



# Isochronous oscillatory motions and the quantum spectrum

Abd Raouf Chouikha

## ► To cite this version:

Abd Raouf Chouikha. Isochronous oscillatory motions and the quantum spectrum. International Journal on Applied Physics and Engineering, 2022, 1, pp.95-101. 10.37394/232030.2022.1.10 . hal-02532457

**HAL Id: hal-02532457**

**<https://hal.science/hal-02532457>**

Submitted on 5 Apr 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Isochronous oscillatory motions and the quantum spectrum

Abd Raouf Chouikha \*

## Abstract

We had proposed necessary and sufficient conditions for isochrony of oscillatory motions. Thanks to the WKB perturbation method we derived expressions for the corrections to the equally spaced valid for analytic isochronous potentials [1]. In this paper, we bring some improvements of these results and we suggest another quantization of the quantum spectrum.

*Key Words and phrases:* oscillatory motions, isochronicity, WKB method, quantum spectrum.<sup>1</sup>

## 1 Isochronous oscillatory motions

There is a link between classical and quantum transformations. This fact has been established by Eleonskii and al. [2]. They show that the classical limit of the isospectral transformation for the Schrodinger equation is precisely the isochronicity preserving the energy dependence of the oscillation frequency. In quantum mechanics, the energy levels of a parabolic well are regularly spaced by a certain quantity. Moreover, it is possible to construct potentials, essentially different from the parabolic well, whose spectrum is exactly harmonic.

The semiclassical WKB method is one of powerful approximations for computing the energy eigenvalues of the Schrodinger equation. The field of its applicability is larger than standard perturbation theory which is restricted to perturbing potentials with small coupling constants. In particular,

---

\*chouikha@math.univ-paris13.fr. 4, Cour des Quesblais 35430 Saint-Pere

<sup>1</sup>2010 Mathematics Subject Classification : 34A34, 34E20, 34C25, 37C10.

PACS numbers: 45.05; 02.30.Hq

it permits to write the quantisation condition as a power series in  $\hbar$  (such series are generally non convergent). The solvable potentials are those whose series can be explicitly summed.

Consider the scalar equation with a center at the origin 0

$$\ddot{x} + g(x) = 0 \quad (1)$$

or its planar equivalent system

$$\dot{x} = y, \quad \dot{y} = -g(x) \quad (2)$$

where  $\dot{x} = \frac{dx}{dt}$ ,  $\ddot{x} = \frac{d^2x}{dt^2}$  and  $g(x) = \frac{dG(x)}{dx}$  is analytic on  $R$  where  $G(x)$  is the potential of (1).

Suppose system (2) admits a periodic orbit in the phase plane with energy  $E$  and  $g(x)$  has bounded period for real energies  $E$ . Given  $G(x)$ , Let  $T(E)$  denotes the minimal period of this periodic orbit. Its expression is

$$T(E) = 2 \int_a^b \frac{dx}{\sqrt{2E - 2G(x)}}. \quad (3)$$

$T(E)$  is well defined and there is a neighborhood of the real axis for which  $T(E)$  is analytic.

We suppose that the potential  $G(x)$  has one minimum value which, for convenience locate at the origin 0 and  $\frac{d^2G(x)}{dx^2}(0) = 1$ . The turning points  $a, b$  of this orbit are solutions of  $G(x) = E$ . Then the origin 0 is a center of (2). This center is isochronous when the period of all orbits near  $0 \in R^2$  are constant ( $T = \frac{2\pi}{\sqrt{g'(0)}} = 2\pi$ ). The corresponding potential  $G(x)$  is also called isochronous.

Since the potential  $G(x)$  has a local minimum at 0, then we may consider an involution  $A$  by

$$G(A(x)) = G(x) \text{ and } A(x)x < 0$$

for all  $x \in [a, b]$ . So, any closed orbit is  $A$ -invariant and  $A$  exchanges the turning points:  $b = A(a)$ .

