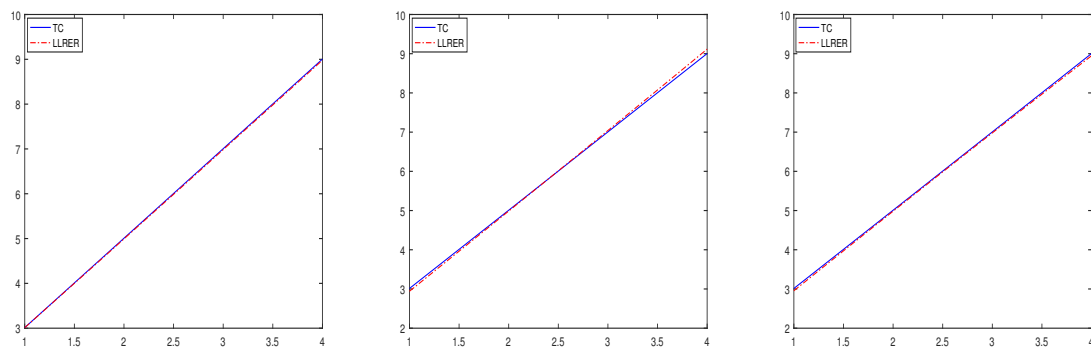


Figure 3. $\mu(\cdot), \widehat{\mu}(\cdot)$ with $n = 300$ for C.P. $\approx 50\%$ and M.C. = 25, 50, and 100 respectively.

4.4. Comparison to other kernel estimators:

4.4.1. CR vs LLRER:

Effect of C.P.:

The proposed estimate shows an improvement over the CR estimate near the right tail where the data points are sparse and mostly uncensored. Figure 4 shows that the LLRER estimator is much more robust to censoring than the CR, in particular for larger samples.

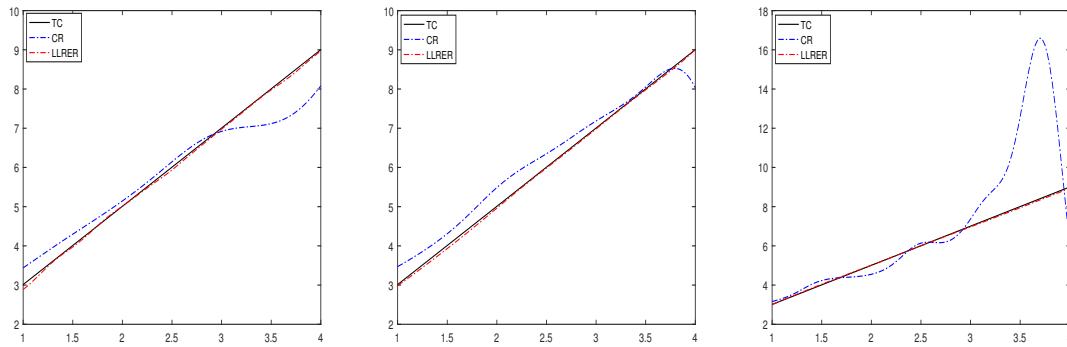


Figure 4. $\mu(\cdot)$, $\widehat{\mu}(\cdot)$ and $\widehat{m}(\cdot)$ with $n = 300$ for C.P. $\approx 35, 50$ and 70% respectively.

Effect of outliers:

We compare the two models when the data contains outliers in the observed response value and we note that there is a significant difference between the two estimators for a fixed C.P. and sample size. As expected, when there are outliers, the relative regression estimator performs better than the Nadaraya-Watson and local linear estimators $m_n(\cdot)$ with respect to the number of outliers (see Figure 5).

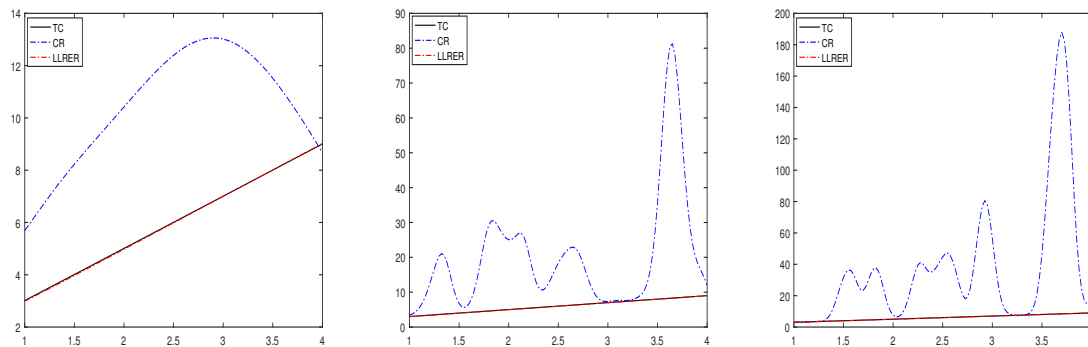


Figure 5. $\mu(\cdot)$, $\widehat{\mu}(\cdot)$ and $\widehat{m}(\cdot)$ with $n = 300$ for C.P. $\approx 35\%$ and M.C. = $25, 50, 100$ respectively.

4.4.2. LLCR vs LLRER

Effect of C.P.:

We observe from Figure 6 that there is no meaningful difference between the LLCR and LL-

RER when the C.P. is low. The two predictors are basically equivalent and both show the good behavior. However for high censorship rate our estimator remains resistant unlike its competitor which moves away from the edges.

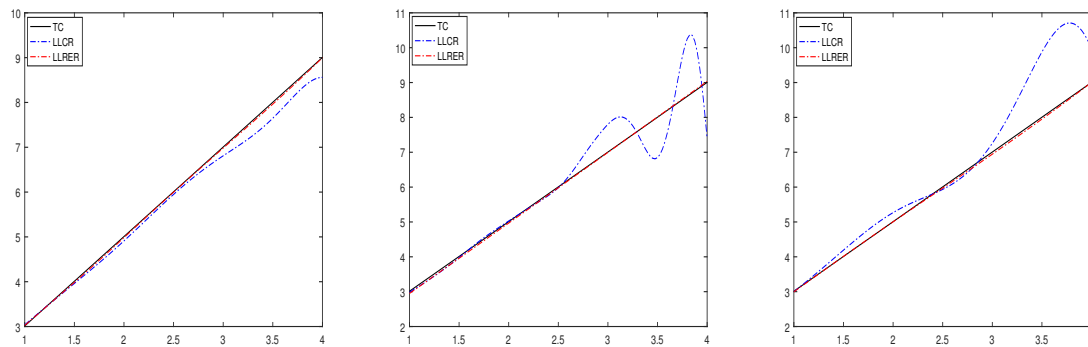


Figure 6. $\mu(\cdot)$, $\widehat{\mu}(\cdot)$ and $m_n(\cdot)$ with $n = 300$ for C.P. $\approx 35, 50$ and 66% respectively.

