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# NONPARAMETRIC ESTIMATION FOR I.I.D. PATHS OF FRACTIONAL SDE

FABIENNE COMTE\* AND NICOLAS MARIE†

ABSTRACT. This paper deals with nonparametric projection estimators of the drift function computed from independent continuous observations, on a compact time interval, of the solution of a stochastic differential equation driven by the fractional Brownian motion. A projection least-squares estimator is defined and a  $L^2$ -type risk bound is proved for it. The consistency and rate of convergence are established for these estimators in the case of the compactly supported trigonometric basis or the  $\mathbb{R}$ -supported Hermite basis.

**Keywords.** Fractional Brownian motion. Nonparametric projection estimator. Stochastic differential equation.

**MS classification 2020.** 62M09 - 62G08 - 60G22 - 60H07

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## 1. INTRODUCTION

Consider the stochastic differential equation

$$(1) \quad X(t) = x_0 + \int_0^t b(X(s))ds + \sigma B(t) ; t \in [0, T],$$

where  $\sigma, T > 0$ ,  $B$  is a fractional Brownian motion of Hurst index  $H \in ]1/2, 1[$ ,  $b : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous map and  $x_0 \in \mathbb{R}^*$ .

In this work, we assume that we observe  $N$  i.i.d. paths of the solution  $X$  of Equation (1). Our aim is to propose and study a nonparametric estimator of the drift function  $b$  based on these observations. This problem is related to functional data analysis, and more specifically, there are various recent contributions about i.i.d. parametric models of (non fractional) stochastic differential equations with mixed effects (see, e.g., Ditlevsen and De Gaetano [23], Overgaard et al. [42], Picchini, De Gaetano and Ditlevsen [43], Picchini and Ditlevsen [44], Comte, Genon-Catalot and Samson [13], Delattre and Lavielle [19], Delattre, Genon-Catalot and Samson [16], Dion and Genon-Catalot [22], Delattre, Genon-Catalot and Larédo [17]). Also, i.i.d. samples of stochastic differential equations have been recently considered in the framework of multiclass classification of diffusions (see Denis, Dion and Martinez [20]). The need of flexibility to deal with the information contained in functional data analysis make it interesting to use a nonparametric approach.

Along the last two decades, many authors studied statistical inference from observations drawn from stochastic differential equations driven by fractional Brownian motion, considering the observation of one path, either in continuous time, or in discrete time with fixed or small step.

Most references on the estimation of the trend component in Equation (1) deal with parametric estimators. Let us start by papers considering continuous time observations. In Kleptsyna and Le Breton [27] and Hu and Nualart [29], estimators of the trend component in Langevin's equation are studied. Kleptsyna and Le Breton [27] provide a maximum likelihood estimator, where the stochastic integral with respect to the solution of Equation (1) returns to an Itô integral. In [49], Tudor and Viens extend this estimator to equations with a drift function depending linearly on the unknown parameter. Hu and Nualart [29] provide a least squares estimator, where the stochastic integral with respect to the solution of Equation (1) is taken in the sense of Skorokhod. In [30], Hu, Nualart and Zhou extend this estimator to equations with a drift function depending linearly on the unknown parameter. Finally, in [36], Marie and Raynaud de Fitte extend this estimator to non-homogeneous semi-linear equations with almost periodic coefficients.

Now, considering discrete time observations, still in parametric context, Tindel and Neuenkirch [39] study a least squares-type estimator defined by an objective function, tailor-made with respect to the main result of Tudor and Viens [50] on the rate of convergence of the quadratic variation of the fractional Brownian motion. In [46], Panloup, Tindel and Varvenne extend the results of [39] under much more flexible conditions. In [8], Chronopoulou and Tindel provide a likelihood based numerical procedure to estimate a parameter involved in both the drift and the volatility functions in a stochastic differential equation with multiplicative fractional noise.

Concerning nonparametric methods for the estimation of the function  $b$  in Equation (1), there are only few references. Saussereau [47] and Comte and Marie [14] study the consistency of Nadaraya-Watson type estimators of the drift function  $b$  in Equation (1). In [37], Mishra and Prakasa Rao established the consistency and a rate of convergence of a nonparametric estimator of the whole trend of the solution to Equation (1) extending that of Kutoyants [31]. Marie [35] deals with the same estimator but for reflected fractional SDE. For nonparametric kernel-based estimators in Itô's calculus framework, the reader is referred to Kutoyants [31] and [32].

The present paper deals with nonparametric projection estimators of  $b$ , computed from  $N$  independent continuous time observations of the solution of Equation (1) on  $[0, T]$ . Let us mention that it became usual that such functional data are available and can be processed thanks to the improvements of computers. The question of nonparametric drift estimation in stochastic differential equations from such data has been studied in Comte and Genon-Catalot [12] who consider an Itô's calculus framework. Here, we propose to extend their functional least squares strategy to fractional SDE in Malliavin's calculus framework. Almost all the references cited above on the statistical inference for fractional SDE are based on long-time behavior properties of the solution which are often difficult to check in practice, but not

required here.

The estimator studied in this paper is introduced in Sections 3.1 and 3.2. Section 3.3 presents the main risk bound results of the paper and Sections 3.4 and 3.5 provide examples of function bases well adapted in our situation. We can in these frameworks obtain convergence results and rates.

Before that, Section 2 deals with some preliminaries on stochastic integration with respect to the fractional Brownian motion. More precisely, the Skorokhod integral with respect to the solution of Equation (1) is required for the definition of the projection estimators studied in Sections 3.1 and 3.2. However, it is difficult in practice to compute Skorokhod's integral when  $H \neq 1/2$ . For this reason, Section 4 proposes an approximated and calculable estimator requiring an observed path of the solution of Equation (1) for two close but different values of the initial condition. Clearly, such a requirement is not possible in any context, but we have in mind the pharmacokinetics application field and explain why it is meaningful in this context.

Lastly, concluding remarks are gathered in Section 5 while most proofs are postponed in Section 6.

**Notations.** The vector space of Lipschitz continuous maps from  $\mathbb{R}$  into itself is denoted by  $\text{Lip}(\mathbb{R})$  and equipped with the usual Lipschitz semi-norm  $\|\cdot\|_{\text{Lip}}$ . Now, consider  $m \in \mathbb{N}^*$ . The Euclidean norm on  $\mathbb{R}^m$  is denoted by  $\|\cdot\|_{2,m}$ ,

$$C_{\mathbf{b}}^m(\mathbb{R}) := \left\{ \varphi \in C^m(\mathbb{R}) : \max_{k \in \llbracket 0, m \rrbracket} \|\varphi^{(k)}\|_{\infty} < \infty \right\}$$

and

$$\text{Lip}_{\mathbf{b}}^m(\mathbb{R}) := \left\{ \varphi \in C^m(\mathbb{R}) : \|\varphi\|_{\text{Lip}_{\mathbf{b}}^m} = \|\varphi\|_{\text{Lip}} \vee \max_{k \in \llbracket 1, m \rrbracket} \|\varphi^{(k)}\|_{\infty} < \infty \right\}.$$

Note that  $C_{\mathbf{b}}^m(\mathbb{R}) \subset \text{Lip}_{\mathbf{b}}^m(\mathbb{R})$ . Finally, for every  $n \in \mathbb{N}^*$ , the vector space of infinitely continuously differentiable maps  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $f$  and all its partial derivatives have polynomial growth is denoted by  $C_{\mathbf{p}}^{\infty}(\mathbb{R}^n, \mathbb{R})$ .

## 2. STOCHASTIC INTEGRALS WITH RESPECT TO THE FRACTIONAL BROWNIAN MOTION

This section presents two different methods to define a stochastic integral with respect to the fractional Brownian motion. The first one is based on the pathwise properties of the fractional Brownian motion. Another stochastic integral with respect to the fractional Brownian motion is defined via the Malliavin divergence operator. This stochastic integral is called Skorokhod's integral with respect to  $B$ . If  $H = 1/2$ , which means that  $B$  is a Brownian motion, the Skorokhod integral defined via the divergence operator coincides with Itô's integral on its domain. This integral is appropriate for the estimation of the drift function  $b$  in Equation (1), while the first one is used in section 4 to propose a calculable estimator.

**2.1. The pathwise stochastic integral.** This subsection deals with some definitions and basic properties of the pathwise stochastic integral with respect to the fractional Brownian motion of Hurst index greater than  $1/2$ .

