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# Fractional $BV$ solutions for $2 \times 2$ systems of conservation laws with a genuinely nonlinear field and a linearly degenerate field

Boris Haspot <sup>\*</sup>, Stéphane Junca <sup>†</sup>

**Abstract:** The class of  $2 \times 2$  nonlinear hyperbolic systems with a genuinely nonlinear field and one linearly degenerate field are considered. Existence of global weak solutions for small initial data in fractional BV spaces  $BV^s$  is proved. The exponent  $s$  is related to the usual fractional Sobolev derivative. Riemann invariants  $w$  and  $z$  corresponding respectively to the genuinely nonlinear field and to the linearly degenerate field play different key roles in this work. We obtain the existence of a global weak solution provided that the initial data written in Riemann coordinates  $(w_0, z_0)$  are small in  $BV^s \times L^\infty$ ,  $1/3 \leq s < 1$ . The restriction on the exponent  $s$  is due to a fundamental result of P.D. Lax, the variation of the Riemann invariant  $z$  on the Lax shock curve depends in a cubic way on the variation of the other Riemann invariant  $w$ .

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# 1 Introduction

In this paper, we study general  $2 \times 2$  hyperbolic systems of the form:

$$\begin{cases} \partial_t U + \partial_x F(U) = 0 \\ U(0, \cdot) = U_0(\cdot) \end{cases} \quad (1.1)$$

with  $U(t, x) \in \Omega \subset \mathbb{R}^2$  an open set,  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ ,  $U_0 = U_0(x)$  is the initial data.  $F$  is the flux of the system and it is regular from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . We assume that the system is strictly hyperbolic on  $\Omega$ , it means that  $DF(U)$  has two different eigenvalues  $\lambda_1$  and  $\lambda_2$ . Without any restriction we can assume that

$$\lambda_1 < 0 < \lambda_2, \quad (1.2)$$

reducing if necessary the open set  $\Omega$ . It implies in particular that we have a basis of eigenvectors of unit norm  $(r_1(U), r_2(U))$  for any  $U \in \Omega \subset \mathbb{R}^2$ . In the sequel we will only be interested in the case of a 1 genuinely nonlinear field and a 2 linearly degenerate field. In particular, it means that for every  $U \in \Omega \subset \mathbb{R}^2$  we have

$$\nabla \lambda_1(U) \cdot r_1(U) \neq 0 \text{ and } \nabla \lambda_2(U) \cdot r_2(U) = 0. \quad (1.3)$$

**Examples** We wish now to give some examples of strictly hyperbolic system with a genuinely nonlinear field and a linearly degenerate field.

1. The classical chromatography system [10, 23] when the velocity is known which is the case for the liquid chromatography.
2. A thin-film model of a perfectly soluble anti-surfactant solution when the capillarity and the diffusion are neglected [35, 51].
3. The Keyfitz-Krantzer system [39] has this structure, it is maybe the first and the most famous known. It is related to some problem of nonlinear elasticity.
4. The  $2 \times 2$  Baiti-Jenssen system [2] with a genuinely nonlinear field. The Baiti-Jenssen systems arise in models for porous media, traffic and gas flows.

5. The Aw-Rascle system is well known for traffic flow [1].

The five first examples are Temple systems [52, 55, 56]. Such systems satisfy a maximum principle which is not generally true for systems of conservation laws. Now the following list provides examples that are not Temple systems.

6. Colombo and Corli consider the class of  $2 \times 2$  system with genuinely nonlinear field and a Temple field [20]. They prove existence of solutions for large  $BV$  data associated to the Temple component. A linear degenerate field is an example of Temple field, the rarefaction and shock curve coincide [10]. Such assumption is not enough to have a Temple system. One interest of our work is to prove existence in  $BV^s$ , so, with possible infinite total variation in  $BV$ .
7. The chromatography system with a sorption effect [6] is a chromatography system with a non constant and unknown velocity. This system is generally not a Temple system [7].
8. There are some triangular systems with a transport equation [5]. This class of systems generalizes the previous one when it is written in Lagrangian coordinates [9].
9. We mention also a system arising in biology [50, 57].

In this paper we would like to extend, for  $2 \times 2$  systems with a genuinely nonlinear field and a linearly degenerate one, the famous result of Glimm [32] concerning the existence of global weak solution for the strictly hyperbolic system with small initial data  $u_0$  in  $BV$ . Indeed we would like to enlarge the set of initial data by working with  $u_0$  belonging to  $BV^s$  with  $0 < s < 1$ ,  $BV^1 = BV$ . We are now going to give a definition of the fractional BV spaces called  $BV^s$ . We refer also to Bruneau [15] for more details.

**Definition 1 ( $TV^s$  variation)** *We say that a function  $u$  is in  $BV^s(\mathbb{R})$  with  $0 < s \leq 1$  and  $p = 1/s \geq 1$  if  $TV^s u < +\infty$  with:*

$$TV^s u := \sup_{n \in \mathbb{N}, x_1 < \dots < x_n} \sum_{i=1}^n |u(x_{i+1}) - u(x_i)|^p \quad (1.4)$$

*The associated semi-norm of the  $TV^s$  variation is,*

$$|u|_{BV^s} := (TV^s u)^s \quad (1.5)$$

*and a norm is*

$$\|u\|_{BV^s} := \|u\|_{L^\infty} + |u|_{BV^s} \quad (1.6)$$

In the same way,  $TV^s u(I)$  is defined as the  $TV^s$  variation of the function  $u$  on the set  $I$ . We note that it is clear that for any  $s \in ]0, 1]$ ,  $BV^s(\mathbb{R}) \subset L^\infty(\mathbb{R})$  [49]. Moreover, if  $u$  belongs to  $L^1(\mathbb{R})$  then the semi-norm  $BV^s$  is a norm. This is due to the fact that a  $BV^s$  function has limits at  $\pm\infty$  and, for a  $L^1(\mathbb{R})$  function, these limits are 0. For  $0 < s_1 < s_2 \leq 1$ , we also have  $BV^{s_2} \subset BV^{s_1}$  [8]. The  $TV^s$  variation was called the  $p$ -variation with  $p = 1/s$  in [49]. We prefer to use the notation  $TV^s$  since it is related to the Sobolev exponent “s”. Indeed,  $BV_{loc}^s$  is close from  $W_{loc}^{s,p}$  but remains different [8], indeed the  $BV^s$  functions are regulated functions [49] as  $BV$  functions.

**Proposition 1 ( $BV^s$  functions are regulated functions [49])** *If  $u \in BV^s$  with  $0 < s \leq 1$  then  $u$  admits only a countable set of discontinuities. Furthermore for every  $x \in \mathbb{R}$ ,  $u$  admits a limit on the left and on the right in  $x$ .*

We would like now to motivate the use of the  $BV^s$  spaces for the study of hyperbolic systems. Actually the most results on the existence of global weak solution for  $2 \times 2$  hyperbolic systems concerned the  $L^\infty$  space and the  $BV$  space. In order to tackle this problem, there exists essentially two different approaches, the first one was developed by Glimm in the 60s [32]. He proved for a general  $n \times n$  strictly hyperbolic system with genuinely nonlinear fields or linearly degenerate fields the existence of global weak entropy solution provided that the initial data is small in  $BV$ . The main difficulty of the proof consists in controlling the  $BV$  norm of the solution all along the time, indeed Glimm has observed that the  $BV$  norm can increase after each interaction between the nonlinear waves. In order to estimate this gain in  $BV$  norm after each interaction, Glimm has introduced a quadratic functional which described the interactions between the nonlinear waves and which allows to evaluate the  $BV$  norm of the solution all along the time. This result has been extended in the 90s by Bressan and his collaborators [10, 12, 13] where they proved the uniqueness of Glimm solution (provided that  $U_0$  belongs also to  $L^1(\mathbb{R})$ ) in a suitable class of solutions which takes into account in particular the Lax conditions for the shocks. The main ingredient to do this is to prove that the wave-front tracking algorithm (we refer to [10] for the definition of the wave-front tracking for general  $n \times n$  systems) generates a Lipschitz semi-group in  $L^1$  [10, 11]. We recall in particular that the solutions resulting from the wave-front tracking method and which are determined via a compactness argument are the same as the solution coming from the Glimm scheme [10].

The second approach was initiated by Di Perna [26, 27] at the beginning of the 80s using the so called compensated compactness which was introduced by Tartar [54]. Roughly speaking this method can be applied for  $2 \times 2$  strictly hyperbolic systems with two genuinely nonlinear fields (see also Serre [52]) when the initial data  $U_0$  is assumed to belong to  $L^\infty(\mathbb{R})$ . The case of the isentropic Euler system has been particularly studied and we refer to [25, 44, 45, 43]. We observe then that this method allows to deal with more general initial data as  $U_0 \in BV$ , however there is generally no result of uniqueness for these solutions. In particular it seems

complicated to select the solution via the Lax conditions on the shocks since we cannot give any sense of traces along a shock for such solutions since they belong only to  $L_{t,x}^\infty$ .

In the  $2 \times 2$  case, Glimm in [32] has obtained a better result of existence of global weak solution inasmuch as he can deal with large initial data  $U_0$  in  $BV$  provided that the  $L^\infty$  norm of  $U_0$  is sufficiently small. It is due to the fact that after an interaction between waves the variation of the  $BV$  norm has a cubic order in terms of the incoming strengths of the waves which interact (in the general case  $n \geq 3$ , this order is only quadratic). This result is a consequence of the existence of Riemann invariants for  $2 \times 2$  systems. Later on, this result has been extended by Glimm and Lax in [33] to the case of small  $L^\infty$  initial data when the fields are genuinely nonlinear. We refer also to the recent work of Bianchini, Colombo and Monti [4]. To do this, the authors use new Glimm functionals to control the  $L^\infty$  norm combined with the method of backward characteristics. In addition they proved a new Oleinik inequality (which is generally restricted to the scalar conservation law with genuinely nonlinear flux) for this  $2 \times 2$  system which gives sufficient compactness informations in order to pass to the limit in the wave-front tracking.

$BV^s$  spaces are intermediate spaces between  $L^\infty$  and  $BV$  spaces, see [8] or the definition 1. We note in particular that the  $BV^s$  spaces admit functions with shocks, from this point of view these spaces are more suitable than Sobolev spaces for dealing with hyperbolic systems of conservation laws [8, 16]. Indeed it is well known that the solution of an hyperbolic system can admit shock in finite time even if the initial data is arbitrary regular. In addition (see [49] and the proposition 1 below),  $BV^s$  functions admit a notion of “traces” as for  $BV$  functions (this is of course not the case for  $L^\infty$  functions). This notion of trace is essential in the result of uniqueness of Bressan et al [10, 12, 13]. Indeed, it gives a sense to the notion of the Lax entropy criterion which enables to select a unique solution (in the results of Bressan et al. a tame oscillation condition is also required). It would be then interesting to prove the existence and the uniqueness of global weak solution for initial data  $U_0$  in  $BV^s$  with  $0 < s < 1$  for strictly hyperbolic systems. It would improve in particular the results of existence of Glimm [32] inasmuch as the initial data  $U_0$  would be less regular as  $BV$ . In addition, working with  $BV^s$  gives a chance to extend the results of uniqueness of Bressan et al. [10, 12, 13] since the notion of trace remains relevant.

In this paper, we will only focus our attention on the existence of global weak solutions for small initial data in  $BV^s$ . Before giving and describing our main results, we would like now to recall some results using the  $BV^s$  space in the framework of conservation laws. For scalar conservation law, the entropy solution corresponding to an initial data belonging in  $BV^s$  remains in  $BV^s$  for all time [8]. Moreover, this result is sharp [30]. We would like to point out that the  $BV^s$  space is also naturally used to describe the regularizing effects of scalar conservation laws. From [40], it is

known that there exists unique global solution for scalar conservation laws when  $U_0$  belongs to  $L^\infty$ . The most famous regularizing effect concerns the uniformly convex flux where the solution becomes instantaneously  $BV_{loc}$ , this is a direct consequence of the so called Oleinik inequality. For a convex flux with a power law degeneracy, the authors in [8] show an optimal regularizing effects on the solution  $u$  inasmuch as the solution  $u$  becomes instantaneously  $BV_{loc}^s$  with  $s$  depending on the power law of the flux. For a nonlinear convex flux, non necessary  $C^1$ , the solution belongs for positive time in a generalized  $BV$  space related to the nonlinearity of the flux, see [34]. These results have been extended for a nonlinear non convex flux, at least  $C^3$ , in a generalized  $BV$  space, and for a more regular flux with polynomial degeneracy in the optimal  $BV^s$  space by Marconi in [46, 47].

The most results dealing with  $BV^s$  initial data concern scalar conservation laws. Indeed it is a priori delicate to prove the stability of the  $BV^s$  norm all along the time, the  $BV^s$  norm is indeed more complicated to compute than the  $BV$  norm. Indeed when we apply to a Glimm scheme, in order to know the  $BV$  norm after an interaction between waves, it is sufficient to estimate locally the strength of the new outgoing waves since we recover the complete  $BV$  norm by summing the absolute value of the different strength on all the euclidean space. In particular using the triangular inequality, we do not need to select subdivisions of the euclidean space in a accurate way in order to control the  $BV$  norm. It is not the case for the  $BV^s$  norm which is reached for particular optimal subdivisions. It implies in particular that for computing the  $BV^s$  norm after a waves interaction, it is not sufficient to know only the values of the outcoming strength. In the scalar case, after each interactions, there exist some zones of monotonicity for the Riemann problem making the analysis simpler to compute the  $BV^s$  norm [8, 37].

In this paper, we would like to extend the analysis of [8] to the case of  $2 \times 2$  strictly hyperbolic system with a genuinely nonlinear field and a linearly degenerate field which corresponds to the case described in (1.3). This case is a particular case of the theory of Glimm [32] on the existence of global weak solution for initial data  $U_0$  in  $(BV(\mathbb{R}))^2$  with a large  $BV$  norm provided that the  $L^\infty$  norm is sufficiently small. We would like also to mention that others authors have yet obtained existence of weak entropy solution for small  $L^\infty$  data and large  $BV$  norm when an eigenvalue is linearly degenerate [4, 24] or a Temple eigenvalue [20]. We extend the results of Glimm by proving the existence of global weak solution for small initial data with  $(w_0, z_0)$  belonging in  $BV^s \times L^\infty$  with  $\frac{1}{3} \leq s < 1$ . Here  $(w, z)$  are the solution of the system (1.1) that we consider in Riemann coordinates respectively in terms of the 1 genuinely nonlinear field and the 2 degenerate field. To do this, we follow the classical method which consists in introducing a wave-front tracking with  $(w^\nu, z^\nu)$  corresponding to the approximate solutions  $(U^\nu)$  of the wave-front tracking written in Riemann coordinates and  $\nu \rightarrow +\infty$  the parameter associated to the wave-front tracking. We are then interested in proving that  $(U^\nu)_{\nu>0}$  converges

to  $U$  a solution of the system (1.1). The main difficulty consists in proving uniform  $BV^s$  estimates on  $(w^\nu, z^\nu)$  and next in verifying that the wave-front tracking is well defined for any time  $t > 0$ . The end of the proof requires to establish compactness argument in order to verify that  $U^\nu$  converges to  $U$  a solution of the system (1.1) (here  $U^\nu$  is the approximated solution associated to the wave-front tracking written in physical coordinates and not in Riemann coordinates).