We proved the following results in [1]

**Theorem A** *Let  $g(x)$  be an analytic function and  $G(x) = \int_0^x g(s)ds$  and  $A$  be the analytic involution defined by  $G(A(x)) = G(x)$ . Suppose that for  $x \neq 0, xg(x) > 0$ . Then the equation*

$$\ddot{x} + g(x) = 0 \quad (1)$$

*has an isochronous center at 0 if and only if the function*

$$\frac{d}{dx}[G(x)/g^2(x)]$$

*is  $A$ -invariant i.e.  $\frac{d}{dx}[G/g^2](x) = \frac{d}{dx}[G/g^2](A(x))$  in some neighborhood of 0.*

**Theorem B** *Let  $G(x) = \int_0^x g(s)ds$  be an analytic potential. Suppose that for  $x \neq 0, xg(x) > 0$ . Then the equation*

$$\ddot{x} + g(x) = 0 \quad (1)$$

*has an isochronous center at 0 if and only if*

$$x - \frac{2G}{g} = F(G) \quad (4)$$

*where  $F$  is an analytic function defined in some neighborhood of 0.*

As consequences we prove the following

**Theorem 1-1** *Let  $G(x) = \int_0^x g(s)ds$  be an analytic potential defined in a neighborhood of 0. Suppose equation*

$$\ddot{x} + g(x) = 0 \quad (1)$$

*has an isochronous center at 0. Let  $g^{(n)}(x)$  be the  $n$ -th derivative of the potential (with respect to  $x$ ):  $g^{(n)}(x) = \frac{d^n}{dx^n}G(x)$ ,  $n \geq 1$  then these derivatives may be expressed under the form*

$$g^{(n)}(x) = a_n(G)x + b_n(G), \quad n \geq 0 \quad (5)$$

*where  $a_n$  and  $b_n$  are analytic functions with respect to  $G$ .*

In fact, as we had see in [1], the functions  $a_n$  and  $b_n$  are only dependent on  $G_1$  the odd part of  $G = G(x)$ .

**Proof of Theorem 1-1** By Proposition 3-4 of [1], condition  $x(G) = \sqrt{2G} + P(G)$  with  $P = P(G)$  is a non-zero analytic function implies that equation (1) has an isochronous center at 0. Deriving with respect to  $G$  one obtains

$$\frac{dx}{dG} = \frac{1}{x} + P'(G) = \frac{1}{g}$$

or equivalently

$$\frac{g}{x} = \frac{1}{1 + xP'(G)} = a_1(G) + \frac{b_1(G)}{x}$$

with

$$a_1(G) = \frac{-1}{2GP'^2 - 1} \quad \text{and} \quad b_1(G) = \frac{2GP'}{2GP'^2 - 1}.$$

Notice that by hypothesis  $G$  is defined in a neighborhood of 0 then  $2GP'^2 - 1$  is necessary non zero.

The functions  $a_1(G)$  and  $b_1(G)$  are analytic since  $P$  and  $P'$  they are too.

Derive now  $g'(x)$  it yields

$$\begin{aligned} g'(x) &= \frac{dg}{dx} = \frac{d}{dx} a_1(G) x + a_1(G) + \frac{d}{dx} b_1(G) = \\ &= \frac{d}{dG} a_1(G) g x + \frac{d}{dG} b_1(G) g + a_1(G). \\ g'(x) &= \left( \left( \frac{d}{dG} a_1 \right) x + \frac{d}{dG} b_1 \right) (a_1 x + b_1) + a_1(G) \end{aligned}$$

where the symbol prime ' means  $\frac{d}{dG}$  and  $a_1$  or  $b_1$  stands for  $a_1(G)$  or  $b_1(G)$ .

After replacing  $g(x) = a_1(G)x + b_1(G)$  one obtains

$$2Ga'_1a_1 + a'_1xb_1 + a_1^2 + 1/2 \frac{a_1\sqrt{2}b_1}{\sqrt{G}} + b'_1a_1x + b'_1b_1$$

By simplifying one find the expression of  $g'(x) = a_2(G)x + b_2(G)$  with

$$a_2(G) = a'_1b_1 + \frac{a_1b_1}{2G} + b'_1a_1$$

$$b_2(G) = 2Ga_1a'_1 + a_1^2 + b_1b'_1$$

Here

$$\frac{a_1b_1}{2G} = \frac{P'}{(2GP'^2 - 1)^2}$$

which is analytic. Then the functions  $a_2(G)$  and  $b_2(G)$  are analytically dependent on the functions  $a_1(G), b_1(G)$  and their derivatives.