**Definition 2.1.** Consider  $x$  and  $w$  two continuous functions from  $[0, T]$  into  $\mathbb{R}$ . Consider a dissection  $D := (t_0, \dots, t_m)$  of  $[s, t]$  with  $m \in \mathbb{N}^*$  and  $s, t \in [0, T]$  such that  $s < t$ . The Riemann sum of  $x$  with respect to  $w$  on  $[s, t]$  for the dissection  $D$  is

$$J_{x,w,D}(s, t) := \sum_{k=0}^{m-1} x(t_k)(w(t_{k+1}) - w(t_k)).$$

**Notation.** With the notations of Definition 2.1, the mesh of the dissection  $D$  is

$$\pi(D) := \max_{k \in \llbracket 0, m-1 \rrbracket} |t_{k+1} - t_k|.$$

The following theorem ensures the existence and the uniqueness of Young's integral (see Friz and Victoir [25], Theorem 6.8).

**Theorem 2.2.** *Let  $x$  (resp.  $w$ ) be a  $\alpha$ -Hölder (resp.  $\beta$ -Hölder) continuous map from  $[0, T]$  into  $\mathbb{R}$  with  $\alpha, \beta \in ]0, 1]$  such that  $\alpha + \beta > 1$ . There exists a unique continuous map  $J_{x,w} : [0, T] \rightarrow \mathbb{R}$  such that for every  $s, t \in [0, T]$  satisfying  $s < t$  and any sequence  $(D_n)_{n \in \mathbb{N}}$  of dissections of  $[s, t]$  such that  $\pi(D_n) \rightarrow 0$  as  $n \rightarrow \infty$ ,*

$$\lim_{n \rightarrow \infty} |J_{x,w}(t) - J_{x,w}(s) - J_{x,w,D_n}(s, t)| = 0.$$

The map  $J_{x,w}$  is the Young integral of  $x$  with respect to  $w$  and  $J_{x,w}(t) - J_{x,w}(s)$  is denoted by

$$\int_s^t x(u)dw(u)$$

for every  $s, t \in [0, T]$  such that  $s < t$ .

For any  $\alpha \in ]1/2, H[$ , the paths of  $B$  are  $\alpha$ -Hölder continuous (see Nualart [41], Section 5.1). So, for every process  $Y = (Y(t))_{t \in [0, T]}$  with  $\beta$ -Hölder continuous paths from  $[0, T]$  into  $\mathbb{R}$  such that  $\alpha + \beta > 1$ , by Theorem 2.2, it is natural to define the pathwise stochastic integral of  $Y$  with respect to  $B$  by

$$\left( \int_0^t Y(s)dB(s) \right) (\omega) := \int_0^t Y(\omega, s)dB(\omega, s)$$

for every  $\omega \in \Omega$  and  $t \in [0, T]$ .

**2.2. The Skorokhod integral.** This subsection deals with some definitions and results on Malliavin calculus.

Consider  $\mathbb{L}^0([0, T], \mathbb{R})$  the space of measurable functions from  $[0, T]$  into  $\mathbb{R}$ , and the reproducing kernel Hilbert space

$$\mathcal{H} := \{h \in \mathbb{L}^0([0, T], \mathbb{R}) : \langle h, h \rangle_{\mathcal{H}} < \infty\}$$

of  $B$ , where  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  is the inner product defined by

$$\langle h, \eta \rangle_{\mathcal{H}} := H(2H - 1) \int_0^T \int_0^T |t - s|^{2H-2} h(s)\eta(s)dsdt$$

for every  $h, \eta \in \mathbb{L}^0([0, T], \mathbb{R})$ . Let  $(\mathbf{B}(h))_{h \in \mathcal{H}}$  be the isonormal Gaussian process defined by

$$\mathbf{B}(h) := \int_0^{\cdot} h(s)dB(s),$$

which is the Wiener integral of  $h \in \mathcal{H}$  with respect to  $B$ .

**Definition 2.3.** *The Malliavin derivative of a smooth functional*

$$F = f(\mathbf{B}(h_1), \dots, \mathbf{B}(h_n))$$

where  $n \in \mathbb{N}^*$ ,  $f \in C_{\mathbf{p}}^{\infty}(\mathbb{R}^n, \mathbb{R})$  and  $h_1, \dots, h_n \in \mathcal{H}$ , is the  $\mathcal{H}$ -valued random variable

$$\mathbf{D}F := \sum_{k=1}^n \partial_k f(\mathbf{B}(h_1), \dots, \mathbf{B}(h_n))h_k.$$

The key property of the operator  $\mathbf{D}$  is the following.

**Proposition 2.4.** *The map  $\mathbf{D}$  is closable from  $\mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$  into  $\mathbb{L}^2(\Omega; \mathcal{H})$ . Its domain in  $\mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ , denoted by  $\mathbb{D}^{1,2}$ , is the closure of the smooth functionals space for the seminorm  $\|\cdot\|_{1,2}$  defined by*

$$\|F\|_{1,2}^2 := \mathbb{E}(|F|^2) + \mathbb{E}(\|\mathbf{D}F\|_{\mathcal{H}}^2) < \infty$$

for every  $F \in \mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ .

For a proof, see Nualart [41], Proposition 1.2.1.

**Definition 2.5.** *The adjoint  $\delta$  of the Malliavin derivative  $\mathbf{D}$  is the divergence operator. The domain of  $\delta$  is denoted by  $\text{dom}(\delta)$ , and  $u \in \text{dom}(\delta)$  if and only if there exists a deterministic constant  $\mathbf{c}_u > 0$  such that for every  $F \in \mathbb{D}^{1,2}$ ,*

$$|\mathbb{E}(\langle \mathbf{D}F, u \rangle_{\mathcal{H}})| \leq \mathbf{c}_u \mathbb{E}(|F|^2)^{1/2}.$$

For any process  $Y = (Y(s))_{s \in [0, T]}$  and every  $t \in ]0, T]$ , if  $Y\mathbf{1}_{[0, t]} \in \text{dom}(\delta)$ , then its Skorokhod integral with respect to  $B$  is defined on  $[0, t]$  by

$$\int_0^t Y(s) \delta B(s) := \delta(Y\mathbf{1}_{[0, t]}),$$

and its Skorokhod integral with respect to  $X$  is defined by

$$\int_0^t Y(s) \delta X(s) := \int_0^t Y(s) b(X(s)) ds + \sigma \int_0^t Y(s) \delta B(s).$$

Note that since  $\delta$  is the adjoint of the Malliavin derivative  $\mathbf{D}$ , the Skorokhod integral of  $Y$  with respect to  $B$  on  $[0, t]$  is a centered random variable. Indeed,

$$(2) \quad \mathbb{E} \left( \int_0^t Y(s) \delta B(s) \right) = \mathbb{E}(1 \cdot \delta(Y\mathbf{1}_{[0, t]})) = \mathbb{E}(\langle \mathbf{D}(1), Y\mathbf{1}_{[0, t]} \rangle_{\mathcal{H}}) = 0.$$

Let  $\mathcal{S}$  be the space of the smooth functionals presented in Definition 2.3 and consider  $\mathbb{D}^{1,2}(\mathcal{H})$ , the closure of

$$\mathcal{S}_{\mathcal{H}} := \left\{ \sum_{j=1}^n F_j h_j ; h_1, \dots, h_n \in \mathcal{H}, F_1, \dots, F_n \in \mathcal{S} \right\}$$

for the seminorm  $\|\cdot\|_{1,2,\mathcal{H}}$  defined by

$$\|u\|_{1,2,\mathcal{H}}^2 := \mathbb{E}(\|u\|_{\mathcal{H}}^2) + \mathbb{E}(\|\mathbf{D}u\|_{\mathcal{H} \otimes \mathcal{H}}^2) < \infty$$

for every  $u \in \mathbb{L}^2(\Omega \times [0, T])$ . The following proposition provides an isometry type property for the Skorokhod integral with respect to  $B$  on  $\mathbb{D}^{1,2}(\mathcal{H})$ , which is a subspace of  $\text{dom}(\delta)$  by Nualart [41], Proposition 1.3.1. This result is useful for our purpose and is proved in Biagini et al. [3] (see Theorem 3.11.1).

**Proposition 2.6.** *For every  $Y, Z \in \mathbb{D}^{1,2}(\mathcal{H})$ ,*

$$\begin{aligned} \mathbb{E}(\delta(Y)\delta(Z)) &= \alpha_H \int_0^T \int_0^T \mathbb{E}(Y(u)Z(v)) |v - u|^{2H-2} dv du \\ &\quad + \alpha_H^2 \int_{[0, T]^4} \mathbb{E}(\mathbf{D}_{u'} Y(v) \mathbf{D}_{v'} Z(u)) |u - u'|^{2H-2} |v - v'|^{2H-2} dudv' dv dv'. \end{aligned}$$

In the sequel, the function  $b$  fulfills the following assumption.

**Assumption 2.7.** *The function  $b$  belongs to  $C^1(\mathbb{R})$  and there exist  $m, M \in \mathbb{R}$  such that*

$$m \leq b'(x) \leq M ; \forall x \in \mathbb{R}.$$

Under Assumption 2.7, the following result is a straightforward application of Proposition 2.6 to functionals of the solution  $X$  of Equation (1).

