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Fractional BV solutions for 2×2 systems of conservation laws with a linearly degenerate field

Boris Haspot *, Stéphane Junca †

Abstract: The class of 2×2 nonlinear hyperbolic systems with one genuinely nonlinear field and one linearly degenerate field are considered. Existence of global weak solutions for small initial data in fractional BV spaces BV^s is proved. The exponent s is related to the usual fractional Sobolev derivative. Riemann invariants w and z corresponding respectively to the genuinely nonlinear component and to the linearly degenerate component play different key roles in this work. We obtain the existence of a global weak solution provided that the initial data written in Riemann coordinates (w_0, z_0) are small in $BV^s \times L^\infty$, $1/3 \leq s < 1$. The restriction on the exponent s is due to a fundamental result of P.D. Lax, the variation of the Riemann invariant z on the Lax shock curve depends in a cubic way of the variation of the other Riemann invariant w.

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1 Introduction

In this paper, we study general 2×2 hyperbolic systems of the form:

$$\begin{cases} \partial_t U + \partial_x F(U) = 0 \\ U(0, \cdot) = U_0 \end{cases} \tag{1.1}$$

with $U(t,x) \in \Omega \subset \mathbb{R}^2$ an open set, $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. F is the flux of the system and it is regular from $\mathbb{R}^2$ to $\mathbb{R}^2$. We assume that the system is strictly hyperbolic on Ω, it means that $DF(U)$ has two different eigenvalues λ_1 and λ_2. Without any restriction we can assume that $\lambda_1 < 0 < \lambda_2$, reducing if necessary the open set Ω. It implies in particular that we have a basis of eigenvector of unit norm $(r_1(U), r_2(U))$ for any $U \in \Omega \subset \mathbb{R}^2$. In the sequel we will only be interested in the case of a 1 genuinely nonlinear field and a 2 linearly degenerate field. In particular, it means that for every $U \in \Omega \subset \mathbb{R}^2$ we have

$$\nabla \lambda_1(U) \cdot r_1(U) \neq 0 \text{ and } \nabla \lambda_2(U) \cdot r_2(U) = 0. \tag{1.2}$$

Examples We wish now to give some examples of strictly hyperbolic system with a genuinely nonliear field and a linearly degenerate field.

- The classical chromatography system [10, 18] when the velocity is known which is the case for the liquid chromatography.
- The Keyfitz-Krantzer system [29] has this structure, it is maybe the first and the most famous known. It is related to some problem of nonlinear elasticity.
- The 2×2 Baiti-Jenssen system [2] with a genuinely nonlinear field. The Baiti-Jenssen systems arise in models for porous media, traffic and gas flows.
- The Aw-Rascle system is well known for traffic flow [1].

  The four first examples are Temple systems [40, 43, 44]. Such systems satisfy a maximum principle which is not generally true for systems of conservation laws. Now the following list provides examples that are not Temple systems.

- Colombo and Corli consider the class of 2×2 system with genuinely nonlinear field and a Temple field [16]. They prove existence of solutions for large BV data associated to the Temple component. A linear degenerate field is an example of Temple field, the rarefaction and shock curve coincide [10]. Such assumption is not enough to have a Temple system. One interest of our work is to prove existence in BV^s, so, with possible infinite total variation in BV.

- The chromatography system with a sorption effect [6] is a chromatography system with a non constant and unknown velocity. This system is generally not a Temple system [7].
- We mention also some triangular systems with a transport equation [5]. This class of systems generalizes the previous one when it is written in Lagrangian coordinates [9].

In this paper we would like to extend, for 2×2 systems with a genuinely nonlinear filed and a linearly degenerate one, the famous result of Glimm [25] concerning the existence of global weak solution for the strictly hyperbolic system with small initial data u_0 in BV. Indeed we would like to enlarge the set of initial data by working with u_0 belonging to BV^s with $0 < s < 1$, $BV^1 = BV$. We are now going to give a definition of the fractional BV spaces called BV^s. We refer also to Bruneau [14] for more details.

Definition 1 (TV^s variation) *We say that a function u is in $BV^s(\mathbb{R})$ with $0 < s \leq 1$ and $p = 1/s \geq 1$ if $TV^s u < +\infty$ with:*

$$TV^s u := \sup_{n \in \mathbb{N},\, x_1 < \cdots < x_n} \sum_{i=1}^{n} |u(x_{i+1}) - u(x_i)|^p \tag{1.3}$$

The associated semi-norm of the TV^s variation is,

$$|u|_{BV^s} := (TV^s u)^s \tag{1.4}$$

and a norm is

$$\|u\|_{BV^s} := \|u\|_{L^\infty} + |u|_{BV^s} \tag{1.5}$$

In the same way, $TV^s u(I)$ is defined as the TV^s variation of the function u on the set I. We note that it is clear that for any $s \in]0, 1]$, $BV^s(\mathbb{R}) \subset L^\infty(\mathbb{R})$ [39]. Moreover, if u belongs to $L^1(\mathbb{R})$ then the semi-norm BV^s is a norm. This is due to the fact that a BV^s function has limts at $\pm\infty$ and, for a $L^1(\mathbb{R})$ function, these limits are 0. For $0 < s_1 < s_2 \leq 1$, we also have $BV^{s_2} \subset BV^{s_1}$ [8]. The TV^s variation was called the $p-$variation with $p = 1/s$ in [39]. We prefer to use the notation TV^s since it is related to the Sobolev exponent "s". Indeed, BV^s_{loc} is close from $W^{s,p}_{loc}$ but remains different [8], indeed the BV^s functions are regulated functions [39] as BV functions.

Proposition 1 (BV^s functions are regulated functions [39]) *If $u \in BV^s$ with $0 < s \leq 1$ then u admits only a countable set of discontinuity. Futhermore for every $x \in \mathbb{R}$, u admits a limit on the left and on the right in x.*

We would like now to motivate the use of the BV^s spaces for the study of hyperbolic systems. Actually the most of the results on the existence of global weak

solution for 2×2 hyperbolic systems concerned the L^∞ space and the BV space. In order to tackle this problem, there exists essentially two different approaches, the first one was developed by Glimm in the 60s [25]. He proved for a general $n \times n$ strictly hyperbolic system with genuinely non linear field or linearly degenerate field the existence of global weak entropy solution provided that the initial data is small in BV. The main difficulty of the proof consists in controlling the BV norm of the solution all along the time, indeed Glimm has observed that the BV norm can increase after each interaction between the nonlinear waves. In order to estimate this gain in BV norm after each interaction, Glimm has introduced a quadratic functional which described the interactions between the nonlinear waves and which allows to evaluate the BV norm of the solution all along the time. This result has been extended in the 90s by Bressan and his collaborators [10, 12, 13] where they proved the uniqueness of Glimm solution (provided that U_0 belongs also to $L^1(\mathbb{R})$) in a suitable class of solution which takes into account in particular the Lax conditions for the shocks. The main ingredient to do this is to prove that the wave front tracking algorithm (we refer to [10] for the definition of the wave front tracking for general $n \times n$ systems) generates a Lipschitz semigroup in L^1 [10, 11]. We recall in particular that the solutions which are issue of the wave front tracking method and which are determined via a compactness argument are the same as the solution coming from the Glimm scheme [10].
The second approach was initiated by Di Perna [20, 21] at the beginning of the 80s using the so called compensated compactness which was introduced by Tartar [42]. Roughly speaking this method can be applied for 2×2 strictly hyperbolic systems with two genuinely non linear fields (see also Serre [40]) when the initial data U_0 is assumed to belong to $L^\infty(\mathbb{R})$. The case of the isentropic Euler system has been particularly studied and we refer to [19, 34, 35, 33]. We observe then that this method allows to deal with more general initial data as $U_0 \in BV$, however there is generally no result of uniqueness for these solutions. In particular it seems complicated to select the solution via the Lax conditions on the shocks since we can not give any sense of traces along a shock for such solutions since they belong only to $L^\infty_{t,x}$.
In the 2×2 case, Glimm in [25] has obtained a better result of existence of global weak solution inasmuch as he can deal with large initial data U_0 in BV provided that the L^∞ norm of U_0 is sufficiently small. It is due to the fact that after an interaction between waves the variation of the BV norm has a cubic order in terms of the incoming strengths of the waves which interact (in the general case $n \geq 3$, this order is only quadratic). This result is a consequence of the existence of Riemann invariants for 2×2 systems. Later on, this result has been extended by Glimm and Lax in [26] to the case of small L^∞ initial data when the fields are genuinely non linear. We refer also to the recent work of Bianchini, Colombo and Monti [4]. To do this, the authors use new Glimm functionals to control the L^∞ norm combined with the method of backward characteristics. In addition they proved a new Oleinik inequality (which is generally restricted to the scalar conservation law with

genuinely non linear flux) for this 2×2 system which gives sufficient compactness informations in order to pass to the limit in the wave front tracking.

BV^s spaces are intermediary spaces between L^∞ and BV spaces, see [8] or the definition 1. We note in particular that the BV^s spaces admits functions with shocks, from this point of view these spaces are suitable for dealing with hyperbolic systems of conservation laws. Indeed it is well known that the solution of an hyperbolic system can admits shock in finite time even if the initial data is arbitrary regular. In addition (see [39] and the proposition 1 below), BV^s functions admit a notion of "traces" as for BV functions (this is of course not the case for L^∞ functions). This notion of trace is essential in the result of uniqueness of Bressan et al [10, 12, 13]. Indeed, it gives a sense to the notion of the Lax entropy criterion which enables to select a unique solution (in the results of Bressan & al. a tame oscillation condition is also required). It would be then interesting to prove the existence and the uniqueness of global weak solution for initial data U_0 in BV^s with $0 < s < 1$ for strictly hyperbolic systems. It would improve in particular the results of existence of Glimm [25] inasmuch as the initial data U_0 would be less regular as BV. In addition, working with BV^s gives a chance to extend the results of uniqueness of Bressan & al. [10, 12, 13] since the notion of trace remains relevant.

In this paper, we will only focus our attention on the existence of global weak solutions for small initial data in BV^s. Before giving and describing our main results, we would like now to recall some results using the BV^s space in the framework of conservation laws. For scalar conservation law, the entropy solution corresponding to an initial data belonging in BV^s remains in BV^s for all time [8]. Moreover, this result is sharp [23]. We would like to point out that the BV^s space is also naturally used to describe the regularizing effects of scalar conservation laws. From [30], it is known that there exists unique global solution for scalar conservation laws when U_0 belongs to L^∞. The most famous regularizing effect concerns the uniformly convex flux where the solution becomes instantaneously BV_{loc}, this is a direct consequence of the so called Oleinik inequality. For a convex flux with a power law degeneracy, the authors in [8] show an optimal regularizing effects on the solution u inasmuch as the solution u becomes instantaneously BV^s_{loc} with s depending on the power law of the flux. For a general nonlinear convex flux locally Lipschitz, the solution belongs for positive time in a generalized BV space related to the nonlinearity of the flux, see [27]. These results have been extended for a nonlinear non convex flux, at least C^3, in a generalized BV space, and for a more regular flux with polynomial degeneracy in the optimal BV^s space by Marconi in [36, 37].

The most of the result dealing with BV^s initial data concerned scalar conservation laws. Indeed it is a priori delicate to prove the stability of the BV^s norm all along the time, the BV^s norm is indeed more complicated to compute than the BV norm.

Indeed when we apply a Glimm scheme, in order to know the BV norm after an interaction between waves, it is sufficient to estimate locally the strength of the new outgoing waves since we recover the complete BV norm by summing the absolute value of the different strength on all the euclidean space. In particular using the triangular inequality, we do not need to select subdivisions of the euclidean space in a accurate way in order to control the BV norm. It is not the case for the BV^s norm which is reached for particular optimal subdivisions. It implies in particular that for computing the BV^s norm after a waves interaction, it is not sufficient to knows only the values of the outcoming strength. In the scalar case, the analysis is simpler since after each interactions, there exists some zone of monotonicity for the Riemann problem making the analysis simpler to compute the BV^s norm [8].