By recurrence we easily prove that

$$g^{(p)}(x) = a_p(G)x + b_p(G)$$

where the function  $a_p(G)$  and  $b_p(G)$  are analytic with respect to  $G$ . Thank to *Maple* we are able to carry out the calculations.

**Theorem 1-2** *Let  $G(x) = \int_0^x g(s)ds$  be an analytic potential and  $\phi(x)$  a function defined in a neighborhood of 0. A be the analytic involution defined by  $G(A(x)) = G(x)$ . Then for  $a < 0 < b = A(a)$  and  $G(a) = G(b) = E$  the following integrals equality holds*

$$\int_a^b \frac{\phi(x)}{\sqrt{E - G(x)}} g(x) dx = \int_0^b \frac{\phi(x) - \phi(A(x))}{\sqrt{E - G(x)}} g(x) dx$$

In particular, if we may expressed  $\phi(x) = u(G)x + v(G)$  then

$$\int_a^b \frac{\phi(x)}{\sqrt{E - G(x)}} g(x) dx = \int_0^b \frac{2u(G)x}{\sqrt{E - G(x)}} g(x) dx$$

**Proof of Theorem 1-2** It suffices to split the integral

$$\int_a^b \frac{\phi(x)}{\sqrt{E - G(x)}} g(x) dx = \int_a^0 \frac{\phi(x)}{\sqrt{E - G(x)}} g(x) dx + \int_0^b \frac{\phi(x)}{\sqrt{E - G(x)}} g(x) dx$$

Recall that  $a < 0 < b$ . By definition when  $x \in [a, 0]$  then  $A(x) \in [0, b]$  and conversely. By a change of variable  $x = A(y)$  the integral becomes

$$\int_a^0 \frac{\phi(x)}{\sqrt{E - G(x)}} g(x) dx = - \int_0^b \frac{\phi(A(y))}{\sqrt{E - G(y)}} g(A(y)) A'(y) dy = - \int_0^b \frac{\phi(A(y))}{\sqrt{E - G(y)}} g(y) dy$$

since  $g(A(y))A'(y) = g(y)$ . Therefore

$$\int_a^b \frac{\phi(x)}{\sqrt{E - G(x)}} g(x) dx = \int_0^b \frac{\phi(x)}{\sqrt{E - G(x)}} g(x) dx - \int_0^b \frac{\phi(A(y))}{\sqrt{E - G(y)}} g(y) dy = .$$

On the other hand, suppose  $\phi(x) = u(G)x + v(G)$ . Then the following integral may be written

$$\begin{aligned} \int_a^b \frac{\phi(x)}{\sqrt{E - G(x)}} g(x) dx &= \int_a^b \frac{u(G)x + v(G)}{\sqrt{E - G(x)}} g(x) dx = \\ &= \int_a^b \frac{u(G)x}{\sqrt{E - G(x)}} g(x) dx + \int_a^b \frac{v(G)}{\sqrt{E - G(x)}} g(x) dx \end{aligned}$$

The last integral can be written

$$\int_a^b \frac{v(G)}{\sqrt{E-G(x)}} g(x) dx = \int_0^E \frac{v(G)}{\sqrt{E-G}} dG = 0$$

since  $v(G)$  is analytic. The other integral can be written

$$\begin{aligned} \int_a^b \frac{u(G)x}{\sqrt{E-G(x)}} g(x) dx &= \int_a^0 \frac{u(G)x}{\sqrt{E-G(x)}} g(x) dx + \int_0^b \frac{2u(G)x}{\sqrt{E-G(x)}} g(x) dx \\ &= \int_0^b \frac{u(G)y}{\sqrt{E-G(y)}} g(A(y)) A'(y) dy = \int_0^b \frac{-u(G)y}{\sqrt{E-G(y)}} g(y) dy = \int_0^b \frac{u(G)y}{\sqrt{E-G(y)}} g(y) dy \end{aligned}$$

since  $y = A(x)$ . Finally,

$$\int_a^b \frac{\phi(x)}{\sqrt{E-G(x)}} g(x) dx = \int_0^b \frac{u(G)x}{\sqrt{E-G(x)}} g(x) dx + \int_0^b \frac{u(G)x}{\sqrt{E-G(x)}} g(x) dx$$