**Corollary 2.8.** *Let  $X$  be the solution of Equation (1). Under Assumption 2.7,  $X \in \mathbb{D}^{1,2}(\mathcal{H})$  and for every  $\varphi, \psi \in \text{Lip}_b^1(\mathbb{R})$ ,*

$$\mathbb{E}(\delta(\varphi(X))\delta(\psi(X))) = \alpha_H \int_0^T \int_0^T \mathbb{E}(\varphi(X(u))\psi(X(v))) |v - u|^{2H-2} dv du + R_{\varphi, \psi}$$

where

$$\begin{aligned} R_{\varphi, \psi} &:= \alpha_H^2 \int_{[0, T]^4} \mathbb{E}(\varphi'(X(v))\psi'(X(v')) \mathbf{D}_{u'} X(v) \mathbf{D}_u X(v')) |u - u'|^{2H-2} |v - v'|^{2H-2} dudv' dv dv' \\ &= \alpha_H^2 \sigma^2 \int_{[0, T]^2} \int_0^v \int_0^{v'} |u - u'|^{2H-2} |v - v'|^{2H-2} \\ &\quad \times \mathbb{E} \left( \varphi'(X(v))\psi'(X(v')) \exp \left( \int_{u'}^v b'(X(s)) ds + \int_u^{v'} b'(X(s)) ds \right) \right) dudv' dv dv'. \end{aligned}$$

The following theorem provides suitable controls of the moments of Skorokhod's integral.

**Theorem 2.9.** *Under Assumption 2.7, for every  $p > 1/H$ , there exists a deterministic constant  $\mathfrak{c}_{p,H,\sigma} > 0$ , only depending on  $p, H$  and  $\sigma$ , such that for every  $\varphi \in \text{Lip}_{\mathbf{b}}^1(\mathbb{R})$ ,*

$$\mathbb{E} \left( \left| \int_0^T \varphi(X(s)) \delta B(s) \right|^p \right) \leq \mathfrak{c}_{p,H,\sigma} \bar{\mathfrak{m}}_{p,H,M}(T) \left[ \left( \int_0^T \mathbb{E}(|\varphi(X(s))|^{1/H}) ds \right)^{pH} + \left( \int_0^T \mathbb{E}(|\varphi'(X(s))|^p)^{1/(pH)} ds \right)^{pH} \right] < \infty$$

where  $\bar{\mathfrak{m}}_{p,H,M}(T) := \mathfrak{m}_{p,H,M}(T) \vee 1$  and

$$\mathfrak{m}_{p,H,M}(T) := \left( -\frac{H}{M} \right)^{pH} \mathbf{1}_{M < 0} + T^{pH} \mathbf{1}_{M=0} + \left( \frac{H}{M} \right)^{pH} e^{pMT} \mathbf{1}_{M > 0}.$$

Note that if  $M < 0$ , then Theorem 2.9 has been already proved in Hu, Nualart and Zhou [30] (see Proposition 4.4.(2)).

**Remark 2.10.** *On the one hand, note that the control of the variance of Skorokhod's integral provided in Theorem 2.9 is a straightforward consequence of Corollary 2.8. On the other hand, with the notations of Corollary 2.8, note that for  $H = 1/2$ , the solution  $X$  of Equation (1) is adapted and then*

$$R_{\varphi,\psi} = \int_0^T \int_0^T \mathbb{E}(\mathbf{D}_u X(v) \mathbf{D}_v X(u)) dudv = 0.$$

*This reduces importantly the order of the variance of Skorokhod's integral with respect to the case  $H > 1/2$ .*

Lastly, the following proposition provides an expression and a bound for the density of the solution to Equation (1).

**Proposition 2.11.** *Under Assumption 2.7, for any  $t \in ]0, T]$ , the probability distribution of  $X^*(t) := X(t) - \mathbb{E}(X(t))$  has a smooth density with respect to Lebesgue's measure  $p_t^*(x_0, \cdot)$  such that for every  $x \in \mathbb{R}$ ,*

$$p_t^*(x_0, x) := \frac{\mathbb{E}(|X^*(t)|)}{2g_t^*(x_0, x)} \exp \left( - \int_0^x \frac{z dz}{g_t^*(x_0, z)} \right)$$

where

$$g_t^*(x_0, x) := \mathbb{E}(\langle \mathbf{D}X(t), -\mathbf{D}\mathbf{L}^{-1}X(t) \rangle_{\mathcal{H}} | X^*(t) = x)$$

and  $\mathbf{L}$  is the Ornstein-Uhlenbeck operator. Moreover, for every  $x \in \mathbb{R}$ ,

$$p_t^*(x_0, x) \leq \frac{\mathbb{E}(|X^*(t)|)}{2\sigma(m, t)^2} \exp \left( - \frac{x^2}{2\sigma(M, t)^2} \right)$$

where

$$\sigma(\mu, t)^2 := \alpha_H \sigma^2 \int_0^t \int_0^t |v - u|^{2H-2} e^{\mu(2t-v-u)} dudv > 0; \forall \mu \in \mathbb{R}.$$

A straightforward consequence of Proposition 2.11 is that for any  $t \in ]0, T]$ , the probability distribution of  $X(t)$  has a smooth density with respect to Lebesgue's measure  $p_t(x_0, \cdot)$  such that for every  $x \in \mathbb{R}$ ,

$$p_t(x_0, x) = p_t^*(x_0, x - \mathbb{E}(|X(t)|))$$

and

$$(3) \quad p_t(x_0, x) \leq \mathfrak{m}_t(x_0, x, m, M),$$

where

$$\mathfrak{m}_t(x_0, x, m, M) := \frac{\mathbb{E}(|X^*(t)|)}{2\sigma(m, t)^2} \exp \left[ - \frac{(x - \mathbb{E}(|X(t)|))^2}{2\sigma(M, t)^2} \right].$$

Since the paths of  $X$  are  $\alpha$ -Hölder continuous for any  $\alpha \in ]0, H[$ ,

$$\begin{aligned} \mathbb{E}(|X^*(t)|) &= \mathbb{E}(|X(t) - x_0 - \mathbb{E}(X(t) - x_0)|) \\ &\leq 2\mathbb{E}(|X(t) - X(0)|) \leq 2\mathbb{E}(\|X\|_{\alpha\text{-Hölder}, T}) t^\alpha \end{aligned}$$

where

$$\|X\|_{\alpha\text{-HöL},T} := \sup_{0 \leq s < t \leq T} \frac{|X(t) - X(s)|}{|t - s|^\alpha},$$

which has a finite first order moment because  $\mathbb{E}(\|B\|_{\alpha\text{-HöL},T}) < \infty$  and  $b$  is Lipschitz continuous. Then, since  $\sigma(m, t)^2 \geq \sigma^2 e^{-2\|b'\|_\infty T} t^{2H}$ ,

$$\mathfrak{m}_t(x_0, x, m, M) \leq \mathfrak{c}_T t^{\alpha-2H}$$

with

$$\mathfrak{c}_T := \frac{\mathbb{E}(\|X\|_{\alpha\text{-HöL},T})}{\sigma^2 e^{-2\|b'\|_\infty T}}.$$

Therefore, by taking  $\alpha \in ]2H - 1, H[$ , Inequality (3) implies that for every  $x \in \mathbb{R}$ ,  $p_\cdot(x_0, x) \in \mathbb{L}^1([0, T], dt)$ .

### 3. PROJECTION ESTIMATORS OF THE DRIFT FUNCTION

Under Assumption 2.7,  $b$  is Lipschitz continuous on  $\mathbb{R}$  and its derivative is bounded. So, Equation (1) has a unique solution  $X$  and the associated Itô map  $\mathcal{I}$  is continuously differentiable from  $\mathbb{R} \times C^0([0, T], \mathbb{R})$  into  $C^0([0, T], \mathbb{R})$ .

**3.1. The objective function.** Let  $f_T$  be the density function defined by

$$f_T(x) := \frac{1}{T} \int_0^T p_s(x_0, x) ds ; \forall x \in \mathbb{R},$$

where  $p_s(x_0, \cdot)$  is the smooth density with respect to Lebesgue's measure of the probability distribution of  $X(s)$  introduced in Proposition 2.11 for any  $s \in ]0, T[$ . Consider also  $N \in \mathbb{N}^*$  independent copies  $B_1, \dots, B_N$  of  $B$ ,  $X^i := \mathcal{I}(x_0, B^i)$  for every  $i \in \{1, \dots, N\}$ , and the objective function  $\gamma_N$  defined by

$$\gamma_N(\tau) := \frac{1}{NT} \sum_{i=1}^N \left( \int_0^T \tau(X^i(s))^2 ds - 2 \int_0^T \tau(X^i(s)) \delta X^i(s) \right)$$

for every function  $\tau : \mathbb{R} \rightarrow \mathbb{R}$ .