More precisely we show that the  $BV^s$  norm of  $w^\nu$  is uniformly conserved all along the time essentially because the waves interactions do not increase the  $BV^s$  norm for  $w^\nu$ . The proof is reminiscent of the scalar case for a convex flux. However it is more complicated to control uniformly the  $L^\infty$  norm of  $z^\nu$ . Indeed the norm of  $z^\nu$  can increase after two types of interactions, interaction between 1-shocks and interaction between a 1-shock and a 2-contact discontinuity. To do this, we consider the  $L^\infty$  norm of  $z^\nu$  along any forward generalized 2-characteristic and we observe that this  $L^\infty$  norm depends on the  $BV^{\frac{1}{3}}$  norm of  $w_0$ . Indeed the  $L^\infty$  norm of  $z^\nu$  along a forward generalized 2-characteristic increases only when the forward generalized 2-characteristic meets a 1-shock, furthermore this increase depends on the cubic strength in the variation of  $w^\nu$  on this 1-shock ( it is important to point out that this increase is directly related to the regularity of the Lax shock curve). It suffices then to follow these 1-shock in a backward manner in order to estimate the  $L^\infty$  norm of  $z^\nu$  in terms of  $\|w_0\|_{BV^{\frac{1}{3}}}$ . It explains why we need to assume that  $w_0$  is in  $BV^s$  with  $s$  at least equal to  $\frac{1}{3}$ . The last step of the proof consist in proving that  $z^\nu$  converges to  $z$  up to a subsequence in  $L^1_{loc,t,x}$ . This part is a priori delicate since we have only a uniform control of the  $L^\infty$  norm of  $z^\nu$ . We observe however that we have additional regularity property if we study the unknown  $z'_L(t, x) = z^\nu(t, \gamma'_2(t, x))$  with  $\gamma'_2(t, x)$  the forward generalized 2-characteristic such that  $\gamma'_2(0, x) = x$ . Here  $z'_L$  is the value of  $z$  in Lagrangian coordinates, following the same idea as for the control of the  $L^\infty$  norm of  $z^\nu$ , we can prove that  $z'_L$  is uniformly bounded in  $L^\infty_x(BV_t)$  and that the speed of propagation is finite. It is then sufficient to prove that  $z'_L$  converges up to a subsequence strongly in  $L^1_{loc,t,x}$ , we prove next that the Lagrangian transformation  $(t, \gamma'_2(t, x))$  is a uniformly bi-Lipschitz homeomorphism in  $\nu$  that is sufficient to ensure that  $z^\nu$  converges also strongly in  $L^1_{loc,t,x}$ . It allows to conclude that the solution  $(U^\nu)$  converges to a solution of (1.1).

## 2 Presentation of the results

We would like to state now our main result. For this purpose we use a distinguished coordinate system called Riemann invariants, which in general exists only for  $2 \times 2$  systems (chapter 20, [53]). This allow to perform a nonlinear diagonalization of the hyperbolic system for smooth solutions. This diagonalization is not valid for discontinuous solutions but the Riemann invariants have still some advantages. The Riemann problem and the interaction of waves is also simpler to study in



these coordinates than in the initial coordinates. The following notations  $(w, z)$  are chosen for the Riemann invariants [53]. Indeed, there exists a change of coordinates  $U \mapsto (w, z) = (w(U), z(U))$  (here  $(w, z) = (w(U), z(U))$  is a standard abuse of notations), reducing the open set  $\Omega$  if necessary, such that

$$\nabla w \cdot r_2 = 0, \quad \nabla z \cdot r_1 = 0. \quad (2.7)$$

In all the sequel,  $U$  is written in this system of coordinates. In particular, the initial data  $U_0$  of the system 1.1 reads  $w_0 = w(U_0)$  and  $z_0 = z(U_0)$ . Our main theorem states as follows.

**Theorem 2.1 (Existence in  $BV^{1/3} \times L^\infty$ )** *Let  $w_0 \in BV^s(\mathbb{R})$  with  $\frac{1}{3} \leq s \leq 1$  and  $z_0 \in L^\infty(\mathbb{R})$  then there exists  $\epsilon_0 > 0$  such that if:*

$$\|w_0\|_{BV^s} + \|z_0\|_{L^\infty} \leq \epsilon_0$$

*then there exists a bounded global weak entropy solution  $U$  for the system (1.1). The Riemann coordinates  $(w, z)$  belong to  $L^\infty(\mathbb{R}^+, BV^s \times L^\infty)$ . Moreover, the Riemann invariant  $z$  can be decomposed in the Lagrangian coordinates associated to the linearly degenerate field as follows:*

$$z(t, \gamma_2(t, x)) = z_0(x) + \eta(t, x) \quad (2.8)$$

*where  $\lambda_2$  is the linearly degenerate eigenvalue which depends only on  $w$  and  $\gamma_2$  represents the generalized 2-characteristics,*

$$\begin{cases} \frac{d\gamma_2}{dt}(t, x) &= \lambda_2(w(t, \gamma_2(t, x))) \\ \gamma_2(0, x) &= x \end{cases}$$

*with  $\eta \in L_x^\infty(\mathbb{R}, BV_t(\mathbb{R}^+)) \cap Lip_x(\mathbb{R}, L_{loc,t}^1(\mathbb{R}^+))$ .*

**Remark 1 (Fractional exponent  $s = 1/3$ )** *The exponent  $s = 1/3$  corresponds to the classic cubic variation of a Riemann invariant through the Lax curve [41]. It is also related to cubic variation of the entropy through a shock wave. This important fact was first pointed out in hydrodynamics studies at least one century ago [28].*

**Remark 2 (Wave front tracking algorithm)** *The usual method to obtain the existence of a solution is to build a sequence of approximate solutions via a wave front tracking algorithm (WFT) [10]. The compactness of the sequence with less regularity than  $BV$  needs different estimates than the usual ones. Moreover the convergence toward a weak entropy solution usually needs  $BV$  estimates to control the error in the weak formulation and in the entropy inequalities [10, 24, 36]. Here a simplified and more accurate WFT is proposed to handle all these difficulties.*

**Remark 3 (Large data)** *The theorem can be extended for large data which values in a domain  $\Omega$  if three conditions are fulfilled in  $\Omega$  namely, a strong strictly hyperbolicity, a solvability of the Riemann and the existence of Riemann invariants. The strong strictly hyperbolicity assumption is a strengthened transversality condition on the characteristics fields, if  $\lambda_1 < \lambda_2$  are the two wave speeds then,  $\sup_{\Omega} \lambda_1(U) < \inf_{\Omega} \lambda_2(U)$ . For instance, in almost all examples cited in introduction, Theorem 2.1 can be extended for larger data.*

**Remark 4 (Optimality)** *The existence result is optimal. That means the fractional regularity  $s \geq 1/3$  is mandatory. For a subclass of such systems it is proven in [5] that a blow up in  $L^\infty$  can occur if  $s < 1/3$ .*

**Remark 5** *In this Theorem, we assume that  $(w_0, z_0) \in BV^s \times L^\infty$ ,  $\frac{1}{3} \leq s < 1$ . A sufficient condition on the initial physical coordinates  $U_0$  to ensure such regularity is to take  $U_0 \in BV^s$ .*

**Remark 6** *The decomposition of  $z$  in (2.8) provides also a stability results in  $BV^\sigma$  for all  $0 \leq \sigma \leq 1$ . This means that  $z$  is in  $L^\infty(\mathbb{R}^+, BV^\sigma)$  if  $z_0 \in BV^\sigma$ .*

This decomposition (2.8) has already been obtained in a different form as a factorization of the gas velocity for a chromatography system (Theorem 7.2 in [7]).

Up to our knowledge, Theorem 2.1 is the first general result concerning the stability of the  $BV^s$  norm in the framework of strictly hyperbolic systems, except for some particular physical systems [5, 9, 38]. It extends in particular the results of Glimm [32] since the initial data are not necessary  $BV$ . Furthermore if we compare this result with the works of Glimm, Lax [33] and Bianchini, Colombo, Monti [4] which deal with initial data in  $L^\infty$ , we can only say that the framework is different. Indeed in [33, 4] the two fields are genuinely nonlinear, in particular the authors extend the Oleinik inequality to their case that allows them to obtain sufficient information in terms of compactness to pass to the limit respectively in their scheme and their wave-front tracking. In our case, the methods are quite different especially on the arguments of compactness which enables us to consider the limit of the approximated solutions  $(w^\nu, z^\nu)$  which are issue of the wave-front tracking. Indeed, an observation is to remark that the solution  $z^\nu$  can be splitted into  $z_0$  the initial data and a function  $\eta^\nu$  which is more regular as  $z^\nu$  itself. We can then pass to the limit in  $\nu$  for  $\eta^\nu$  in the wave-front tracking.

**Remark 7** *For  $2 \times 2$  Temple system with a genuinely nonlinear field and a linearly degenerate field, these results improve the classical existence in  $BV$ . The existence for  $L^\infty$  data in [12] needs that all fields are genuinely nonlinear.*

**Remark 8** *Since the  $BV^s$  norm has a notion of trace it would be interesting to prove the uniqueness of the solution.*

In Section 4, the Lax curve and the different interactions between 1-waves and 2-waves are described. Furthermore a simplified wave-front tracking well adapted to our case which concerns a genuinely nonlinear field and a linearly degenerate is proposed. In Section 5 the compactness of the approximate sequences build by the WFT is performed. The main existence Theorem 2.1 is finally proven in Section 6. This section deals with the consistency error in the WFT without  $BV$  bounds. In Appendix A, a self-contained discussion and a useful Lemma to compute the generalized total variation are given.

### 3 On the Riemann problem

In this section, the wave-front tracking algorithm (WFT) used is presented to solve the initial value problem (1.1). Simplifying the (WFT) is useful to simplify the estimates on the approximate solutions [3]. Taking advantage of the linearly degenerate field, we present a simpler wave-front tracking algorithm (WFT) as the one used by Bressan and Colombo in [11] for general  $2 \times 2$  systems. For this purpose, the Riemann problem and the interaction of waves is first studied.

In the sequel, we denote by  $A(U)$  the  $2 \times 2$  hyperbolic matrix  $DF(U)$  and without loss of generality by  $\lambda_1 < 0 < \lambda_2$  its eigenvalues and by  $l_1, l_2$  (respectively  $r_1, r_2$ ) its left (respectively right) eigenvectors, normalized so that:

$$\|r_i(U)\| = 1, \langle l_j(U); r_i(U) \rangle = \delta_{i,j}, \quad i, j = 1, 2.$$

Furthermore we recall that for all  $U \in \Omega(1.3)$ :

$$\nabla \lambda_1(U) \cdot r_1(U) \neq 0 \quad \text{and} \quad \nabla \lambda_2(U) \cdot r_2(U) = 0. \quad (3.9)$$

In the sequel we assume that we have  $\nabla \lambda_1(U) \cdot r_1(U) > 0$  for all  $U$  in  $\Omega$ . Furthermore for  $\Omega = B(0, r)$  sufficiently small, we have.

$$\sup_{U \in B(0, r)} \lambda_1(U) < 0 < \inf_{U \in B(0, r)} \lambda_2(U). \quad (3.10)$$

#### 3.1 Riemann invariants and Lax curves

An important feature for  $2 \times 2$  systems is the existence (at least locally) of coordinates in the state space, the Riemann invariants. All properties of the solutions  $U$  are stated in the Riemann invariants coordinates. We call  $w$  and  $z$  the Riemann invariant associated to genuinely nonlinear (GNL) eigenvalue  $\lambda_1$  and the linearly degenerate one  $\lambda_2$ . More precisely we have  $\nabla w(U) \cdot r_2(U) = 0$  and  $\nabla z(U) \cdot r_1(U) = 0$  for any  $U \in \Omega \subset \mathbb{R}^2$ . When  $U$  is a solution of system (1.1),  $(w(t, x), z(t, x))$  denotes  $(w(U(t, x)), z(U(t, x)))$ . With this usual notation, where the solution  $U$  is smooth, the Riemann invariants satisfy with here by abuse of notation  $\lambda_1(w, z) = \lambda_1(U)$  and  $\lambda_2(w, z) = \lambda_2(U)$ :

$$\begin{cases} \partial_t w + \lambda_1(w, z) \partial_x w = 0 \\ \partial_t z + \lambda_2(w, z) \partial_x z = 0. \end{cases} \quad (3.11)$$

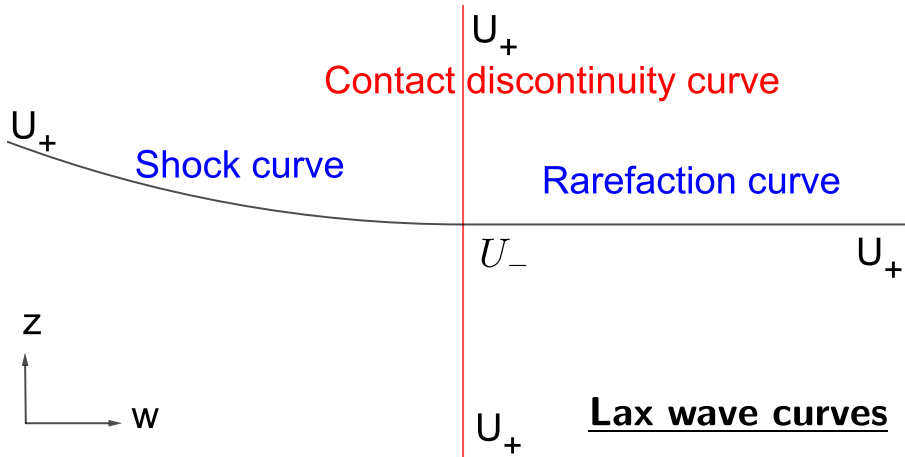


Figure 1: Lax waves curves where the state on the left of the wave  $U_- = (w_-, z_-)$  is fixed.  $U_+$  is the right state connected by a 1-wave when  $w$  varies,  $w_+ \neq w_-$ , or a 2-wave when  $w_+ = w_-$  is constant.

**Remark 9** Notice that  $\partial_w \lambda_1 > 0$ , since the first field is GNL, and  $\lambda_2$  is independent on  $z$  since the second field is linearly degenerate (see Theorem 8.2.5 [24]). That is  $\partial_z \lambda_2 = 0$  and  $\lambda_2$  depends only on  $w$ .

The map  $U \rightarrow (w(U), z(U))$  is a local diffeomorphism and we can assume that the origin in  $U$  coordinates corresponds to the origin in  $(w, z)$  coordinates. We are going now to define the Lax curves in these new coordinates  $(w, z)$ . For a fixed state  $U_-$ , the Lax curves describe the set of state  $U_+$  such that the Riemann problem with the left state  $U_-$  and the right state  $U_+$  is a simple wave [53]. For each  $U_-$  fixed there are two Lax curves, one for the 1-waves and another one for the 2-waves.

**Lax Curves** The picture (see the figure 1) of the Lax curves  $L(U_-)$  will be used systematically throughout the paper. The same convexity of the shock curves is fixed for all the figures. It simplify the study of waves interactions. However, with a linearly degenerate field, the convexity of shock curves can not be fixed. Thus, below, for the estimates in Section 3.3 the convexity is not used.

For the rarefaction  $R_1$  we have:

$$\begin{cases} w = w_- + \sigma & \text{with } \sigma \geq 0 \\ z = z_- \end{cases} \quad (3.12)$$

For the  $S_1$  shock we have:

$$\begin{cases} w = w_- + \sigma & \text{with } \sigma \leq 0 \\ z = z_- + \mathcal{O}(\sigma^3) \end{cases} \quad (3.13)$$

Notice, with the choice of the convexity for the shock curve,  $z$  increases through a shock wave. (For the concave case,  $z$  decreases).

For the 2-wave there is only a contact discontinuity (CD):

$$\begin{cases} w = w_- \\ z = z_- + \sigma, \quad \sigma \in \mathbb{R} \end{cases} \quad (3.14)$$

### 3.2 The Riemann problem

The solution of the Riemann problem is given in the plane  $(w, z)$  in figure 2. The initial data is  $U(0, x) = U_{\pm}$ ,  $\pm x \geq 0$ .  $U_0$  is the intermediate constant state between  $U_-$  and  $U_+$  when we solve the Riemann problem (do not confuse  $U_0$  with the initial data).

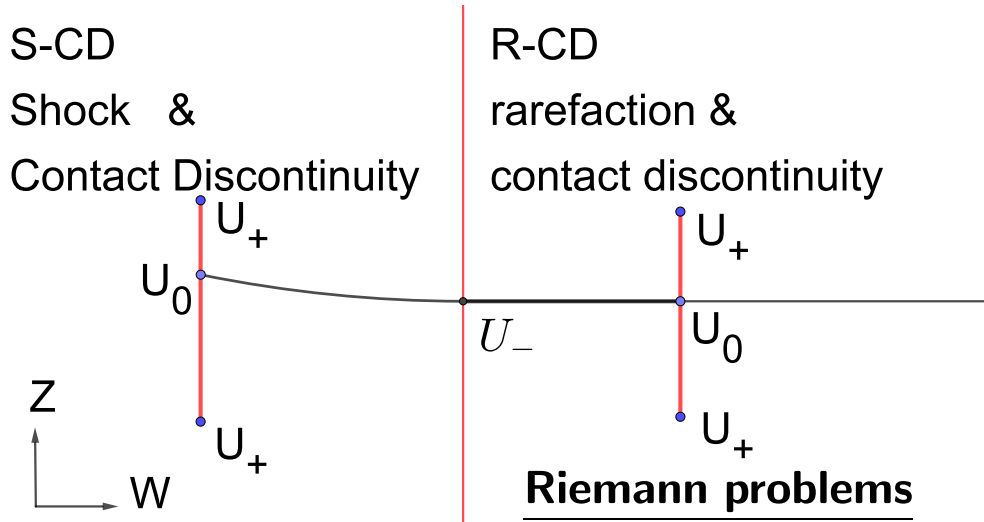


Figure 2: Solutions of the Riemann problems for a left state  $U_-$  fixed and all the possible configurations for  $U_+$ . The solutions are represented in the plane  $(w, z)$  of Riemann invariants

### 3.3 Exact interactions

Next we consider the different interactions that we can have. We will note  $CD$  for the 2-contact discontinuity wave,  $S_1$  for the 1-shock wave and  $R_1$  for the 1

rarefaction wave. We recall that the only possible interactions are:

$$CD - R_1, CD - S_1, R_1 - S_1, S_1 - R_1, S_1 - S_1,$$

where  $L - R$  means the interaction between a left wave and a right wave. The left wave is a 2-wave, a contact discontinuity (CD), or a 1-wave, a rarefaction (R) or a shock (S) wave. The right wave is always a 1-wave since a 2-wave is a contact discontinuity and goes to the right with the speed  $\lambda_2 > 0$  and cannot interact with a left 1-wave which goes to the left with the speed  $\lambda_1 < 0$  or a contact discontinuity which moves with the same speed  $\lambda_2$ .