In this paper, we would like to extend the analysis of [8] to the case of 2×2 strictly hyperbolic system with one genuinely nonlinear field and one linearly degenerate field which corresponds to the case described in (1.2). This case is a particular case of the theory of Glimm [25] on the existence of global weak solution for initial data U_0 in $(BV(\mathbb{R}))^2$ with a large BV norm provided that the L^∞ norm is sufficiently small. We would like also to mention that others authors have yet obtained existence of weak entropy solution for small L^∞ data and large BV norm when an eigenvalue is linearly degenerate [4, 18] or a Temple eigenvalue [16]. We extend the results of Glimm by proving the existence of global weak solution for small initial data with (w_0, z_0) belonging in $BV^s \times L^\infty$ with $\frac{1}{3} \leq s < 1$. Here (w, z) are the solution of the system (1.1) that we consider in Riemann coordinates respectively in terms of the 1 genuinely nonlinear field and the 2 degenerate field. To do this, we follow the classical method which consists in introducing a wave front tracking with (w^ν, z^ν) corresponding to the appoximate solutions (U^ν) of the wave front tracking written in Riemann coordinates and $\nu \to +\infty$ the parameter associated to the wave front tracking. We are then interested in proving that $(U^\nu)_{\nu>0}$ converges to U a solution of the system (1.1). The main difficulty consists in proving uniform BV^s estimates on (w^ν, z^ν) and next in verifying that the wave front tracking is well defined for any time $t > 0$. The end of the proof requires to establish compactness argument in order to verify that U^ν converges to U a solution of the system (1.1) (here U^ν is the approximated solution associated to the wave front tracking written in physical coordinates and not in Riemann coordinates).
More precisely we show that the BV^s norm of w^ν is uniformly conserved all along the time essentially because the waves interactions do not increase the BV^s norm for w^ν. The proof is reminiscent of the scalar case for a convex flux. However it is more complicated to control uniformly the L^∞ norm of z^ν. Indeed the norm of z^ν can increase after two types of interactions, interaction between 1-shocks and interaction between 1-shock and a 2-contact discontinuity. To do this, we consider the L^∞ norm of z^ν along any forward generalized 2-characteristic and we observe that this L^∞ norm depends on the $BV^{\frac{1}{3}}$ norm of w_0. Indeed the L^∞ norm of z^ν along a forward generalized 2-characteristic increases only when the forward

generalized 2-characteristic meets a 1-shock, furthermore this increase depends on the cubic strength in the variation of w^ν on this 1-shock (it is important to point out that this increase is directly related to the regularity of the Lax shock curve). It suffices then to follows these 1-shock in a backward manner in order to estimate the L^∞ norm of z^ν in terms of $\|w_0\|_{BV^{\frac{1}{3}}}$. It explains why we need to assume that w_0 is in BV^s with s at least equal to $\frac{1}{3}$. The last step of the proof consist in proving that z^ν converges to z up to a subsequence in $L^1_{loc,t,x}$. This part is a priori delicate since we have only a uniform control of the L^∞ norm of z^ν. We observe however that we have additional regularity property if we study the unknown $z^\nu_L(t,x) = z^\nu(t,\gamma^\nu_2(t,x))$ with $\gamma^\nu_2(t,x)$ the forward generalized 2-characteristic such that $\gamma^\nu_2(0,x) = x$. Here z^ν_L is the value of z in Lagrangian coordinates, following the same idea as for the control of the L^∞ norm of z^ν, we can prove that z^ν_L is uniformly bounded in $L^\infty_x(BV_t)$ and that the speed of propagation is finite. It is then sufficient to prove that z^ν_L converges up to a subsequence strongly in $L^1_{loc,t,x}$, we prove next that the Lagrangian transformation $(t,\gamma^\nu_2(t,x))$ is a uniformly bi-Lipschitz homeomorphism in ν what is sufficient to ensure that z^ν converges also strongly in $L^1_{loc,t,x}$. It allows to conclude that the solution (U^ν) converges to a solution of (1.1).

2 Presentation of the results

We would like to state now our main result. For this purpose we use a distinguished coordinate system called Riemann invariants, which in general exists only for 2×2 systems (chapter 20, [41]). This allow to perform a nonlinear diagonalization of the hyperbolic system for smooth solutions. This diagonalization is not valid for discontinuous solutions but the Riemann invariants have still some advantages. The Riemann problem and the interaction of waves is also simpler to study in these coordinates than in the initial coordinates. The following notations (w,z) are chosen for the Riemann invariants. Thanks to Riemann [41], there exists a change of coordinates $U \mapsto (w,z) = (w(U),z(U))$ (here $(w,z) = (w(U),z(U))$ is a standard abuse of notations), reducing the open set Ω if necessary, such that

$$\nabla w \cdot r_2 = 0, \qquad \nabla z \cdot r_1 = 0. \tag{2.6}$$

In all the sequel, U is written in this system of coordinates. In particular, the initial data U_0 of the system 1.1 reads $w_0 = w(U_0)$ and $z_0 = z(U_0)$. Our main theorem states as follows.

Theorem 2.1 (Existence in $BV^{1/3}\times L^\infty$) *Let $w_0 \in BV^s(\mathbb{R})$ with $\frac{1}{3} \le s \le 1$ and $z_0 \in L^\infty(\mathbb{R})$ then there exists $\epsilon_0 > 0$ such that if:*

$$\|w_0\|_{BV^s} + \|z_0\|_{L^\infty} \le \epsilon_0$$

then there exists a global weak solution U for the system (1.1). The Riemann coordinates (w,z) belong to $L^\infty(\mathbb{R}^+, BV^s\times L^\infty)$.

Moreover, the Riemann invariant z can be decomposed in the Lagrangian coordinates associated to the linearly degnerate field as follows:

$$z(t, \gamma_2(t,x)) = z_0(x) + \eta(t,x) \tag{2.7}$$

where λ_2 is the linearly degenerate eigenvalue which depends only on w and γ_2 represents the generalized 2-characterisics,

$$\begin{cases} \dfrac{d\gamma_2}{dt}(t,x) & = \lambda_2(w(t,\gamma_2(t,x))) \\ \gamma_2(0,x) & = x \end{cases}$$

with $\eta \in L^\infty_x(\mathbb{R}, BV_t(\mathbb{R}^+)) \cap Lip_x(\mathbb{R}, L^1_{loc,t}(\mathbb{R}^+))$.

Remark 1 *In this Theorem, we assume that $(w_0, z_0) \in BV^s \times L^\infty$, $\frac{1}{3} \leq s < 1$. A sufficient condition on the initial physical coordinates U_0 to ensure such regularity is to take $U_0 \in BV^s$.*

Remark 2 *The decomposition of z in (2.7) provides also a stability results in BV^σ for all $0 \leq \sigma \leq 1$. This means that z is in $L^\infty(\mathbb{R}^+, BV^\sigma)$ if $z_0 \in BV^\sigma$.*

This decomposition (2.7) has already been obtained as a factorization of the gas velocity for a chromatography system (Theorem 7.2 in [7]).

Up to our knowledge, Theorem 2.1 is the first general result concerning the stability of the BV^s norm in the framework of strictly hyperbolic systems, except for some particular physical systems [9, 28]. It extends in particular the results of Glimm [25] since the initial data are not necessary BV. Furthermore if we compare this result with the works of Glimm, Lax and Bianchini,Colombo, Monti (see [26] and [4]) which deal with initial data in L^∞, we can only say that the framework is different. Indeed in [26, 4] the two fields are genuinely nonlinear, in particular the authors extend the Oleinik inequality to their case what allows them to obtain sufficient information in terms of compactness to pass to the limit respectively in their scheme and their wave front tracking. In our case, the technics are quite different especially on the arguments of compactness which enables us to consider the limit of the approximated solutions (w^ν, z^ν) which are issue of the wave front tracking. Indeed, an observation is to remark that the solution z^ν can be splited into z_0 the initial data and a function η^η which is more regular as z^ν itself. We can then pass to the limit in ν for η^ν in the wave front tracking.

Remark 3 *For 2×2 Temple system with a genuinely nonlinear field and a linearly degenerate field, these results improve the classical existence in BV. The existence for L^∞ data in [12] needs that all fields are genuinely nonlinear.*

Remark 4 *Since the BV^s norm has a notion of trace it would be interesting to prove the uniqueness of the solution.*

In the section 3, we start by giving one useful Lemma to compute in a simple way the BV^s norm of a sequence $(u_n)_{n\in\mathbb{N}}$. In the section 4, we describe the Lax curve and the different interactions between 1 and 2-waves. Furthermore we define a simplified wave front tracking well adapted to our case which concerns a one genuinely nonlinear field and one linearly degenerate. In the section 5 we prove the Theorem 2.1.

3 Local monotonicity and computation of $TV^s u$

The computation of the TV^s variation can be more complicate than the usual TV variation [8]. Here, we prove in a self contained and in a detailled way an important tool to compute the BV^s norm for a function which is piecewise constant.

For a sequence $(u_n)_{n=1,\cdots,N}$, a subdivision σ is considered as a subset of $\{1,\cdots,N\}$ or as an increasing application from $\{1,\cdots,|\sigma|\}$ to $\{1,\cdots,N\}$ where $|\sigma|\leq N$ is the the cardinal of the subdivision. This means that the subdivision σ can be written in terms of the bijection σ as $\{\sigma(1),\cdots,\sigma(|\sigma|)\}$

Lemma 1 *Assume that we have a sequence $(u_n)_{n=1,\cdots,N}$ with $i_0\in\{2,\cdots,N-1\}$:*

$$u_{i_0-1}<u_{i_0}<u_{i_0+1}$$

and assume that for $0<s\leq 1$:

$$TV^s u=\sum_{1\leq i<|\sigma|}|u_{\sigma(i+1)}-u_{\sigma(i)}|^{\frac{1}{s}}$$

with σ a subdivision of $\{1,\cdots,N\}$ which is optimal to compute $TV^s u$. Then i_0 is not in the subdivision σ. In particular if we set $v=(u_1,\cdots,u_{i_0-1},u_{i_0+1},\cdots,u_N)$ then we have:

$$TV^s u=TV^s v.$$

The conclusion is the same if we have:

$$u_{i_0-1}>u_{i_0}>u_{i_0+1}.$$

Proof: We assume here by absurd that $i_0\in\sigma$ with σ an optimal subdivision for the BV^s norm of $(u_n)_{n=1,\cdots N}$. Furthermore the first term of the subdivision σ before i_0 is i_{-1} and the next one is i_1, we have in particular $i_{-1}<i_0<i_1$. It is important to note that for example i_{-1} exists, indeed if we assume that i_0 is the first term of the subdivision σ it is easy to observe that $\widetilde{\sigma}=\{i_0-1\}\cup\sigma$ is a subdivision which produces a larger BV^s norm as the subdivision σ. The same argument is also true for i_1. We are now going to consider different cases. We note in the sequel $p=\frac{1}{s}$.

- $i_{-1} < i_0 - 1$ and $i_1 > i_0 + 1$.
  In this case if we have $u_{i_{-1}} < u_{i_0} < u_{i_1}$ then i_0 can not be in the subdivision since:
  $$|u_{i_0} - u_{i_{-1}}|^p + |u_{i_0} - u_{i_1}|^p < |u_{i_{-1}} - u_{i_1}|^p. \tag{3.8}$$
  Indeed the subdivision σ would be not optimal for the BV^s norm.
  If we have $u_{i_{-1}} < u_{i_0}$, $u_{i_1} \leq u_{i_0}$, then we observe that:
  $$|u_{i_{-1}} - u_{i_0}|^p + |u_{i_1} - u_{i_0}|^p < |u_{i_{-1}} - u_{i_0+1}|^p + |u_{i_1} - u_{i_0+1}|^p.$$
  It implies that σ is not optimal for the TV^s norm since the subdivision $\widetilde{\sigma}$ $\cdots, i_{-1}, i_0 + 1, i_1, \cdots$ produces a larger BV^s norm.
  The other cases $u_{i_{-1}} > u_{i_0} > u_{i_1}$ and $u_{i_{-1}} > u_{i_0} \leq u_{i_1}$ can be treated similarly, in particular for the last case it suffices to consider the subdivision $\widetilde{\sigma}$ defined by $\cdots, i_{-1}, i_0 - 1, i_1, \cdots$. We note that the case $u_{i_1} = u_{i_0}$ has no interest since we can omit in the subdivision σ the term i_0.

- $i_{-1} < i_0 < i_0 + 1 = i_1$ with $i_{-1} < i_0 - 1$.
  If $u_{i_{-1}} < u_{i_0}$ then i_0 is not in the subdivision again using (3.8).
  If $u_{i_{-1}} > u_{i_0}$ then we have:
  $$|u_{i_{-1}} - u_{i_0}|^p + |u_{i_0+1} - u_{i_0}|^p < |u_{i_{-1}} - u_{i_0-1}|^p + |u_{i_0-1} - u_{i_0+1}|^p.$$
  It means that σ is not an optimal subdivision for the BV^s norm, indeed we can consider the subdivision $\widetilde{\sigma}$ $\cdots, i_{-1}, i_0 - 1, i_0 + 1, \cdots$ which produces a larger BV^s norm.

- We proceed similarly for the last case $i_{-1} = i_0 - 1 < i_0$.