Note that for any bounded function  $\tau$  from  $\mathbb{R}$  into itself, thanks to Equality (2),

$$\begin{aligned} \mathbb{E}(\gamma_N(\tau)) &= \frac{1}{T} \int_0^T \mathbb{E}(\tau(X(s))^2 - 2\tau(X(s))b(X(s))) ds + \frac{\sigma}{T} \mathbb{E} \left( \int_0^T \tau(X(s)) \delta B(s) \right) \\ &= \frac{1}{T} \int_0^T \mathbb{E}((\tau(X(s)) - b(X(s)))^2) ds - \frac{1}{T} \int_0^T \mathbb{E}(b(X(s))^2) ds. \end{aligned}$$

Then, the definition of  $f_T$  gives

$$(4) \quad \mathbb{E}(\gamma_N(\tau)) = \int_{-\infty}^{\infty} (\tau(x) - b(x))^2 f_T(x) dx - \int_{-\infty}^{\infty} b(x)^2 f_T(x) dx.$$

Equality (4) shows that  $\mathbb{E}(\gamma_N(\tau))$  is the smallest for  $\tau$  the nearest of  $b$ . Therefore, minimizing its empirical version  $\gamma_N(\tau)$  should provide a functional estimator near of  $b$ .

**Remark 3.1.** *The pathwise stochastic integral with respect to  $B$ , defined in Section 2.1, is not centered in general. For instance, if  $H = 1/2$ , then it coincides with Stratonovich's integral. So this is not even the case for  $H = 1/2$ . This is the main reason why the objective function above is defined via Skorokhod's integral.*



**3.2. Projection estimators.** Consider  $A \in \mathcal{B}(\mathbb{R})$  and assume that  $\mathbb{L}^2(A, dx)$  (resp.  $\mathbb{L}^2(A, f_T(x)dx)$ ) is equipped with its usual inner product  $\langle \cdot, \cdot \rangle$  (resp.  $\langle \cdot, \cdot \rangle_{f_T}$ ). For any  $m \in \mathbb{N}^*$ , consider also

$$\mathcal{S}_m := \text{span}\{\varphi_0, \dots, \varphi_{m-1}\},$$

where  $(\varphi_0, \dots, \varphi_{m-1})$  is an orthonormal family of  $\mathbb{L}^2(A, dx)$ . Moreover, assume that the functions  $\varphi_j$ ,  $j \in \mathbb{N}$  are bounded. So,  $\mathcal{S}_m \subset \mathbb{L}^2(A, f_T(x)dx)$ .

Let

$$\widehat{b}_m = \arg \min_{\tau \in \mathcal{S}_m} \gamma_N(\tau)$$

be the projection estimator of  $b_A := b|_A$  on  $\mathcal{S}_m$ . As in Comte and Genon-Catalot [12], section 2.2,

$$\widehat{b}_m = \sum_{j=0}^{m-1} \widehat{\theta}_j \varphi_j$$

where

$$\widehat{\theta}(m) := (\widehat{\theta}_0, \dots, \widehat{\theta}_{m-1})^* = \widehat{\Phi}(m)^{-1} \widehat{\mathbf{x}}(m)$$

with

$$\widehat{\mathbf{x}}(m) := \left( \frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X^i(s)) \delta X^i(s) \right)_{j=0, \dots, m-1}^*$$

and

$$\widehat{\Phi}(m) := \left( \frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X^i(s)) \varphi_k(X^i(s)) ds \right)_{j, k=0, \dots, m-1}.$$

Note that

$$\widehat{\mathbf{x}}(m) = (\langle \varphi_j, b \rangle_N)_{j=0, \dots, m-1}^* + \mathbf{e}(m)$$

and

$$\widehat{\Phi}(m) = (\langle \varphi_j, \varphi_k \rangle_N)_{j, k=0, \dots, m-1},$$

where

$$\langle \tau_1, \tau_2 \rangle_N := \frac{1}{NT} \sum_{i=1}^N \int_0^T \tau_1(X^i(s)) \tau_2(X^i(s)) ds$$

for every bounded and measurable functions  $\tau_1, \tau_2 : \mathbb{R} \rightarrow \mathbb{R}$ , and

$$\mathbf{e}(m) := \left( \frac{\sigma}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X^i(s)) \delta B^i(s) \right)_{j=0, \dots, m-1}^*.$$

By Equality (2),  $\mathbf{e}(m)$  is centered, as expected for an error term in regression.

**3.3. Risk of the projection estimators.** Throughout this subsection,  $f_T$  and the functions  $\varphi_j$ ,  $j \in \mathbb{N}$  fulfill the following assumption.

**Assumption 3.2.**  $\lambda(A \cap \text{supp}(f_T)) > 0$  and, for  $m \leq NT$ ,

- (1)  $(\varphi_0, \dots, \varphi_{m-1})$  is an orthonormal family of  $\mathbb{L}^2(A, dx)$ .
- (2) The functions  $\varphi_j$ ,  $j = 0, \dots, m-1$  are bounded and belong to  $C_{\mathbf{b}}^1(A)$ .
- (3) There exist  $x_0, \dots, x_{m-1} \in A \cap \text{supp}(f_T)$  such that

$$\det[(\varphi_j(x_k))_{j, k=0, \dots, m-1}] \neq 0.$$

By Comte and Genon-Catalot [12], Lemma 1, which remains true for  $H > 1/2$  without additional arguments,

$$\Phi(m) := \mathbb{E}(\widehat{\Phi}(m)) = \left( \int \varphi_j(x) \varphi_k(x) f_T(x) dx \right)_{0 \leq j, k \leq m-1}$$

is invertible under Assumption 3.2. In addition, we impose that

$$L(m) := \sup_{x \in A} \sum_{j=0}^{m-1} \varphi_j(x)^2 \text{ and } R(m) := \sup_{x \in A} \sum_{j=0}^{m-1} [\varphi'_j(x)]^2$$

fulfill the following assumption.

**Assumption 3.3.** *There exists  $\rho > 0$  and  $\kappa \geq 1$  such that  $R(m) \leq \rho L(m)^\kappa$  and*

$$L(m)(\|\Phi(m)^{-1}\|_{\text{op}} \vee 1) \leq \frac{\mathbf{c}_{\kappa,T}}{2} \cdot \frac{NT}{\log(NT)} \quad \text{with} \quad \mathbf{c}_{\kappa,T} := \frac{3 \log(3/2) - 1}{(7 + \kappa)T}.$$

The above condition is a generalization of the so-called *stability condition* introduced for standard regression by Cohen et al. [9, 10], also considered in Comte and Genon-Catalot [12]. In order to ensure the existence and the stability of the estimator,  $\widehat{b}_m$  is replaced by

$$\widetilde{b}_m := \widehat{b}_m \mathbf{1}_{\widehat{\Lambda}(m)},$$

where

$$\widehat{\Lambda}(m) := \left\{ L(m)(\|\widehat{\Phi}(m)^{-1}\|_{\text{op}} \vee 1) \leq \mathbf{c}_{\kappa,T} \frac{NT}{\log(NT)} \right\}.$$

The two following results provide controls of the empirical risk and of the  $f_T$ -weighted integrated risk of  $\widetilde{b}_m$  respectively.

**Theorem 3.4.** *Under Assumptions 2.7, 3.2 and 3.3,*

$$\mathbb{E}(\|\widetilde{b}_m - b_A\|_N^2) \leq \inf_{\tau \in \mathcal{S}_m} \|\tau - b_A\|_{f_T}^2 + \frac{2}{NT} \text{trace}(\Phi(m)^{-1} \Phi(m, \sigma)) + \mathbf{c}_{\rho, \kappa, \sigma, \mathbf{b}} \frac{\overline{\mathbf{m}}_{2,H,M}(T)}{NT}$$

where

$$\Phi(m, \sigma) := \left( \frac{\sigma^2}{T} \mathbb{E} \left( \left( \int_0^T \varphi_j(X(s)) \delta B(s) \right) \left( \int_0^T \varphi_k(X(s)) \delta B(s) \right) \right) \right)_{j,k=0, \dots, m-1}$$

and  $\mathbf{c}_{\rho, \kappa, \sigma, \mathbf{b}} > 0$  is a deterministic constant depending only on  $\rho, \kappa, \sigma$  and  $\mathbf{b} := \|b_A\|_\infty^4$ .