We study all the possible interactions in this section. The strength of an i-wave is quantified by the variation of associated Riemann invariant through the wave:  $\Delta w$  for a 1-wave and  $\Delta z$  for a 2-wave.

Let us summarize important features of such interactions where the 1-wave can be only a rarefaction or a shock wave and the 2-wave can be only a contact discontinuity. The following list of claims are verified just after by the exhaustive study of all possible interactions.

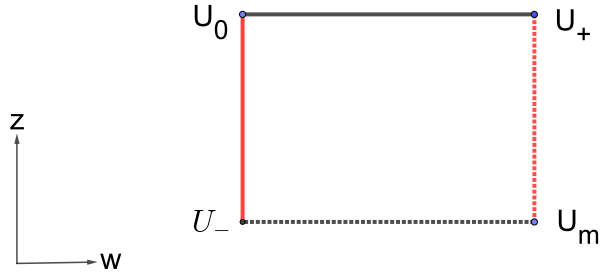
1. When two waves interact then there are two resulting waves which are a 1-wave on the left and a 2-wave on the right.
2. The strength of a 1-wave does not change after an interaction with a 2-wave.
3. Assume that two 1-waves interact with respectively the strength  $\sigma_1$  and  $\tilde{\sigma}_1$  then the outgoing 1-wave has the strength  $\sigma'_1 = \sigma_1 + \tilde{\sigma}_1$ .
4. The variation of  $w$  after an interaction behaves like the variation of the solution of a scalar conservation law. It means that  $TVw$  and  $TV^s w$  is not increasing as for a scalar conservation laws.
5. The  $L^\infty$  norm of  $z$  can increase only when there is an interaction with a shock. The increase of  $\|z\|_\infty$  is at most the cube of the strength of the shock,  $\mathcal{O}(w_+ - w_-)^3$ .

In all the following pictures,

- $U_- = (w_-, z_-)$  is the left state,
- $U_+ = (w_+, z_+)$  is the right state,
- $U_0 = (w_0, z_0)$  is the intermediate state before the interaction,
- $U_m = (w_m, z_m)$  is the intermediate state after the interaction.

All the previous claims can be checked on the pictures. However, the pictures assume that the shock curves are convex. The convexity assumption of shock curves is not used in the computations to deal with the general case.

To study the possible increase of the total variation or the  $L^\infty$  norm of  $z$  such notations are introduced.

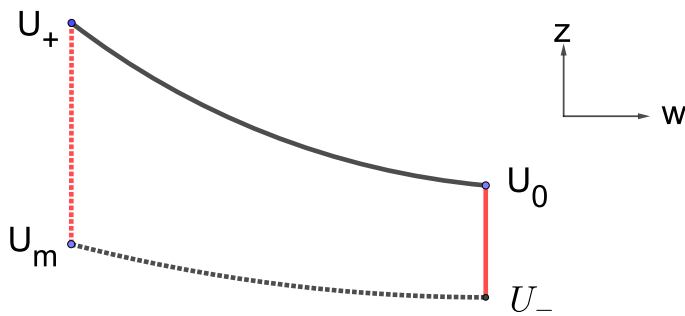


**Interaction Contact Discontinuity & Rarefaction**

Figure 3: Interaction of a contact discontinuity with a rarefaction. The interacting waves are represented by full lines, a 2-wave or a 1-wave followed by a 1-wave. The dotted lines represent the resulting waves, a 1-wave (horizontal) followed by a 2-wave (vertical).

- The local total variation of  $z$  just before the interaction is ,  
 $TV^{before} z = |z_+ - z_0| + |z_0 - z_-|$ .
- The local total variation of  $z$  just after the interaction is ,  
 $TV^{after} z = |z_+ - z_m| + |z_m - z_-| \leq TV^{before} z + 2|z_m - z_0|$ .
- The  $L^\infty$  norm of  $z$  just before the interaction is denoted by  $\|z\|_\infty^{before}$  and just after by  $\|z\|_\infty^{after} \leq \|z\|_\infty^{before} + |z_m - z_0|$ .

An important point is the control of the  $BV$  or  $BV^s$  norm of  $w$  after an interaction. There are two cases. First case, after the interaction, the solution has



**Interaction Contact Discontinuity & Shock**

Figure 4: Interaction of a contact discontinuity with a shock

only three different values the state  $U^-, U_m, U^+$ . This is true if the 1 outgoing wave is a 1-shock. Second case, there is a 1 rarefaction outgoing wave, so the solution has a continuum of values (when we will deal with the wave-front tracking this continuum of values will be split in a finite number of values depending on the parameter  $\nu$  with  $\nu$  goes to  $+\infty$ ). However since the 1 rarefaction wave create a zone of monotonicity in  $w$ , using the Lemma 5 it does not change all the arguments which are related to estimating the  $BV$  or  $BV^s$  norm of  $w$  (we can apply the Lemma 5 because the wave-front tracking will have a finite number of values).

$CD - R_1$ . Let us consider the simplest interaction  $CD - R_1$ , figure 3, which generate waves  $R_1 - CD$ . We observe that:

$$w_- = w_0 < w_+, \quad z_- < z_0 = z_+ \quad \text{and} \quad w_- < w_m = w_+, \quad z_- = z_m < z_+.$$

In particular the functions  $w(t, \cdot)$  and  $z(t, \cdot)$  have the same values before and after the interaction and these values are in the same order, it implies then that  $BV$  and  $BV^s$  norm does not change for this interaction both for  $z$  and  $w$ . Furthermore we have  $w_+ - w_0 = w_m - w_-$  then the strength of the 1-wave is invariant after this interaction with this 2-wave. Notice also that  $\|z\|_\infty^{after} = \|z\|_\infty^{before}$  and  $TV^{before} z = TV^{after} z$ .

$CD - S_1$ . Now, consider the interaction  $CD - S_1$ , figure 4 and 5, which generates waves  $S_1 - CD$ . We observe that:

$$w_- = w_0 > w_+ \quad \text{and} \quad w_0 > w_m = w_+$$

In particular it implies again that the  $BV$  and the  $BV^s$  norm of  $w$  does not change after this type of interaction. Furthermore the strength of the 1 outgoing wave is

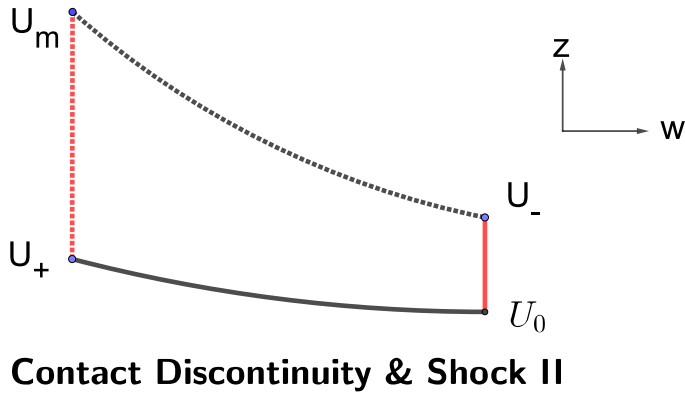


Figure 5: Interaction between a contact discontinuity and a shock with an augmentation of  $\|z\|_\infty$ .



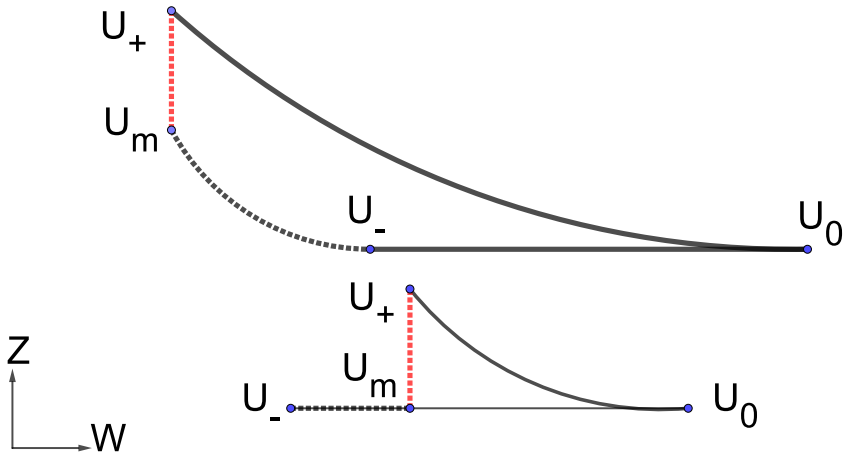


Figure 6: Interaction of a rarefaction with a shock wave: two cases depending on the relative strength of the waves

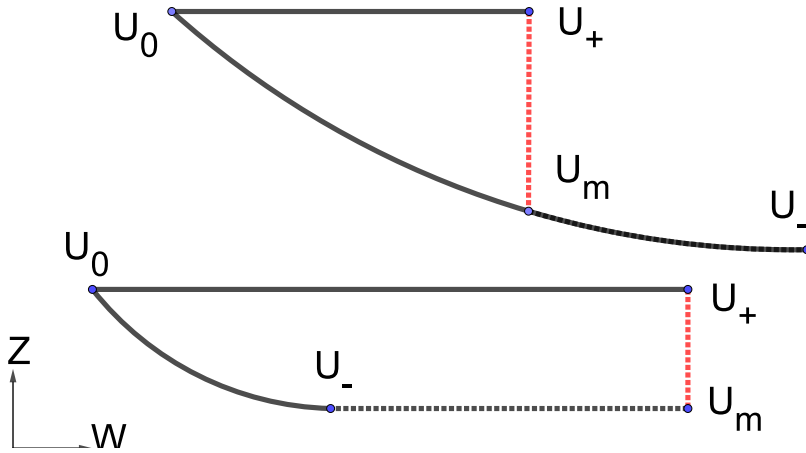


Figure 7: Interaction of a shock wave with a rarefaction: two cases depending on the relative strength of the waves

the same as the strength of the 1 incoming wave. We observe however that the  $L^\infty$  norm of  $z$  can increase in the figure 5,  $|z_m|$  is larger than  $|z_-|$ ,  $|z_0|$  and  $|z_+|$ . Similarly the  $BV$  norm can increase for  $z$ . Thus, there is no maximum principle for  $z$ . However, the increase of  $\|z\|_\infty$  is controlled as in the last case, the shock-shock interaction. To be more precise and without using the convexity of the shock curves in the figure we always have  $z_m - z_- = \mathcal{O}(w_m - w_-)^3 = \mathcal{O}(w_+ - w_-)^3$ , thus  $\|z\|_\infty^{after} \leq \|z\|_\infty^{before} + \mathcal{O}(w_+ - w_-)^3$ . For the local total variation we proceed as

follows,

$$\begin{aligned}
TV^{after} z &= (|z_+ - z_m|) + |z_m - z_-| \\
&\leq (|z_+ - z_0| + |z_0 - z_-| + |z_- - z_m|) + |z_m - z_-| \\
&\leq TV^{before} z + 2|z_m - z_-| \leq TV^{before} z + \mathcal{O}(w_+ - w_-)^3,
\end{aligned}$$

which concludes these interaction estimates.

$R_1 - S_1$ . The interaction of 1-waves  $R_1 - S_1$  generates  $R_1 - CD$  or  $S_1 - CD$ , figure 6, and we have in each case  $w_m = w_+$ . It implies in particular that  $w(t, \cdot)$  has the same values after the interaction excepted the value  $w_0$  and some values of the incoming 1 rarefaction. Furthermore the values are in the same order, we deduce then that the  $BV$  and the  $BV^s$  norm decreases since we restrict in some sense the number of possible subdivision. We can observe that for this interaction the  $L^\infty$  norm of  $z$  does not increase.

For the general cases, without the convexity assumption, the following general inequalities are used,

$$\begin{aligned}
TV^{after} z &\leq TV^{before} z + 2|z_m - z_0|, \\
\|z\|_\infty^{after} &\leq \|z\|_\infty^{before} + |z_m - z_0|.
\end{aligned}$$

Thus, all the point is to control  $|z_m - z_0|$ . If the interaction  $R_1 - S_1$  produces  $CD - R_1$  then  $z_m = z_+$  and  $z_0 = z_+ + \mathcal{O}(w_+ - w_-)^3$ , thus  $z_m - z_0 = +\mathcal{O}(w_+ - w_-)^3$  and

$$\begin{aligned}
TV^{after} z &\leq TV^{before} z + \mathcal{O}(w_+ - w_-)^3, \\
\|z\|_\infty^{after} &\leq \|z\|_\infty^{before} + \mathcal{O}(w_+ - w_-)^3.
\end{aligned}$$

The case when the interaction produces a shock is similar because  $z_m = z_+ + \mathcal{O}(w_+ - w_-)^3$  and the similar estimates follow.

$S_1 - R_1$ . The interaction  $S_1 - R_1$  is similar to the previous case, figure 7, and yields to the same estimates.

$S_1 - S_1$ . We finish with the interaction  $S_1 - S_1$  which generate the waves  $S_1 - CD$ , figure 8. This is the most interesting case with a non-scalar type interaction.

Here,  $w$  continues to behave like a solution for a convex scalar conservation law since  $w(t, \cdot)$  has the same values after the interaction except  $w_0$ , the order of local minimal and local maximal values of  $w$  does not change then the  $BV$  and  $BV^s$  norms do not increase for  $w$ .  $z$  is not monotonous after the shock interactions in particular the  $L^\infty$  and  $BV$  norm can increase. We use again (3.13) one obtain (see also the Glimm estimate in [24] for a  $2 \times 2$  system).  $U_m$  is connected to  $U_-$  by a shock wave thus,  $z_m = z_- + \mathcal{O}(w_+ - w_-)^3$ .  $U_0$  is also connected to  $U_-$  by a shock wave,  $z_0 = z_- + \mathcal{O}(w_+ - w_-)^3$ . Thus,  $z_m - z_0 = \mathcal{O}(w_+ - w_-)^3$  and we can conclude as for the case  $CD - S_1$ .

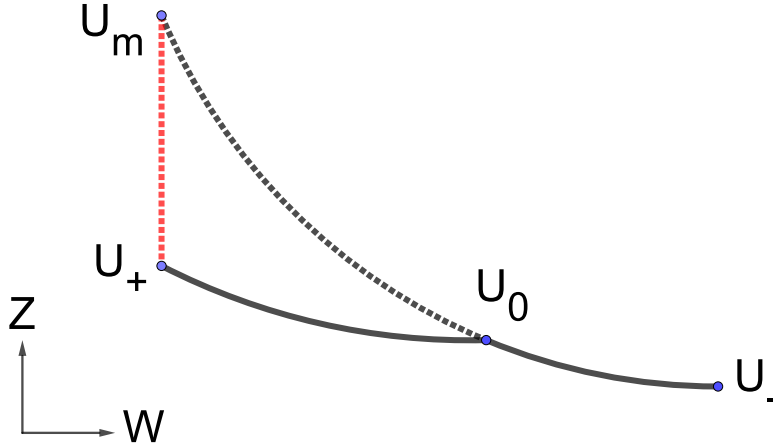


Figure 8: Interaction of two shock waves

In conclusion in all cases,  $w$  always behaves like a solution of a convex scalar law (in particular the  $BV$  and the  $BV^s$  norms decrease) and the estimates on  $z$  can only increase when at least one shocks interacts. The estimates are,

$$\|w\|_{\infty}^{after} \leq \|w\|_{\infty}^{before}, \quad \|z\|_{\infty}^{after} \leq \|z\|_{\infty}^{before} + \mathcal{O}(w_+ - w_-)^3, \quad (3.15)$$

$$TV^{after} w \leq TV^{before} w, \quad TV^{after} z \leq TV^{before} z + \mathcal{O}(w_+ - w_-)^3. \quad (3.16)$$

## 4 A wave-front tracking algorithm

In this section, the wave-front tracking algorithm (WFT) used is presented to solve the initial value problem (1.1). Simplifying the WFT is useful to simplify the estimates on the approximate solutions [3]. Taking advantage of the linearly degenerate field, we present a simpler WFT as the one used by Bressan and Colombo in [11] for general  $2 \times 2$  systems.