In conclusion in all the case i_0 can not belong to σ, it concludes the proof. □

4 A wave front tracking algorithm

In this section, the wave front tracking algorithm (WFT) used is presented to solve the initial value problem (1.1). Simplifying the (WFT) is useful to simplify the estimates on the approximate solutions [3]. Taking advantage of the linearly degenerate field, we present a simpler wave front tracking algorithm (WFT) as the one used by Bressan and Colombo in [11] for general 2×2 systems. For this purpose, the Riemann problem and the interaction of waves is first studied.
In the sequel, we denote by $A(U)$ the 2×2 hyperbolic matrix $DF(U)$ and without loss of generality by $\lambda_1 < 0 < \lambda_2$ its eigenvalues and by l_1, l_2 (respectively r_1, r_2) its left (repectively right) eigenvectors, normalized so that:
$$\|r_i(U)\| = 1, \ \langle l_j(U); r_i(U) \rangle = \delta_{i,j}, \ i, j = 1, 2.$$

Furthermore we recall that for all $U \in \Omega(1.2)$:

$$\nabla\lambda_1(U) \cdot r_1(U) \neq 0 \text{ and } \nabla\lambda_2(U) \cdot r_2(U) = 0. \tag{4.9}$$

In the sequel we assume that we have $\nabla\lambda_1(U)\cdot r_1(U) > 0$ for all U in Ω. Furthermore for $\Omega = B(0,r)$ sufficiently small, we have.

$$\sup_{U\in B(0,r)} \lambda_1(U) < 0 < \inf_{U\in B(0,r)} \lambda_2(U). \tag{4.10}$$

4.1 Riemann invariants and Lax curves

An important feature for 2×2 systems is the existence (at least locally) of coordinates in the state space, the Riemann invariants. All properties of the solutions U are stated in the Riemann invariants coordinates. We call w and z the Riemann invariant associated to genuinely nonlinear (GNL) eigenvalue λ_1 and the linearly degenerate one λ_2. More precisely we have $\nabla w(U)\cdot r_2(U) = 0$ and $\nabla z(U)\cdot r_1(U) = 0$ for any $U \in \Omega \subset \mathbb{R}^2$. When U is a solution of system (1.1), $(w(t,x), z(t,x)))$ denotes $(w(U(t,x)), z(U(t,x)))$. With this usual notation, the Riemann invariant satisfies with here by abuse of notation $\lambda_1(w,z) = \lambda_1(U)$ and $\lambda_2(w,z) = \lambda_2(U)$:

$$\begin{cases} \partial_t w + \lambda_1(w,z)\partial_x w = 0 \\ \partial_t z + \lambda_2(w)\partial_x z = 0. \end{cases}$$

Remark 5 *Notice that $\partial_w\lambda_1 > 0$, since the first field is GNL, and λ_2 is independent of z since the second field is linearly degenerate (see Theorem 8.2.5 [18]). That is $\partial_z\lambda_2 = 0$ and λ_2 depends only on w.*

The map $U \to (w(U), z(U))$ is a local diffeomorphism and we can assume that the origin in U coordinates corresponds to the origin in (w,z) coordinates. We are going now to define the Lax curves in these new coordinates (w,z). For a fixed state U_-, the Lax curves describe the set of state U_+ such that the Riemann problem with the left state U_- and the right state U_+ is a simple wave [41]. For each U_- fixed there are two Lax curves, one for the 1-waves and another one for the 2-waves.

Lax Curves The picture (see the figure 1) of the Lax curves $L(U_-)$ will be used systematically throughout the paper. The convexity (or the concavity) of the shock curve is constant at least on a small ball $B(0,r)$ with $r > 0$ and does not depend on the point U_- and U_+ which are both in $B(0,r)$. It simply corresponds to fix a sign of the third derivative of the shock curve (we refer to [18] Theorem 8.2.3). The other sign can be studied in a similar way.

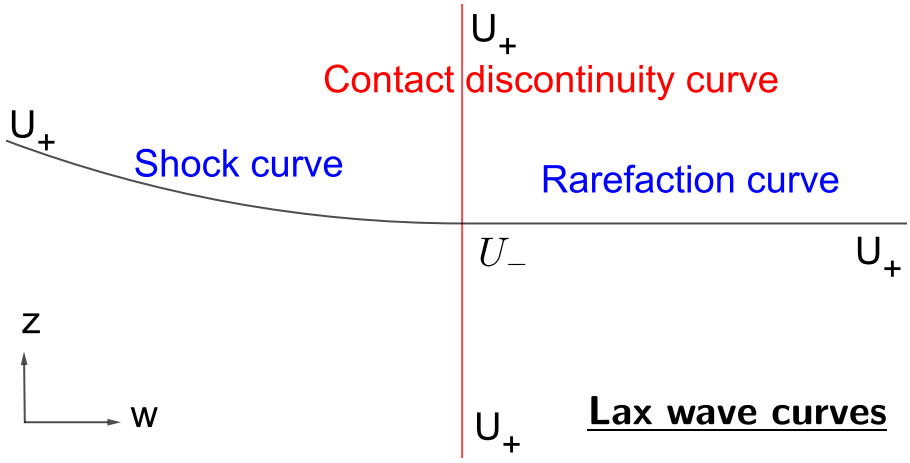

Figure 1: Lax waves curves where the state on the left of the wave $U_- = (w_-, z_-)$ is fixed. U_+ is the right state connected by a 1-wave when w varies, $w_+ \neq w_-$, or a 2-wave when $w_+ = w_-$ is constant.

For the rarefaction R_1 we have:

$$\begin{cases} w = w_- + \sigma \ \text{ with } \sigma \geq 0. \\ z = z_- \end{cases} \tag{4.11}$$

For the S_1 shock we have:

$$\begin{cases} w = w_- + \sigma \ \text{ with } \sigma \leq 0. \\ z = z_- + \mathcal{O}(\sigma^3) \end{cases} \tag{4.12}$$

Notice, with the choice of the convexity for the shock curve, z increases through a shock wave. (For the concave case, z decreases).

For the 2-wave there is only a contact discontinuity (CD):

$$\begin{cases} w = w_- \\ z = z_- + \sigma, \quad \sigma \in \mathbb{R} \end{cases} \tag{4.13}$$

4.2 The Riemann problem

The solution of the Riemann problem is given in the plane (w, z) in figure 2. The initial data is $U(0, x) = U_\pm, \quad \pm x \geq 0$. U_0 is the intermediate constant state between U_- and U_+ when we solve the Riemann problem (do not confuse U_0 with the initial data).

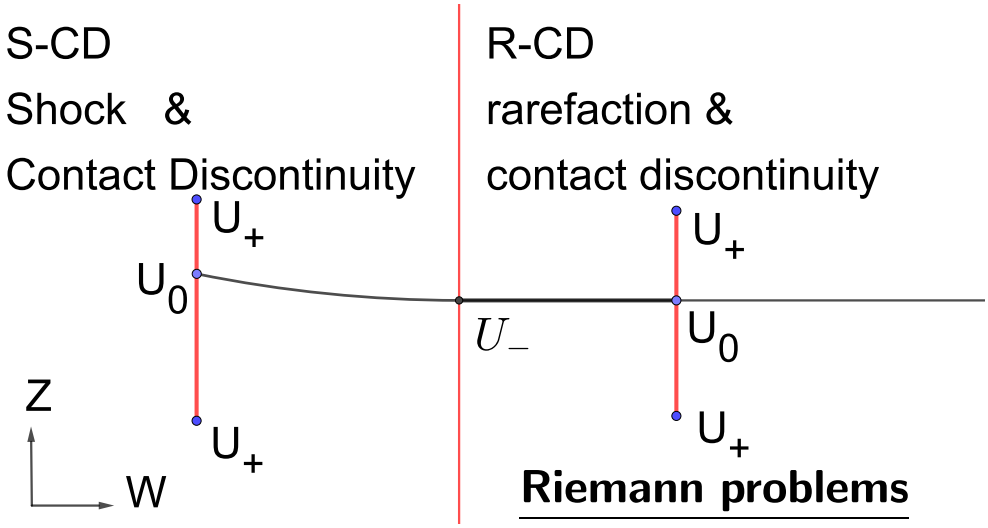

Figure 2: Solutions of the Riemann problems for a left state U_- fixed and all the possible configurations for U_+. The solutions are represented in the plane (w, z) of Riemann invariants

4.3 Nonlinear interactions

Next we consider the different interactions that we can have. We will note CD for the 2-contact discontinuity wave, $S1$ for the 1-shock wave and $R1$ for the 1 rarefaction wave. We recall that the only possible interactions are:

$$CD - R1, CD - S_1, R_1 - S_1, S_1 - R_1, S_1 - S_1,$$

where $L - R$ means the interaction between a left wave an a right wave. The left wave is a 2-wave, a contact discontinuity (CD), or a 1-wave, a rarefaction (R) or a shock (S) wave. The right wave is always a 1-wave since a 2-wave is a contact discontinuity and goes to the right with the speed $\lambda_2 > 0$ and cannot interact with a left 1-wave which goes to the left with the speed $\lambda_1 < 0$ or a contact discontinuity which moves with the same speed λ_2.

We study all the possible interactions in this section. The strength of an i-wave is quantified by the variation of associated Riemann invariant through the wave: Δw for a 1-wave and Δz for a 2-wave.

Let us summarize important features of such interaction where the 1-wave can be only a rarefaction or a shock wave and the 2-wave can be only a contact discontinuity. The following list of claims are verified just after by the exhaustive study of all possible interactions.

1. When two waves interact then there are two resulting waves which are a 1-wave on the left and a 2-wave on the right.

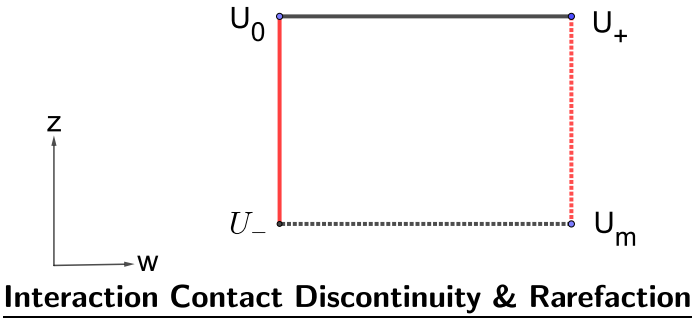

Figure 3: Interaction of a contact discontinuity with a rarefaction. The interacting waves are represented by FULL lines, a 2-wave or a 1-wave followed by a 1-wave. The dotted lines represent the resulting waves, a 1-wave (horizontal) followed by a 2-wave (vertical).

2. The strength of a 1-wave does not change after an interaction with a 2-wave.
3. Assume that two 1-wave interacts with respectively the strength σ_1 and $\widetilde{\sigma}_1$ then the outgoing 1-wave has the strength $\sigma_1' = \sigma_1 + \widetilde{\sigma}_1$.
4. The variation of w after an interaction behaves like the variation of the solution of a scalar conservation law. It means that TVw and $TV^s w$ is not increasing as for a scalar conservation laws.
5. The L^∞ norm of z can increase only when there is an interaction $CD - S_1$ or $S_1 - S_1$.

In all the following pictures,

- $U_- = (w_-, z_-)$ is the left state,
- $U_+ = (w_+, z_+)$ is the right state,
- $U_0 = (w_0, z_0)$ is the intermediary state before the interaction,
- $U_m = (w_m, z_m)$ is the intermediary state after the interaction.

An important point is the control of the BV or BV^s norm of w after an interaction. There are two cases. First case, after the interaction, the solution has only three different values the state U^-, U_m, U^+. This is true if the 1 outgoing wave is a 1-shock. Second case, there is a is a 1 rarefaction outgoing wave, so the solution has a continuum of values (when we will deal with the wave front tracking this continuum of values will be split in a finite number of values depending on

the parameter ν with ν goes to $+\infty$). However since the 1 rarefaction wave create a zone of monotonicity in w, using the Lemma 1 it does not change all the arguments which are related to estimating the BV or BV^s norm of w (we can apply the Lemma 1 because the wave front tracking will have a finite number of values).