We may also derive

**Corollary 1-3** *Under hypotheses of Proposition C, consider the derivatives of  $g : g^{(j)}(x) = \frac{d^j g}{dx^j}$ . Then the analytic function*

$$V_{m,\nu}(x) = \prod_{j=1}^m \left( \frac{d^j g}{dx^j} \right)_j^\nu$$

*may be expressed under the form :*

$$V_{m,\nu}(x) = u_{m,\nu}(G)x + v_{m,\nu}(G) \quad (6)$$

*where  $\nu = (\nu_1, \nu_2, \dots, \nu_m)$  and  $u_{m,\nu}$  and  $v_{m,\nu}$  are analytic functions with respect to  $G$ .*

*Proof* By Proposition 1-1 any derivative of  $g$  may be written when  $G$  is isochronous

$$g^{(n)}(x) = a_n(G)x + b_n(G), n \geq 0$$

where  $a_n$  and  $b_n$  being analytic functions. It is easy to realize that it is the same for any power of any derivative  $(g^{(n)}(x))_n^\nu$ . We may prove that by recurrence

$$(g^{(n)}(x))_n^\nu = a_{n,\nu}(G)x + b_{n,\nu}(G).$$

More generally, we may also prove by recurrence that a product of power of derivatives have the similar expression

$$(g^{(n)}(x))_1^\nu (g^{(p)}(x))_2^\nu = a_{n,p,\nu}(G)x + b_{n,p,\nu}(G).$$

Thus we may write for any product

$$V_{m,\nu}(x) = \prod_{j=1}^m \left( \frac{d^j g}{dx^j} \right)_j^\nu = u_{m,\nu}(G)x + v_{m,\nu}(G)$$

## 2 Application: The quantum spectrum and WKB quantization

Consider the Schrodinger equation

$$\left[ -\frac{\hbar^2}{2} \frac{d^2}{dx^2} + G(x) \right] \psi(x) = E\psi(x). \quad (7)$$

The Hamiltonian of the system is given by

$$H = \frac{p^2}{m} + G(x)$$

where the mass  $m = 1$ .

This Hamiltonian is a constant of motion, whose value is equal to the total energy  $E$ .

The wave function can always be written as

$$\psi(x) = \exp\left(\frac{i}{\hbar}\sigma(x)\right)$$

The WKB expansion for the phase is a power series in  $\hbar$  :

$$\sigma(x) = \sum_0^\infty \left(\frac{\hbar}{i}\right)^k \sigma_k(x).$$

Following [1], [2] rewrite the quantisation condition as

$$\sum_0^\infty I_{2k}(E) = \left(n + \frac{1}{2}\right)\hbar, \quad n \in \mathbb{N}$$

1where

$$I_{2k}(E) = \frac{1}{2\pi} \left(\frac{\hbar}{i}\right)^{2k} \int_\gamma d\sigma_{2k}, \quad k \in \mathbb{N}. \quad (8)$$



When  $G(x)$  is analytic and  $xg(x) > 0$ , it has been proved that the contour integrals can be replaced by equivalent Rieman integrals between the two turning points. More precisely,

$$I_2(E) = -\frac{\hbar^2}{24\sqrt{2\pi}} \frac{\partial^2}{\partial E^2} \int_a^b \frac{g^2(x)}{\sqrt{E-G(x)}} dx$$

and

$$I_4(E) = -\frac{\hbar^4}{4\sqrt{2\pi}} \left[ \frac{1}{120} \frac{\partial^3}{\partial E^3} \int_a^b \frac{g'^2(x)}{\sqrt{E-G(x)}} dx - \frac{1}{288} \frac{\partial^4}{\partial E^4} \int_a^b \frac{g^2(x)g'(x)}{\sqrt{E-G(x)}} dx \right]$$

One establishes the following

$$\oint_{\gamma} d\sigma_m = 2 \sum_{L(\nu)=m} \frac{2^{\frac{m}{2}-1+|\nu|} i}{(m-3+2|\nu|)!!} \frac{\partial^{\frac{m}{2}-1+|\nu|}}{\partial E^{\frac{m}{2}-1+|\nu|}} \int_a^b \frac{U_{\nu} G^{(\nu)}}{\sqrt{E-G}} dx \quad (9)$$

where

$$G^{(\nu)}(x) = \prod_{j=1}^m \left( \frac{d^j G}{dx^j} \right)_j^{\nu}$$

Coefficients  $U_{\nu}$  are defined by a recurrence equation.