Moreover, the classical proof of convergence towards a weak solution uses uniform  $BV$  estimates [10][7.4, 143-146], [24][14.7, 535]. To work only with less regularity,  $w \in BV^{1/3}, z \in L^{\infty}$ , a more accurate scheme is necessary to prove the consistency of the scheme in Section 6. We define now the wave-front tracking that we will use in the sequel. First we shall work with initial data  $(w_{0,\nu}, z_{0,\nu})$  which are piece-wise constant approximation of  $(w_0, z_0)$  such that:

$$\begin{aligned} \|(w_0, z_0) - (w_{0,\nu}, z_{0,\nu})\|_{L^1} &\leq \nu^{-1}, \\ \text{Osc}(w_{0,\nu}) &\leq \text{Osc}(w_0). \end{aligned} \quad (4.17)$$

with  $\nu > 0$  and  $\nu$  goes to  $+\infty$ . Here  $\text{Osc}(w_0) = \sup_{x,y \in \mathbb{R}} |w_0(x) - w_0(y)|$  denote the oscillation of  $w_0$ . Furthermore we assume that:

$$w_{0,\nu} \in \nu^{-1}\mathbb{Z} \text{ and } z_{0,\nu} \in \mathbb{R}. \quad (4.18)$$

We define now  $N_\nu$  as the number of discontinuities that the initial data  $(w_{0,\nu}, z_{0,\nu})$  has. We start the wave-front tracking by solving the  $N_\nu$  first Riemann problems.

## 4.1 Approximate Riemann solver

We would like to explain how we describe the solution of a Riemann problem between  $(w^-, z^-)$  and  $(w^+, z^+)$  in our algorithm of wave-front tracking (for the beginning we assume that  $(w^-, z^-)$  and  $(w^+, z^+)$  are some values of  $(w_{0,\nu}, z_{0,\nu})$ ). The Riemann problem is the combination of a 1-shock or a 1 rarefaction with a 2-contact discontinuity. If we get a 1-shock and a contact discontinuity the solution is the exact solution of the Riemann problem with respectively the speed  $\lambda_1((z^-, w^-); (z^m, w^m))$ ,  $\lambda_2(w^-, w^+)$  where  $\lambda_1((z^-, w^-); (z^m, w^m))$  to the speed defined by the Rankine Hugoniot relation.  $(z^m, w^m)$  corresponds here to the intermediate state and we know that  $w^m = w^+$  because the value of  $w$  is constant along the 2 contact discontinuity. Notice that the shocks and the contact discontinuities are solved exactly with the exact speeds.

To stay with piece-wise constant solutions, an approximate solver is needed only for rarefaction waves. If the solution of the Riemann problem is a combination of a 1 rarefaction wave and a 2-contact discontinuity, we have to define the solution corresponding to the rarefaction. We note again  $(z^m, w^m)$  the intermediate state of the exact solution. We observe using (3.12) and (3.14) that  $w_m = w_+$ , it implies in particular that since  $w_+$  is in  $\nu^{-1}\mathbb{Z}$  that  $w_m$  is again in  $\nu^{-1}\mathbb{Z}$ . We have in particular:

$$w_m = w_- + k_+ \nu^{-1},$$

with  $k_+ \in \mathbb{N}$  since  $w$  increases through a rarefaction wave. The approximate rarefaction is no longer a continuous rarefaction but a sequence of consecutive constant states separated by non-entropic jumps. Shortly speaking such a non-entropic jump is called a jump. To distinguish this jump from a contact-discontinuity jump of the second family, this jump is also called a 1-jump.

We define now the intermediate state  $w_k = w_- + k \nu^{-1}$  with  $0 \leq k \leq k_+$ . The solution of the rarefaction for our wave-front tracking at time  $t > 0$  with the initial discontinuity at time  $t = 0$  in  $y$  is:

$$(w, z)t, x) = \begin{cases} (w_-, z_-) & \text{if } x < x_1(t) \\ (w_k, z_k) & \text{if } x_k(t) < x < x_{k+1}(t) \\ (w_m, z_m) & \text{if } x > x_{k_+}(t) \end{cases} \quad (4.19)$$

Notice that  $z$  is not constant along the approximate rarefaction wave. This error is done to obtain a weak solution of the exact system (1.1) for the approximate

solution  $U_\nu$ . Thus,  $(w_k, z_k)$  are on the non entropic part of the exact Rankine-Hugoniot of the first family which corresponds to the genuinely nonlinear field. This method has already been used for a subclass of System (1.1) in [5]. Now  $\dot{x}_k$ , the speed of the front, has to be defined. Again, to get a weak solution, the Rankine-Hugoniot' speed is chosen,  $\dot{x}_k = \lambda_1(w_{k-1}, w_k)$ . But this jump is not an entropic jump, thus an error occurs in the entropy inequalities. Hopefully this error is cubic due to the classical theorem of the variation of the production of the entropy near a weak shock [24][Thm 8.5.1, 314]. Again the space  $BV^{1/3}$  appears naturally to study the sharp error of consistency of the WFT. This point will be discussed later in Section 6 to prove the convergence towards an entropy solution. In conclusion the front chosen is the line  $x_k(t) = y + t\lambda_1(w_{k-1}, w_k)$  before an interaction with another wave.

The approximate solution  $u$  can be prolonged until a time  $t_1$  when the first interaction between two or more waves front takes place.

**Remark 10** *It is important to observe that for  $t \in ]0, t_1[$ , the solution  $w(t, x)$  take his values in  $\nu^{-1}\mathbb{Z}$ .*

Since  $u(t_1, \cdot)$  is still a piece-wise constant function, the corresponding Riemann problem can again be approximately solved. The solution  $(z, w)$  is then continued up to a time  $t_2$  when the second set of wave interactions takes place, and so on.

## 4.2 No three-wave interaction

An important feature of the classical Bressan's WFT [10] is to avoid too much interactions at the same time, in order to simplify the study of interacting waves. For this purpose Bressan uses the unstability of the event of the interaction of three waves at the same time and the stability of the interaction of only two waves. In this section, these properties are used for the modified WFT used here.

The front-speed is the exact Rankine-Hugoniot' speed, so, it is possible that an interaction with three waves or more can occur at the same time. There are at least two ways to handle this difficulty. The first one is to not avoid the interaction with three waves or more. Since such interactions are only between the first family and at most only one wave for the second family it is possible to study all this case as for scalar conservation laws as in [36]. This is a particularity of System (1.1) that the waves of the first family behaves (almost) like in the scalar case. Nevertheless, a second method keeps the nice property that interactions occur only between two fronts.

Instead of modifying slightly the speeds as Bressan [10] to avoid the interaction of three waves, the initial data can be slightly modified. The initial data is slightly modified such that the two fundamental requirements (4.18) are satisfied. The approximate initial data becomes a control variable to obtain polygonal fronts which should only interact in pairs. For instance, the initial data can be modified

as follows. Let

$$U_{0,\nu}(x) = \sum_k \chi_k(x) U_{0,\nu}^k,$$

be the piece-wise constant initial data where  $\chi_k$  is the indicatrix of the interval  $I_k = (a_k, a_{k+1})$  and  $U_{0,\nu}^k$  is the mean value of  $U_0$  on the interval  $I_k$ . Consider a slight modification of the interval by replacing  $a_k$  by  $b_k = a_k + e_k$  where

$$|e_k| < \min_j |a_{j+1} - a_j|/2, \quad \|e\| := \sum_k |e_k| \leq 1/\nu.$$

Now the  $I_k$  are replaced by  $J_k = (b_k, b_{k+1})$  and the new piece-wise constant initial data keeps the value  $U_{0,\nu}^k$  on  $J_k$ . Since  $U_{0,\nu}$  is uniformly bounded in  $L^\infty$  such modification of the initial data implies an error in  $L^1$  of order  $1/\nu$ .

It turns to prove that there exists such a vector  $e = (e_k)_k$  such that there is never three waves interacting at the same time. An algorithm is proposed to build  $e$  by a sequences of vector  $e^0, e^1, \dots$ . The initial vector is simply the null vector  $e^0 = (0, \dots, 0)$ .

The fronts are at the beginning before the first interaction a finite numbers of segments,  $x_i(t) = x_i(0) + s_i^1 t$ , where  $t \in (0, t_1)$ ,  $x_i(0)$  is equal to one  $a_k$  and  $t_1 > 0$  is the first interaction time.

If three waves interact at the first interaction time  $t_1$ . That means that at least three initial Riemann problems interact. Such event is *unstable* with respect to initial points of the wave fronts  $(t, x) = (0, a_k)$ . So there exists a small modification parameterized by the non null vector  $e^1$ ,  $\|e^1\| > 0$ ,  $e^1$  belongs in the ball

$$B(e^0, 1/\nu) = \{ e, \|e - e^0\| < r_0 \}, \quad r_0 = 1/\nu,$$

such that there are only two waves interacting for the first time  $t_1(e^1)$  which is a slight modification of  $t_1 = t_1(e^0)$ .

Moreover the property to have only one interaction with only two waves at time  $t_1$  is a *stable* property. That means there exists an open ball around  $e^1$  such that for all  $e \in B(e^1, r_1)$ ,  $r_1 > 0$ , only two waves interact at the first interaction time  $t_1(e)$  when  $e^1$  is replaced by  $e$ . Moreover, with  $r_0 = 1/\nu$ , it can be assumed that  $B(e^1, r_1) \subset B(e^0, r_0)$ .

Now, the second interaction time  $t_2(e^1)$  is considered. At most two curves supported the fronts become polygonal lines and stay lines in the the interval  $(t_1(e^1), t_2(e^1))$ , issuing from the points  $x_i(t_1(e^1))$ . If there are only two waves interacting, nothing is changed,  $e^2 = e^1$ ,  $r_2 = r_1$ . Else, the values  $(x_i(t_1(e^1)))_i$  are two be modified in order to forbid the interaction of three waves. These is again possible due to the unstability of such event by modifying  $e^1$  by  $e^2 \in B(e^1, r_1)$ . The initial values  $(x_i(0))_i$  are slightly modified to have only two waves interacting at time  $t_2$  and also at time  $t_1$  since  $e^2$  belongs to  $B(e^1, r_1)$ .

Again, the nice property at time  $t_2$  is stable, all the picture depend continuously of the initial data, so a small positive  $r_2$  can be chosen such that for all  $e \in B(e^2, r_2)$

the non interaction of three waves is still true at time  $t_2(e)$ . Moreover,  $r_2$  is chosen small enough to satisfies  $B(e^2, r_2) \subset B(e^1, r_1)$  in such way no no interaction of three waves remains also true at time  $t_1(e)$ .

The process can be continued for consecutives interaction times  $t_3, \dots, t_n$  such sthat  $B(e^n, r_n) \subset B(e^{n-1}, r_{n-1})$ .

An important fact proven below in the next Section 4.4 is that such WFT has a finite number of interactions for all time. Thus the process stops in finite number of iterations, at  $n = n^\#$  and  $e = e^{n^\#}$  is chosen. Therefore, it is assumed in all the sequel that at most two waves can interact at the same time.

### 4.3 Interactions in the WFT

The study of interactions proceeds as for the exact solutions in Section 3.3 except for one kind of waves, the rarefactions. There are no rarefaction waves in the WFT. Each rarefaction is replaced by a successive series of weak non entropic jumps. Thus, the previous interactions in Section 3.3 involving a rarefaction has no now to be considered with an interaction with a weak non entropic jump. Such jump is exactly computed on the Rankine-Hugoniot curve. More precisely the half part of the Rankine-Hugoniot curve which is not admissible for the entropy criterium. The Rankine-Hugoniot curve is a smooth curve [10, 52]. Hence, the cubic contact of the Rankine-Hugoniot curve with the rarefaction curve is still true. That means that for such interaction the same computations hold as for the case with an interaction with a real shock. That means that  $z$  does not remain constant as for a real rarefaction. Thus,  $z$  has a cubic jump through such non entropic jump wave. All the others estimates in Section 3.3 remains the same so there are not repeated here.

To summarize the estimates on the  $L^\infty$  norm (3.15) and the total variation (3.16) after an interaction are still valid for the WFT. That is to say the  $L^\infty$  norm and the variation of  $w_\nu$  cannot increase after an interaction. For  $z_\nu$  the increase is possible but at the most of the cubic order of the variation of  $w$ .

### 4.4 Finite number of interactions for all time

We are going to prove in the sequel that there is a finite number of interactions so that we can define the wave-front tracking on the time interval  $(0, +\infty)$ . Assume for the moment that we can only define the wave-front tracking on a time interval  $(0, T^*)$  with  $T^* < +\infty$  such that there is an infinite number of wave interactions. We will prove in fact that necessary  $T^* = +\infty$ .

**Remark 11** *It is important to verify that  $(w, z)(t, \cdot)$  remains bounded in a ball  $B(0, r)$  with  $r > 0$  sufficiently small for all time  $t \in (0, T^*)$ . Indeed using the Lax Theorem, we can solve the Riemann problem only if the oscillation between two states  $(w_-, z_-)$  and  $(w_+, z_+)$  is sufficiently small.*

We define now  $N_1(t)$  as:

$$N_1(t) = \text{number of 1-wave at the time } t.$$

We observe easily that for  $t \in ]0, t_1[$ , we have from (3.12):

$$N_1(t) \leq \nu N_\nu \text{Osc}(w_0) + N_\nu.$$

The first term on the right hand side corresponds to the maximal number of rarefaction and  $N_\nu$  to the maximal number of 1-shock. Let us estimate  $N_1(t)$  after an interaction at the time  $t_k$ . We start by recalling that if we have a 1 rarefaction wave by definition of the Riemann problem (see (4.19)), his strength is necessary of size  $\frac{1}{\nu}$ . Assume now that we have an interaction between a 1 rarefaction and a 1-shock then the strength of the 1 rarefaction is  $\frac{1}{\nu}$  and the strength of the 1-shock is  $\frac{k}{\nu}$  with  $k \in -\mathbb{N}^*$  then we have seen in the section 3.3 that the 1-wave has the strength  $\sigma'_1 = \frac{1}{\nu} + \frac{k}{\nu} \leq 0$ . It implies in particular that the outgoing 1-wave disappears or is a shock. In particular after such interactions the number of 1-waves decreases of 1 or 2 units. Similarly if we have an interaction between two shocks, we know that the strength of the outgoing 1-wave is the sum of the two strength of the incoming waves, then this strength is negative and the outgoing 1-wave is a shock. It implies in particular that after such interactions the number of 1-waves decreases of 1 unit. If we have an interaction between a 2-wave and a 1-wave since this interaction is transparent the outgoing wave is a unique rarefaction if the incoming one is also a rarefaction and otherwise a 1-shock. It implies in particular that  $N_1(t)$  does not change after such interactions. We have then proved that  $N_1(t)$  is a decreasing function of the time and then that:

$$N_1(t) \leq \nu N_\nu \text{Osc}(w_0) + N_\nu,$$

for all  $t \in (0, T^*)$ . We define now  $N'_1$  as:

$$N'_1 = \text{Number of interactions between 1-waves on } (0, T^*)$$

We have seen that, when we have an interaction between 1-waves, the number of 1-waves decreases, at least of one unit. It implies then that  $N'_1$  is inferior to the maximal number of 1-waves:

$$N'_1 \leq \nu N_\nu \text{Osc}(w_0) + N_\nu. \quad (4.20)$$

We define now  $N'_2(t)$  as:

$$N'_2 = \text{Number of interactions between 1-waves and 2-waves on } (0, T^*)$$

For a one wave we can define a 1 polygonal line which is an extension of the one wave. Indeed the one wave are created at the time  $t = 0^+$  and after each interaction it can be prolonged by a unique 1-wave (or even the one wave can disappear, in



this case the polygonal line is stopped). We can then define a polygonal 1-wave line. We note that two different 1 polygonal lines merge after an interaction time corresponding to an interaction between 1-waves if they meet us. Their number is finite and bounded by  $\nu N_\nu \text{Osc}(w_0) + N_\nu$ . Similarly we can define some polygonal lines for 2-waves. At the difference with the 1 polygonal line, we can create a 2 polygonal line after an interaction between 1-waves. Their number is finite and bounded by  $N_\nu + (N_\nu \text{Osc}(w_0) + N_\nu)$ ,  $N_\nu$  corresponds to the number of polygonal line issued from the time  $t = 0$  and  $(N_\nu \text{Osc}(w_0) + N_\nu)$  is the maximal number of interaction between 1-waves and then the maximum number of 2 polygonal lines that we can create. Since  $\lambda_1(w, z) < 0 < \lambda_2(w, z)$  for any  $(w, z) \in B(0, r)$  we deduce by transversality that a polygonal 2-wave can interact with a polygonal 1-wave only one time. It implies in particular that the number of interaction on  $(0, T^*)$  between polygonal 1-wave and polygonal 2-wave is at most  $(2N_\nu + \nu N_\nu \text{Osc}(w_0))(\nu N_\nu \text{Osc}(w_0) + N_\nu)$ . It implies in particular that:

$$N'_2 \leq (2N_\nu + \nu N_\nu \text{Osc}(w_0))(\nu N_\nu \text{Osc}(w_0) + N_\nu). \quad (4.21)$$

From (4.20) and (4.21), we deduce that the number of interaction on  $(0, T^*)$  is finite and then  $T^* = +\infty$ .