Effect of outliers:

Figure 7 shows clearly that the curve of the LLCR estimator is moves away from the TC when the M.C. increases which reflect the effectiveness of the procedure in presence of outliers.

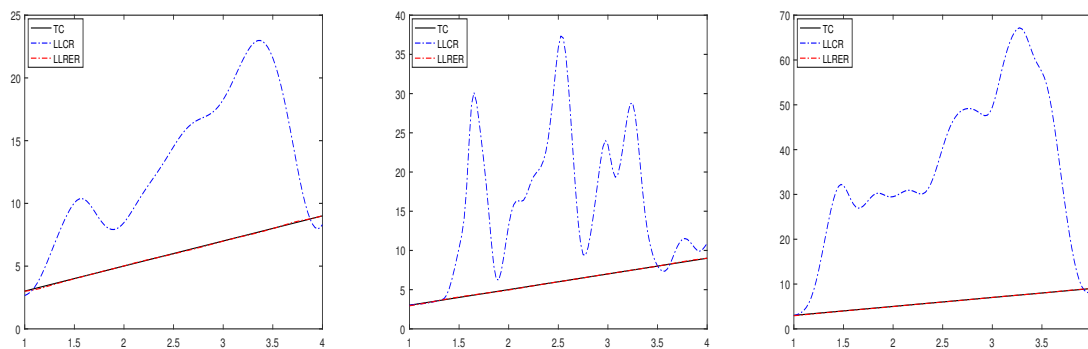


Figure 7. $\mu(\cdot)$, $\widehat{\mu}(\cdot)$ and $m_n(\cdot)$ with $n = 300$ for C.P. $\approx 35\%$ and M.C. = 25, 50, 100 respectively.

4.4.3. LLRER versus CR ans LLCR

Finally, in this figure, we can clearly see that in the presence of outliers, the new estimator obtained by combining the RER and LL methods is much more efficient compared to the two methods treated separately as that has been treated by many authors.

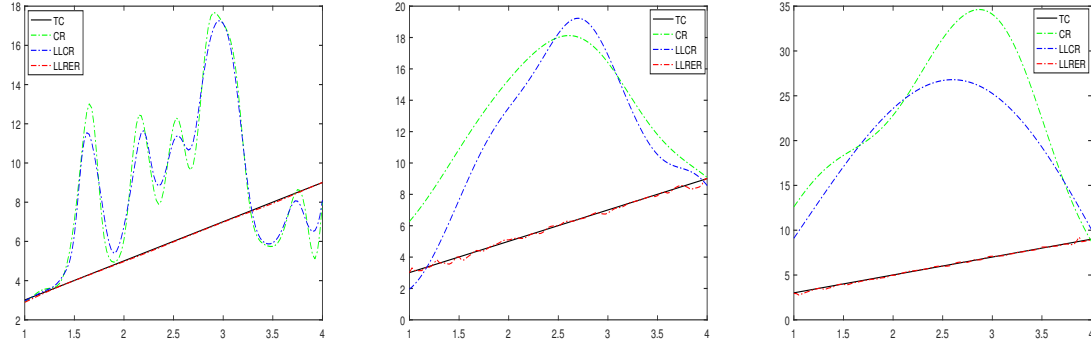


Figure 8. $\mu(\cdot)$, $\widehat{m}(\cdot)$, $\widehat{\mu}(\cdot)$ and $m_n(\cdot)$ with $n = 300$ for C.P. $\approx 35\%$ and M.C.= 25, 50, 100 respectively.

5. Proofs and auxiliary results

Proof of the Proposition 1. Let introduce some notations for $\ell = 1, 2$ and $\gamma = 0, 1, 2$:

$$\widehat{S}_{\ell,\gamma}(x) = \frac{1}{nh} \sum_{i=1}^n \widehat{T}_i^{\star,-\ell} (X_i - x)^\gamma K_h(X_i - x) \quad \text{and} \quad \widetilde{S}_{\ell,\gamma}(x) = \frac{1}{nh} \sum_{i=1}^n T_i^{\star,-\ell} (X_i - x)^\gamma K_h(X_i - x).$$

We use the following decomposition:

$$\begin{aligned} \widehat{\mu}_\ell(x) - \widetilde{\mu}_\ell(x) &= \widehat{S}_{2,2}(x) \widehat{S}_{\ell,0}(x) - \widetilde{S}_{2,1}(x) \widehat{S}_{\ell,1}(x) - \left(\widetilde{S}_{2,2}(x) \widetilde{S}_{\ell,0}(x) - \widetilde{S}_{2,1}(x) \widetilde{S}_{\ell,1}(x) \right) \\ &= \widehat{S}_{2,2}(x) \widehat{S}_{\ell,0}(x) - \widetilde{S}_{2,2}(x) \widetilde{S}_{\ell,0}(x) - \left(\widehat{S}_{2,1}(x) \widehat{S}_{\ell,1}(x) - \widetilde{S}_{2,1}(x) \widetilde{S}_{\ell,1}(x) \right) \\ &=: \mathcal{B}_{\ell,1}(x) - \mathcal{B}_{\ell,2}(x). \end{aligned}$$