Notice that higher order corrections quickly increase in complexity and we know only few cases where a WKB expansion can be worked to all orders. Which resulting in a convergent series whose sum is identical to the exact spectrum.

It is known, the spectrum of a potential is generally not strictly regularly spaced, except for the harmonic  $G(x) = \frac{1}{2}x^2$  and the isotonic ones:

$$G(x) = \frac{1}{8\alpha^2} [\alpha x + 1 - \frac{1}{\alpha x + 1}]^2$$

There are isochronous potentials with a strictly equally spaced (harmonic) spectrum.

By Theorem 1-1, we may write  $g = \frac{dG}{dx} = a(G)x + b(G)$ . Writing

$$I_2(E) = \frac{-\hbar^2}{24\sqrt{2\pi}} \frac{\partial^2}{\partial E^2} \int_a^b \frac{g^2(x)}{\sqrt{E-G(x)}} dx = \frac{-\hbar^2}{24\sqrt{2\pi}} \frac{\partial^2}{\partial E^2} \int_a^b \frac{g(x)}{\sqrt{E-G(x)}} g(x) dx$$

and by Theorem 1-2 we may express

$$-\frac{\hbar^2}{24\sqrt{2}\pi} \frac{\partial^2}{\partial E^2} \int_0^b \frac{2a(G)x}{\sqrt{E-G(x)}} g(x) dx = -\frac{\hbar^2}{24\sqrt{2}\pi} \frac{\partial^2}{\partial E^2} \int_0^E \frac{2a(v)\sqrt{x(v)}}{\sqrt{E-v}} dv.$$

Then making the change of variables  $u = \frac{v}{E}$  (we suppose here  $\omega = 1$ )

$$I_2(E) = -\frac{\hbar^2}{24\pi} \frac{\partial^2}{\partial E^2} [E \int_0^1 \frac{2a(uE)x(u)}{\sqrt{1-u}} du].$$

A similar calculation gives the fourth order correction

$$I_4(E) = -\frac{\hbar^4}{4\sqrt{2}\pi} [\frac{1}{120} \frac{\partial^3}{\partial E^3} \int_a^b \frac{g'^2(x)}{g(x)\sqrt{E-G(x)}} g(x) dx \\ - \frac{1}{288} \frac{\partial^4}{\partial E^4} \int_a^b \frac{g(x)g'(x)}{\sqrt{E-G(x)}} g(x) dx].$$

By Theorem 1-1 one gets  $g = \frac{dG}{dx} = a(G)x + b(G)$   
and  $g' = \frac{dg}{dx} = a_1(G)\sqrt{2G} + b_1(G)$ . However, we may easily prove

$$\frac{g'^2}{g} = a_{1,2}(G)x + b_{1,2}(G)$$

where  $a_{1,2}(G)$  and  $b_{1,2}(G)$  are analytic functions. As well as

$$g(x)g'(x) = c_{1,2}(G)x + d_{1,2}(G)$$

where  $c_{1,2}(G)$  and  $d_{1,2}(G)$  are analytic functions. Similar to  $I_2, I_4$  may be expressed through Abel integrals :

$$I_4(E) = -\frac{\hbar^4}{4\pi} [\frac{E^{-3}}{120} \int_0^E \frac{(x(v))^5}{\sqrt{E-v}} \frac{\partial^3}{\partial v^3} a_{1,2}(v) dv \\ - \frac{E^{-4}}{288} \int_0^E \frac{(x(v))^7}{\sqrt{E-v}} \frac{\partial^4}{\partial v^4} c_{1,2}(v) dv].$$

Thus one can choose  $I_2(E)$  and deduce the corresponding analytic isochronous potential such that, its asymptotic decay is faster than the asymptotic decay of  $I_4(E)$ . Therefore,  $I_2(E)$  and  $I_4(E)$  grow exponentially fast as  $E$  grows to  $\infty$ . We will get the similar for higher order corrections.