**Remark 12** *The only point to verify is to ensure that all along the algorithm of wave-front tracking,  $(w(t, \cdot), z(t, \cdot))$  must remains in a set  $[-r', r']^2$  with  $r' > 0$  sufficiently small such that we are able to solve any Riemann problem. It will be verified in the sequel when we will prove the Theorem 2.1. We mention however that the  $L^\infty$  norm of  $w(t, \cdot)$  is not increasing. After each interaction, we have proved that the  $L^\infty$  norm of  $w$  does not increase. It is not the case for  $z$  since after the interaction between two 1-shocks or the interaction between a contact discontinuity and a 1-shock, the  $L^\infty$  norm of  $z$  can increase.*

## 5 Uniform estimates and compactness for $(w_\nu, z_\nu)$

In this section we are going to obtain key estimates to prove the Theorem 2.1 with the initial data  $(w_0, z_0)$  belonging to  $BV^s \times L^\infty$  with  $s \geq 1/3$ . We consider again the solution of the wave-front tracking  $(w_\nu, z_\nu)$  defined in the section 4 on a time interval  $(0, T_\nu^*)$  with  $T_\nu^* > 0$ . In addition we construct  $(w_{0,\nu}, z_{0,\nu})_{\nu>0}$  verifying (4.17) and such that for any  $\nu > 0$  we have:

$$\|(w_{0,\nu}, z_{0,\nu})\|_{BV^s \times L^\infty} \leq \|(w_0, z_0)\|_{BV^s \times L^\infty}. \quad (5.22)$$

### 5.1 Control of $\|w_\nu(t, \cdot)\|_{BV^s}$

We are now going to obtain uniform estimate in  $\nu$  in  $BV^s$  for the solution  $w_\nu$ . More precisely we wish to prove that for any  $t \in (0, T_\nu^*)$ :

$$\|w_\nu(t, \cdot)\|_{BV^s(\mathbb{R})} \leq \|w_0\|_{BV^s}.$$

The point is that  $w_\nu$  behaves (almost) like the entropy solution of a scalar conservation law. The Riemann invariant  $w$  is monotonic for the solution of the Riemann problem and not affected by the other Riemann invariant, except only for the velocity of the 1-waves which does not affect the strength of the 1-waves. These properties remain valid for  $w_\nu$ , Section 4.1. The decay of the  $BV^s$  variation is well known for scalar conservation laws [8, 37]. The point is the monotonicity of  $w$  through elementary waves to compute the  $BV^s$  variation, see Appendix A.

Now, let us explain in detail the decay of  $TV^s w_\nu$ . Assume that  $t_1 \in (0, T_\nu^*)$  is the first time where we have a wave interaction in our wave-front tracking. Let us prove now that for any  $t \in (0, t_1)$  we have:

$$\|w_\nu(t, \cdot)\|_{BV^s(\mathbb{R})} = \|w_{0,\nu}\|_{BV^s(\mathbb{R})}. \quad (5.23)$$

If we come back to the estimate (5.23), it suffices to observe that the solution  $w_\nu(t, \cdot)$  for  $t \in (0, t_1)$  is the combination of the solutions of different Riemann problems which deal with all the initial discontinuities. We obtain then a combination of a 1-wave and a 2-CD wave. If we have a 1-shock and a 2-CD wave the values of  $w_\nu(t, \cdot)$  do not change compared with  $w_\nu(0, \cdot)$  and conserve the same order, then the  $BV^s$  norm remains the same, Lemma 5 in Appendix A. If the 1-wave is an approximate rarefaction, we get different 1-jump fronts and  $w_\nu(t, \cdot)$  takes new values. For example if we have a discontinuity in  $x_\alpha$  at the time  $t = 0$  with the values  $w_{0,\nu}(x_\alpha^-)$  and  $w_{0,\nu}(x_\alpha^+)$  then the 1-jump fronts produce the following new values at time  $t \in (0, t_1)$  which are  $w_{0,\nu}(x_\alpha^-) + k/\nu$  with  $k \in \{0, \dots, k^+\}$  and with  $w_{0,\nu}(x_\alpha^+) = w_{0,\nu}(x_\alpha^-) + k^+/\nu$ . However even if we have new values for  $w_\nu(t, \cdot)$  we have a zone of monotonicity for the 1-jump fronts and using again Lemma 5, we conclude again that the  $BV^s$  norm does not change. It proves the estimate (5.23). Next we would like to understand how the  $BV^s$  norm varies after each interactions. Assume that we have an interaction at a time  $t_k$  with  $U^-, U_0, U^+$  the incoming states and  $U^-, U_m, U^+$  the outgoing states (here for simplicity of notation we have skipped the index  $\nu$ ), we can observe that the number of different values in  $w_\nu$  decreases or remains constant after the interaction. Indeed even when the outgoing 1-wave is one approximated rarefaction, we recall that there is no more than one jump front (see statements 2 and 3 in Section 3.3), furthermore we have:

$$U^- = (w^-, z^-), \quad U_m = (w^+, z^m) \quad \text{and} \quad U^+ = (w^+, z^+).$$

If we note  $t_k$  the time of interaction, it implies in particular that the value  $w_0$  disappears at the time  $t_k^+$  and that the values of  $w_\nu(t_k^+, \cdot)$  have the same order as the values of  $w_\nu(t_k^-, \cdot)$ . We deduce then that for any interaction, we have:

$$\|w_\nu(t_k^+, \cdot)\|_{TV^s} \leq \|w_\nu(t_k^-, \cdot)\|_{TV^s}. \quad (5.24)$$

We have then two possibilities the outgoing waves are a 1-shock wave and a 2-CD wave or a 1-jump front and a 2-CD wave. In both cases,  $w_\nu(t_k^+, \cdot)$  has the

same values as  $w_\nu(t_k^-, \cdot)$  excepted the value  $w_0$ , furthermore these values have the same order. It implies again (5.24). We deduce from (5.23), (5.24) and (5.22) that the norm  $\|w_\nu(t, \cdot)\|_{BV^s(\mathbb{R})}$  is decreasing all along the time and in particular we have for any  $t \in (0, T_\nu^*)$ :

$$\|w_\nu(t, \cdot)\|_{BV^s(\mathbb{R})} \leq \|w_{0,\nu}\|_{BV^s(\mathbb{R})} \leq \|w_0\|_{BV^s(\mathbb{R})}. \quad (5.25)$$

## 5.2 Control of $\|z_\nu(t, \cdot)\|_{L^\infty}$

We are now going to bound uniformly  $z_\nu(t, \cdot)$  in  $\nu$  in  $L^\infty$  norm for any  $t \in (0, T_\nu^*)$ . For smooth solution  $z$  satisfies a transport equation (3.11), thus  $z$  is constant along 2-characteristic. The main idea to control the  $L^\infty$  norm of  $z_\nu$  is to follow any generalized 2-characteristic where  $z_\nu$  is essentially constant as for the smooth solution except when the characteristic crosses a 1-wave. In this case, the variation of  $z_\nu$  increases at most by the cubic order of the strength of the 1-wave. Thus the total variation of  $z_\nu$  along a generalized 2-characteristic is of the order of the  $BV^{1/3}$  semi-norm of  $w_\nu$  on the same curve. This curve is space-like [21, 33] (transverse) for the characteristics of the first family, so such  $BV^{1/3}$  semi-norm is essentially  $TV^{1/3}w_{0,\nu}$ . The purpose of this section is to detail and prove precisely these previous assertion.

We recall in particular that this is important to control the  $L^\infty$  norm of  $z_\nu$  in order to prove that the wave-front is well defined, i.e.  $U^\nu$  stays in  $\Omega$  (we can solve Riemann problem only if the oscillation of  $z_\nu$  and  $w_\nu$  are sufficiently small, we refer to the Remark 12). To do this, we define  $\gamma_2^\nu(t, x_0)$  the forward generalized 2-characteristic (see [24]) which is an absolutely continuous solution of the differential inclusion:

$$\begin{aligned} \frac{d}{dt} \gamma_2^\nu(t, x_0) \in & [\min(\lambda_2((w_\nu, z_\nu)(t, \gamma_2^\nu(t, x_0)^+), \lambda_2((w_\nu, z_\nu)(t, \gamma_2^\nu(t, x_0)^-)) \\ & , \max(\lambda_2((w_\nu, z_\nu)(t, \gamma_2^\nu(t, x_0)^+), \lambda_2((w, z)(t, \gamma_2(t, x_0)^-)))]), \end{aligned}$$

and such that  $\gamma_2^\nu(0, x_0) = x_0$ . In the sequel, in order to simplify the notation, we just will denote by  $\gamma_2^\nu(t)$  a forward generalized 2-characteristic. Now we are interested in estimating the  $L^\infty$  norm of  $z_\nu$  along a forward generalized 2-characteristic such that  $\gamma_2^\nu(0) \neq x_\alpha$  with  $x_\alpha$  the points where  $(w_{0,\nu}, z_{0,\nu})$  is discontinuous. In order to follow the evolution of the  $L^\infty$  norm of  $z_\nu$  along a forward generalized 2-characteristic, it is important to understand how the  $L^\infty$  norm of  $z_\nu$  can vary. It is the case only when the forward generalized 2-characteristic meets a 1-wave, a 2-wave or an interaction point. We recall that since  $\lambda_2 > 0$  and  $\lambda_1 < 0$  a forward generalized 2-characteristic and a 1-wave are necessary transversal. In Section 4.3 it was shown that the variation of  $z_\nu$  across a 1-wave is of cubic order in the strength of the shock (3.15), (3.16),

$$z_\nu^+ - z_\nu^- = \mathcal{O}(w_\nu^+ - w_\nu^-)^3.$$

Now it is important to mention that a generalized forward 2-characteristic cannot meet a 2-CD wave-front. Indeed since the second wave is degenerate, we know that

$\lambda_2$  does not depend on  $z$ . In particular using (3.14) we deduce that  $\lambda_2((w_\nu^-, z_\nu^-)) = \lambda_2((w_\nu^+, z_\nu^+))$  along a 2-CD, it implies that 2 generalized forward characteristic cannot meet a 2-CD wave (indeed they should be parallel to the 2-CD front). Now let us deal with the last case when the forward generalized 2-characteristics meets an interaction point. The only case is when the interaction point is between a 1-jump front and a 1-shock front, or between two 1-shock fronts. Let us start with the case of two 1-shock fronts, we define then by  $\sigma_1$  and  $\tilde{\sigma}_1$  the strength of the 2 incoming wave fronts with  $U^-, U_0, U^+$  the incoming states such that for  $U^- = (w^-, z^-)$ :

$$U_0 = (w^- + \sigma_1, z^- + \mathcal{O}(\sigma_1^3)), U^+ = (w^- + \sigma_1 + \tilde{\sigma}_1, z^- + \mathcal{O}(\sigma_1^3) + \mathcal{O}(\tilde{\sigma}_1^3)). \quad (5.26)$$

We know that the outgoing intermediate state is:

$$U_m = (w^- + \sigma_1', z^m) = (w^- + \sigma_1 + \tilde{\sigma}_1, z^- + \mathcal{O}((\sigma_1 + \tilde{\sigma}_1)^3)).$$

Now since  $\lambda_2$  does not depend on  $z$ , we deduce that the forward 2-characteristic follows the outgoing 2-CD wave after the interaction.

**Remark 13** *By convention, we assume that the value of  $z_\nu$  on the 2-CD wave-front corresponds to  $z_\nu^+$  the value on the right of the 2-CD wave-front.*

It implies then that after the interaction point the value of  $z_\nu$  has increased on  $\gamma_2'$  and is such that:

$$z_\nu^+ - z_\nu^- = \mathcal{O}(\sigma_1^3) + \mathcal{O}(\tilde{\sigma}_1^3). \quad (5.27)$$

Here  $z_\nu^+$  is the value of  $z_\nu$  on  $\gamma_2'$  just after the interaction point and  $z_\nu^-$  the value of  $z_\nu$  just before. Let us consider the case now of an interaction between a 1-jump front and a 1-shock front (the case of an interaction between a 1-shock front and a 1-jump front is similar), we have then:

$$U^- = (w^-, z^-), U_0 = (w^- + \sigma_1, z^-), U^+ = (w^- + \sigma_1 + \tilde{\sigma}_1, z^- + \mathcal{O}(\tilde{\sigma}_1^3)). \quad (5.28)$$

We know that the outgoing intermediate state is:

$$U_m = (w^- + \sigma_1', z^m) = (w^- + \sigma_1 + \tilde{\sigma}_1, z^- + \mathcal{O}((\sigma_1 + \tilde{\sigma}_1)^3)). \quad (5.29)$$

Again we deduce that the forward 2-characteristic follows the outgoing 2-CD wave-front after the interaction. And we have in addition:

$$z_\nu^+ - z_\nu^- = \mathcal{O}(\tilde{\sigma}_1^3). \quad (5.30)$$

In conclusion  $z_\nu$  has a variation of order  $(\Delta w)^3$  when the forward 2-characteristic meets an interaction point and that we have (5.27) or (5.30).

Assume now that  $\gamma_2'(0) = x_\alpha$ , then  $\gamma_2'$  is the 2-wave polygonal front which is issued from  $x_\alpha$ . If  $\gamma_2'$  meets 1-jump front then the conclusion is similar as meeting a 1-shock front with a cubic variation for  $z_\nu$ .

Here  $z_\nu^0$  is the intermediate state before the interaction and  $\sigma_1$  is the strength of the 1-shock wave. We recall that the value of  $z_\nu$  on  $\gamma_2$  before the interaction is by convention  $z_\nu^0$  since we consider the value on the right for a 2 CD wave-front. We observe then that if  $t_k$  is the interaction point we have:

$$z_\nu(t_k^+, \gamma_2^\nu(t_k^+)) = z(t_k^-, \gamma_2^\nu(t_k^-)) + O(\sigma_1^3).$$

**Remark 14** *If  $\gamma_2^\nu(0) \neq x_\alpha$  and that the 2 forward characteristic  $\gamma_2^\nu$  becomes after meeting an interaction point a 2-CD polygonal wave-front, we can estimate the evolution of the  $L^\infty$  norm of  $z_\nu$  as in the case where  $\gamma_2^\nu(0) = x_\alpha$ .*

We can now calculate the value of  $z_\nu$  at the point  $(T, \gamma_2^\nu(T))$  with  $T > 0$ . We have seen using the fact that along  $\gamma_2^\nu(t)$ ,  $z_\nu$  has a variation of  $O(\sigma_1^3)$  after each interaction with a 1-shock front, 1-jump front or an interaction point, we obtain then from (5.27) and (5.30):

$$z_\nu(T, \gamma_2^\nu(T)) = z_\nu(0, \gamma_2^\nu(0)) + \sum_{\alpha \in J} \mathcal{O}(\sigma_\alpha^3). \quad (5.31)$$

Here  $J$  corresponds to the set of 1-wave fronts which have met  $\gamma_2^\nu$  on the time interval  $[0, T]$  including the interaction points. In particular it exists  $C > 0$  independent on  $\nu$  such that:

$$\sum_{\alpha \in J} \mathcal{O}(\sigma_\alpha^3) \leq C TV^s w_\nu(\cdot, \gamma_2^\nu(\cdot))([0, T]) + \mathcal{O}(\nu^{-3}), \quad (5.32)$$

with  $s = 1/3$  (we deal always in the sequel with  $s = 1/3$ ).