On the one hand, for $\ell = 1, 2$, we get

$$\begin{aligned} \mathcal{B}_{\ell,1}(x) &= \left(\widehat{S}_{2,2}(x) - \widetilde{S}_{2,2}(x) \right) \left(\widehat{S}_{\ell,0}(x) - \widetilde{S}_{\ell,0}(x) \right) + \left(\widetilde{S}_{\ell,0}(x) - \mathbb{E}[\widetilde{S}_{\ell,0}(x)] \right) \left(\widehat{S}_{2,2}(x) - \widetilde{S}_{2,2}(x) \right) \\ &\quad + \mathbb{E}[\widetilde{S}_{\ell,0}(x)] \left(\widehat{S}_{2,2}(x) - \widetilde{S}_{2,2}(x) \right) + \left(\widetilde{S}_{2,2}(x) - \mathbb{E}[\widetilde{S}_{2,2}(x)] \right) \left(\widehat{S}_{\ell,0}(x) - \widetilde{S}_{\ell,0}(x) \right) \\ &\quad + \mathbb{E}[\widetilde{S}_{2,2}(x)] \left(\widehat{S}_{\ell,0}(x) - \widetilde{S}_{\ell,0}(x) \right). \end{aligned} \tag{5.1}$$

On the other hand, for $\ell = 1, 2$, we get

$$\begin{aligned} \mathcal{B}_{\ell,2}(x) &= \left(\widehat{S}_{2,1}(x) - \widetilde{S}_{2,1}(x) \right) \left(\widehat{S}_{\ell,1}(x) - \widetilde{S}_{\ell,1}(x) \right) + \left(\widetilde{S}_{2,1}(x) - \mathbb{E}[\widetilde{S}_{2,1}(x)] \right) \left(\widehat{S}_{\ell,1}(x) - \widetilde{S}_{\ell,1}(x) \right) \\ &\quad + \mathbb{E}[\widetilde{S}_{2,1}(x)] \left(\widehat{S}_{\ell,1}(x) - \widetilde{S}_{\ell,1}(x) \right) + \left(\widetilde{S}_{\ell,1}(x) - \mathbb{E}[\widetilde{S}_{\ell,1}(x)] \right) \left(\widehat{S}_{2,1}(x) - \widetilde{S}_{2,1}(x) \right) \\ &\quad + \mathbb{E}[\widetilde{S}_{\ell,1}(x)] \left(\widehat{S}_{2,1}(x) - \widetilde{S}_{2,1}(x) \right). \end{aligned} \tag{5.2}$$

It remains to study each term of the decomposition (5.1) and (5.2). For this, we will state and proof the following three **Lemma 5.1-5.3**.

LEMMA 5.1 *Under hypotheses H2 i) and H3, for $\ell = 1, 2$, $\gamma = 0, 1, 2$, and n large enough, we have*

$$\sup_{x \in C} \left| \widehat{S}_{\ell,\gamma}(x) - \widetilde{S}_{\ell,\gamma}(x) \right| = O_{a.s.} \left(\sqrt{\frac{\log \log n}{n}} \right).$$

Proof of Lemma 5.1. For $\ell = 1, 2$, $\gamma = 0, 1, 2$, we have

$$\begin{aligned} \sup_{x \in C} |\widehat{S}_{\ell, \gamma}(x) - \widetilde{S}_{\ell, \gamma}(x)| &= \sup_{x \in C} \left| \frac{1}{nh} \sum_{i=1}^n \widehat{T}_i^{\star, -\ell}(X_i - x)^\gamma K_h(X_i - x) - \frac{1}{nh} \sum_{i=1}^n T_i^{\star, -\ell}(X_i - x)^\gamma K_h(X_i - x) \right| \\ &= \sup_{x \in C} \left| \frac{1}{nh} \left(\sum_{i=1}^n \frac{\delta_i Y_i^{-\ell}}{\overline{G}_n(Y_i)} (X_i - x)^\gamma K_h(X_i - x) - \sum_{i=1}^n \frac{\delta_i Y_i^{-\ell}}{\overline{G}(Y_i)} (X_i - x)^\gamma K_h(X_i - x) \right) \right| \\ &= \sup_{x \in C} \left| \frac{1}{nh} \sum_{i=1}^n \delta_i T_i^{-\ell}(X_i - x)^\gamma K_h(X_i - x) \left(\frac{1}{\overline{G}_n(Y_i)} - \frac{1}{\overline{G}(Y_i)} \right) \right| \\ &\leq \frac{1}{\overline{G}^2(\tau_F)} \sup_{t \leq \tau_F} |\overline{G}_n(t) - \overline{G}(t)| \times \sup_{x \in C} \left| \frac{1}{nh} \sum_{i=1}^n T_i^{-\ell}(X_i - x)^\gamma K_h(X_i - x) \right| \\ &=: \sup_{t \leq \tau_F} \mathcal{D}_1(t) \times \sup_{x \in C} |\mathcal{D}_2(x)|. \end{aligned}$$

From Lemma 4.2. in [Deuhevles and Einmahl \(2000\)](#), the first term of the right hand side is equal to:

$$\sup_{t \leq \tau_F} \mathcal{D}_1(t) = O_{a.s.} \left(\sqrt{\frac{\log \log n}{n}} \right) \quad \text{as } n \rightarrow \infty. \quad (5.3)$$

For the second term and using the strong law of large numbers we have

$$\sup_{x \in C} |\mathcal{D}_2(x)| \leq C \sup_{x \in C} \left| \mathbb{E} \left[h^{-1}(X_1 - x)^\gamma K_h(X_1 - x) \right] \right|.$$

By a change of variable, Taylor expansion and with the condition [\(3.1\)](#), we get

$$\begin{aligned} \mathbb{E} \left[h^{-1}(X_1 - x)^\gamma K_h(X_1 - x) \right] &= h^{-1} \int (u - x)^\gamma K_h(u - x) f(u) du \\ &= h^{-1} \int (vh)^\gamma K(v) f(x + vh) h dv \\ &= h^\gamma f(x) \int v^\gamma K(v) dv + h^{\gamma+1} \int v^{\gamma+1} K(v) f'(\xi) dv. \end{aligned}$$

Under the kernel hypothesis [H2 i\)](#) and the regularity hypothesis [H3](#), we get

$$\sup_{x \in C} |\mathcal{D}_2(x)| = O(h^\gamma). \quad (5.4)$$

Combining the results [\(5.3\)](#) and [\(5.4\)](#), the proof of [Lemma 5.1](#) is achieved. ■
LEMMA 5.2 Under hypotheses [H1](#), [H2 i\)](#), [H3](#) and [H4](#) for $\ell = 1, 2$, $\gamma = 0, 1, 2$, and n large enough, we have

$$\sup_{x \in C} |\widetilde{S}_{\ell, \gamma}(x) - \mathbb{E}[\widetilde{S}_{\ell, \gamma}(x)]| = O_{a.s.} \left(\sqrt{\frac{\log n}{nh}} \right).$$