$\underline{CD - R_1}$. Let us consider the simplest interaction $CD - R_1$, figure 3, which generate waves $R_1 - CD$. We observe that:

$$w_- = w_0 < w_+,\ z_- < z_0 = z_+ \ \text{ and } \ w_- < w_m = w_+,\ z_- = z_m < z_+.$$

In particular the functions $w(t, \cdot)$ and $z(t, \cdot)$ have the same values before and after

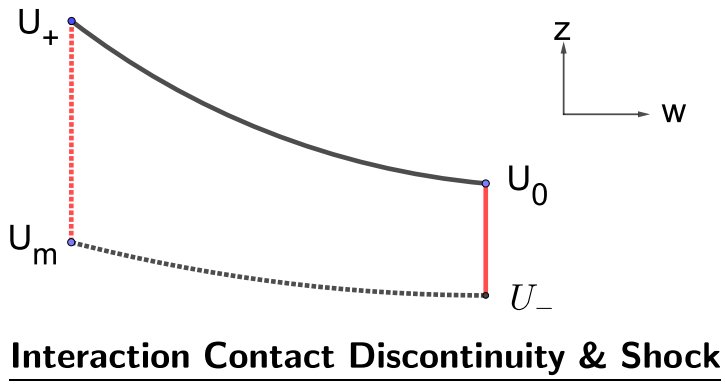

Figure 4: Interaction of a contact discontinuity with a shock

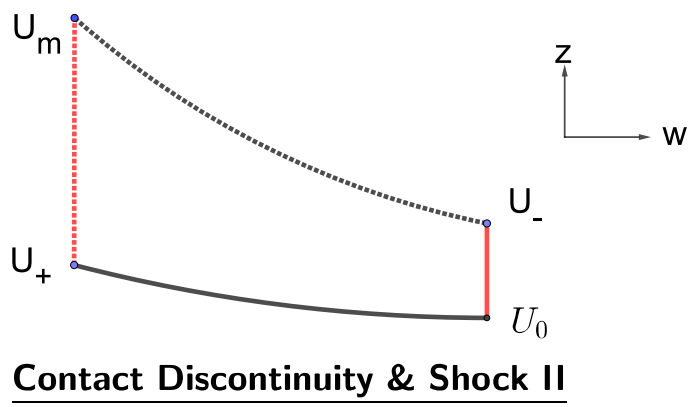

Figure 5: Interaction between a contact discontinuity and a shock with an augmentation of $\|z\|_\infty$.

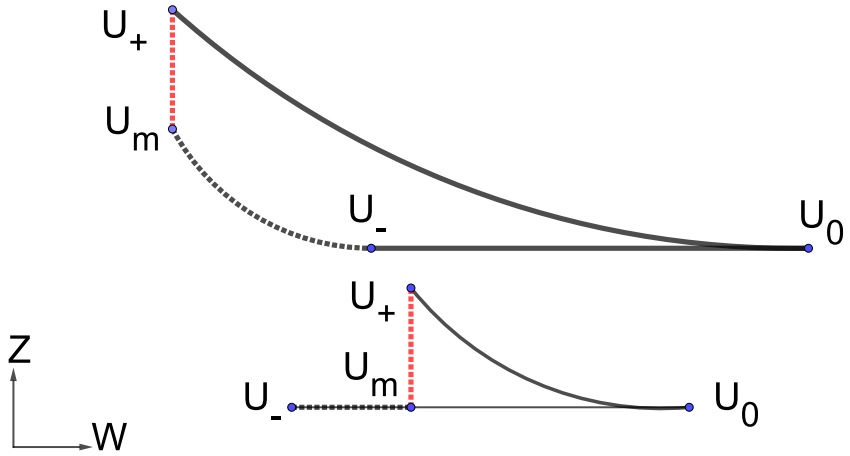

Figure 6: Interaction of a rarefaction with a shock wave: two cases depending on the relative strength of the waves

the interaction and these values are in the same order, it implies then that BV and BV^s norm does not change for this interaction both for z and w. Furthermore we have $w_+ - w_0 = w_m - w_-$ then the strength of the 1-wave is invariant after this interaction with this 2-wave.

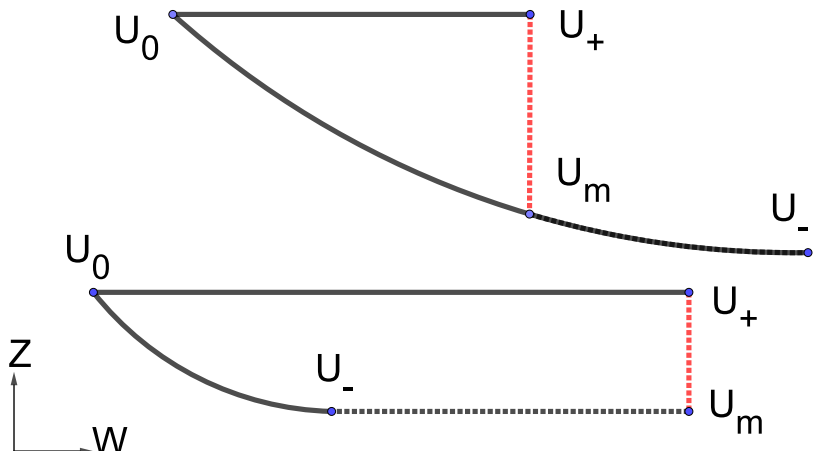

Figure 7: Interaction of a shock wave with a rarefaction: two cases depending on the relative strength of the waves

$\underline{CD - S_1}$. Now, consider the interaction $CD - S_1$, figure 4 and 5, which generates waves $S_1 - CD$. We observe that:

$$w_- = w_0 > w_+ \text{ and } w_0 > w_m = w_+$$

In particular it implies again that the BV and the BV^s norm of w does not change after this type of interaction. Furthermore the strength of the 1 outgoing wave is the same as the strength of the 1 incoming wave. We observe however that the L^∞ norm of z can increase in the figure 5, $|z_m|$ is larger than $|z_-|$, $|z_0|$ and $|z_+|$. Similarly the BV norm can increase for z. Thus, there is no maximum principle for z. However, the increase of $\|z\|_\infty$ is controlled as in the last case (4.14), the shock-shock interaction.

$\underline{R_1 - S_1}$. The interaction of 1-waves $R_1 - S_1$ generates $R_1 - CD$ or $S_1 - CD$, figure 6, and we have in each case $w_m = w_+$. It implies in particular that $w(t, \cdot)$ has the same values after the interaction excepted the value w_0 and some values of the incoming 1 rarefaction. Furthermore the values are in the same order, we deduce then that the BV and the BV^s norm decreases since we restrict in some sense the number of possible subdivision. We can observe that for this interaction the L^∞ norm of z does not increase.

$\underline{S_1 - R_1}$. The interaction $S_1 - R_1$ is similar to the previous case, figure 7, decay of TVw and invariance of TVz.

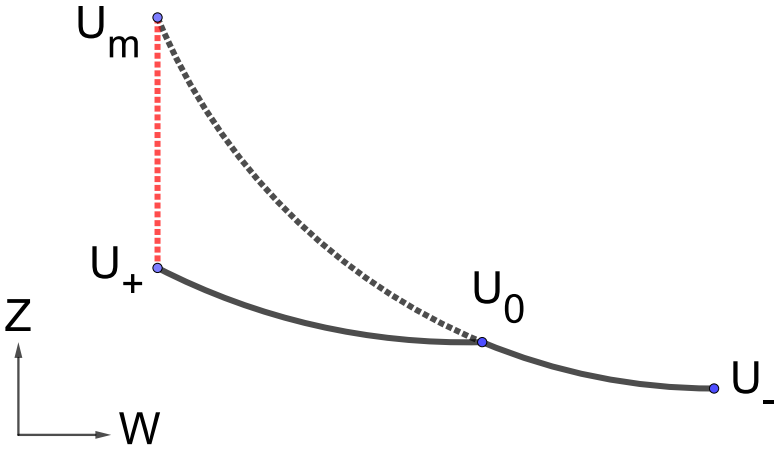

Figure 8: Interaction of two shock waves

$\underline{S_1 - S_1}$. We finish with the interaction $S_1 - S_1$ which generate the waves $S_1 - CD$, figure 8. This is the most interesting case with a non-scalar type interaction.

Here, w continues to behave like a solution for a convex scalar conservation law since $w(t, \cdot)$ has the same values after the interaction except w_0, the order of local minimal and local maximal values of w does not change then the BV and BV^s norms do not increase for w. z is not monotonous after the shock interactions in

particular the L^∞ and BV norm can increase. From (4.12) one obtain (see also the Glimm estimate in [18] for a 2×2 system):

$$\begin{aligned}
&|z_m - z_+| + |z_m - z_-| \\
=&|z_+ - z_0| + |z_0 - z_-| + 2|z_m - z_+| \\
=&|z_+ - z_0| + |z_0 - z_-| + 2(|z_m - z_-| - |z_0 - z_+| - |z_0 - z_-|) \\
=&|z_+ - z_0| + |z_0 - z_-| + 2(O(|w_+ - w_-|^3) - O(|w_0 - w_+|^3) - O(|w_0 - w_-|^3))
\end{aligned} \tag{4.14}$$

In conclusion in all cases, w always behaves like a solution of a convex scalar law (in particular the BV and the BV^s norm decreases) and z can only increase (or decrease for concave shock curves) when two shocks interact. In particular the L^∞ and TV norm of z can increase.

4.4 A simplified WFT for such 2×2 systems

Due to the special structure of the 2×2 systems studied, the wave front tracking can be simplified in the plane (w, z). We define now the wave front tracking that we will use in the sequel. First we shall work with initial data $(w_{0,\nu}, z_{0,\nu})$ which are piecewise constant approximation of (w_0, z_0) such that:

$$\begin{aligned}
&\|(w_0, z_0) - (w_{0,\nu}, z_{0,\nu})\|_{L^1} \leq \nu^{-1}, \\
&\mathrm{Osc}(w_{0,\nu}) \leq \mathrm{Osc}(w_0).
\end{aligned} \tag{4.15}$$

with $\nu > 0$ and ν goes to $+\infty$. Here $\mathrm{Osc}(w_0) = \sup_{x,y\in\mathbb{R}} |w_0(x) - w_0(y)|$ denote the oscillation of w_0. Furthermore we assume that:

$$w_{0,\nu} \in \nu^{-1}\mathbb{Z} \text{ and } z_{0,\nu} \in \mathbb{R}. \tag{4.16}$$

We define now N_ν as the number of discontinuities that the initial data $(w_{0,\nu}, z_{0,\nu})$ has. We start the wave front tracking by solving the N_ν first problem of Riemann. We would like to explain how we describe the solution of a Riemann problem between (w^-, z^-) and (w^+, z^+) in our algorithm of wave front tracking (for the beginning we assume that (w^-, z^-) and (w^+, z^+) are some values of $(w_{0,\nu}, z_{0,\nu})$). The Riemann problem is the combination of a 1-shock or a 1 rarefaction with a 2-contact discontinuity. If we get a 1-shock and a contact discontinuity the solution is the exact solution of the Riemann problem with respectively the speed $\lambda_1((z^-, w^-), (z^m, w^m))$, $\lambda_2((z^+, w^+))$ where $\lambda_1((z^-, w^-), (z^m, w^m))$ corresponds to the speed define by the Rankine Hugoniot relation. (z^m, w^m) corresponds here to the intermediary state and we know that $w^m = w^+$ because the value of w is constant along the 2 discontinuity of contact.

Approximate Riemann solver To stay with piecewise constant solutions, an approximate solver is needed only for rarefaction wave. If the solution of the

Riemann problem is a combination of a 1 rarefaction wave and a 2-contact discontinuity, we have to define the solution corresponding to the rarefaction. We note again (z^m, w^m) the intermediary state of the exact solution. We observe using (4.11) and (4.13) that $w_m = w_+$, it implies in particular that since w_+ is in $\nu^{-1}\mathbb{Z}$ that w_m is again in $\nu^{-1}\mathbb{Z}$. We have in particular:

$$w_m = w_- + k_+ \nu^{-1},$$

with $k_+ \in \mathbb{N}$ since w increases through a rarefaction wave. We define now the intermediary state $w_k = w_- + k\,\nu^{-1}$ with $0 \leq k \leq k_+$. The solution of the rarefaction for our wave front tracking at time $t > 0$ with the initial discontinuity at time $t = 0$ in y is:

$$\begin{aligned} w(t,x) &= w_- \quad \text{if } \ x < x_1(t) \\ &= w_k \quad \text{if } \ x_k(t) < x < x_{k+1}(t) \\ &= w_m \quad \text{if } \ x > x_{k^+}(t) \\ z(t,x) &= z^-, \end{aligned} \tag{4.17}$$

with $x_i(t) = y + t\lambda_1(w_i)$. The approximate solution u can be prolonged until a time t_1 when the first interaction between two or more waves front takes place.

Remark 6 *It is important to observe that for $t \in]0, t_1[$, the solution $w(t,x)$ take his values in $\nu^{-1}\mathbb{Z}$.*

Since $u(t_1, \cdot)$ is still a piecewise constant function, the corresponding Riemann problem can again be approximately solved. The solution (z, w) is then continued up to a time t_2 when the second set of wave interactions takes place, and so on.