**Remark 15** *It is important to point out that if  $\gamma_2^\nu(0) \neq x_\alpha$  then the forward 2-characteristic  $\gamma_2^\nu$  can meet only one time an interaction point. Indeed after this interaction the forward 2-characteristic becomes a 2 polygonal line and we have constructed a wave-front tracking where the interactions concern only two fronts. It means that the forward 2-characteristic after the meeting with an interaction point can cross after only 1-jump fronts and 1-shock fronts. Similarly if  $\gamma_2^\nu(0) = x_\alpha$  then the forward 2-characteristic  $\gamma_2^\nu$  which is a 2 polygonal front will meet only 1-jump fronts and 1-shock fronts.*

In order to prove (5.32), we only consider the case where  $\gamma_2^\nu(0) \neq x_\alpha$  and the case where the forward generalized 2-characteristic  $\gamma_2^\nu$  meets one interaction point. The other case are simple to treat. Let us start with the case where  $\gamma_2^\nu$  meets an interaction point with two 1-shock wave fronts at the time  $t_k$ , from (5.26), (5.27) we deduce that:

$$|z_\nu(t_k^+, \gamma_2^\nu(t_k^+)) - z(t_k^-, \gamma_2^\nu(t_k^-))| \leq \mathcal{O}(\sigma_1^3) + \mathcal{O}(\tilde{\sigma}_1^3) \leq \mathcal{O}(|\sigma_1 + \tilde{\sigma}_1|^3).$$

And in particular it says that:

$$|z_\nu(t_k^+, \gamma_2^\nu(t_k^+)) - z(t_k^-, \gamma_2^\nu(t_k^-))| \leq \mathcal{O}(|w_\nu(t_k^+, \gamma_2^\nu(t_k^+)) - w_\nu(t_k^-, \gamma_2^\nu(t_k^-))|^3). \quad (5.33)$$

Let us deal now with the more tricky case of the cross of  $\gamma_2^\nu$  with an interaction point comprising 1-jump front and 1-shock front at the time  $t'_k$ , we have obtained from (5.28), (5.30) that:

$$|z_\nu((t'_k)^+, \gamma_2^\nu((t'_k)^+)) - z((t'_k)^-, \gamma_2^\nu((t'_k)^-))| \leq \mathcal{O}(\tilde{\sigma}_1^3). \quad (5.34)$$

We recall that we have:

$$\tilde{\sigma}_1 = w_\nu((t'_k)^+, \gamma_2^\nu((t'_k)^+)) - w_\nu((t'_k)^-, \gamma_2^\nu((t'_k)^-)) - \sigma_1 \leq 0. \quad (5.35)$$

It is important to note that  $\sigma_1 = 1/\nu$ , indeed our wave-front tracking ensures that all the rarefaction fronts have the strength  $1/\nu$ . Similarly we know from (5.29) that the intermediate outgoing state is  $U_m = (w_+, z_m)$  and that the 1 outgoing wave is a 1-shock or is canceled out. For the moment assume that the 1 outgoing wave-front is not canceled out, in particular since  $U_m = (w_+, z_m)$ , it implies that  $w_+ < w_-$ . But  $w^\nu(t'_k, \cdot)$  takes only values in  $\nu^{-1}\mathbb{Z}$ , thus:

$$w_\nu((t'_k)^+, \gamma_2^\nu((t'_k)^+)) - w_\nu((t'_k)^-, \gamma_2^\nu((t'_k)^-)) = -\frac{k}{\nu}, \quad (5.36)$$

with  $k \in \mathbb{N}^*$ . From (5.35) and (5.36), we have:

$$|\tilde{\sigma}|^3 = \left(\frac{k+1}{\nu}\right)^3 \leq 2^3 \left(\frac{k}{\nu}\right)^3 = 2^3 |w_\nu((t'_k)^+, \gamma_2^\nu((t'_k)^+)) - w_\nu((t'_k)^-, \gamma_2^\nu((t'_k)^-))|^3. \quad (5.37)$$

We again deduce then using (5.34) and (5.37) that:

$$\begin{aligned} & |z_\nu((t'_k)^+, \gamma_2^\nu((t'_k)^+)) - z((t'_k)^-, \gamma_2^\nu((t'_k)^-))| \\ & \leq \mathcal{O}(|w_\nu((t'_k)^+, \gamma_2^\nu((t'_k)^+)) - w_\nu((t'_k)^-, \gamma_2^\nu((t'_k)^-))|^3). \end{aligned} \quad (5.38)$$

We finish now with the case where the 1 outgoing wave is canceled out, it corresponds to the following situation:

$$U^- = (w^-, z^-), \quad U_0 = (w^- + \sigma_1, z^-), \quad U^+ = (w^-, z^- + \mathcal{O}(\sigma_1^3)). \quad (5.39)$$

In this case since the strength of a 1-jump  $\sigma_1 = 1/\nu$  because this is the strength of a 1-jump front, we have:

$$|z_\nu((t'_k)^+, \gamma_2^\nu((t'_k)^+)) - z((t'_k)^-, \gamma_2^\nu((t'_k)^-))| \leq \mathcal{O}(\nu^{-3}). \quad (5.40)$$

We proceed similarly for an interaction between a 1-shock wave and a 1-jump wave. From the Remark 15, we know that the forward generalized 2-characteristic  $\gamma_2^\nu$  can meet only one time an interaction point then combining (5.33), (5.38) and (5.40)

allows to prove the estimate (5.32).

We wish now to estimate  $TV^s w_\nu(\cdot, \gamma_2^\nu(\cdot))([0, T])$  in terms of the  $BV^{\frac{1}{3}}$  norm of  $w_\nu(0, \cdot)$ . To do this we are going to consider a zone of dependence of the forward generalized 2-characteristic  $\gamma_2^\nu$ . We now choose the 1 minimal backward generalized characteristic issued from  $(\gamma_2^\nu(T), T)$  that we note  $\gamma_1^\nu$  (note that  $\gamma_1^\nu$  is defined on  $[0, T]$ , we refer to [24] Chapter X for the notion of minimal backward generalized characteristics). We now define the sequence of following functions  $\gamma_{2,\alpha}^\nu$  with  $\alpha \in [0, 1]$  as follows with  $\tau \in [0, T]$  (here  $\tau$  does not correspond to the physical time  $t$ ),

$$\gamma_{2,\alpha}^\nu(\tau) = \alpha\gamma_2^\nu(\tau) + (1 - \alpha)\gamma_2^\nu(0) + C_\alpha^\nu\tau,$$

with

$$C_\alpha^\nu = \frac{\gamma_1^\nu(\alpha T) - \alpha\gamma_2^\nu(T) - (1 - \alpha)\gamma_2^\nu(0)}{T} \geq 0$$

since  $\gamma_1^\nu(\alpha T) \geq \gamma_1^\nu(T) = \gamma_2^\nu(T) \geq \alpha\gamma_2^\nu(T) + (1 - \alpha)\gamma_2^\nu(0)$  for any  $\alpha \in [0, 1]$  (indeed we recall that the backward generalized characteristic goes on the right since  $\lambda_1 < 0$ ). We observe also that:

$$\gamma_{2,\alpha}^\nu(T) = \gamma_1^\nu(\alpha T) \quad \text{and} \quad (\gamma_{2,\alpha}^\nu)'(t) \geq 0 \quad \text{for } t \in [0, T].$$

The derivative of  $\gamma_{2,\alpha}^\nu$  is in fact defined on the point where  $\gamma_2^\nu$  is differentiable.

**Remark 16** *It is important to note that the forward generalized 2-characteristic is defined in a unique way. The second point is that the domain delimited by the curves  $\{(\gamma_2^\nu(t), t), t \in [0, T]\}$ ,  $\{(\gamma_1^\nu(t), t), t \in [0, T]\}$  and  $\{(y, 0), y \in [\gamma_2^\nu(0), \gamma_1^\nu(0)]\}$  is the union of all the curves  $\{(\gamma_{2,\alpha}^\nu(t), \alpha t), t \in [0, T]\}$  with  $\alpha \in [0, 1]$ . We denote by  $\Gamma_2$  this domain. There is no 1-wave-front which enters in  $\Gamma_2$  on the right since we have taken the minimal backward 1-characteristic.*

We define now  $(x_1, t_1)$  as the point where there is for the first time an interaction between wave fronts inside the domain  $\Gamma_2$  with  $t_1 \in ]0, T]$ . We denote now by  $t_{1,1}$  the first time where there is an interaction in our wave-front tracking, it implies in particular that  $0 < t_{1,1} \leq t_1$ . Furthermore we know that for  $\alpha$  such that  $\alpha T < t_1$  there is no interaction point on the curve  $\{(\gamma_{2,\alpha}^\nu(t), \alpha t), t \in [0, T]\}$ . We denote now by  $\alpha_0$  the first  $\alpha$  where the curve  $\{(\gamma_{2,\alpha}^\nu(t), \alpha t), t \in [0, T]\}$  meets an interaction point  $(x_k, t_k)$  inside  $\Gamma_2$ . In particular it implies that there is no interaction point in the open domain  $\Gamma_0$  delimited by  $\{(\gamma_{2,\alpha_0}^\nu(t), \alpha_0 t), t \in [0, T]\}$ ,  $\{(\gamma_1^\nu(t), t), t \in [0, T]\}$  and  $\{(y, 0), y \in [\gamma_2^\nu(0), \gamma_1^\nu(0)]\}$ . We know that the values of  $w$  change on the curve  $\{(\gamma_{2,\alpha}^\nu(t), \alpha t), t \in [0, T]\}$  only when this curve meets a 1-wave-front. Furthermore, the only 1-wave-front which can cross  $\{(\gamma_{2,\alpha}^\nu(t), \alpha t), t \in [0, T]\}$  for  $0 < \alpha < \alpha_0$  are the 1-wave-front which are issued from the set  $\{(y, 0), y \in [\gamma_2^\nu(0), \gamma_1^\nu(0)]\}$ . Indeed, there is no interaction point in the open domain  $\Gamma_0$  and the 1-wave which are outside from  $\Gamma_0$  cannot enter in  $\Gamma_0$ . It implies then, since the curve  $\{(\gamma_{2,\alpha}^\nu(t), \alpha t), t \in [0, T]\}$  are transversal to the 1-wave-front, that the values of  $w$  on  $\{(\gamma_{2,\alpha}^\nu(t), \alpha t), t \in [0, T]\}$

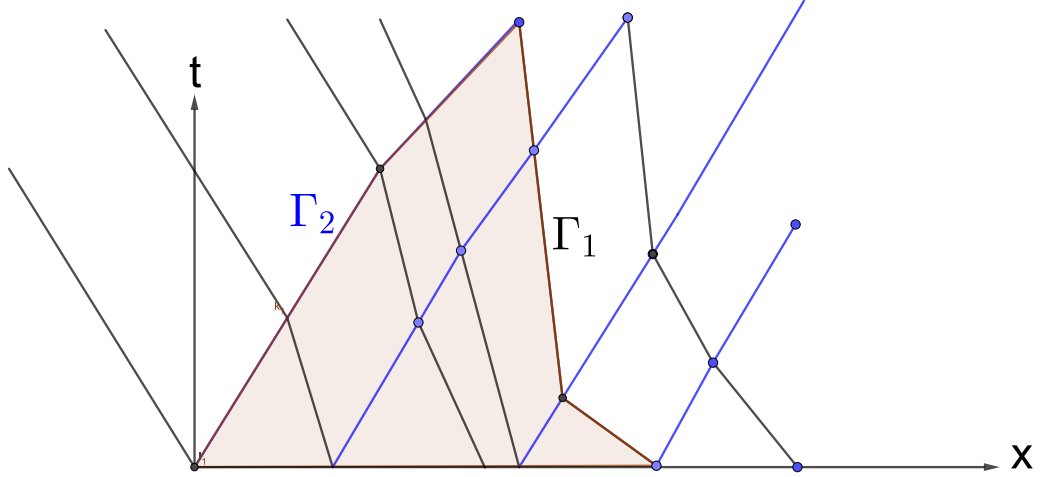


Figure 9: The wave-front Tracking and the dependence zone delimited by the 2-characteristic  $\Gamma_2$  on the left, and the 1-characteristic  $\Gamma_1$  on the right

for  $\alpha \in ]0, \alpha_0[$  are included in the set of the values of  $w_\nu(t, \cdot)$  at the time  $t = 0^+$  (or in other word  $t \in (0, t_{1,1})$ ) and that they are ranged in the same order. In particular it implies, using (5.25), that for any  $\alpha \in (0, \alpha_0)$  we have:

$$TV^s w_\nu(\alpha \cdot, \gamma_{2,\alpha}^\nu(\cdot))(]0, T]) \leq TV^s w_\nu(t, \cdot)(\mathbb{R}) \leq TV^s w_{0,\nu}(\mathbb{R}) \leq \|w_0\|_{BV^s}^{\frac{1}{s}}, \quad (5.41)$$

with  $t \in (0, t_{1,1})$ .

Next we wish to estimate the  $BV^s$  norm of  $w^\nu$  along  $\{(\gamma_{2,\alpha}(t), \alpha t), t \in [0, T]\}$  for  $\alpha = \alpha_0^+$ . We note that the interaction point  $(x_k, t_k)$  on  $\{(\gamma_{2,\alpha_0}(t), \alpha_0 t), t \in [0, T]\}$  is by definition inside  $\Gamma_2$ . In any case of interaction, if we have incoming states  $(U^-, U_0, U^+)$  and outgoing states  $(U^-, U_m, U^+)$  then the values of  $w^\nu$  around  $(x_k, t_k^-)$  are  $(w_-, w_0, w_+)$  and the the values of  $w^\nu$  around  $(x_k, t_k^+)$  are  $(w_-, w_+)$ . It means that there is one value  $w_\nu$  in less  $w_0$  on the curves  $\{(\gamma_{2,\alpha}^\nu(t), \alpha t), t \in [0, T]\}$  for  $\alpha = \alpha_0^+$  compared with the values of  $w_\nu$  on the curves  $\{(\gamma_{2,\alpha}^\nu(t), \alpha t), t \in [0, T]\}$  for  $\alpha = \alpha_0^-$ . Furthermore the order of the values of  $w_\nu$  on the curves  $\{(\gamma_{2,\alpha}^\nu(t), \alpha t), t \in [0, T]\}$  with  $\alpha = \alpha_0^+$  and on the curves  $\{(\gamma_{2,\alpha}^\nu(t), \alpha t), t \in [0, T]\}$  with  $\alpha = \alpha_0^-$  does not change. It implies that the  $BV^s$  norm is decreasing after the interaction along the curves  $\{(\gamma_{2,\alpha}^\nu(t), \alpha t), t \in [0, T]\}$  with  $\alpha - \alpha_0 > 0$  sufficiently small. It gives then using (5.41) that:

$$TV^s w_\nu(\alpha_0^+ \cdot, \gamma_{2,\alpha_0^+}^\nu(\cdot))(]0, T]) \leq TV^s w_\nu(\alpha_0^- \cdot, \gamma_{2,\alpha_0^-}^\nu(\cdot))(]0, T]) \leq \|w_0\|_{BV^s}^{\frac{1}{s}}, \quad (5.42)$$

with  $t \in (0, t_{1,1})$ . The previous argument is again true if there is more than 1 interaction on the curve  $\{(\alpha_0 t, \gamma_{2,\alpha_0}^\nu(t)), t \in [0, T]\}$ . Now we define  $\alpha_1 > \alpha_0$  the next



$\alpha$  where there is an interaction inside  $\Gamma_2$  on the curve  $\{(\alpha_1 t, \gamma_{2,\alpha_1}^\nu(t)), t \in [0, T]\}$  and we define by  $\Gamma_1$  the open domain delimited by the curves  $\{(\alpha_1 t, \gamma_{2,\alpha_1}^\nu(t)), t \in [0, T]\}$ ,  $\{(\alpha_0 t, \gamma_{2,\alpha_0}^\nu(t)), t \in [0, T]\}$  and  $\{(\gamma_1^\nu(t), t), t \in [0, T]\}$ . We observe then that all the 1-waves which meet a curve  $\{(\alpha t, \gamma_{2,\alpha}^\nu(t)), t \in [0, T]\}$  with  $\alpha \in (\alpha_0, \alpha_1)$  are issued from the curve  $\{(\alpha_0 t, \gamma_{2,\alpha_0}^\nu(t)), t \in [0, T]\}$ . It implies that the values of  $w_\nu$  on  $\{(\alpha t, \gamma_{2,\alpha}^\nu(t)), t \in [0, T]\}$  are included in set of values of  $w_\nu$  on the curve  $\{(\alpha' t, \gamma_{2,\alpha'}^\nu(t)), t \in [0, T]\}$  with  $\alpha' = \alpha_0^+$ , furthermore by transversality the values keep the same order. We deduce then that for any  $\alpha \in (\alpha_0, \alpha_1)$  we have using in addition (5.42):

$$TV^s w_\nu(\alpha \cdot, \gamma_{2,\alpha}^\nu(\cdot))( [0, T] ) \leq TV^s w_\nu(\alpha_0^+ \cdot, \gamma_{2,\alpha_0^+}^\nu(\cdot))( [0, T] ) \leq \|w_0\|_{BV^s}^{\frac{1}{s}}. \quad (5.43)$$