Proof of Lemma 5.2. Let

$$C_n = \{x_i - b_n, x_i + b_n, 1 \leq i \leq d_n\}$$

is the intervals extremities grid where $b_n = n^{-1/2q}$ for $q > 0$ and cover the compact set C by $\cup_{i=1}^{d_n} [x_i - b_n, x_i + b_n]$ with $d_n = O(n^{1/2q})$.

$$\sup_{x \in C} |\widetilde{S}_{\ell,\gamma}(x) - \mathbb{E}[\widetilde{S}_{\ell,\gamma}(x)]| \leq \max_{1 \leq i \leq d_n} \max_{x \in C_n} |\widetilde{S}_{\ell,\gamma}(x) - \mathbb{E}[\widetilde{S}_{\ell,\gamma}(x)]| + 2^q C b_n^q.$$

using $b_n = n^{-1/2q}$ then

$$b_n^q = O\left(\sqrt{\frac{\log n}{nh}}\right).$$

For this, observe that for all $\varepsilon > 0$,

$$\mathbb{P}\left(\max_{x \in C_n} |\widetilde{S}_{\ell,\gamma}(x) - \mathbb{E}[\widetilde{S}_{\ell,\gamma}(x)]| > \varepsilon\right) \leq \sum_{x \in C_n} \mathbb{P}\left(|\widetilde{S}_{\ell,\gamma}(x) - \mathbb{E}[\widetilde{S}_{\ell,\gamma}(x)]| > \varepsilon\right).$$

Let us write for $\ell = 1, 2$, $\gamma = 0, 1, 2$ and $x \in C_n$

$$\begin{aligned} \widetilde{S}_{\ell,\gamma}(x) - \mathbb{E}[\widetilde{S}_{\ell,\gamma}(x)] &= \frac{1}{nh} \sum_{i=1}^n T_i^{\star,-\ell}(X_i - x)^\gamma K_h(X_i - x) - \mathbb{E}\left[\frac{1}{nh} \sum_{i=1}^n T_i^{\star,-\ell}(X_i - x)^\gamma K_h(X_i - x)\right] \\ &= \frac{1}{n} \sum_{i=1}^n \frac{T_i^{\star,-\ell}(X_i - x)^\gamma K_h(X_i - x) - \mathbb{E}[T_i^{\star,-\ell}(X_i - x)^\gamma K_h(X_i - x)]}{h} \\ &=: \frac{1}{n} \sum_{i=1}^n \mathcal{A}_{\gamma,i}^\ell(x). \end{aligned}$$

In view of **Corollary A.8.** (see Appendix), we focus on the absolute moments of order ν of $\mathcal{A}_{\gamma,i}^\ell(x)$

$$\begin{aligned} \mathbb{E}|\mathcal{A}_{\gamma,i}^\ell(x)|^\nu &= \mathbb{E}\left|h^{-\nu} \left(T_i^{\star,-\ell}(X_i - x)^\gamma K_h(X_i - x) - \mathbb{E}[T_i^{\star,-\ell}(X_i - x)^\gamma K_h(X_i - x)]\right)^\nu\right| \\ &= h^{-\nu} \mathbb{E}\left|\sum_{k=0}^{\nu} c_{k,\nu} \left(T_i^{\star,-\ell}(X_i - x)^\gamma K_h(X_i - x)\right)^k \mathbb{E}[T_i^{\star,-\ell}(X_i - x)^\gamma K_h(X_i - x)]^{\nu-k}\right| \\ &\leq h^{-\nu} \sum_{k=0}^{\nu} c_{k,\nu} \left|\mathbb{E}\left[\left(T_i^{\star,-\ell}(X_i - x)^\gamma K_h(X_i - x)\right)^k\right]\right| \mathbb{E}\left[T_i^{\star,-\ell}(X_i - x)^\gamma K_h(X_i - x)\right]^{\nu-k}. \end{aligned}$$

On the one hand, using the conditional expectation property, Taylor expansion and under **H2 i)** and **H5**, we have

$$\begin{aligned} \mathbb{E}\left[\left(T_1^{\star,-\ell}(X_1 - x)^\gamma K_h(X_1 - x)\right)^k\right] &= \mathbb{E}\left[T_1^{\star,-k\ell}(X_1 - x)^{\gamma k} K_h^k(X_1 - x)\right] \\ &= \mathbb{E}\left[(X_1 - x)^{\gamma k} K_h^k(X_1 - x) \mathbb{E}[T_1^{\star,-k\ell} | X_1]\right] \\ &= \int (u - x)^{\gamma k} K_h^k(u - x) \mathbb{E}[T_1^{\star,-k\ell} | X_1 = u] f(u) du \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{\overline{G}^{k-1}(\tau_F)} \int (u-x)^{\gamma k} K_h^k(u-x) \int t^{-\ell k} f_{T|X}(t|u) dt f(u) du \\
 &= \frac{1}{\overline{G}^{k-1}(\tau_F)} \int (u-x)^{\gamma k} K_h^k(u-x) v_{\ell,k}(u) du \\
 &= \frac{h^{\gamma k+1}}{\overline{G}^{k-1}(\tau_F)} \int s^{\gamma k} K^k(s) v_{\ell,k}(x+sh) ds \\
 &= \frac{h^{\gamma k+1}}{\overline{G}^{k-1}(\tau_F)} v_{\ell,k}(x) \int s^{\gamma k} K^k(s) ds + \frac{h^{\gamma k+2}}{\overline{G}^{k-1}(\tau_F)} \int s^{\gamma k+1} K^k(s) v'_{\ell,k}(\xi) ds.
 \end{aligned}$$