Remark 7 *We assume in the sequel that we have only one interaction at the same time and that the interaction concerns only two fronts and no more (indeed as in [10], if three fronts or more meets at the same point, we can avoiding this situation by changing the speed of one of the incoming fronts. Of course this change of speed can be chosen arbitrary small.)*

We are going to prove in the sequel that there is a finite number of interaction so that we can define the wave front tracking on the time interval $(0, +\infty)$. Assume for the moment that we can only define the wave-front tracking on a time interval $(0, T^*)$ with $T^* < +\infty$ such that there is an infinite number of wave interactions. We will prove in fact that necessary $T^* = +\infty$.

Remark 8 *It is important to verify that $(w, z)(t, \cdot)$ remains bounded in a ball $B(0, r)$ with $r > 0$ sufficiently small for all time $t \in (0, T^*)$. Indeed using the Lax Theorem, we can solve the Riemann problem only if the oscillation between two states (w_-, z_-) and (w_+, z_+) is sufficiently small.*

We define now $N_1(t)$ as:

$$N_1(t) = \text{number of 1-wave at the time } t.$$

We observe easily that for $t \in]0, t_1[$, we have from (4.11):

$$N_1(t) \leq \nu\, N_\nu\, \mathrm{Osc}(w_0) + N_\nu.$$

The first term on the right hand side corresponds to the maximal number of rarefaction and N_ν to the maximal number of 1-shock. Let us estimate $N_1(t)$ after an interaction at the time t_k. We start by recalling that if we have a 1 rarefaction wave by definition of the Riemann problem (see (4.17)), his strength is necessary of size $\frac{1}{\nu}$. Assume now that we have an interaction between a 1 rarefaction and a 1-shock then the strength of the 1 rarefaction is $\frac{1}{\nu}$ and the strength of the 1-shock is $\frac{k}{\nu}$ with $k \in -\mathbb{N}^*$ then we have seen in the section 4.3 that the 1-wave has the strength $\sigma_1' = \frac{1}{\nu} + \frac{k}{\nu} \leq 0$. It implies in particular that the outgoing 1-wave disappears or is a shock. In particular after such interaction the number of 1-waves decrease of 1 or 2 units. Similarly if we have an interaction between two shock, we know that the strength of the outgoing 1-wave is the sum of the two strength of the incoming waves, then this strength is negative and the outgoing 1-wave is a shock. It implies in particular that after such interaction the number of 1-waves decrease of 1 unit. If we have an interaction between a 2-wave and a 1-wave since this interaction is transparent the outgoing wave is a unique rarefaction if the incoming one is also a rarefaction and otherwise a 1-shock. It implies in particular that $N_1(t)$ does not change after such interactions. We have then proved that $N_1(t)$ is a decreasing function of the time and then that:

$$N_1(t) \leq \nu\, N_\nu\, \mathrm{Osc}(w_0) + N_\nu,$$

for all $t \in (0, T^*)$. We define now N_1' as:

$$N_1' = \text{Number of interactions between 1-waves on } (0, T^*)$$

We have seen that when we have an interaction between 1-waves the number of 1-waves decrease at least of one unit it implies then that N_1' is inferior to the maximal number of 1-waves:

$$N_1' \leq \nu\, N_\nu\, \mathrm{Osc}(w_0) + N_\nu. \tag{4.18}$$

We define now $N_2'(t)$ as:

$$N_2' = \text{Number of interactions between 1-waves and 2-waves on } (0, T^*)$$

For a one wave we can define a 1 polygonal line which is an extension of the one wave. Indeed the one wave are created at the time $t = 0^+$ and after each interaction it can be prolongated by a unique 1-wave (or even the one wave can disappear, in this case the polygonal line is stopped). We can then define a polygonal 1-wave line.

We note that two different 1 polygonal lines are similar after an interaction time corresponding to an interaction between 1-waves if they meet us. Their number is finite and bounded by $\nu\, N_\nu \operatorname{Osc}(w_0) + N_\nu$. Similarly we can define some polygonal 2-wave. At the difference with the 1 polygonal line, we can create a 2 polygonal line after an interaction between 1-waves. Their number is finite and bounded by $N_\nu + (N_\nu \operatorname{Osc}(w_0) + N_\nu)$, N_ν corresponds to the number of polygonal line issue of the time $t = 0$ and $(N_\nu \operatorname{Osc}(w_0) + N_\nu)$ is the maximal number of interaction between 1-waves and then the maximum number of 2 polygonal line that we can create. Since $\lambda_1(w, z) < 0 < \lambda_2(w, z)$ for any $(w, z) \in B(0, r)$ we deduce by transversality that a polygonal 2-wave can interact with a polygonal 1-wave only one time. It implies in particular that the number of interaction on $(0, T^*)$ between polygonal 1-wave and polygonal 2-wave is at more $(2N_\nu + N_\nu \operatorname{Osc}(w_0))(\nu\, N_\nu \operatorname{Osc}(w_0) + N_\nu)$. It implies in particular that:

$$N_2' \leq (2N_\nu + N_\nu \operatorname{Osc}(w_0))(\nu\, N_\nu \operatorname{Osc}(w_0) + N_\nu). \tag{4.19}$$

From (4.18) and (4.19), we deduce that the number of interaction on $(0, T^*)$ is finite and then $T^* = +\infty$.

Remark 9 *The only point to verify is to ensure that all along the algorithm of wave front tracking, $(w(t, \cdot), z(t, \cdot))$ must remains in a set $[-r', r']^2$ with $r' > 0$ sufficiently small such that we are able to solve any Riemann problem. It will be verified in the sequel when we will prove the Theorem 2.1. We mention however that the L^∞ norm of $w(t, \cdot)$ is not increasing. After each interaction, we have proved that the L^∞ norm of w does not increase. It is not the case for z since after the interaction betwenn two 1-shocks or the interaction between a contact discontinuity and a 1-shock, the L^∞ norm of z can increase.*

5 Existence for $s \geq 1/3$ in $BV^s \times L^\infty$

In this section we are going to prove the Theorem 2.1 with the initial data (w_0, z_0) belonging to $BV^s \times L^\infty$ with $s \geq 1/3$. We consider again the solution of the wave front tracking (w_ν, z_ν) defined in the section 4.4 on a time interval $(0, T_\nu^*)$ with $T_\nu^* > 0$. In addition we construct $(w_{0,\nu}, z_{0,\nu})_{\nu>0}$ verifying (4.15) and such that for any $\nu > 0$ we have:

$$\|(w_{0,\nu}, z_{0,\nu})\|_{BV^s \times L^\infty} \leq \|(w_0, z_0)\|_{BV^s \times L^\infty}. \tag{5.20}$$

We are now going to obtain uniform estimate in ν in BV^s for the solution w_ν. More precisely we wish to prove that for any $t \in (0, T_\nu^*)$:

$$\|w_\nu(t, \cdot)\|_{BV^s(\mathbb{R})} \leq \|w_0\|_{BV^s}.$$

Control of $\|w_\nu(t,\cdot)\|_{BV^s}$

Assume that $t_1 \in (0, T_\nu^*)$ is the first time where we have a wave interaction in our wave front tracking. Let us prove now that for any $t \in (0, t_1)$ we have:

$$\|w_\nu(t,\cdot)\|_{BV^s(\mathbb{R})} = \|w_{0,\nu}\|_{BV^s(\mathbb{R})}. \tag{5.21}$$

If we come back to the estimate (5.21), it suffices to observe that the solution $w_\nu(t,\cdot)$ for $t \in (0, t_1)$ is the combination of the solutions of different Riemann problems which deal with all the initial discontinuities. We obtain then a combination of a 1-wave and a 2 CD wave. If we have a 1-shock and a 2 CD wave the values of $w_\nu(t,\cdot)$ does not change compared with $w_\nu(0,\cdot)$ and conserves the same order, then the BV^s norm remains the same. It the 1-wave is a rarefaction, we get different 1 rarefaction fronts and $w_\nu(t,\cdot)$ takes new values. For example if we have a discontinuity in x_α at the time $t = 0$ with the values $w_{0,\nu}(x_\alpha^-)$ and $w_{0,\nu}(x_\alpha^+)$ then the 1 rarefactions fronts produce the following new values at time $t \in (0, t_1)$ which are $w_{0,\nu}(x_\alpha^-) + \frac{k}{\nu}$ with $k \in \{0, \cdots, k^+\}$ and with $w_{0,\nu}(x_\alpha^+) = w_{0,\nu}(x_\alpha -) + \frac{k^+}{\nu}$. However even if we have new values for $w_\nu(t,\cdot)$ we have a zone of monotonicity for the 1 rarefaction fronts and using again the Lemma 1, we conclude again that the BV^s norm does not change. It proves the estimate (5.21).
Next we would like to understand how the BV^s norm vary after each interactions. Assume that we have an interaction at a time t_k with U^-, U_0, U^+ the incoming states and U^-, U_m, U^+ the outgoing states (here for simplicity of notation we have skip the index ν), we can observe that the number of different values in w_ν decrease or remains constant after the interaction. Indeed even when the outcoming 1-wave is one rarefaction, we recall that there is no more that one rarefaction front (this is due to the statements 2 and 3 p 13), furthermore we have:

$$U^- = (w^-, z^-), U_m = (w^+, z^m) \text{ and } U^+ = (w^+, z^+).$$

If we note t_k the time of interaction, it implies in particular that the value w_0 disappears at the time t_k^+ and that the values of $w_\nu(t_k^+,\cdot)$ have the same order as the values of $w_\nu(t_k^-,\cdot)$. We deduce then that for any interaction, we have:

$$\|w_\nu(t_k^+,\cdot)\|_{TV^s} \leq \|w_\nu(t_k^-,\cdot)\|_{TV^s}. \tag{5.22}$$

we have then two possibility the outcoming waves are a 1-shock wave and a 2 CD wave or a set of 1 rarefaction fronts and a 2 CD wave. In the first case, we observe that $w_\nu(t_k^+,\cdot)$ has the same values as $w_\nu(t_k^-,\cdot)$ excepted the value w_0, furthermore these values have the same order. It implies in particular that:

$$\|w_\nu(t_k^+,\cdot)\|_{TV^s} \leq \|w_\nu(t_k^-,\cdot)\|_{TV^s}. \tag{5.23}$$

We deduce from (5.21), (5.23) and (5.20) that the norm $\|w_\nu(t,\cdot)\|_{BV^s(\mathbb{R})}$ is decreasing all along the time and in particular we have for any $t \in (0, T_\nu^*)$:

$$\|w_\nu(t,\cdot)\|_{BV^s(\mathbb{R})} \leq \|w_{0,\nu}\|_{BV^s(\mathbb{R})} \leq \|w_0\|_{BV^s(\mathbb{R})}. \tag{5.24}$$

We are now going to bound uniformly $z_\nu(t,\cdot)$ in ν in L^∞ norm for any $t \in (0, T_\nu^*)$.

Control of $\|z_\nu(t,\cdot)\|_{L^\infty}$

We recall in particular that this is important to control the L^∞ norm of z_ν in order to prove that the wave front is well defined, i.e. U^ν stays in Ω (we can solve Riemann problem only if the oscillation of z_ν and w_ν are sufficiently small, we refer to the Remark 9). To do this, we define $\gamma_2^\nu(t,x_0)$ the forward generalized 2-characteristic (see [18]) which is an absolutely continuous solution of the differential inclusion:

$$\frac{d}{dt}\gamma_2^\nu(t,x_0) \in [\min(\lambda_2((w_\nu,z_\nu)(t,\gamma_2^\nu(t,x_0)^+),\lambda_2((w_\nu,z_\nu)(t,\gamma_2^\nu(t,x_0)^-) \\ ,\max(\lambda_2((w_\nu,z_\nu)(t,\gamma_2^\nu(t,x_0)^+),\lambda_2((w,z)(t,\gamma_2(t,x_0)^-)],$$

and such that $\gamma_2^\nu(0,x_0)=x_0$. In the sequel, in order to simplify the notation, we just will denote by $\gamma_2^\nu(t)$ a forward generalized 2-characteristic. Now we are interesting in estimating the L^∞ norm of z_ν along a forward generalized 2-characteristic such that $\gamma_2^\nu(0)\neq x_\alpha$ with x_α the points where $(w_{0,\nu},z_{0,\nu})$ is discontinuous. In order to follow the evolution of the L^∞ norm of z_ν along a forward generalized 2-characteristic, it is important to understand when the L^∞ norm of z_ν can vary. It is the case only when the forward generalized 2-characteristic meets a 1-wave, a 2-wave or an interaction point. We recall that since $\lambda_2>0$ and $\lambda_1<0$ a forward generalized 2-characteristic and a 1-wave are necessary transversal. First let us assume that γ_2 meets a 1 rarefaction, from (4.11) we know that the value of z_ν is the same along a 1 rarefaction. In particular it implies that the value of z_ν on γ_2 does not change when the forward generalized 2-characteristic meets a 1 rarefaction wave. Now we assume that the forward generalized 2-characteristic γ_2^ν meets a 1-shock. Again using (4.12) and the figure 1, we observe two things. First we remark that $z_\nu^+>z_\nu^-$, it induces in particular that z_ν is increasing in this case, second we know that:

$$z_\nu^+-z_\nu^-=0((w_\nu^+-w_\nu^-)^3).$$

Now it is important to mention that a generalized forward 2-characteristic can not meet a 2 CD wave front. Indeed since the second wave is degenerate, we know that λ_2 does not depends on z. In particular using (4.13) we deduce that $\lambda_2((w_\nu^-,z_\nu^-))=\lambda_2((w_\nu^+,z_\nu^+))$ along a 2 CD, it implies that 2 generalized forward characteristic can not meet a 2 CD wave (indeed they should be parallel to the 2 CD front).