**Corollary 3.5.** *Under Assumptions 2.7, 3.2 and 3.3,*

$$\mathbb{E}(\|\widetilde{b}_m - b_A\|_{f_T}^2) \leq \left( 1 + \frac{4T \mathbf{c}_{\kappa,T}}{\log(NT)} \right) \inf_{\tau \in \mathcal{S}_m} \|\tau - b_A\|_{f_T}^2 + \frac{8}{NT} \text{trace}(\Phi(m)^{-1} \Phi(m, \sigma)) + \bar{\mathbf{c}}_{\rho, \kappa, \sigma, \mathbf{b}} \frac{\overline{\mathbf{m}}_{2,H,M}(T)}{NT}$$

where  $\bar{\mathbf{c}}_{\rho, \kappa, \sigma, \mathbf{b}} > 0$  is a deterministic constant depending only on  $\rho, \kappa, \sigma$  and  $\mathbf{b}$ .

**Remark 3.6.** *Note that*

$$\begin{aligned} \text{trace}(\Phi(m)^{-1} \Phi(m, \sigma)) &= \text{trace}(\Phi(m)^{-1/2} \Phi(m, \sigma) \Phi(m)^{-1/2}) \\ &\leq m \|\Phi(m)^{-1/2} \Phi(m, \sigma) \Phi(m)^{-1/2}\|_{\text{op}} \\ &= m \sup_{\tau \in \mathcal{S}_m, \|\tau\|_{f_T}=1} \mathbb{E} \left[ \left( \int_0^T \tau(X(s)) \delta B(s) \right)^2 \right]. \end{aligned}$$

The risk decompositions given in Theorem 3.4 and Corollary 3.5 both involve the same types of terms:

- The first one is equal or proportional to  $\inf_{\tau \in \mathcal{S}_m} \|\tau - b_A\|_{f_T}^2$  and is a squared bias term due to the projection strategy. It is decreasing when  $m$  increases, because then the projection space grows.
- The second one,  $\text{trace}(\Phi(m)^{-1} \Phi(m, \sigma)) / (NT)$ , is a variance term. From the remark above, it is bounded by  $m \|\Phi(m)^{-1} \Phi(m, \sigma)\|_{\text{op}} / (NT)$  which is increasing with  $m$ .
- The last one is a residual negligible term, which is small when  $N$  is large. Note that if the upper-bound  $M$  of  $b'$  is nonnegative, then  $\overline{\mathbf{m}}_{2,H,M}(T)$  explodes for large values of  $T$ .

The order of the bias generally depends on the regularity of the function, and the order of the trace term is discussed below. Both quantities imply that a choice of  $m$  ensuring a compromise between the bias and the variance is required, to obtain the consistency of the estimator and a rate.

Finally, let us provide a control for  $\text{trace}(\Phi(m)^{-1}\Phi(m, \sigma))$  which allows comparison with non fractional results.

**Proposition 3.7.** *Under Assumptions 2.7 and 3.2,*

$$\frac{1}{NT} \text{trace}(\Phi(m)^{-1}\Phi(m, \sigma)) \leq \mathfrak{c}_{2,H,\sigma} \sigma^2 \frac{\bar{\mathfrak{m}}_{2,H,M}(T)}{NT^{2-2H}} m(1 + R(m) \|\Phi(m)^{-1}\|_{\text{op}}).$$

In the standard case, with  $H = 1/2$  and constant volatility function  $\sigma$ , it holds that

$$\frac{1}{NT} \text{trace}(\Phi(m)^{-1}\Phi(m, \sigma)) = \sigma^2 \frac{m}{NT}$$

as established in Comte and Genon-Catalot [12]. Here, for  $M < 0$ ,  $NT$  becomes  $NT^{2-2H}$  which is coherent. However, the additional term  $R(m) \|\Phi(m)^{-1}\|_{\text{op}}$  may have an important order in  $m$  and substantially increase the variance. Thus, it will deteriorate the rate of the estimators. So, there is a *discontinuity* between the cases  $H = 1/2$  and  $H > 1/2$ , which is explained in Remark 2.10.

Now, for projection estimators, different bases can be considered. In the present setting, the bases have to be differentiable. We present two examples in the sequel.

**3.4. Rates on Fourier-Sobolev spaces for trigonometric basis.** A first example is the compactly supported trigonometric basis. For  $A = [\ell, \mathbf{r}]$ , it is defined by

$$\begin{aligned} \varphi_0(x) &:= \frac{1}{\sqrt{\mathbf{r} - \ell}} \mathbf{1}_{[\ell, \mathbf{r}]}(x), \\ \varphi_{2j+1}(x) &:= \sqrt{\frac{2}{\mathbf{r} - \ell}} \cos\left(2\pi j \frac{x - \ell}{\mathbf{r} - \ell}\right) \mathbf{1}_{[\ell, \mathbf{r}]}(x) \text{ and} \\ \varphi_{2j}(x) &:= \sqrt{\frac{2}{\mathbf{r} - \ell}} \sin\left(2\pi j \frac{x - \ell}{\mathbf{r} - \ell}\right) \mathbf{1}_{[a, b]}(x) \end{aligned}$$

for every  $x \in \mathbb{R}$  and  $j \geq 1$ . This basis satisfies, for  $m$  odd and any  $x \in [\ell, \mathbf{r}]$ ,

$$\sum_{j=0}^{m-1} \varphi_j^2(x) = m \text{ and } \sup_{x \in [\ell, \mathbf{r}]} \sum_{j=0}^{m-1} [\varphi_j'(x)]^2 \leq \frac{(2\pi)^2}{(\mathbf{r} - \ell)^3} m^3.$$

So,

$$L(m) = m \text{ and } R(m) = \rho(\ell, \mathbf{r}) m^3$$

where  $\rho(\ell, \mathbf{r}) = (2\pi)^2 / (\mathbf{r} - \ell)^3$ .

In the Brownian setting, where  $H = 1/2$ , for a constant volatility function  $\sigma(x) \equiv \sigma$ , as recalled above, the variance term is  $\sigma^2 m / (NT)$  (see Comte and Genon-Catalot [12]). Here, if we assume that  $f_T$  is lower bounded on  $A$  by  $f_0 > 0$ , then  $\|\Phi(m)^{-1}\|_{\text{op}} \leq 1/f_0$  and the bound of Proposition 3.7 becomes

$$\frac{1}{NT} \text{trace}(\Phi(m)^{-1}\Phi(m, \sigma)) \leq \mathfrak{c}_{2,H,\sigma} \sigma^2 \frac{\bar{\mathfrak{m}}_{2,H,M}(T)}{NT^{2-2H}} m \left[ 1 + \frac{\rho(\ell, \mathbf{r})}{f_0} m^3 \right].$$

The additional term  $R(m) \|\Phi(m)^{-1}\|_{\text{op}}$  discussed after Proposition 3.7 has here order  $m^3$ .

Now, let us evaluate the bias term. Consider  $\beta \in \mathbb{N}^*$  and the Sobolev space

$$W_2^\beta([\ell, \mathbf{r}]) := \left\{ \varphi : [\ell, \mathbf{r}] \rightarrow \mathbb{R} : \int_\ell^\mathbf{r} |\varphi^{(\beta)}(x)|^2 dx < \infty \right\}.$$

If  $b_A \in W_2^\beta([\ell, \mathbf{r}])$ , by DeVore and Lorentz [21], Theorem 2.3 p. 205, then there exists a deterministic constant  $\mathbf{c}_{\beta, \ell, \mathbf{r}} > 0$ , not depending on  $m$ , such that

$$\|p_{\mathcal{S}_m}^\perp(b_A) - b_A\|^2 \leq \mathbf{c}_{\beta, \ell, \mathbf{r}} m^{-2\beta},$$

where  $p_{\mathcal{S}_m}^\perp$  is the orthogonal projection from  $\mathbb{L}^2(A, dx)$  onto  $\mathcal{S}_m$ . If in addition  $f_T$  is upper bounded on  $A$  by  $f_1$ , then

$$\inf_{\tau \in \mathcal{S}_m} \|\tau - b_A\|_{f_T}^2 \leq f_1 \|p_{\mathcal{S}_m}^\perp(b_A) - b_A\|^2 \leq f_1 \mathbf{c}_{\beta, \ell, \mathbf{r}} m^{-2\beta}.$$

As a consequence, the inequality of Theorem 3.4 can be written

$$\mathbb{E}(\|\tilde{b}_m - b_A\|_N^2) \leq f_1 \mathbf{c}_{\beta, \ell, \mathbf{r}} m^{-2\beta} + \mathbf{c}_{2, H, \sigma} \sigma^2 \frac{\bar{\mathbf{m}}_{2, H, M}(T)}{NT^{2-2H}} m \left[ 1 + \frac{\rho(\ell, \mathbf{r})}{f_0} m^3 \right] + \mathbf{c}_{\rho, \kappa, \sigma, \mathbf{b}} \frac{\bar{\mathbf{m}}_{2, H, M}(T)}{NT}.$$

We obtain the following result.