On the other hand, using the same arguments as previously and under **H2 i)**, **H4** we have

$$\begin{aligned}
 \mathbb{E} \left[T_1^{\star,-\ell} (X_1 - x)^\gamma K_h(X_1 - x) \right]^{v-k} &= \left(\int (u-x)^\gamma K_h(X_1 - x) \mathbb{E} \left[T_1^{\star,-\ell} | X_1 = u \right] f(u) du \right)^{v-k} \\
 &= \left(\int (hv)^\gamma K(v) \mu_\ell(u) f(u) du \right)^{v-k} \\
 &= \left(h^{\gamma+1} \int v^\gamma K(v) r_\ell(x+vh) dv \right)^{v-k} \\
 &= \left(h^{\gamma+1} r_\ell(x) \int v^\gamma K(v) dv + h^{\gamma+2} \int v^{\gamma+1} K(v) r'_\ell(\xi) dv \right)^{v-k}.
 \end{aligned}$$

Then, for $\ell = 1, 2$, $\gamma = 0, 1, 2$ and for all $v \geq 2$, we get easily

$$\begin{aligned}
 \mathbb{E} |\mathcal{A}_{\gamma,1}^\ell(x)|^v &\leq O(h^{-v}) \times O(h^{\gamma k+1}) \times O(h^{(\gamma+1)(v-k)}) \\
 &= O(h^{\gamma v-k+1}) \\
 &= O(\max_{1 \leq k \leq v} h^{-k+1}) \\
 &= O(h^{-v+1}).
 \end{aligned}$$

Now, we can apply the exponential inequality in **Corollary A.8**. by choosing $a^2 = h^{-1}$, we get

$$\mathbb{P} \left(\left| \widetilde{S}_{\ell,\gamma}(x) - \mathbb{E}[\widetilde{S}_{\ell,\gamma}(x)] \right| > \varepsilon \right) = \mathbb{P} \left(\left| \sum_{i=1}^n \mathcal{A}_{\gamma,i}^\ell(x) \right| > \varepsilon n \right) \leq 2 \exp \left(-\frac{\varepsilon^2 n h}{2(1+\varepsilon)} \right).$$

Hence, for a fixed ε_0 , choosing $\varepsilon = \varepsilon_0 \left(\frac{\log n}{nh} \right)^{1/2}$, we get

$$\mathbb{P} \left(\left| \widetilde{S}_{\ell,\gamma}(x) - \mathbb{E}[\widetilde{S}_{\ell,\gamma}(x)] \right| > \varepsilon \right) \leq 2 \exp \left\{ -\frac{\varepsilon_0^2 \log n}{2 \left(1 + \varepsilon_0 \sqrt{\frac{\log n}{nh}} \right)} \right\}$$

and for n large enough, we have

$$\mathbb{P} \left(\left| \widetilde{S}_{\ell,\gamma}(x) - \mathbb{E}[\widetilde{S}_{\ell,\gamma}(x)] \right| > \varepsilon \right) \leq 2 \exp \left(-\frac{\varepsilon_0^2}{4} \log n \right) = 2n^{-\frac{\varepsilon_0^2}{4}}$$

which gives

$$\sum_{x \in C_n} \mathbb{P} \left(\left| \widetilde{S}_{\ell, \gamma}(x) - \mathbb{E}[\widetilde{S}_{\ell, \gamma}(x)] \right| > \varepsilon \right) \leq 4d_n n^{-\frac{\varepsilon_0^2}{4} + \frac{\nu}{2}}.$$

Finally, an appropriate choice of ε_0 yields to an upper bound of order $n^{-3/2}$ and by Borel-Cantelli's lemma we get the result. ■

LEMMA 5.3 *Under Hypotheses H1, H2 i) and H4, for $\ell = 1, 2$ and $\gamma = 0, 1, 2$, we have*

$$\sup_{x \in C} \left| \mathbb{E}[\widetilde{S}_{\ell, \gamma}(x)] \right| = O(h^\gamma).$$

Proof of Lemma 5.3. Using the conditional expectation property, Taylor expansion, under Hypotheses H1, H2 i) and H4 and using the fact that $\mathbb{E}[T_1^{\star, -\ell} | X_1 = u] = \mu_\ell(u)$ with $\mu_\ell(u) = r_\ell(u)/f(u)$, we get

$$\begin{aligned} \left| \mathbb{E}[\widetilde{S}_{\ell, \gamma}(x)] \right| &= \left| \frac{1}{nh} \mathbb{E} \left[\sum_{i=1}^n T_i^{\star, -\ell} (X_i - x)^\gamma K_h(X_i - x) \right] \right| \\ &= \left| \frac{1}{h} \mathbb{E} \left[T_1^{\star, -\ell} (X_1 - x)^\gamma K_h(X_1 - x) \right] \right| \\ &= \left| \frac{1}{h} \int (u - x)^\gamma K_h(u - x) r_\ell(u) du \right| \\ &= \left| \int (vh)^\gamma K(v) r_\ell(x + vh) dv \right| \\ &= h^\gamma \left| \int v^\gamma K(v) \{r_\ell(x) + vhr'_\ell(\xi)\} dv \right| \\ &\leq h^\gamma r_\ell(x) \left| \int v^\gamma K(v) dv \right| + h^{\gamma+1} \left| \int v^{\gamma+1} K(v) r'_\ell(\xi) dv \right|. \\ &= O(h^\gamma). \end{aligned}$$

■

Now, combining on the one hand Lemma 5.1 and Lemma 5.3 and on the other hand Lemma 5.2 and Lemma 5.3, we conclude the proof of Proposition 1 ■

Proof of Proposition 2. Let remark the decomposition for $\ell = 1, 2$:

$$\begin{aligned} \widetilde{\mu}_\ell(x) - \mathbb{E}[\widetilde{\mu}_\ell(x)] &= \widetilde{S}_{2,2}(x)\widetilde{S}_{\ell,0}(x) - \widetilde{S}_{2,1}(x)\widetilde{S}_{\ell,1}(x) - \mathbb{E} \left[\widetilde{S}_{2,2}(x)\widetilde{S}_{\ell,0}(x) - \widetilde{S}_{2,1}(x)\widetilde{S}_{\ell,1}(x) \right] \\ &= \widetilde{S}_{2,2}(x)\widetilde{S}_{\ell,0}(x) - \mathbb{E} \left[\widetilde{S}_{2,2}(x)\widetilde{S}_{\ell,0}(x) \right] - \left\{ \widetilde{S}_{2,1}(x)\widetilde{S}_{\ell,1}(x) - \mathbb{E} \left[\widetilde{S}_{2,1}(x)\widetilde{S}_{\ell,1}(x) \right] \right\} \\ &=: \mathcal{E}_{\ell,1}(x) - \mathcal{E}_{\ell,2}(x). \end{aligned}$$