Now let us deal with the last case when the forward generalized 2-characteristics meets an interaction point. The only case is when the interaction point is between a 1 rarefaction front and a 1-shock front, or between two 1-shock fronts. Let us start with the case of two 1-shock fronts, we define then by σ_1 and $\widetilde{\sigma}_1$ the strength of the 2 incoming wave fronts with U^-,U_0,U^+ the incoming states such that for $U^-=(w^-,z^-)$:

$$U_0=(w^-+\sigma_1,z^-+0(\sigma_1^3)),\ U^+=(w^-+\sigma_1+\widetilde{\sigma}_1,z^-+O(\sigma_1^3)+O(\widetilde{\sigma}_1^3)). \quad (5.25)$$

We know that the outcoming intermediary state is:

$$U_m = (w^- + \sigma_1', z^m) = (w^- + \sigma_1 + \widetilde{\sigma}_1, z^- + O((\sigma_1 + \widetilde{\sigma}_1)^3).$$

Now since λ_2 does not depend on z, we deduce that the forward 2-characteristic follow the outcoming 2 CD wave after the interaction.

Remark 10 *By convention, we assume that the value of z_ν on the 2 CD wave front corresponds to z_ν^+ the value on the right of the 2 CD wave front.*

It implies then that after the interaction point the value of z_ν has increased on γ_2^ν and is such that:

$$z_\nu^+ - z_\nu^- = O(\sigma_1^3) + O(\widetilde{\sigma}_1^3). \tag{5.26}$$

Here z_ν^+ is the value of z_ν on γ_2^ν just after the interaction point and z_ν^- the value of z_ν just before. Let us consider the case now of an interaction between a 1 rarefaction front and a 1-shock front (the case of an interaction between a 1-shock front and a 1 rarefaction front is similar), we have then:

$$U^- = (w^-, z^-),\ U_0 = (w^- + \sigma_1, z^-),\ U^+ = (w^- + \sigma_1 + \widetilde{\sigma}_1, z^- + O(\widetilde{\sigma}_1^3)). \tag{5.27}$$

We know that the outcoming intermediary state is:

$$U_m = (w^- + \sigma_1', z^m) = (w^- + \sigma_1 + \widetilde{\sigma}_1, z^- + O((\sigma_1 + \widetilde{\sigma}_1)^3)). \tag{5.28}$$

Again we deduce that the forward 2-characteristic follow the outcoming 2 CD wave front after the interaction. And we have in addition:

$$z_\nu^+ - z_\nu^- = 0(\widetilde{\sigma}_1^3). \tag{5.29}$$

In conclusion we have seen that z_ν is increasing when the forward 2-characteristic meets an interaction point and that we have (5.26) or (5.29).
Assume now that $\gamma_2^\nu(0) = x_\alpha$, then γ_2^ν is the 2-wave polygonal front which is issue from x_α. If γ_2^ν meets 1 rarefaction then the value of z_ν does not change along the forward generalized 2-characteristic (see figure 3). If the interaction is with a 1-shock front (see figure 4), we observe that:

$$z_\nu^+ = (z_\nu^+ - z_0^\nu) + z_0^\nu = O(\sigma_1^3) + z_0^\nu.$$

Here z_0^ν is the intermediary state before the interaction and σ_1 is the strength of the 1-shock wave. We recall that the value of z_ν on γ_2 before the interaction is by convention z_ν^0 since we consider the value on the right for a 2 CD wave front. We observe then that if t_k is the interaction point we have:

$$z_\nu(t_k^+, \gamma_2^\nu(t_k^+)) = z(t_k^-, \gamma_2^\nu(t_k^-)) + O(\sigma_1^3).$$

Remark 11 *If $\gamma_2^\nu(0) \neq x_\alpha$ and that the 2 forward γ_2^ν becomes after meeting an interaction point a 2 CD polygonal wave front, we can estimate the evolution of the L^∞ norm of z_ν as in the case where $\gamma_2^\nu(0) = x_\alpha$.*

We can now calculate the value of z_ν at the point $(T, \gamma_2^\nu(T))$ with $T > 0$. We have seen using the fact that z_ν is increasing along $\gamma_2^\nu(t)$ and that z_ν increases of $O(\sigma_1^3)$ after each interaction with a 1-shock front or an interaction point where there is a 1-shock front, we obtain then from (5.26) and (5.29):

$$z_\nu(T, \gamma_2^\nu(T)) = z_\nu(0, \gamma_2^\nu(0)) + \sum_{\alpha \in J} O(\sigma_\alpha^3). \tag{5.30}$$

Here J corresponds to the set of 1-shock wave fronts which have meet γ_2^ν on the time interval $[0, T]$ including the interaction points. In particular it exists $C > 0$ independent on ν such that:

$$\sum_{\alpha \in J} O(\sigma_\alpha^3) \leq C\, TV^s w_\nu(\cdot, \gamma_2^\nu(\cdot))([0, T]) + O\left(\nu^{-3}\right), \tag{5.31}$$

with $s = \frac{1}{3}$ (we deal always in the sequel with $s = \frac{1}{3}$).

Remark 12 *It is important to point out that if $\gamma_2^\nu(0) \neq x_\alpha$ then the forward 2-characteristic γ_2^ν can meet only one time an interaction point. Indeed after this interaction the forward 2-characteristic becomes a 2 polygonal line and we have constructed a wave front tracking where the interactions concern only two fronts. It means that the forward 2-characteristic after the meeting with an interaction point can cross after only 1 rarefaction fronts and 1-shock fronts. Similarly if $\gamma_2^\nu(0) = x_\alpha$ then the forward 2-characteristic γ_2^ν which is a 2 polygonal front will meet only 1 rarefaction fronts and 1-shock fronts.*

In order to prove (5.31), we only consider the case where $\gamma_2^\nu(0) \neq x_\alpha$ and the case where the forward generalized 2-characteristic γ_2^ν meets one interaction point. The other case are simple to treat. Let us start with the case where γ_2^ν meets an interaction point with two 1-shock wave fronts at the time t_k, from (5.25), (5.26) we deduce that:

$$z_\nu(t_k^+, \gamma_2^\nu(t_k^+)) = z(t_k^-, \gamma_2^\nu(t_k^-)) + O(\sigma_1^3) + 0(\widetilde{\sigma}_1^3) \leq z(t_k^-, \gamma_2^\nu(t_k^-)) + O(|\sigma_1 + \widetilde{\sigma}_1|^3).$$

And in particular it says that:

$$z_\nu(t_k^+, \gamma_2^\nu(t_k^+)) \leq z(t_k^-, \gamma_2^\nu(t_k^-)) + O(|w_\nu(t_k^+, \gamma_2^\nu(t_k^+)) - w_\nu(t_k^-, \gamma_2^\nu(t_k^-)|^3). \tag{5.32}$$

Let us deal now with the more tricky case of the cross of γ_2^ν with an interaction point comprising 1 rarefaction wave front and 1-shock front at the time t'_k, we have obtained from (5.27), (5.29) that:

$$z_\nu((t'_k)^+, \gamma_2^\nu((t'_k)^+)) \leq z((t'_k)^-, \gamma_2^\nu((t'_k)^-)) + O(\widetilde{\sigma}_1^3). \tag{5.33}$$

We recall that we have:

$$\widetilde{\sigma}_1 = w_\nu((t'_k)^+, \gamma_2^\nu((t'_k)^+)) - w_\nu((t'_k)^-, \gamma_2^\nu((t'_k)^-) - \sigma_1 \leq 0. \tag{5.34}$$

It is important to note that $\sigma_1 = \frac{1}{\nu}$, indeed our wave front tracking ensures that all the rarefaction fronts have the strength $\frac{1}{\nu}$. Similarly we know from (5.28) that the intermediary outcoming state is $U_m = (w_+, z_m)$ and that the 1 outcoming wave is a 1-shock or is cancelled out. For the moment assume that the 1 outcoming wave front is not cancelled out, in particular since $U_m = (w_+, z_m)$, it implies that $w_+ < w_-$. But we know that $w^\nu(t'_k, \cdot)$ takes only values in $\frac{\mathbb{Z}}{\nu}$, we deduce then that:

$$w_\nu((t'_k)^+, \gamma_2^\nu((t'_k)^+)) - w_\nu((t'_k)^-, \gamma_2^\nu((t'_k)^-) = -\frac{k}{\nu}, \tag{5.35}$$

with $k \in \mathbb{N}^*$. From (5.34) and (5.35), we have:

$$|\widetilde{\sigma}|^3 = \left(\frac{k+1}{\nu}\right)^3 \leq 2^3 \left(\frac{k}{\nu}\right)^3 = 2^3 |w_\nu((t'_k)^+, \gamma_2^\nu((t'_k)^+)) - w_\nu((t'_k)^-, \gamma_2^\nu((t'_k)^-)|^3. \tag{5.36}$$

We deduce then using (5.33) and (5.36) that:

$$\begin{aligned} z_\nu((t'_k)^+, \gamma_2^\nu((t'_k)^+)) &\leq z((t'_k)^-, \gamma_2^\nu((t'_k)^-)) \\ &\quad + O(|w_\nu((t'_k)^+, \gamma_2^\nu((t'_k)^+)) - w_\nu((t'_k)^-, \gamma_2^\nu((t'_k)^-)|^3). \end{aligned} \tag{5.37}$$

We finish now with the case where the 1 outcoming wave is cancelled out, it corresponds to the following situation:

$$U^- = (w^-, z^-),\ U_0 = (w^- + \sigma_1, z^-),\ U^+ = \left(w^-, z^- + O\left(\sigma_1^3\right)\right). \tag{5.38}$$

In this case since $\sigma_1 = \frac{1}{\nu}$ because this is the strength of a 1 rarefaction front, we have:

$$z_\nu((t'_k)^+, \gamma_2^\nu((t'_k)^+)) \leq z((t'_k)^-, \gamma_2^\nu((t'_k)^-)) + O\left(\nu^{-3}\right). \tag{5.39}$$

We proceed similarly for an interaction between a 1-shock wave and a 1 rarefaction wave. From the Remark 12, we know that the forward generalized 2-characteristic γ_2^ν can meet only one time an interaction point then combining (5.32), (5.37) and (5.39) allows to prove the estimate (5.31).

We wish now to estimate $TV^s w_\nu(\cdot, \gamma_2^\nu(\cdot))([0,T])$ in terms of the $BV^{\frac{1}{3}}$ norm of $w_\nu(0,\cdot)$. To do this we are going to consider a zone of dependence of the forward generalized 2-characteristic γ_2^ν. We now choose the 1 minimal backward generalized characteristic issue from $(\gamma_2^\nu(T), T)$ that we note γ_1^ν (note that γ_1^ν is defined on $[0,T]$, we refer to [18] Chapter X for the notion of minimal backward generalized characteristics). We now define the sequence of following functions $\gamma_{2,\alpha}^\nu$ with $\alpha \in [0,1]$ as follows with $\tau \in [0,T]$ (here τ does not correspond to the physical time t),

$$\gamma_{2,\alpha}^\nu(\tau) = \alpha\gamma_2^\nu(\tau) + (1-\alpha)\gamma_2^\nu(0) + C_\alpha^\nu \tau,$$

with

$$C_\alpha^\nu = \frac{\gamma_1^\nu(\alpha T) - \alpha\gamma_2^\nu(T) - (1-\alpha)\gamma_2^\nu(0)}{T} \geq 0$$

since $\gamma_1^\nu(\alpha T) \geq \gamma_1^\nu(T) = \gamma_2^\nu(T) \geq \alpha\gamma_2^\nu(T) + (1-\alpha)\gamma_2^\nu(0)$ for any $\alpha \in [0,1]$ (indeed we recall that the backward generalized characteristic goes on the right since $\lambda_1 < 0$). We observe also that:

$$\gamma_{2,\alpha}^\nu(T) = \gamma_1^\nu(\alpha T) \text{ and } (\gamma_{2,\alpha}^\nu)'(t) \geq 0 \text{ for } t \in [0,T].$$

The derivative of $\gamma_{2,\alpha}^\nu$ is in fact defined on the point where γ_2^ν is differentiable.