Repeating the argument, we deduce finally that the function:

$$\alpha \rightarrow \|w(\alpha \cdot, \gamma_{2,\alpha}^\nu(\cdot))\|_{BV^s([0,T])} \quad (5.44)$$

is decreasing in  $\alpha$ . It implies from (5.32) and (5.22) since  $\gamma_2^\nu = \gamma_{2,\alpha}^\nu$  with  $\alpha = 1$  that it exists  $C > 0$  such that for any  $\nu > 0$  we have:

$$\sum_{\alpha \in J} \mathcal{O}(\sigma_\alpha^3) \leq C \|w_0\|_{BV^s(\mathbb{R})}^{\frac{1}{s}} + \mathcal{O}(\nu^{-3}), \quad (5.45)$$

with  $s = \frac{1}{3}$ . From (5.31) and (5.45), we deduce that it exists  $C > 0$  independent on  $\nu$  such that for any  $T \in (0, T_\nu^*)$  and any forward generalized 2-characteristic  $\gamma_2^\nu$ :

$$|z_\nu(T, \gamma_2^\nu(T))| \leq |z_\nu(0, \gamma_2^\nu(0))| + C \|w_0\|_{BV^s(\mathbb{R})}^{\frac{1}{s}} + \mathcal{O}(\nu^{-3}). \quad (5.46)$$

By the way, a  $BV$  bound for  $z_\nu$  along the generalized 2-characteristic is also obtained,

$$TV(z_\nu(\cdot, \gamma_2^\nu(\cdot)))( [0, T] ) \leq \mathcal{O}\left(\|w_0\|_{BV^s(\mathbb{R})}^{\frac{1}{s}}\right) + \mathcal{O}(\nu^{-3}). \quad (5.47)$$

Since the forward generalized 2-characteristics describe all the space  $(0, T_\nu^*) \times \mathbb{R}$ , we deduce from (5.46) and (5.22) that for any  $t \in (0, T_\nu^*)$  we get for  $C > 0$  independent on  $\nu$ :

$$\|z_\nu(t, \cdot)\|_{L^\infty} \leq \|z_0\|_{L^\infty} + C \|w_0\|_{BV^{\frac{1}{3}}(\mathbb{R})}^3 + o(1).$$

We deduce now that the  $L^\infty$  norm of  $z_\nu$  is uniformly bounded in  $\nu$  all along the time interval  $(0, T_\nu^*)$  and remains small for large  $\nu$ , then using the Remark 12 we deduce that  $T_\nu^* = +\infty$ . The wave-front tracking is then globally defined in time. To summarize we have obtained uniform bounds in  $\nu$  on  $z_\nu$  in  $L_t^\infty(L^\infty)$  and on  $w_\nu$  in  $L_t^\infty(BV^{\frac{1}{3}}(\mathbb{R}))$ , we wish now to develop some compactness argument in order to pass to the limit when  $\nu$  goes to  $+\infty$ . The difficulty is to prove in particular that  $z_\nu$  converges strongly to  $z$  in  $L_{loc,t,x}^1$  since we cannot directly use Helly Theorem as it is the case for  $w_\nu$ .

### 5.3 Compactness argument for $(z_\nu)_{\nu>0}$

We consider now the Lipschitz homeomorphism:

$$\phi^\nu(t, x) = (t, \gamma_2^\nu(t, x)),$$

with  $\gamma_2^\nu(t, x)$  the forward generalized 2-characteristic such that  $\gamma_2^\nu(0, x) = x$ . Furthermore we define  $\eta_L^\nu$  and  $z_L^\nu$  as follows:

$$\eta_\nu(t, x) = z_\nu(t, \gamma_2^\nu(t, x)) - z_0(x) \quad \text{and} \quad z_L^\nu(t, x) = z_\nu(t, \gamma_2^\nu(t, x)).$$

We observe in particular that:

$$z_\nu(t, x) = z_L^\nu((\phi^\nu)^{-1}(t, x)),$$

with  $(\phi^\nu)^{-1}$  the inverse of the Lipschitz homeomorphism  $\phi^\nu$  (see [18, 19] for the notion of a Lipschitz homeomorphism and bi-Lipschitz homeomorphism, when the inverse is also Lipschitz, in geometric measure theory see [31, 48]). We are going now to prove a succession of different Lemmas.

**Lemma 1** *Up to a subsequence, we have:*

$$\lim_{\nu \rightarrow +\infty} z_L^\nu = z_L \quad \text{in } L_{loc,t,x}^1.$$

**Proof:** We have seen that  $z_\nu$  is *BV* along the curve  $\{(t, \gamma_2^\nu(t, x)), t \in [0, T]\}$  (5.47). It implies that  $z_L^\nu$  is uniformly bounded in  $\nu$  in  $L_x^\infty(BV([0, T]))$  for any  $T > 0$ . Now we know that the speed of propagation of  $z_L^\nu$  in the plane  $(x, t)$  is finite, indeed the speed is bounded by:

$$\sup_{(t,x) \in S_1} \left| \frac{1}{\lambda_1((w^\nu(t, x), z^\nu(t, x))^-, (w^\nu(t, x), z^\nu(t, x))^+)} \right| < M, \quad (5.48)$$

with  $S_1$  the set of the 1-shock and  $\lambda_1((w^\nu(t, x), z^\nu(t, x))^-, (w^\nu(t, x), z^\nu(t, x))^+)$  the speed of the shock defined by the Rankine Hugoniot relation. Here  $M$  does not depend on  $\nu$  and (5.48) is true because on a small square  $[-r, r]^2$  with  $r > 0$  sufficiently small we have:

$$\sup_{x \in [-r, r]^2} \lambda_1(x) < 0.$$

It is well known [53, 34] that it implies that  $z_L^\nu$  is uniformly bounded in  $Lip_x(L_{loc,t}^1)$ . Using the Kolmogorov theorem, we deduce that up to a subsequence in  $\nu$   $z_L^\nu$  converges to  $z$  in  $L_{loc,t,x}^1$ :

$$\lim_{\nu \rightarrow +\infty} z_L^\nu = z_L \quad \text{in } L_{loc,t,x}^1.$$

□

Next we use the following lemma.

**Lemma 2** Let  $\psi^\nu$  a homeomorphism uniformly Lipschitz in  $\nu$  from  $\mathbb{R}^+ \times \mathbb{R}$  to  $\mathbb{R}^+ \times \mathbb{R}$  such that there exists  $M > 1$  verifying for any  $\nu > 0$  and almost everywhere:

$$0 \leq \frac{1}{M} \leq |\det D_{t,x}\psi^\nu| \leq M, \quad 0 \leq \|D_{t,x}\psi^\nu\| \leq M. \quad (5.49)$$

We assume that

$$\lim_{\nu \rightarrow +\infty} y^\nu = y \text{ in } L^1_{t,x,loc}, \quad \text{and} \quad \lim_{\nu \rightarrow +\infty} \psi^\nu = \psi \text{ in } L^\infty_{t,x,loc} \quad (5.50)$$

then,

$$\lim_{\nu \rightarrow +\infty} y^\nu(\psi^\nu) = y(\psi) \text{ in } L^1_{t,x,loc}.$$

**Proof:** Let  $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$  a positive regular function with compact support and with values in  $[0, 1]$ . For  $\epsilon > 0$  we take  $\tilde{y}$  a continuous function in  $L^1_{t,x}$  such that:

$$\|\tilde{y} - y\|_{L^1_{t,x}(K_\varphi)} \leq \epsilon. \quad (5.51)$$

with  $K_\varphi$  a compact of  $\mathbb{R}^+ \times \mathbb{R}$  sufficiently large such that for any  $\nu > 0$  we have  $\text{supp } \varphi((\psi^\nu)^{-1})$  which is included in  $K_\varphi$ . Using (5.49) and (5.51) we have:

$$\begin{aligned} & \int_{\mathbb{R}^+} \int_{\mathbb{R}} \varphi |y^\nu \circ \psi^\nu - y \circ \psi| dx dt \\ & \leq \int_{\mathbb{R}^+} \int_{\mathbb{R}} \varphi |y^\nu \circ \psi^\nu - y \circ \psi^\nu| dx dt + \int_{\mathbb{R}^+} \int_{\mathbb{R}} \varphi |y \circ \psi^\nu - \tilde{y} \circ \psi^\nu| dx dt \\ & \quad + \int_{\mathbb{R}^+} \int_{\mathbb{R}} \varphi |\tilde{y} \circ \psi^\nu - \tilde{y} \circ \psi| dx dt + \int_{\mathbb{R}^+} \int_{\mathbb{R}} \varphi |\tilde{y} \circ \psi - y \circ \psi| dx dt \\ & \leq 2\epsilon M + M \|y - y^\nu\|_{L^1_{t,x}(K_\varphi)} + \int_{\mathbb{R}^+} \int_{\mathbb{R}} \varphi |\tilde{y} \circ \psi^\nu - \tilde{y} \circ \psi| dx dt. \end{aligned}$$

Using dominated convergence we can deal with the last integral and prove that for any positive  $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ , we have  $\varphi y^\nu \circ \psi^\nu \rightarrow \varphi y \circ \psi$  in  $L^1_{t,x}$ . We deduce then that  $y^\nu \circ \psi^\nu \rightarrow y \circ \psi$  in  $L^1_{t,x,loc}$   $\square$

**Lemma 3**  $\phi^\nu$  and  $(\phi^\nu)^{-1}$  verify the assumption (5.49) of the Lemma 2.

**Proof:** It suffices to verify that there exist  $M > 1$  such that:

$$0 \leq \frac{1}{M} \leq |\det D_{t,x}\phi^\nu| \leq M, \quad 0 \leq \|D_{t,x}\phi^\nu\| \leq M. \quad (5.52)$$

We observe that:

$$D_{t,x}\phi^\nu(t, x) = \begin{pmatrix} 1 & 0 \\ \partial_t \gamma_2^\nu(t, x) & \partial_x \gamma_2^\nu(t, x) \end{pmatrix} \quad (5.53)$$

and:

$$D_{t,x}(\phi^\nu)^{-1}(\phi^\nu(t, x)) = \begin{pmatrix} 1 & 0 \\ -\frac{\partial_t \gamma_2^\nu(t, x)}{\partial_x \gamma_2^\nu(t, x)} & \frac{1}{\partial_x \gamma_2^\nu(t, x)} \end{pmatrix} \quad (5.54)$$

We know that:

$$\partial_t \gamma_2^\nu(t, x) \in [\min_{\pm}(\lambda_2(w^\nu(t, \gamma_2^\nu(t, x)^\pm))), \max_{\pm}(\lambda_2(w^\nu(t, \gamma_2^\nu(t, x)^\pm)))]$$

It implies in particular that  $\partial_t \gamma_2^\nu$  is uniformly bounded in  $\nu$  since we have seen that  $w_\nu$  is uniformly bounded in  $L_{t,x}^\infty$ . Similarly for  $\partial_x \gamma_2^\nu(t, x)$ , we observe that  $v_\nu(t, x) = \partial_x \gamma_2^\nu(t, x)$  verifies:

$$\partial_t v_\nu(t, x) = \partial_x(\lambda_2(w_\nu))(t, \gamma_2^\nu(t, x))v_\nu(t, x), \quad v_\nu(0, x) = x.$$

Indeed we can observe in fact that  $\gamma_2^\nu$  verify except at the point where  $\gamma_2^\nu$  meets a 1-wave-front:

$$\partial_t \gamma_2^\nu(t, x) = \lambda_2(w_\nu(t, \gamma_2^\nu(t, x))).$$

We deduce that:

$$v_\nu(t, x) = \exp\left(\int_0^t \partial_x(\lambda_2(w_\nu))(s, \gamma_2^\nu(s, x))ds\right). \quad (5.55)$$

and:

$$\det D_{t,x} \phi^\nu = v_\nu(t, x). \quad (5.56)$$

We have now:

$$\int_0^t \partial_x(\lambda_2(w_\nu))(s, \gamma_2^\nu(s, x))ds = \sum_{\alpha \in J_1} [\lambda_2(w_{\alpha,\nu})]. \quad (5.57)$$

$J_1$  is the set of point where a 1-wave-front meets the curve  $\{(\alpha t, \gamma_{2,\alpha}^\nu(t)), t \in [0, T]\}$ . Since in (5.57), we have a telescopic sum, we deduce that:

$$-2\|\lambda_2(w_\nu(\cdot, \cdot))\|_{L_{t,x}^\infty} \leq \int_0^t \partial_x(\lambda_2(w_\nu))(s, \gamma_2^\nu(s, x))ds \leq 2\|\lambda_2(w_\nu(\cdot, \cdot))\|_{L_{t,x}^\infty} \quad (5.58)$$

From (5.55) and (5.58) we deduce that  $\partial_x \gamma_2^\nu$  and  $\frac{1}{\partial_x \gamma_2^\nu}$  is uniformly bounded in  $\nu$ . Furthermore it implies also that the determinant of  $D_{x,t} \phi^\nu$  and  $D_{x,t}(\phi^\nu)^{-1}$  satisfies the assumption (5.52) using (5.56). Since  $\partial_t \gamma_2^\nu$  is uniformly bounded in  $\nu$  we deduce finally that  $\phi^\nu$  and  $(\phi^\nu)^{-1}$  verifies uniformly in  $\nu$  (5.52) using the formula (5.53) and (5.54).  $\square$

**Lemma 4**  $((\phi^\nu)^{-1})_{\nu>0}$  converges up to a subsequence to  $\phi^{-1}$  in  $L_{t,x,loc}^\infty$ .

**Proof:** Indeed for any compact  $K$  of  $\mathbb{R}^+ \times \mathbb{R}$  we observe that  $(\phi^\nu)^{-1}$  is a continuous function from  $K$  to  $\mathbb{R}^+ \times \mathbb{R}$  because  $(\phi^\nu)^{-1}$  is Lipschitz using the Lemma 3. Since the sequence  $((\phi^\nu)^{-1})_{\nu>0}$  is uniformly Lipschitz in  $C(K, \mathbb{R}^+ \times \mathbb{R})$  from the Lemma 3, using the Ascoli Theorem up to a subsequence  $((\phi^\nu)^{-1})_{\nu>0}$  converges uniformly to  $\phi^{-1}$  on  $K$ . Using a standard argument of diagonal process we obtain that  $((\phi^\nu)^{-1})_{\nu>0}$  converges up to a subsequence to  $(\phi)^{-1}$  in  $L^\infty_{t,x,loc}$ .  $\square$

Using the Lemmas 1, 2, 3 and 4 we deduce that  $z_L^\nu o(\phi^\nu)^{-1} = z_\nu$  converges strongly to  $z = z_L(\phi^{-1})$  in  $L^1_{loc,t,x}$  up to a subsequence. It implies that up to a subsequence  $z_\nu$  converges almost everywhere up to a subsequence to  $z = z_L(\phi^{-1})$  when  $\nu$  goes to  $+\infty$ . Concerning the convergence of the sequel  $(w_\nu)_{\nu>0}$ , the proof is more simple since  $(w_\nu)_{\nu>0}$  is uniformly bounded in  $L^\infty(\mathbb{R}^+, BV^{\frac{1}{3}})$ . Thus, as in [8, 37] for the scalar case or [5, 9] for a  $2 \times 2$  system  $w^\nu$  is also bounded in  $Lip_t^s([0, +\infty[, L^p_{loc}(\mathbb{R}, \mathbb{R}))$  with  $p = 1/s$  and the compactness follows.

We deduce then that the sequence  $u_\nu = (w, z)^{-1}(w_{\epsilon_\nu}, z_{\epsilon_\nu})$  with  $(w, z)^{-1}$  the inverse of the local diffeomorphism  $(w, z)$  converges also almost everywhere to  $u$  with  $u_\nu$  uniformly bounded in  $L^\infty_{t,x}$ .