On the one side $\{\mathcal{E}_{\ell,1}, \text{ for } \ell = 1, 2\}$, we have

$$\begin{aligned} \mathcal{E}_{\ell,1}(x) &= \left(\widetilde{S}_{2,2}(x) - \mathbb{E} \left[\widetilde{S}_{2,2}(x) \right] \right) \left(\widetilde{S}_{\ell,0}(x) - \mathbb{E} \left[\widetilde{S}_{\ell,0}(x) \right] \right) + \mathbb{E} \left[\widetilde{S}_{2,2}(x) \right] \left(\widetilde{S}_{\ell,0}(x) - \mathbb{E} \left[\widetilde{S}_{\ell,0}(x) \right] \right) \\ &\quad + \mathbb{E} \left[\widetilde{S}_{\ell,0}(x) \right] \left(\widetilde{S}_{2,2}(x) - \mathbb{E} \left[\widetilde{S}_{2,2}(x) \right] \right) - Cov \left(\widetilde{S}_{\ell,0}(x), \widetilde{S}_{2,2}(x) \right). \end{aligned} \quad (5.5)$$

On the other side $\{\mathcal{E}_{\ell,2}, \text{ for } \ell = 1, 2\}$, we have

$$\begin{aligned} \mathcal{E}_{\ell,2}(x) &= (\widetilde{S}_{2,1}(x) - \mathbb{E}[\widetilde{S}_{2,1}(x)])(\widetilde{S}_{\ell,1}(x) - \mathbb{E}[\widetilde{S}_{\ell,1}(x)]) + \mathbb{E}[\widetilde{S}_{2,1}(x)](\widetilde{S}_{\ell,1}(x) - \mathbb{E}[\widetilde{S}_{\ell,1}(x)]) \\ &\quad + \mathbb{E}[\widetilde{S}_{\ell,1}(x)](\widetilde{S}_{2,1}(x) - \mathbb{E}[\widetilde{S}_{2,1}(x)]) - \text{Cov}(\widetilde{S}_{2,1}(x), \widetilde{S}_{\ell,1}(x)). \end{aligned} \quad (5.6)$$

It remains to study each term of the decomposition (5.5) and (5.6). We want to mention that most of the terms are studied in Lemma 5.2 and Lemma 5.3. The covariance terms are studied in the two following Lemmas.

LEMMA 5.4 Under Hypotheses H1, H2 and H4, for $\ell = 1, 2$ and n large enough, we have

$$\text{Cov}(\widetilde{S}_{\ell,0}(x), \widetilde{S}_{2,2}(x)) = o\left(\sqrt{\frac{\log n}{nh}}\right).$$

Proof of Lemma 5.4. By definition for $\ell = 1, 2$, we have

$$\text{Cov}(\widetilde{S}_{\ell,0}(x), \widetilde{S}_{2,2}(x)) = \mathbb{E}[\widetilde{S}_{\ell,0}(x)\widetilde{S}_{2,2}(x)] - \mathbb{E}[\widetilde{S}_{\ell,0}(x)]\mathbb{E}[\widetilde{S}_{2,2}(x)].$$

The proof will be made in three steps.

Step 1. It is easy to see that under H2 and H4 for $\ell = 1, 2$ and using Lemma 5.3, we get $\mathbb{E}[\widetilde{S}_{\ell,0}(x)] = O(1)$. Similarly, under H2 i) and H4 for $\ell = 2$ we have $\mathbb{E}[\widetilde{S}_{2,2}(x)] = O(h^2)$. Now, it remains to study the quantity $\mathbb{E}[\widetilde{S}_{\ell,0}(x)\widetilde{S}_{2,2}(x)]$. For that, it suffices to remark that

$$\begin{aligned} \mathbb{E}[\widetilde{S}_{\ell,0}(x)\widetilde{S}_{2,2}(x)] &= \frac{1}{(nh)^2} \mathbb{E}\left[\sum_{j=1}^n T_j^{*\,-\ell} K_h(X_j - x) \sum_{i=1}^n T_i^{*\,-2} (X_i - x)^2 K_h(X_j - x)\right] \\ &= \frac{1}{(nh)^2} \left\{ n \mathbb{E}\left[T_1^{*\,-\ell-2} (X_1 - x)^2 K_h^2(X_1 - x)\right] \right. \\ &\quad \left. + n(n-1) \mathbb{E}\left[T_1^{*\,-\ell} K_h(X_1 - x)\right] \mathbb{E}\left[T_1^{*\,-2} (X_1 - x)^2 K_h(X_1 - x)\right] \right\}. \end{aligned}$$

Step 2. Here after denote by $\varrho = \ell + 2$, for $\ell = 1, 2$. First, we have to calculate

$$\begin{aligned} \mathbb{E}\left[T_1^{*\,-\varrho} | X_1 = u\right] &= \mathbb{E}\left[\frac{\delta_1 Y_1^{-\varrho}}{\overline{G}^2(Y_1)} | X_1 = u\right] \\ &= \mathbb{E}\left[\frac{T_1^{-\varrho}}{\overline{G}^2(T_1)} \mathbb{E}[\mathbb{1}_{\{T_1 \leq R_1\}} | T_1] | X_1 = u\right] \\ &= \mathbb{E}\left[\frac{T_1^{-\varrho}}{\overline{G}(T_1)} | X_1 = u\right] \\ &\leq \frac{1}{\overline{G}(\tau_F)} \int t^{-\varrho} f_{T_1|X_1}(t|u) dt. \end{aligned} \quad (5.7)$$