Remark 13 *It is important to note that the forward generalized 2-characteristic is defined in a unique way. The second point is that the domain delimited by the curves* $\{(\gamma_2^\nu(t), t), t \in [0,T]\}$, $\{(\gamma_1^\nu(t), t), t \in [0,T]\}$ *and* $\{(y,0), y \in [\gamma_2^\nu(0), \gamma_1^\nu(0)]\}$ *is the union of all the curves* $\{(\gamma_{2,\alpha}^\nu(t), \alpha t), t \in [0,T]\}$ *with* $\alpha \in [0,1]$. *We denote by* Γ_2 *this domain. There is no 1-wave front which enters in* Γ_2 *on the right since we have taken the minimal backward 1-characteristic.*

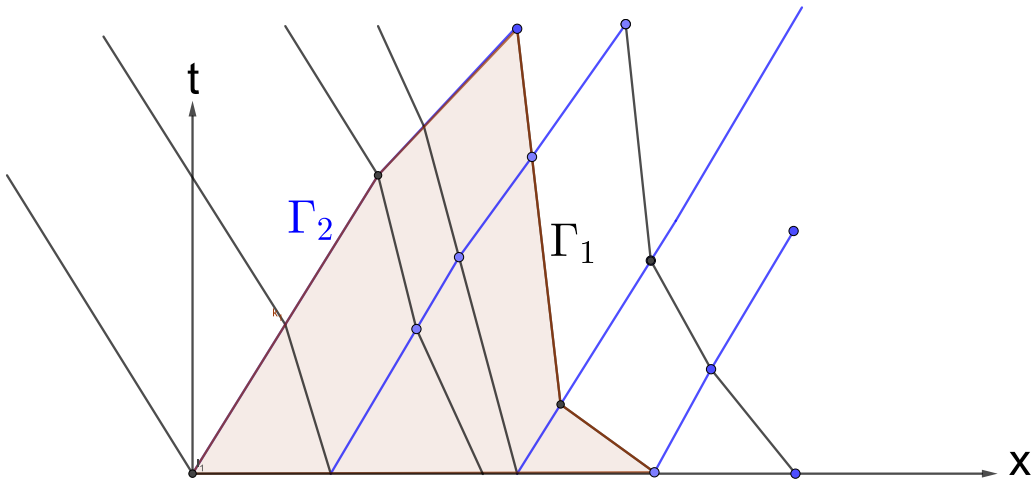

Figure 9: The Wave Front Tracking and the dependance zone delimited by the 2-characteristic Γ_2 on the left, and the 1-characteristic Γ_1 on the right

We define now (x_1, t_1) as the point where there is for the first time an interaction between wave fronts inside the domain Γ_2 with $t_1 \in]0,T]$. We denote now by $t_{1,1}$ the first time where there is an interaction in our wave front tracking, it implies in particular that $0 < t_{1,1} \leq t_1$. Furthermore we know that for α such that $\alpha T < t_1$ there is no interaction point on the curve $\{(\gamma_{2,\alpha}^\nu(t), \alpha t), t \in [0,T]\}$. We denote now by α_0 the first α where the curve $\{(\gamma_{2,\alpha}^\nu(t), \alpha t), t \in [0,T]\}$ meets an interaction

point (x_k, t_k) inside Γ_2. In particular it implies that there is no interaction point in the open domain Γ_0 delimited by $\{(\gamma^\nu_{2,\alpha_0}(t), \alpha_0 t), t \in [0,T]\}$, $\{(\gamma^\nu_1(t), t), t \in [0,T]\}$ and $\{(y,0), y \in [\gamma^\nu_2(0), \gamma^\nu_1(0)]\}$. We know that the values of w change on the curve $\{(\gamma^\nu_{2,\alpha}(t), \alpha t), t \in [0,T]\}$ only when this curve meets a 1-wave front. Furthermore the only 1-wave front which can cross $\{(\gamma^\nu_{2,\alpha}(t), \alpha t), t \in [0,T]\}$ for $0 < \alpha < \alpha_0$ are the 1-wave front which are issue of the set $\{(y,0), y \in [\gamma^\nu_2(0), \gamma^\nu_1(0)]\}$ (indeed there is no interaction point in the open domain Γ_0 and the 1-wave which are outside from Γ_0 can not enter in Γ_0), it implies then since the curve $\{(\gamma^\nu_{2,\alpha}(t), \alpha t), t \in [0,T]\}$ are transversal to the 1-wave front that the values of w on $\{(\gamma^\nu_{2,\alpha}(t), \alpha t), t \in [0,T]\}$ for $\alpha \in]0, \alpha_0[$ are included in the set of the value of $w_\nu(t, \cdot)$ at the time $t = 0^+$ (or in other word $t \in (0, t_{1,1})$ and that they are ranged in the same order. In particular it implies using (5.24) that for any $\alpha \in (0, \alpha_0)$ we have:

$$TV^s w_\nu(\alpha\cdot, \gamma^\nu_{2,\alpha}(\cdot))([0,T]) \leq TV^s w_\nu(t,\cdot)(\mathbb{R}) \leq TV^s w_{0,\nu}(\mathbb{R}) \leq \|w_0\|^{\frac{1}{s}}_{BV^s}, \quad (5.40)$$

with $t \in (0, t_{1,1})$.
Next we wish to estimate the BV^s norm of w^ν along $\{(\gamma_{2,\alpha}(t), \alpha t), t \in [0,T]\}$ for $\alpha = \alpha_0^+$. We note that the interaction point (x_k, t_k) on $\{(\gamma_{2,\alpha_0}(t), \alpha_0 t), t \in [0,T]\}$ is by definition inside Γ_2. In any case of interaction, if we have incoming states (U^-, U_0, U^+) and outgoing states (U^-, U_m, U^+) then the values of w^ν around (x_k, t_k^-) are (w_-, w_0, w_+) and the the values of w^ν around (x_k, t_k^+) are (w_-, w_+). It means that there is one value w_ν in less w_0 on the curves $\{(\gamma^\nu_{2,\alpha}(t), \alpha t), t \in [0,T]\}$ for $\alpha = \alpha_0^+$ compared with the values of w_ν on the curves $\{(\gamma^\nu_{2,\alpha}(t), \alpha t), t \in [0,T]\}$ for $\alpha = \alpha_0^+$. Furthermore the order of the values of w_ν on the curves $\{(\gamma^\nu_{2,\alpha}(t), \alpha t), t \in [0,T]\}$ with $\alpha = \alpha_0^+$ and on the curves $\{(\gamma^\nu_{2,\alpha}(t), \alpha t), t \in [0,T]\}$ with $\alpha = \alpha_0^-$ does not change. It implies that the BV^s norm is decreasing after the interaction along the curves $\{(\gamma^\nu_{2,\alpha}(t), \alpha t), t \in [0,T]\}$ with $\alpha - \alpha_0 > 0$ sufficiently small. It gives then using (5.40) that:

$$TV^s w_\nu(\alpha_0^+\cdot, \gamma^\nu_{2,\alpha_0^+}(\cdot))([0,T]) \leq TV^s w_\nu(\alpha_0^-\cdot, \gamma^\nu_{2,\alpha_0^-}(\cdot))([0,T]) \leq \|w_0\|^{\frac{1}{s}}_{BV^s}, \quad (5.41)$$

with $t \in (0, t_{1,1})$. The previous argument is again true if there is more than 1 interaction on the curve $\{(\alpha_0 t, \gamma^\nu_{2,\alpha_0}(t)), t \in [0,T]\}$. Now we define $\alpha_1 > \alpha_0$ the next α where there is an interaction inside Γ_2 on the curve $\{(\alpha_1 t, \gamma^\nu_{2,\alpha_1}(t)), t \in [0,T]\}$ and we define by Γ_1 the open domain delimited by the curves $\{(\alpha_1 t, \gamma^\nu_{2,\alpha_1}(t)), t \in [0,T]\}$, $\{(\alpha_0 t, \gamma^\nu_{2,\alpha_0}(t)), t \in [0,T]\}$ and $\{(\gamma^\nu_1(t), t), t \in [0,T]\}$. We observe then that all the 1-wave which meet a curve $\{(\alpha t, \gamma^\nu_{2,\alpha}(t)), t \in [0,T]\}$ with $\alpha \in (\alpha_0, \alpha_1)$ are issue of the curve $\{(\alpha_0 t, \gamma^\nu_{2,\alpha_0}(t)), t \in [0,T]\}$. It implies that the values of w_ν on $\{(\alpha t, \gamma^\nu_{2,\alpha}(t)), t \in [0,T]\}$ are included in set of values of w_ν on the curve $\{(\alpha' t, \gamma^\nu_{2,\alpha'}(t)), t \in [0,T]\}$ with $\alpha' = \alpha_0^+$, furthermore by transversality the values keep the same order. We deduce then that for any $\alpha \in (\alpha_0, \alpha_1)$ we have using in addition (5.41):

$$TV^s w_\nu(\alpha\cdot, \gamma^\nu_{2,\alpha}(\cdot))([0,T]) \leq TV^s w_\nu(\alpha_0^+\cdot, \gamma^\nu_{2,\alpha_0^+}(\cdot))([0,T]) \leq \|w_0\|^{\frac{1}{s}}_{BV^s}. \quad (5.42)$$

Repeating the argument, we deduce finally that the function:

$$\alpha \to \|w(\alpha\cdot, \gamma^\nu_{2,\alpha}(\cdot))\|_{BV^s([0,T])} \tag{5.43}$$

is decreasing in α. It implies from (5.31) and (5.20) since $\gamma^\nu_2 = \gamma^\nu_{2,\alpha}$ with $\alpha = 1$ that it exists $C > 0$ such that for any $\nu > 0$ we have:

$$\sum_{\alpha\in J} O(\sigma^3_\alpha) \leq C\|w_0\|^{\frac{1}{s}}_{BV^s(\mathbb{R})} + O\left(\nu^{-3}\right), \tag{5.44}$$

with $s = \frac{1}{3}$. From (5.30) and (5.44), we deduce that it exists $C > 0$ independent on ν such that for any $T \in (0, T^*_\nu)$ and any forward generalized 2-characteristic γ^ν_2:

$$|z_\nu(T, \gamma^\nu_2(T))| \leq |z_\nu(0, \gamma^\nu_2(0))| + C\|w_0\|^{\frac{1}{s}}_{BV^s(\mathbb{R})} + O\left(\nu^{-3}\right). \tag{5.45}$$

Since the forward generalized 2-characteristics describe all the space $(0, T^*_\nu)\times\mathbb{R}$, we deduce from (5.45) and (5.20) that for any $t \in (0, T^*_\nu)$ we get for $C > 0$ independent on ν:

$$\|z_\nu(t,\cdot)\|_{L^\infty} \leq \|z_0\|_{L^\infty} + C\|w_0\|^3_{BV^{\frac{1}{3}}(\mathbb{R})} + o(1).$$

We deduce now that the L^∞ norm of z_ν is uniformly bounded in ν all along the time interval $(0, T^*_\nu)$ and remains small for large ν, then using the Remark 9 we deduce that $T^*_\nu = +\infty$. The wave front tracking is then globally defined in time.
To summarize we have obtained uniform bound in ν on z_ν in $L^\infty_t(L^\infty)$ and on w_ν in $L^\infty_t(BV^{\frac{1}{3}}(\mathbb{R}))$, we wish now to develop some compactness argument in order to pass to the limit when ν goes to $+\infty$. The difficulty is to prove in particular that z_ν converges strongly to z in $L^1_{loc,t,x}$ since we can not use Helly Theorem as it is the case for w_ν.

Compactness argument for $(z_\nu)_{\nu>0}$

We consider now the Lipschitz homeomorphism:

$$\phi^\nu(t,x) = (t, \gamma^\nu_2(t,x)),$$

with $\gamma^\nu_2(t,x)$ the forward generalized 2-characteristic such that $\gamma^\nu_2(0,x) = x$. Furthermore we define η^ν_L and z^ν_L as follows:

$$\eta_\nu(t,x) = z_\nu(t, \gamma^\nu_2(t,x)) - z_0(x) \text{ and } z^\nu_L(t,x) = z_\nu(t, \gamma^\nu_2(t,x)).$$

We observe in particular that:

$$z_\nu(t,x) = z^\nu_L((\phi^\nu)^{-1}(t,x)),$$

with $(\phi^\nu)^{-1}$ the inverse of the Lipschitz homeomorphism ϕ^ν (for instance see [24, 38] for the notion of a Lipschitz homeomorphism and bi-Lipschitz homeomorphism, when the inverse is also Lipschitz, in geometric measure theory). We are going now to prove a succession of different Lemmas.