Turn now to upper order WKB correction. Following [1], [3] the explicit expression for  $I_{2n}(E)$  is given by

$$I_{2n}(E) = -\frac{\sqrt{2}}{\pi} \hbar^{2n} \sum_{L(\nu)=2n} \frac{2^{|\nu|}}{(2n-3+2|\nu|)!!} J_\nu(E) \quad (10)$$

where

$$J_\nu(E) = \frac{\partial^{n-1+|\nu|}}{\partial E^{n-1+|\nu|}} \int_a^b \frac{U_\nu G^{(\nu)}}{\sqrt{E-G}} dx$$

where

$$G^{(\nu)}(x) = \prod_{j=1}^{2n} \left( \frac{d^j G}{dx^j} \right)_j^\nu$$

and where  $\nu = (\nu_1, \nu_2, \dots, \nu_{2n})$ ,  $\nu_j \in N$ ,  $L(\nu) = \sum_{j=1}^{2n} j\nu_j$  and  $|\nu| = \sum_{j=1}^{2n} \nu_j$ . The coefficients  $U_\nu$  satisfy a certain recurrence relation.

By Corollary 1-2  $G^{(\nu)}$  may be expressed under the form

$$G^{(\nu)}(x) = \prod_{j=1}^n \left( \frac{d^j G}{dx^j} \right)_j^\nu = u_{n,\nu}(G)x + v_{n,\nu}(G)$$

where  $u_{n,\nu}$  and  $v_{n,\nu}$  are analytic functions with respect to  $G$ . Therefore,

$$J_\nu(E) = \frac{\partial^{n-1+|\nu|}}{\partial E^{n-1+|\nu|}} \int_a^b \frac{U_\nu u_{n,\nu}(G)x}{\sqrt{E-G}} g(x) dx$$

By Corollary 1-1, we can write

$$J_\nu(E) = \frac{\partial^{n-1+|\nu|}}{\partial E^{n-1+|\nu|}} \int_0^b \frac{2U_\nu u_{n,\nu}(G)x}{\sqrt{E-G}} g(x) dx$$

$$J_\nu(E) = \frac{\partial^{n-1+|\nu|}}{\partial E^{n-1+|\nu|}} \int_0^E \frac{2U_\nu u_{n,\nu}(v)x(v)}{\sqrt{E-v}} dv$$

Another equivalent formulation via Abel integrals

$$J_\nu(E) = 2U_\nu E^{-n+1-|\nu|} \int_0^E \frac{(x(v))^{n-2+|\nu|}}{\sqrt{E-v}} \frac{\partial^{n-1+|\nu|}}{\partial E^{n-1+|\nu|}} u_{n,\nu}(v) dv$$

Similar to  $I_2$  and  $I_4$  the  $n$ th correction  $I_{2n}$  will be expressed through Abel integrals :

$$I_{2n}(E) = \frac{-\hbar^{2n}}{\pi} \sum_{L(\nu)=2n} \frac{2^{|\nu|+1} U_\nu E^{-n+1-|\nu|}}{(2n-3+2|\nu|)!!} \int_0^E \frac{(x(v))^{n-2+|\nu|}}{\sqrt{E-v}} \frac{\partial^{n-1+|\nu|}}{\partial E^{n-1+|\nu|}} u_{n,\nu}(v) dv$$

where  $u_{n,\nu}(G)$  is such that

$$G^{(\nu)}(x) = \prod_{j=1}^m \left( \frac{d^j G}{dx^j} \right)_j^\nu = u_{n,\nu}(G)x + v_{n,\nu}(G).$$

**Concluding remarks** We have therefore highlighted another no less complicated expression of  $I_{2n}(E)$  in the general case. The natural question is which of the two is more appropriate to use. In fact, it will depend on the type of isochronous potentials. In some situations, it may be easier to use either of these formulas. As we remarked higher order corrections quickly increase in complexity. Here too the WKB corrections  $I_{2n}(E)$  grow exponentially fast as  $E$  grows to  $\infty$ . The WKB series should be summed for any isochronous potential and would be finite as  $E$  grows to  $\infty$ .

#### DATA AVAILABILITY STATEMENTS :

Data available on request from the author

#### REFERENCES

- [1] Chouikha A.R. 2019, *On isochronous analytic motions and the quantum spectrum* Physica Scripta, Volume 94, Number 12.
- [2] Eleonskii VM, Korolev VG and Kulagin NE 1997, *On a classical analog of the isospectral Schrodinger problem*, JETP Lett., 65 (11), 889.
- [3] Robnik M and Salasnich L 1997, *WKB to all orders and the accuracy of the semiclassical quantization*, J. Phys. A: Math. Gen., 30, 1711.