**Proposition 3.8.** *Under Assumption 2.7, if  $f_0 \leq f_T(x) \leq f_1$  for every  $x \in [\ell, \mathbf{r}]$ ,  $b_A \in W_2^\beta([\ell, \mathbf{r}])$  and the estimator is computed in the trigonometric basis on  $[\ell, \mathbf{r}]$ , then there exists a deterministic constant  $\mathbf{c}_{\beta, \ell, \mathbf{r}, f_0, f_1} > 0$ , not depending on  $N$  and  $T$ , such that with  $m_{\text{opt}} := [(NT^{2-2H})^{1/(2\beta+4)}]$ ,*

$$\mathbb{E}(\|\tilde{b}_{m_{\text{opt}}} - b_A\|_N^2) \leq \mathbf{c}_{\beta, \ell, \mathbf{r}, f_0, f_1} \bar{\mathbf{m}}_{2, H, M}(T) (NT^{2-2H})^{-2\beta/(2\beta+4)} + \mathbf{c}_{\rho, \kappa, \sigma, \mathbf{b}} \frac{\bar{\mathbf{m}}_{2, H, M}(T)}{NT}.$$

We obtain the consistency of the estimator with respect to the empirical risk for a fixed  $T$  and  $N \rightarrow +\infty$ , and a rate of convergence which degrades from the rate  $N^{-2\beta/(2\beta+1)}$  found in Comte and Genon-Catalot [12] for  $H = 1/2$  and  $\sigma$  constant, to the rate  $N^{-2\beta/(2\beta+4)}$ .

The choice of  $m_{\text{opt}}$  above has the interest to provide a rate, but it is not possible in practice, as it depends on  $\beta$  which is unknown.

Finally, note that the function  $b : x \mapsto \mu x$  with  $\mu \in \mathbb{R}^*$  fulfills the conditions of Proposition 3.8. Indeed, since  $b' = \mu$ , the function  $b$  satisfies Assumption 2.7 with  $m = M = \mu$ , and  $b_A \in W_2^1([\ell, \mathbf{r}])$  for every  $\mathbf{r} > \ell$ . Moreover, since the solution of Equation (1) in this case is the fractional Ornstein-Uhlenbeck process, which is a Gaussian process, then for every  $\mathbf{r} > \ell$ , there exist  $f_0, f_1 > 0$  such that  $f_0 \leq f_T \leq f_1$ . In fact, under Assumption 2.7, thanks to Inequality (3),  $f_T$  is still upper-bounded for nonlinear drift functions.

**3.5. Discussion on the Hermite example.** The second example is the non-compactly supported Hermite basis. Here,  $A = \mathbb{R}$ , and the Hermite polynomial and the Hermite function of order  $j \geq 0$  are given by

$$(5) \quad H_j(x) := (-1)^j e^{x^2} \frac{d^j}{dx^j} (e^{-x^2}) \text{ and } h_j(x) := \mathbf{c}_j H_j(x) e^{-x^2/2}; \forall x \in \mathbb{R},$$

where  $\mathbf{c}_j = (2^j j! \sqrt{\pi})^{-1/2}$ .

The sequence  $(h_j)_{j \geq 0}$  is an orthonormal basis of  $\mathbb{L}^2(\mathbb{R}, dx)$ . By Abramowitz and Stegun [1], and In-dritz [26],

$$\|h_j\|_\infty \leq \Phi_0 \text{ with } \Phi_0 = 1/\pi^{1/4}.$$

So that, for  $\varphi_j = h_j$  the Hermite basis,  $L(m) \leq \Phi_0^2 m$ . Moreover, as

$$h'_j(x) = \sqrt{\frac{j}{2}} h_{j-1}(x) - \frac{\sqrt{j+1}}{h_{j+1}(x)},$$

we find

$$\sup_{x \in \mathbb{R}} \sum_{j=0}^{m-1} h'_j(x)^2 \leq 2\sqrt{\pi} m^2.$$

Thus,  $R(m) \leq 2\sqrt{\pi} m^2$ . Here, the bound of Proposition 3.7 becomes

$$\frac{1}{NT} \text{trace}(\Phi(m)^{-1} \Phi(m, \sigma)) \leq \mathbf{c}_{2, H, \sigma} \sigma^2 \frac{\bar{\mathbf{m}}_{2, H, M}(T)}{NT^{2-2H}} m (1 + 2\sqrt{\pi} \|\Phi(m)^{-1}\|_{\text{op}} m^2),$$

where  $\mathfrak{c} > 0$  is a universal constant.

This case is more complicated since  $f_T$  can no longer be assumed lower bounded on  $\mathbb{R}$ , otherwise it would not be integrable. Therefore, the order of the variance and specifically of  $\|\Phi(m)^{-1}\|_{\text{op}}$  is more difficult to evaluate in general context. What is known is that it is growing with  $m$  and with order larger than order  $\sqrt{m}$  (see [11]). However, we can still assume that  $f_T$  is upper-bounded by a constant  $f_1 > 0$ , and thus, we can evaluate the bias in a similar way as previously by considering Sobolev-Hermite spaces (see Bongioanni and Torrea [5] or Belomestny et al. [4]) and balls. The Sobolev-Hermite space with regularity  $s > 0$  is given by

$$(6) \quad W_H^s := \left\{ \theta \in \mathbb{L}^2(\mathbb{R}) : \sum_{k \geq 0} k^s a_k(\theta)^2 < \infty \right\},$$

where  $a_k(\theta) := \langle \theta, h_k \rangle$ ,  $k \in \mathbb{N}$ . The Sobolev-Hermite ball is given by

$$W_H^s(D) := \left\{ \theta \in \mathbb{L}^2(\mathbb{R}) : \sum_{k \geq 0} k^s a_k(\theta)^2 \leq D \right\}; D > 0.$$

Thus, if  $b$  belongs to  $W_H^s(D)$ , then we have

$$\|p_{S_m}^\perp(b) - b\|^2 \leq Dm^{-s}.$$

For details, and especially for regularity properties of functions in these spaces, we refer to Section 4.1 of Belomestny et al. [4].

**Proposition 3.9.** *Under Assumption 2.7, if  $f_T(x) \leq f_1$  for every  $x \in \mathbb{R}$ ,  $b \in W_H^s(D)$ , the estimator is computed in the Hermite basis on  $\mathbb{R}$ , and  $\|\Phi(m)^{-1}\|_{\text{op}} \leq m^\kappa$  with  $\kappa \geq 1$ , then there exists a deterministic constant  $\mathfrak{c}_{s,D,f_1} > 0$ , not depending on  $N$  and  $T$ , such that with  $m_{\text{opt}} := \lceil (NT^{2-2H})^{1/(s+3+\kappa)} \rceil$ ,*

$$\mathbb{E}(\|\tilde{b}_{m_{\text{opt}}} - b\|_N^2) \leq \mathfrak{c}_{s,D,f_1} \bar{\mathfrak{m}}_{2,H,M}(T) (NT^{2-2H})^{-s/(s+3+\kappa)} + \mathfrak{c}_{\rho,\kappa,\sigma,b} \frac{\bar{\mathfrak{m}}_{2,H,M}(T)}{NT}.$$

The rate depends on the unknown  $\kappa$ , and we mention that if  $\|\Phi(m)^{-1}\|_{\text{op}}$  grows exponentially with  $m$ , then the rate will become logarithmic, except if the bias also decreases exponentially.

#### 4. TOWARDS A CALCULABLE ESTIMATOR

Even if  $\hat{b}_m$  is a good estimator of  $b_A$  in theory, the Skorokhod integral is difficult to compute in practice. So, assume that the process  $X$  has been observed  $N$  times for two close initial conditions  $x_0$  and  $x_0 + \varepsilon$  with  $\varepsilon > 0$ , a situation which can occur in pharmacology. We first present a possible application context, and then build an approximate but computable estimator.

**4.1. Application context.** Let us give details about the application field we have in mind. If  $t \mapsto X_{x_0}(t)$  denotes the concentration of a drug along time during its elimination by a patient with initial dose  $x_0 > 0$ ,  $t \mapsto X_{x_0+\varepsilon}(t)$  could be approximated by replicating the exact same protocol on the same patient, but with the initial dose  $x_0 + \varepsilon$  after the complete elimination of the previous dose.

This is an interesting perspective because differential equations driven by the fractional Brownian motion with  $H > 1/2$  are well adapted to model the concentration process in pharmacokinetics. Indeed, D'Argenio and Park [15] showed that the elimination process has both a deterministic and a random components. A natural way to take into account these two components is to add a stochastic noise in the linear differential equation which classically models the concentration. It has been studied in the Itô calculus framework by many authors (see e.g. Donnet and Samson [24]). However, as mentioned in Delattre and Lavielle [18], the extension of the deterministic concentration model as a diffusion process is not realistic on the biological side because its paths are too rough.