Lemma 2 *Up to a subsequence, we have:*

$$\lim_{\nu\to+\infty} z_L^\nu = z_L \ \ in\ L^1_{loc,t,x}.$$

Proof: We have seen that z_ν is BV along the curve $\{(t,\gamma_2^\nu(t,x)),\ t\in[0,T]\}$, indeed we have seen that z_ν is increasing along $\{(t,\gamma_2^\nu(t,x)),\ t\in[0,T]\}$ and that z_ν is uniformly bounded in $L^\infty_{t,x}$ then z_ν is uniformly bounded in ν in BV along the curve $\{(t,\gamma_2^\nu(t,x)),\ t\in[0,T]\}$. It implies that z_L^ν is uniformly bounded in ν in $L^\infty_x(BV([0,T]))$ for any $T>0$. Now we know that the speed of propagation of z_L^ν in the plane (x,t) is finite, indeed the speed is bounded by:

$$\sup_{(t,x)\in S_1}\left|\frac{1}{\lambda_1((w^\nu(t,x),z^\nu(t,x))^-,(w^\nu(t,x),z^\nu(t,x))^+)}\right| < M, \tag{5.46}$$

with S_1 the set of the 1-shock and $\lambda_1((w^\nu(t,x),z^\nu(t,x))^-,(w^\nu(t,x),z^\nu(t,x))^+)$ the speed of the shock defined by the Rankine Hugoniot relation. Here M does not depend on ν and (5.46) is true because on a small square $[-r,r]^2$ with $r>0$ sufficiently small we have:

$$\sup_{x\in[-r,r]^2}\lambda_1(x)<0.$$

It is well known [41, 27] that it implies that z_L^ν is uniformly bounded in $Lip_x(L^1_{loc,t})$. Using the Kolmogorov theorem, we deduce that up to a subsequence in ν z_L^ν converges to z in $L^1_{loc,t,x}$:

$$\lim_{\nu\to+\infty} z_L^\nu = z_L \ \text{ in } L^1_{loc,t,x}.$$

□

Next we use the following lemma.

Lemma 3 *Let ψ^ν a homeomorphism uniformly Lipschitz in ν from $\mathbb{R}^+\times\mathbb{R}$ to $\mathbb{R}^+\times\mathbb{R}$ such that there exists $M>1$ verifying for any $\nu>0$ and almost everywhere:*

$$0\le\frac{1}{M}\le|\det D_{t,x}\psi^\nu|\le M,\ 0\le\|D_{t,x}\psi^\nu\|\le M. \tag{5.47}$$

We assume that

$$\lim_{\nu\to+\infty} y^\nu = y \ in\ L^1_{t,x,loc},\ \ and\ \lim_{\nu\to+\infty}\psi^\nu=\psi\ in\ L^\infty_{t,x,loc} \tag{5.48}$$

then,

$$\lim_{\nu\to+\infty} y^\nu(\psi^\nu) = y(\psi)\ in\ L^1_{t,x,loc}.$$

Proof: Let $\varphi\in C_c^\infty(\mathbb{R}^+\times\mathbb{R})$ a positive regular function with compact support and with values in $[0,1]$. For $\epsilon>0$ we take $\widetilde{y}$ a continuous function in $L^1_{t,x}$ such that:

$$\|\widetilde{y}-y\|_{L^1_{t,x}(K_\varphi)}\le\epsilon. \tag{5.49}$$

with K_φ a compact of $\mathbb{R}^+ \times \mathbb{R}$ sufficiently large such that for any $\nu > 0$ we have supp $\varphi\left((\psi^\nu)^{-1}\right)$ which is included in K_φ. Using (5.47) and (5.49) we have:

$$\begin{aligned}
&\int_{\mathbb{R}^+}\int_{\mathbb{R}} \varphi|y^\nu \circ \psi^\nu - y \circ \psi| dxdt \\
\leq &\int_{\mathbb{R}^+}\int_{\mathbb{R}} \varphi|y^\nu \circ \psi^\nu - y \circ \psi^\nu| dxdt + \int_{\mathbb{R}^+}\int_{\mathbb{R}} \varphi|y \circ \psi^\nu - \widetilde{y} \circ \psi^\nu| dxdt \\
&+ \int_{\mathbb{R}^+}\int_{\mathbb{R}} \varphi|\widetilde{y} \circ \psi^\nu - \widetilde{y} \circ \psi| dxdt + \int_{\mathbb{R}^+}\int_{\mathbb{R}} \varphi|\widetilde{y} \circ \psi - y \circ \psi| dxdt \\
\leq &2\epsilon M + M\|y - y^\nu\|_{L^1_{t,x}(K_\varphi)} + \int_{\mathbb{R}^+}\int_{\mathbb{R}} \varphi|\widetilde{y} \circ \psi^\nu - \widetilde{y} \circ \psi| dxdt.
\end{aligned}$$

Using dominated convergence we can deal with the last integral and prove that for any positive $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$, we have $\varphi y^\nu \circ \psi^\nu \to \varphi y \circ \psi$ in $L^1_{t,x}$. We deduce then that $y^\nu \circ \psi^\nu \to y \circ \psi$ in $L^1_{t,x,loc}$ □

Lemma 4 *ϕ^ν and $(\phi^\nu)^{-1}$ verify the assumption (5.47) of the Lemma 3.*

Proof: It suffices to verify that there exist $M > 1$ such that:

$$0 \leq \frac{1}{M} \leq |\det D_{t,x}\phi^\nu| \leq M,\ 0 \leq \|D_{t,x}\phi^\nu\| \leq M. \tag{5.50}$$

We observe that:

$$D_{t,x}\phi^\nu(t,x) = \begin{pmatrix} 1 & 0 \\ \partial_t\gamma_2^\nu(t,x) & \partial_x\gamma_2^\nu(t,x) \end{pmatrix} \tag{5.51}$$

and:

$$D_{t,x}(\phi^\nu)^{-1}(\phi^\nu(t,x)) = \begin{pmatrix} 1 & 0 \\ -\dfrac{\partial_t\gamma_2^\nu(t,x)}{\partial_x\gamma_2^\nu(t,x)} & \dfrac{1}{\partial_x\gamma_2^\nu(t,x)} \end{pmatrix} \tag{5.52}$$

We know that:

$$\partial_t\gamma_2^\nu(t,x) \in [\min_\pm(\lambda_2(w^\nu(t,\gamma_2^\nu(t,x)^\pm))), \max_\pm(\lambda_2(w^\nu(t,\gamma_2^\nu(t,x)^\pm)))]$$

It implies in particular that $\partial_t\gamma_2^\nu$ is uniformly bounded in ν since we have seen that w_ν is uniformly bounded in $L^\infty_{t,x}$. Similarly for $\partial_x\gamma_2^\nu(t,x)$, we observe that $v_\nu(t,x) = \partial_x\gamma_2^\nu(t,x)$ verifies:

$$\partial_t v_\nu(t,x) = \partial_x(\lambda_2(w_\nu))(t,\gamma_2^\nu(t,x))v_\nu(t,x),\ v_\nu(0,x) = x.$$

Indeed we can observe in fact that γ_2^ν verify except at the point where γ_2^ν meets a 1-wave front:

$$\partial_t\gamma_2^\nu(t,x) = \lambda_2(w_\nu(t,\gamma_2^\nu(t,x))).$$

We deduce that:

$$v_\nu(t,x) = \exp\left(\int_0^t \partial_x(\lambda_2(w_\nu))(s,\gamma_2^\nu(s,x))ds\right). \tag{5.53}$$

and:

$$\det D_{t,x}\phi^\nu = v_\nu(t,x). \tag{5.54}$$

We have now:

$$\int_0^t \partial_x(\lambda_2(w_\nu))(s,\gamma_2^\nu(s,x))ds = \sum_{\alpha\in J_1}[\lambda_2(w_{\alpha,\nu})]. \tag{5.55}$$

J_1 is the set of point where a 1-wave front meets the curve $\{(\alpha t, \gamma_{2,\alpha}^\nu(t)),\ \ t\in[0,T]\}$. Since in (5.55), we have a telescopic sum, we deduce that:

$$-2\|\lambda_2(w_\nu(\cdot,\cdot))\|_{L^\infty_{t,x}} \leq \int_0^t \partial_x(\lambda_2(w_\nu))(s,\gamma_2^\nu(s,x))ds \leq 2\|\lambda_2(w_\nu(\cdot,\cdot))\|_{L^\infty_{t,x}} \tag{5.56}$$

From (5.53) and (5.56) we deduce that $\partial_x\gamma_2^\nu$ and $\frac{1}{\partial_x\gamma_2^\nu}$ is uniformly bounded in ν. Furthermore it implies also that the determinant of $D_{x,t}\phi^\nu$ and $D_{x,t}(\phi^\nu)^{-1}$ satisfies the assumption (5.50) using (5.54). Since $\partial_t\gamma_2^\nu$ is uniformly bounded in ν we deduce finally that ϕ^ν and $(\phi^\nu)^{-1}$verifies uniformly in ν (5.50) using the formula (5.51) and (5.52). □

Lemma 5 *$((\phi^\nu)^{-1})_{\nu>0}$ converges up to a subsequence to ϕ^{-1} in $L^\infty_{t,x,loc}$.*

Proof: Indeed for any compact K of $\mathbb{R}^+\times\mathbb{R}$ we observe that $(\phi^\nu)^{-1}$ is a continuous function from K to $\mathbb{R}^+\times\mathbb{R}$ because $(\phi^\nu)^{-1}$ is Lipschitz using the Lemma 4. Since the sequence $((\phi^\nu)^{-1})_{\nu>0}$ is uniformly Lipschitz in $C(K,\mathbb{R}^+\times\mathbb{R})$ from the Lemma 4, using the Ascoli Theorem up to a subsequence $((\phi^\nu)^{-1})_{\nu>0}$ converges uniformly to ϕ^{-1} on K. Using a standard argument of diagonal process we obtain that $((\phi^\nu)^{-1})_{\nu>0}$ converges up to a subsequence to $(\phi)^{-1}$ in $L^\infty_{t,x,loc}$. □

Using the Lemmas 2, 3, 4 and 5 we deduce that $z_L^\nu o(\phi^\nu)^{-1} = z_\nu$ converges strongly to $z = z_L(\phi^{-1})$ in $L^1_{loc,t,x}$ up to a subsequence. It implies that up to a subsequence z_ν converges almost everywhere up to a subsequence to $z = z_L(\phi^{-1})$ when ν goes to $+\infty$. concerning the convergence of the sequel $(w_\nu)_{\nu>0}$, the proof is more simple since $(w_\nu)_{\nu>0}$ is uniformly bounded in $L^\infty(\mathbb{R}^+, BV^{\frac{1}{3}})$. Thus, as in [8] for the scalar case or [9] for a 2×2 system w^ν is also bounded in $Lip_t^s([0,+\infty[, L^p_{loc}(\mathbb{R},\mathbb{R}))$ with $p = 1/s$ and the compactness follows.
We deduce then that the sequence $u_\nu = (w,z)^{-1}(w_{\epsilon_\nu}, z_{\epsilon_\nu})$ with $(w,z)^{-1}$ the inverse of the local diffeomorphism (w,z) converges also almost everywhere to u with u_ν uniformly bounded in $L^\infty_{t,x}$. It is then classical to verify that u is a global weak solution of the system (1.1) using dominated convergence (see [10, 18, 41] for more details).

Decomposition of z in Theorem 2.1

Finally, the decomposition of z (2.7) for the weak solution can be proved. We already use this decomposition for the approximate sequence $(z^\nu)_{\nu>0}$ with:

$$z_\nu(t, \gamma_2^\nu(t,x)) = z_{0,\nu}(x) + \eta_\nu(t,x).$$

The 2-characteristics (γ_2^ν) are equi-Lipschitz. Thus, up to a subsequence, we can pass to the limit when $\nu \to +\infty$. At the limit, γ_2 satisfies the differential equation for the generalized 2-characteristics [18, 22]. Moreover, $z_{0,\nu}$ and η_ν converge in L^1_{loc}, the whole sequence for $(z_{0,\nu})$ and only a subsequence for (η_ν). Now, we can pass to the limit in $z_\nu(t,x)$ using the bi-Lipschitz homeomorphism, as previously with Lemma 3, to obtain the decomposition (2.7). The proof of Theorem (2.1) is achieved.

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