Step 3. Then, using the conditional expectation property, Taylor expansion and under H2 ii) and H4, we have

$$\begin{aligned}
 \mathbb{E} \left[T_1^{\star, -\ell} (X_1 - x)^2 K_h^2 (X_1 - x) \right] &= \mathbb{E} \left[(X_1 - x)^2 K_h^2 (X_1 - x) \mathbb{E} \left[T_1^{\star, -\ell} | X_1 \right] \right] \\
 &= \int (u - x)^2 K_h^2 (u - x) \mathbb{E} \left[T_1^{\star, -\ell} | X_1 = u \right] f(u) du \\
 &\leq \frac{1}{\overline{G}(\tau_F)} \int (u - x)^2 K_h^2 (u - x) \int t^{-\ell} f_{T_1 | X_1}(t|u) dt f(u) du \\
 &= \frac{1}{\overline{G}(\tau_F)} \int (u - x)^2 K_h^2 (u - x) r_\varrho(u) du \\
 &= \frac{h^3}{\overline{G}(\tau_F)} \int v^2 K^2(v) r_\varrho(x + vh) dv \\
 &= \frac{h^3}{\overline{G}(\tau_F)} \int v^2 K^2(v) r_\varrho(x) dv + \frac{h^4}{\overline{G}(\tau_F)} \int v^3 K^2(v) r'_\varrho(\xi) dv.
 \end{aligned}$$

Finally, combining the three steps, we get

$$\text{Cov} \left(\widetilde{S}_{\ell, 0}(x), \widetilde{S}_{2, 2}(x) \right) = O \left(\frac{h}{n} \right)$$

which is negligible with respect to $\sqrt{\frac{\log n}{nh}}$. ■

LEMMA 5.5 Under Hypotheses **H1**, **H2 i**) and **H4**, for $\ell = 1, 2$ and n large enough, we have

$$\text{Cov} \left(\widetilde{S}_{\ell, 1}(x), \widetilde{S}_{2, 1}(x) \right) = o \left(\sqrt{\frac{\log n}{nh}} \right).$$

Proof of Lemma 5.5. In the same way, for $\ell = 1, 2$, write

$$\text{Cov} \left(\widetilde{S}_{\ell, 1}(x), \widetilde{S}_{2, 1}(x) \right) = \mathbb{E} \left[\widetilde{S}_{\ell, 1}(x) \widetilde{S}_{2, 1}(x) \right] - \mathbb{E} \left[\widetilde{S}_{\ell, 1}(x) \right] \mathbb{E} \left[\widetilde{S}_{2, 1}(x) \right].$$

We will use the following steps:

Step 4. It is easy to see that from **Lemma 5.3** under **H2** and **H4** for $\ell = 1, 2$ we have $\mathbb{E} \left[\widetilde{S}_{\ell, 1}(x) \right] = O(h)$. Similarly, under **H2 i**) and **H4** for $\ell = 2$ we have $\mathbb{E} \left[\widetilde{S}_{2, 1}(x) \right] = O(h)$. Now, it remains to study the quantity $\mathbb{E} \left[\widetilde{S}_{\ell, 1}(x) \widetilde{S}_{2, 1}(x) \right]$. To do that, Let us remark that

$$\begin{aligned}
 \mathbb{E} \left[\widetilde{S}_{\ell, 1}(x) \widetilde{S}_{2, 1}(x) \right] &= \frac{1}{(nh)^2} \mathbb{E} \left[\sum_{j=1}^n T_j^{\star, -\ell} (X_j - x) K_h(X_j - x) \sum_{i=1}^n T_i^{\star, -2} (X_i - x) K_h(X_i - x) \right] \\
 &= \frac{1}{(nh)^2} \left\{ n \mathbb{E} \left[T_1^{\star, -\ell-2} (X_1 - x)^2 K_h^2 (X_1 - x) \right] \right. \\
 &\quad \left. + n(n-1) \mathbb{E} \left[T_1^{\star, -\ell} (X_1 - x) K_h(X_1 - x) \right] \times \mathbb{E} \left[T_1^{\star, -2} (X_1 - x) K_h(X_1 - x) \right] \right\}.
 \end{aligned}$$

Step 5. We use the same notation $\varrho = \ell + 2$ to avoid any confusion and by (5.7) we have

$$\begin{aligned}
 \mathbb{E} \left[T_1^{\star, -\varrho} (X_1 - x)^2 K_h^2(X_1 - x) \right] &= \mathbb{E} \left[(X_1 - x)^2 K_h^2(X_1 - x) \mathbb{E} \left[T_1^{\star, -\varrho} | X_1 \right] \right] \\
 &= \int (u - x)^2 K_h^2(u - x) \mathbb{E} \left[T_1^{\star, -\varrho} | X_1 = u \right] f(u) du \\
 &= \frac{1}{\overline{G}(\tau_F)} \int (u - x)^2 K_h^2(u - x) \int t^{-\varrho} f_{T_1, X_1}(t, u) dt du \\
 &= \frac{1}{\overline{G}(\tau_F)} \int (u - x)^2 K_h^2(u - x) r_\varrho(u) du \\
 &= \frac{h}{\overline{G}(\tau_F)} \int (vh)^2 K^2(v) r_\varrho(x + vh) dv \\
 &= \frac{h^3}{\overline{G}(\tau_F)} \int v^2 K^2(v) r_\varrho(x + vh) dv,
 \end{aligned}$$

under H2 ii) and H4, we get $O(h^3)$.
 Combining steps 4 and 5 we have

$$\text{Cov}(\widetilde{S}_{\ell, 1}(x), \widetilde{S}_{2, 1}(x)) = O\left(\frac{h}{n}\right)$$

which is negligible with respect to $\sqrt{\frac{\log n}{nh}}$. ■

Finally, combining Lemma 5.2 and Lemma 5.3 in the proof of Proposition 1 with Lemma 5.4 and Lemma 5.5, we get the result of the Proposition 2. ■