So, as mentioned in Marie [34], Section 5, a way to increase the regularity of the paths of the concentration process is to replace the Brownian motion by a fractional Brownian motion of Hurst index close to 1 as driving signal.

**4.2. Generalized results for approximate estimator.** Throughout this subsection,  $b$  fulfills the following reinforced assumption.

**Assumption 4.1.** *The function  $b$  belongs to  $\text{Lip}_{\mathbf{b}}^2(\mathbb{R})$  and fulfills Assumption 2.7.*

Under Assumption 4.1, the following proposition allows to compute the Skorokhod integral of the solution  $X$  to Equation (1) with respect to  $B$  if two paths of  $X$  can be observed with different but close initial conditions.

**Proposition 4.2.** *Under Assumption 4.1, for every  $\varphi \in \text{Lip}_{\mathbf{b}}^1(\mathbb{R})$ ,  $x \in \mathbb{R}$ ,  $\varepsilon > 0$  and  $t \in ]0, T]$ ,*

$$\left| \int_0^t \varphi(X_{x_0}(u)) \delta X_{x_0}(u) - S_\varphi(x_0, \varepsilon, t) \right| \leq \alpha_H \sigma^2 \frac{\|b''\|_\infty \|\varphi'\|_\infty}{2} \varepsilon t^{2H-1} \mathbf{m}_{H,M}(t)$$

where

$$\begin{aligned} S_\varphi(x_0, \varepsilon, t) &:= \int_0^t \varphi(X_{x_0}(u)) dX_{x_0}(u) \\ &\quad - \alpha_H \sigma^2 \int_0^t \int_0^u \varphi'(X_{x_0}(u)) \frac{X_{x_0+\varepsilon}(u) - X_{x_0}(u)}{X_{x_0+\varepsilon}(v) - X_{x_0}(v)} |u-v|^{2H-2} dv du, \end{aligned}$$

$X_x$  is the solution to Equation (1) with initial condition  $x \in \mathbb{R}$ , and

$$\mathbf{m}_{H,M}(t) := \frac{1}{M^2(2H-1)} \mathbf{1}_{M<0} + \frac{t^2}{2H(2H+1)} \mathbf{1}_{M=0} + \frac{e^{2Mt}}{M^2(2H-1)} \mathbf{1}_{M>0}.$$

Note that if  $M < 0$ , then Theorem 4.2 has been already established in Comte and Marie [14] (see Corollary 2.8).

By Proposition 4.2, for every  $j \in \mathbb{N}$  and  $i \in \{1, \dots, N\}$ ,

$$\begin{aligned} S_{\varphi_j}^i(x_0, \varepsilon, T) &:= \int_0^T \varphi_j(X_{x_0}^i(u)) dX_{x_0}^i(u) \\ &\quad - \alpha_H \sigma^2 \int_0^T \int_0^u \varphi_j'(X_{x_0}^i(u)) \frac{X_{x_0+\varepsilon}^i(u) - X_{x_0}^i(u)}{X_{x_0+\varepsilon}^i(v) - X_{x_0}^i(v)} |u-v|^{2H-2} dv du \end{aligned}$$

provides a good approximation of

$$\int_0^T \varphi_j(X_{x_0}^i(s)) \delta X_{x_0}^i(s).$$

This legitimates to consider the alternative *more computable* estimator

$$\widehat{b}_{m,\varepsilon} := \sum_{j=0}^{m-1} \widehat{\theta}_{j,\varepsilon} \varphi_j,$$

where

$$\widehat{\theta}(m, \varepsilon) := (\widehat{\theta}_{0,\varepsilon}, \dots, \widehat{\theta}_{m-1,\varepsilon})^* = \widehat{\mathbf{\Phi}}(m)^{-1} \widehat{\mathbf{x}}(m, \varepsilon)$$

and

$$\widehat{\mathbf{x}}(m, \varepsilon) := \left( \frac{1}{NT} \sum_{i=1}^N S_{\varphi_j}^i(x_0, \varepsilon, T) \right)_{j=0, \dots, m-1}^*.$$

**Remark 4.3.** *For any  $i \in \{1, \dots, N\}$ , since  $\mathcal{I}(\cdot, B^i)$  is locally Lipschitz continuous on  $\mathbb{R}$  and  $X_{x_0}^i$  is  $\alpha$ -Hölder continuous on  $[0, T]$  with  $\alpha \in ]1/2, H[$ , there exists a square integrable random variable  $C : \Omega \rightarrow ]0, \infty[$  such that for any  $\eta, \varepsilon \in ]0, 1[$ ,*

$$\begin{aligned} |X_{x_0}^i(t+\eta) - X_{x_0+\varepsilon}^i(t)| &\leq |X_{x_0}^i(t+\eta) - X_{x_0}^i(t)| + |X_{x_0}^i(t) - X_{x_0+\varepsilon}^i(t)| \\ &\leq C(\eta^\alpha + \varepsilon). \end{aligned}$$

By taking  $\eta = \varepsilon^{1/\alpha}$ ,

$$|X_{x_0}^i(t + \varepsilon^{1/\alpha}) - X_{x_0+\varepsilon}^i(t)| \leq 2C\varepsilon.$$

So, despite the lack of information available on the behavior of the quotient involved in  $S_{\varphi_j}^i(x_0, \varepsilon, T)$ , one could replace it by

$$\begin{aligned} \tilde{S}_{\varphi_j}^i(x_0, \varepsilon, T) &:= \int_0^T \varphi(X_{x_0}^i(u)) dX_{x_0}^i(u) \\ &\quad - \alpha_H \sigma^2 \int_0^T \int_0^u \varphi'(X_{x_0}^i(u)) \frac{X_{x_0}^i(u + \varepsilon^{1/\alpha}) - X_{x_0}^i(u)}{X_{x_0}^i(v + \varepsilon^{1/\alpha}) - X_{x_0}^i(v)} |u - v|^{2H-2} dv du \end{aligned}$$

in the expression of  $\hat{\theta}(m, \varepsilon)$ . This would avoid the requirement of the paths  $(X_{x_0+\varepsilon}^i)$  and  $(X_{x_0}^i)$  for each individual  $i$ .

Thus, to be coherent and realistic, we consider the estimator modified as follows:

$$\tilde{b}_{m,\varepsilon} := \hat{b}_{m,\varepsilon} \mathbf{1}_{\hat{\Lambda}(m)}.$$

Then, we can prove the following result as a consequence of Corollary 3.5.

**Corollary 4.4.** *Under Assumptions 4.1, 3.2 and 3.3,*

$$\begin{aligned} \mathbb{E}(\|\tilde{b}_{m,\varepsilon} - b_A\|_{f_T}^2) &\leq \mathfrak{d}_{\kappa,NT} \inf_{\tau \in \mathcal{S}_m} \|\tau - b_A\|_{f_T}^2 + \frac{16}{NT} \text{trace}(\Phi(m)^{-1} \Phi(m, \sigma)) + \bar{\mathfrak{c}}_{\rho,\kappa,\sigma,\bar{\mathfrak{b}}} \frac{\bar{\mathfrak{m}}_{2,H,M}(T)}{NT} \\ &\quad + \mathfrak{c}_{\rho,\kappa,\sigma,H,\bar{\mathfrak{b}}} L(m)^{\kappa-1} \left( \frac{NT}{\log(NT)} \right)^2 \frac{\mathfrak{m}_{H,M}(T)^2}{T^{4-4H}} \varepsilon^2 \end{aligned}$$

where  $\mathfrak{c}_{\rho,\kappa,\sigma,H,\bar{\mathfrak{b}}} > 0$  is a deterministic constant depending only on  $\rho, \kappa, \sigma, H$  and  $\bar{\mathfrak{b}} := \|b''\|_{\infty}^2$ , and

$$\mathfrak{d}_{\kappa,NT} := 2 \left( 1 + 4 \frac{3 \log(3/2) - 1}{(7 + \kappa) \log(NT)} \right).$$

In order to keep the rate of convergence obtained for  $\tilde{b}_m$ , one can assume that  $\varepsilon$  depends on  $N$  and  $T$ , and take

$$\varepsilon_{N,T} = \frac{1}{(NT)^{1/2}} \left( \frac{NT}{\log(NT)} \right)^{-1}.$$

## 5. CONCLUDING REMARKS

In this paper, we propose an estimation strategy for nonparametric reconstruction of the drift function  $b$  from  $N$  i.i.d. observations  $(X_i(t))_{t \in [0,T]}$  drawn in the fractional SDE given by (1). We bound the empirical and  $f_T$ -weighted  $\mathbb{L}^2$ -risk of the estimator, and compare the result with non fractional case. In the case of a specific trigonometric basis, we can, under additional assumptions, prove the consistency of the estimator and evaluate its rate of convergence. However, the choice  $m_{\text{opt}}$  which is proposed to obtain this result is not possible in practice, as it depends on unknown parameters. Therefore, a model selection step in the spirit of Comte and Genon-Catalot [12] would have to be settled.