## 5.4 Additive decomposition of $z$

Finally, the decomposition of  $z$  (2.8) in Theorem 2.1 for the weak solution can be proved as the direct consequence of the structure of  $z_\nu$  to get the compactness in previous Section 5.3. The Riemann invariant  $z$  associated to the linearly degenerate field is decomposed as a sum of a  $BV$  function and the initial data. This representation of  $z$  allowq to get more informations on  $z$  with only an  $L^\infty$  bound on the initial data. Such decomposition in a multiplicative form is used to get stability results for a subclass of systems with a genuinely nonlinear field and a linearly degenerate one in [7]. This is also the key point to get compactness in the previous subsection 5.3 where this additive decomposition for the approximate sequence  $(z^\nu)_{\nu>0}$  is in the approximate lagrangian coordinates,

$$z_\nu(t, \gamma_2^\nu(t, x)) = z_{0,\nu}(x) + \eta_\nu(t, x).$$

The 2-characteristics  $(\gamma_2^\nu)$  are equi-Lipschitz. Thus, up to a subsequence, we can pass to the limit when  $\nu \rightarrow +\infty$ . At the limit,  $\gamma_2$  satisfies the differential equation for the generalized 2-characteristics [23, 29]. Moreover,  $z_{0,\nu}$  and  $\eta_\nu$  converge in  $L^1_{loc}$ , the whole sequence for  $(z_{0,\nu})$  and only a subsequence for  $(\eta_\nu)$ . Now, we can pass to the limit in  $z_\nu(t, x)$  using the bi-Lipschitz homeomorphism, as previously with Lemma 2, to obtain the decomposition (2.8). The proof of Theorem (2.1) is achieved.

## 6 Convergence towards a weak entropy solution

It is classical in  $BV$  framework to verify that  $u$  is a global weak solution of the system (1.1) using  $BV$  bounds and dominated convergence (see [10, 24, 36, 53] for more details). In the  $BV^{1/3}$  framework it is done in [5, 9] for less nonlinear systems. In this section the fundamental difference with the  $BV$  framework is explained.

The  $BV^{1/3}$  framework needs to use a simpler and more precise scheme than the general Bressan scheme [10]. First the convergence towards a weak solution is proven and then that such solution is entropic. For this purpose the approximate solution  $U_\nu$  appears to be a weak solution, but not an entropic one, for the system (1.1) when  $U_0$  is replaced by the approximate initial data  $U_{0,\nu}$ .

More precisely, the consistency error in the weak formulation of system (1.1) with the piece-wise initial data  $U_{0,\nu}$  and the smooth test vectorial function  $\Phi$  is,

$$CE_\nu := \int_0^\infty \int_{\mathbb{R}} U_\nu \Phi_t + F(U_\nu) \Phi_x \, dx \, dt + \int_{\mathbb{R}} U_{0,\nu}(x) \Phi(0, x) \, dx$$

Strictly speaking the consistency error is usually defined in the textbooks with  $U_0$  instead of  $U_{0,\nu}$ . For our purpose, it is better to use the approximate piece-wise-constant initial data as in [22]. Else, there is simply an error of consistency due to the approximation of the initial data.

Due to the continuity in time in  $L^1_{loc}$  in space of  $U_\nu$ , the initial data is recovered strongly. Then, it can be assumed that the test function vanishes at  $t = 0$ . Thus, the last integral term in  $CE_\nu$  is now 0.

Now, Applying the green formula, [24][sec 14.7, p. 534, equation (14,7.1)] or [10][p. 143-144], the consistency error becomes,

$$CE_\nu := \int_0^\infty \sum_{fronts} E_\nu \Phi(t, y(t)) \, dt, \quad E_\nu := (\dot{y}(t)[U_\nu] - [F(U_\nu)])(t, y(t)),$$

where the sum runs over all jump discontinuities which are polygonal lines  $x = y(t)$ ,  $E_\nu$  is the local consistency error and the jumps are

$$[U] = U(t, y(t)+) - U(t, y(t)-), \quad [F(U)] = F(U(t, y(t)+)) - F(U(t, y(t)-)).$$

Now, the speed are the exact speed of the jump discontinuities in our wave front tracking algorithm, thus  $CE_\nu = 0$  as in [22]. With our WFT, the approximate solution  $U_\nu$  is a weak solution of system (1.1) with the approximate initial data as in [22]. Thus the convergence towards a weak solution is proved.

However,  $U_\nu$  is not an entropy solution. It is due to the approximation of rarefaction waves by non entropic jumps in the approximate Riemann solver. Let  $\eta$  be a convex entropy of the system and  $q$  the associated entropy flux,  $\phi$  be a scalar test function with support in  $(0, \infty)$ , now the consistency error is

$$CE'_\nu := \int_0^\infty \int_{\mathbb{R}} \eta(U_\nu) \phi_t + q(U_\nu) \phi_x \, dx \, dt.$$

It has to be proven that  $\lim_{\nu \rightarrow +\infty} CE'_\nu \leq 0$ . For this purpose the global consistency error is rewritten with the Green's formula and the local consistency error  $E'_\nu$ ,

$$CE'_\nu := \int_0^\infty \sum_{\text{fronts}} E'_\nu \phi(t, y(t)) dt, \quad E'_\nu := (\dot{y}(t)[\eta(U_\nu)] - [q(U_\nu)])(t, y(t)).$$

The local consistency error for the entropy inequality needs more work.  $E'_\nu \neq (0, 0)$  when the exact continuous rarefaction is replaced by piece-wise constant states [10, p. 126], [36, p. 305].

For an approximate rarefaction, let  $U_0$  be the left state,  $U_m$  the right one and  $u_i$  the intermediary states,  $i = 1, \dots, m-1$  with  $U_i = (w_i, z_i)$ . Localizing the error at a non entropic jump front  $i$  between  $U_i$  and  $U_{i+1}$ , we see that  $E'_\nu = \mathcal{O}(|w_{i+1} - w_i|^3) = \mathcal{O}(\nu^{-3})$ , due to the fundamental Lax' cubic estimate [41] for the entropy [24][Thm 8.5.1 p. 314]. Adding all these local errors for any fixed time yield the estimate

$$\sum_{\text{fronts}} E'_\nu \leq \mathcal{O}(1) \cdot TV^{1/3} w_0.$$

It shows that the consistency error in the WFT is naturally related to  $BV^{1/3}$  estimates. This is not enough to prove that  $\lim CE'_\nu \leq 0$  when  $\nu \rightarrow +\infty$ . To prove that the consistency error goes to zero two cases are considered, the case  $s > 1/3$  which is handled in a similar way as the case  $s = 1$  ( $BV$ ) in textbooks, and the critical and more complicated case  $s = 1/3$ .

If  $w_0 \in BV^s$  with  $s > 1/3$ , then we can split the local error term as

$$|w_{i+1} - w_i|^3 = |w_{i+1} - w_i|^p |w_{i+1} - w_i|^\eta = |w_{i+1} - w_i|^p \nu^{-\eta},$$

where  $p = 1/s = 3 - \eta$ ,  $\eta > 0$ . Therefore

$$\sum_{\text{fronts}} E'_\nu \leq \mathcal{O}(1) \cdot \sum_{\text{non entropic jumps}} |w_{i+1} - w_i|^p \nu^{-\eta} \leq \mathcal{O}(1) \nu^{-\eta} TV^s w_0.$$

The sum is taken over all non entropic jumps. Thus the right hand side converges towards zero when  $\nu \rightarrow +\infty$  and yields  $\lim CE'_\nu \leq 0$ .

Now, consider the case  $s = 1/3$ . Let  $\bar{m}$  denote the number of consecutive non entropic jumps to approximate a rarefaction wave. Then  $\bar{m} \leq \mathcal{O}(\nu M + 1)$ , where  $M = \|w_0\|_\infty$  and for any such block of approximate rarefaction wave, we have

$$\sum |w_{i+1} - w_i|^3 \leq \bar{m} \nu^{-3} \leq \mathcal{O}(\nu M + 1) \nu^{-3} = \mathcal{O}(\nu^{-2}). \quad (6.59)$$

The number of such blocks is not increasing with time and is bounded by the number  $N_\nu$  of initial Riemann problems at  $t = 0$  in the WFT. Therefore

$$\sum_{\text{fronts}} E'_\nu \leq \mathcal{O}(1) \sum_{\text{blocks}} 2(\nu M + 1) \nu^{-3} \leq 2N_0 (\nu M + 1) \nu^{-3} \cdot \mathcal{O}(1). \quad (6.60)$$

Therefore if we choose  $N_\nu = \nu$  to begin with, we see that the entropy error consistency becomes nonpositive as  $\nu$  tends to  $+\infty$  which concludes this section.

## A Local monotonicity and computation of $TV^s u$

The computation of the  $TV^s$  variation can be more complicate than the usual  $TV$  variation [8]. In this appendix, we explain the difference with the classic  $BV$  case,  $s = 1$ , and also how to easily obtain the decay of  $TV^s w_\nu$  in the wave front tracking algorithm. For this purpose, considering only finite sequence is enough.

For a sequence  $(u_n)_{1 \leq n \leq N}$ ,  $N > 1$ , a subdivision  $\sigma$  is considered as a subset of  $\{1, \dots, N\}$  or as an increasing application from  $\{1, \dots, |\sigma|\}$  to  $\{1, \dots, N\}$  where  $2 \leq |\sigma| \leq N$  is the cardinal of the subdivision. This means that the subdivision  $\sigma$  can be written in terms of the bijection  $\sigma$  on the ordered set  $\{\sigma(1), \dots, \sigma(|\sigma|)\}$ . In short, we note  $\sigma = \{\sigma(1), \dots, \sigma(|\sigma|)\}$ . Let  $p = 1/s \geq 1$ , the  $p$ -variation of a finite sequence is [14, 15],

$$p-TVu = TV^s u := \max_{\sigma \subset \{1, \dots, N\}} \sum_{1 \leq i < |\sigma|} |u_{\sigma(i+1)} - u_{\sigma(i)}|^p,$$

where the maximum is taken on all subdivisions of the ordered set  $\{1, \dots, N\}$ . The number of subdivisions is quite big,  $\mathcal{O}(2^N)$ , but finite so  $p-TVu$  is really a maximum. A subdivision where the maximum is achieved is called and *optimal subdivision* for the sequence  $u$ . Let  $TV^s u[\sigma]$  the  $p$ -variation of  $u$  on  $\sigma$ ,

$$p-TVu[\sigma] = TV^s u[\sigma] := \sum_{1 \leq i < |\sigma|} |u_{\sigma(i+1)} - u_{\sigma(i)}|^p. \quad (1.61)$$

The subdivision  $\sigma$  is *optimal* reads  $TV^s u = TV^s u[\sigma]$ .

For  $p = 1$ , this maximum is easy to compute, it suffices to take the biggest partition  $\sigma = \{1, \dots, N\}$ . For  $p > 1$ , i.e.  $s < 1$ , this is rarely so simple. It is due to the convex inequality,  $p > 1$ ,  $a > 0$ ,  $b > 0$ ,

$$(a + b)^p > a^p + b^p.$$

In other words, if for some  $k$ ,  $1 < k < N$ ,  $u_{\sigma(k-1)} < u_{\sigma(k)} < u_{\sigma(k+1)}$  then

$$|u_{\sigma(k+1)} - u_{\sigma(k-1)}|^p > |u_{\sigma(k+1)} - u_{\sigma(k)}|^p + |u_{\sigma(k)} - u_{\sigma(k-1)}|^p.$$

Thus, a bigger  $p$ -variation is achieved on a smaller subdivision,

$\tau = \sigma - \{\sigma(k)\}$ ,  $TV^s u[\sigma] < TV^s u[\tau]$ . For an increasing sequence, and repeating the previous argument, there is only one optimal subdivision,  $\sigma = \{1, N\}$ ,  $|\sigma| = 2$ .

The previous discussion gives the following useful lemma where some values are added in a sequence  $u$  in a monotonic order without increasing the  $p$ -variation.

**Lemma 5 (No increase of the  $p$ -variation)** *Let  $u = (u_n)_{1 \leq n \leq N}$  a finite sequence. Let  $1 \leq k < N$  such that  $u_k \leq u_{k+1}$  and  $1 \leq j$ . Consider  $v = (v_n)_{1 \leq n \leq N+j}$  a sequence such that,*

- $v_n = u_n$  for  $n \leq k$ ,



- $u_k \leq v_n \leq v_{n+1} \leq u_{k+1}$  for  $k < n < k + j$
- $v_{n+j} = u_n$  for  $k + 1 \leq n \leq N$ ,

then  $TV^s v = TV^s u$ .

If  $j = 1$  only one value  $v_{k+1}$  is added between  $u_k$  and  $u_{k+1}$ .

The same result holds if  $u_k \geq u_{k+1}$  and  $u_k \geq v_n \geq v_{n+1} \geq u_{k+1}$  for  $k < n < k + j$ .

**Application to bound  $TV^s w_\nu$**  After each interaction in the WFT, a new Riemann problem is solved. The new values appearing for  $w_\nu$  are ordered between two values of  $w_\nu$  so  $TV^s w_\nu$  is not increasing. For the details, see Section 5.1.

**Proof of Lemma 5:** It suffices to prove the lemma for  $j = 1$ . The case  $j > 1$  is easily done by induction. The inequality  $TV^s v \geq TV^s u$  is clear,  $v$  has already all the values of  $u$  in the same order than  $u$  except one more value ( $j = 1$ ). The point is to prove that for all subdivision  $\sigma \in \{1, \dots, N + 1\}$ ,  $TV^s v[\sigma] \leq TV^s u$ . If  $k + 1$  does not belong to  $\sigma$  then  $v$  restricted on  $\sigma$  is identical to  $u$  restricted on  $\sigma$  and  $TV^s v[\sigma] = TV^s u[\sigma]$ . So, the only problematic case is when  $k + 1$  belongs to  $\sigma$ . In this case, a subdivision  $\tau$  is now built such that  $TV^s v[\sigma] \leq TV^s u[\tau]$ . For this purpose, consider all the four following cases. Let  $i$  be defined by  $\sigma(i) = k + 1$ ,  $v_{\sigma(i-1)} = u_{\sigma(i-1)}$  and  $v_{\sigma(i+1)} = u_{\sigma(i+1)-1}$ . To have a subdivision related to  $u$ , let us define  $\tilde{\sigma} = \sigma - \{k + 1\}$ , so

$$v_{\sigma(i-1)} = u_{\sigma(i-1)} = u_{\tilde{\sigma}(i-1)}, \quad v_{\sigma(i+1)} = u_{\sigma(i+1)-1} = u_{\tilde{\sigma}(i+1)}$$

For each case,  $u_k \leq v_{k+1} \leq u_{k+1}$  and we have only to bound,

$$V := |v_{\sigma(i+1)} - v_{\sigma(i)}|^p + |v_{\sigma(i)} - v_{\sigma(i-1)}|^p = |u_{\tilde{\sigma}(i+1)} - v_{k+1}|^p + |v_{k+1} - u_{\tilde{\sigma}(i-1)}|^p.$$

1.  $u_{\tilde{\sigma}(i-1)} \leq u_k$  and  $u_{k+1} \leq u_{\tilde{\sigma}(i+1)}$ , it suffices to take  $\tau = \tilde{\sigma}$  since,

$$V \leq |u_{\tilde{\sigma}(i+1)} - u_{\tilde{\sigma}(i-1)}|^p.$$

2.  $u_{\tilde{\sigma}(i-1)} \geq u_k$  and  $u_{k+1} \leq u_{\tilde{\sigma}(i+1)}$ , it suffices to take  $\tau = \{k\} \cup \tilde{\sigma}$  since,

$$V \leq |u_{\tilde{\sigma}(i+1)} - u_k|^p + |u_k - u_{\tilde{\sigma}(i-1)}|^p.$$

3.  $u_{\tilde{\sigma}(i-1)} \leq u_k$  and  $u_{k+1} \geq u_{\tilde{\sigma}(i+1)}$ , it suffices to take  $\tau = \{k + 1\} \cup \tilde{\sigma}$  since,

$$V \leq |u_{\tilde{\sigma}(i+1)} - u_{k+1}|^p + |u_{k+1} - u_{\tilde{\sigma}(i-1)}|^p.$$

4.  $u_{\tilde{\sigma}(i-1)} \geq u_k$  and  $u_{k+1} \geq u_{\tilde{\sigma}(i+1)}$ , it suffices to take  $\tau = \{k, k + 1\} \cup \tilde{\sigma}$  since,

$$V \leq |u_{\tilde{\sigma}(i+1)} - u_{k+1}|^p + |u_{k+1} - u_k|^p + |u_k - u_{\tilde{\sigma}(i-1)}|^p.$$

□

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