Proof of Proposition 3. Using the conditional expectation property for $\ell = 1, 2$ we get $\mathbb{E}[T_1^{\star, -2} T_2^{\star, -\ell} | X_1, X_2] = \mu_2(X_1) \mu_\ell(X_2)$. Then we have

$$\begin{aligned}
 \mathbb{E}[\widetilde{\mu}_\ell(x)] - r_\ell(x) r_2(x) &= \mathbb{E} \left[\widetilde{S}_{2, 2}(x) \widetilde{S}_{\ell, 0}(x) - \widetilde{S}_{2, 1}(x) \widetilde{S}_{\ell, 1}(x) \right] - r_\ell(x) r_2(x) \\
 &= \frac{1}{(nh)^2} \mathbb{E} \left[\left(\sum_{i=1}^n (X_i - x)^2 T_i^{\star, -2} K_h(X_i - x) \right) \left(\sum_{j=1}^n T_j^{\star, -\ell} K_h(X_j - x) \right) \right. \\
 &\quad \left. - \left(\sum_{i=1}^n (X_i - x) T_i^{\star, -2} K_h(X_i - x) \right) \left(\sum_{j=1}^n T_j^{\star, -\ell} (X_j - x) K_h(X_j - x) \right) \right] - r_\ell(x) r_2(x) \\
 &= h^{-2} \mathbb{E} \left[\left((X_1 - x)^2 T_1^{\star, -2} K_h(X_1 - x) \right) \left(T_2^{\star, -\ell} K_h(X_2 - x) \right) \right. \\
 &\quad \left. - \left((X_1 - x) T_1^{\star, -2} K_h(X_1 - x) \right) \left(T_2^{\star, -\ell} (X_2 - x) K_h(X_2 - x) \right) \right] - r_\ell(x) r_2(x) \\
 &= h^{-2} \int \int K(u - x) K(v - x) \left((u - x)^2 - (u - x)(v - x) \right) \\
 &\quad \times \mathbb{E}[T_1^{\star, -2} T_2^{\star, -\ell} | X_1 = u, X_2 = v] f(u) f(v) dudv - r_\ell(x) r_2(x) \\
 &= h^{-2} \int \int K(u - x) K(v - x) \left((u - x)^2 - (u - x)(v - x) \right) r_2(u) r_\ell(v) dudv - r_\ell(x) r_2(x).
 \end{aligned}$$

By a change of variable, we get

$$\begin{aligned} &= \int \int K(t)K(s) \left((th)^2 - (th)(sh) \right) r_2(x+th)r_\ell(x+sh) dt ds - r_\ell(x)r_2(x) \\ &= h^2 \int \int (t^2 - ts)K(t)K(s)r_2(x+th)r_\ell(x+sh) dt ds - r_\ell(x)r_2(x) \\ &= h^2 \int \int (t^2 - ts)K(t)K(s) \{ r_2(x+th)r_\ell(x+sh) - r_\ell(x)r_2(x) \} dt ds \end{aligned}$$

Using Taylor expansion, it is easy to see for $\ell = 1, 2$

$$\begin{aligned} r_2(x+th)r_\ell(x+sh) - r_\ell(x)r_2(x) &= (r_2(x+th) - r_2(x))(r_\ell(x+sh) - r_\ell(x)) \\ &\quad + r_\ell(x)(r_2(x+th) - r_2(x)) + r_2(x)(r_\ell(x+sh) - r_\ell(x)) \\ &= h^2 tsr'_2(\xi_1)r'_\ell(\xi_2) + thr'_2(\xi_1)r_\ell(x) + shr'_\ell(\xi_2)r_2(x) \end{aligned}$$

where $\xi'_1 \in]x, x+th[$ and $\xi'_2 \in]x, x+sh[$. Then, we have

$$\begin{aligned} \mathbb{E}[\widetilde{\mu}_\ell(x)] - r_\ell(x)r_2(x) &= h^2 \int \int (t^2 - ts)K(t)K(s) \{ h^2 tsr'_2(\xi_1)r'_\ell(\xi_2) + thr'_2(\xi_1)r_\ell(x) + shr'_\ell(\xi_2)r_2(x) \} dt ds \\ &= h^4 \int \int ts(t^2 - ts)K(t)K(s)r'_2(\xi_1)r'_\ell(\xi_2) dt ds + h^3 \int \int t(t^2 - ts)K(t)K(s)r'_2(\xi_1)r_\ell(x) dt ds \\ &\quad + h^3 \int \int s(t^2 - ts)K(t)K(s)r'_\ell(\xi_2)r_2(x) dt ds \\ &= h^4 \left(\left(\int t^3 K(t)r'_2(\xi_1) dt \right) \left(\int sK(s)r'_\ell(\xi_2) ds \right) - \left(\int t^2 K(t)r'_2(\xi_1) dt \right) \left(\int s^2 K(s)r'_\ell(\xi_2) ds \right) \right) \\ &\quad + h^3 \left(\left(r_\ell(x) \int t^3 K(t)r'_2(\xi_1) dt \right) \left(\int K(s) ds \right) - \left(r_\ell(x) \int sK(s) ds \right) \left(\int t^2 K(t)r'_2(\xi_1) dt \right) \right) \\ &\quad + h^3 \left(\left(r_2(x) \int t^2 K(t) dt \right) \left(\int sK(s)r'_\ell(\xi_2) ds \right) - \left(r_2(x) \int tK(t) dt \right) \left(\int s^2 K(s)r'_\ell(\xi_2) ds \right) \right). \end{aligned}$$

Under the hypotheses **H2 i)** and **H4** for $\ell = 1, 2$, the result can be deduced directly from the last equality $O(h^3)$. ■

6. Appendix

Corollary A.8. in [Ferraty and Vieu \(2006\)](#) p. 234 .

Let U_i be a sequence of independent r.v. with zero mean. If $\forall m \geq 2, \exists C_m > 0, \mathbb{E}[|U_1^m|] \leq C_m a^{2(m-1)}$, we have

$$\forall \varepsilon > 0, \quad \mathbb{P} \left(\left| \sum_{i=1}^n U_i \right| > n\varepsilon \right) \leq 2 \exp \left\{ -\frac{\varepsilon^2 n}{2a^2(1 + \varepsilon)} \right\}.$$

7. Conclusion

In this paper we establish the uniform strong consistency with rate for the local linear relative error regression estimator over a compact set, when the variable of interest is subject to random

right censoring. A large simulation study was conducted through which our estimator performance was highlighted in spite of well known boundary effects of kernel estimation. On the one hand, for a practical point of view the results indicate the lack of flexibility in estimating a function using traditional approaches. On the other hand, the proposed estimates are closest to the true curve. In conclusion, the LLRER method has more advantage than the CR and LLCR such as the efficiency in presence of outliers and censorship compared to the two other methods. Finally, we point out that the bias term appears to inhabit, however the combination of the two methods LL and RER has revealed several terms which do not allow to obtain a standard result of order one or two. Conversely, we can say that the reduction of the bias is highlighted.

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