The definition of our projection estimator involves a Skorokhod integral which is not computable, so we also describe an approximated estimator, in case two paths with slightly different initial conditions can be available for each individual  $i$ : this context may be realistic in pharmacokinetics experiments. We can provide a new risk bound in this context. We also propose another idea which seems intuitive, does not require two paths per individual but would require a deeper study.

Lastly, the tedious question of discretization may be investigated in the future, to take into account the fact that, for each individual  $i$ , the observation is  $(X_i(k\Delta))_{1 \leq k \leq n}$  where  $n\Delta = T$ .

## 6. PROOFS

**6.1. Proof of Theorem 2.9.** On the one hand, for any  $s, t \in [0, T]$ ,

$$\mathbf{D}_s X(t) = \sigma \mathbf{1}_{[0,t]}(s) \exp \left( \int_s^t b'(X(u)) du \right)$$

and, by Assumption 2.7,

$$|\mathbf{D}_s X(t)| \leq \sigma \mathbf{1}_{[0,t]}(s) e^{M(t-s)}.$$

Then, by the chain rule for Malliavin's derivative and Jensen's inequality,

$$\begin{aligned}
\mathbb{E} \left( \int_0^T \int_0^T |\mathbf{D}_s[\varphi(X(t))]|^{1/H} ds dt \right)^{pH} &= \mathbb{E} \left( \int_0^T \int_0^T |\varphi'(X(t))\mathbf{D}_s X(t)|^{1/H} ds dt \right)^{pH} \\
&\leq \sigma^p \left( \int_0^T \mathbb{E}(|\varphi'(X(t))|^{1/H}) \int_0^t e^{M/H(t-s)} ds dt \right)^{pH} \\
(7) \quad &\leq \sigma^p \mathfrak{m}_{p,H,M}(T) \left( \int_0^T \mathbb{E}(|\varphi'(X(t))|^p)^{1/(pH)} dt \right)^{pH}
\end{aligned}$$

where

$$\mathfrak{m}_{p,H,M}(T) = \left(-\frac{H}{M}\right)^{pH} \mathbf{1}_{M<0} + T^{pH} \mathbf{1}_{M=0} + \left(\frac{H}{M}\right)^{pH} e^{pMT} \mathbf{1}_{M>0}.$$

On the other hand, by Hu et al. [30], Lemma 3.1, there exists a deterministic constant  $\mathfrak{c}_{p,H} > 0$ , depending only on  $p$  and  $H$ , such that for any  $\varphi \in \text{Lip}_b^1(\mathbb{R})$ ,

$$\begin{aligned}
\mathbb{E} \left( \left| \int_0^T \varphi(X(t)) \delta B(t) \right|^p \right) &\leq \mathfrak{c}_{p,H} \left[ \left( \int_0^T \mathbb{E}(|\varphi(X(t))|^{1/H}) dt \right)^{pH} \right. \\
(8) \quad &\left. + \mathbb{E} \left( \int_0^T \int_0^T |\mathbf{D}_s[\varphi(X(t))]|^{1/H} ds dt \right)^{pH} \right].
\end{aligned}$$

Inequalities (7) and (8) together allow to conclude.

**6.2. Proof of Proposition 2.11.** Consider  $t \in ]0, T]$ . On the one hand, for any  $s \in \mathbb{R}_+$ ,

$$\mathbf{D}_s X(t) = \sigma \mathbf{1}_{[0,t]}(s) \exp \left( \int_s^t b'(X(u)) du \right).$$

So, since  $b'$  is a  $[m, M]$ -valued function,

$$\sigma \mathbf{1}_{[0,t]}(s) e^{m(t-s)} \leq \mathbf{D}_s X(t) \leq \sigma \mathbf{1}_{[0,t]}(s) e^{M(t-s)}.$$

On the other hand, by Nourdin and Viens [40], Proposition 3.7,

$$-\mathbf{D}_s \mathbf{L}^{-1} X(t) = \int_0^\infty e^{-u} \mathbf{T}_u(\mathbf{D}_s X(t)) du$$

where  $(\mathbf{T}_u)_{u \in \mathbb{R}_+}$  is the Ornstein-Uhlenbeck semigroup (see Nualart [41], Section 1.4). Moreover, for any  $u \in \mathbb{R}_+$ ,  $(\mathbf{T}_u)|_{\mathbb{R}} = \text{Id}_{\mathbb{R}}$  by Mehler's formula (see Nualart [41], Equation (1.67)), and for every  $U_1, U_2 \in \mathbb{L}^2(\Omega, \mathbb{P})$ ,

$$U_1 \geq U_2 \implies \mathbf{T}_u(U_1) \geq \mathbf{T}_u(U_2)$$

by Nualart [41], Property (i) page 55. Then,

$$\sigma \mathbf{1}_{[0,t]}(s) e^{m(t-s)} \leq -\mathbf{D}_s \mathbf{L}^{-1} X(t) \leq \sigma \mathbf{1}_{[0,t]}(s) e^{M(t-s)}.$$

Therefore,

$$\sigma(m, t)^2 \leq g_t^*(x_0, X^*(t)) \leq \sigma(M, t)^2$$

where

$$\sigma(\mu, t)^2 = \alpha_H \sigma^2 \int_0^t \int_0^t |v - u|^{2H-2} e^{\mu(2t-v-u)} dudv > 0; \forall \mu \in \mathbb{R}.$$

Nourdin and Viens [40], Theorem 3.1 and Corollary 3.5 allow to conclude.



**6.3. Proof of Theorem 3.4.** The proof relies on two lemmas which are stated first.

**Lemma 6.1.** *There exists a deterministic constant  $\mathbf{c}_e > 0$ , not depending on  $m$ ,  $N$  and  $T$ , such that*

$$\mathbb{E}(|\mathbf{e}(m)^* \mathbf{e}(m)|^2) \leq \mathbf{c}_e \sigma^4 \frac{\bar{\mathbf{m}}_{4,H,M}(T)}{N^2 T^{4-4H}} m(L(m)^2 + R(m)^2).$$

**Lemma 6.2.** *Consider*

$$\Omega(m) := \left\{ \sup_{\tau \in \mathcal{S}_m} \left| \frac{\|\tau\|_N^2}{\|\tau\|_{f_T}^2} - 1 \right| \leq \frac{1}{2} \right\}.$$

*Under Assumptions 2.7, 3.2 and 3.3, there exists a deterministic constant  $\mathbf{c}_\Omega > 0$ , not depending on  $m$ ,  $N$  and  $T$ , such that*

$$\mathbb{P}(\widehat{\Lambda}(m)^c) \leq \frac{\mathbf{c}_\Omega}{(NT)^{6+\kappa}} \text{ and } \mathbb{P}(\Omega(m)^c) \leq \frac{\mathbf{c}_\Omega}{(NT)^{6+\kappa}}.$$

**6.3.1. Steps of the proof.** First of all,

$$(9) \quad \|\widehat{b}_m - b_A\|_N^2 = \|\widehat{b}_m - b_A\|_N^2 \mathbf{1}_{\widehat{\Lambda}(m)} + \|b_A\|_N^2 \mathbf{1}_{\widehat{\Lambda}(m)^c} = U_1 + U_2 + U_3$$

where

$$\begin{aligned} U_1 &:= \|b_A\|_N^2 \mathbf{1}_{\widehat{\Lambda}(m)^c}, \\ U_2 &:= \|\widehat{b}_m - b_A\|_N^2 \mathbf{1}_{\widehat{\Lambda}(m) \cap \Omega(m)} \text{ and} \\ U_3 &:= \|\widehat{b}_m - b_A\|_N^2 \mathbf{1}_{\widehat{\Lambda}(m) \cap \Omega(m)^c}. \end{aligned}$$

Let us find suitable bounds for  $\mathbb{E}(U_1)$ ,  $\mathbb{E}(U_2)$  and  $\mathbb{E}(U_3)$ .

- **Bound for  $\mathbb{E}(U_1)$ .** By Lemma 6.2,

$$\mathbb{E}(U_1) \leq \mathbb{E}(\|b_A\|_N^4)^{1/2} \mathbb{P}(\widehat{\Lambda}(m)^c)^{1/2} \leq \frac{\mathbf{c}_1}{NT}$$

where

$$\mathbf{c}_1 := \mathbf{c}_\Omega^{1/2} \left( \int_{-\infty}^{\infty} b_A(x)^4 f_T(x) dx \right)^{1/2}.$$

- **Bound for  $\mathbb{E}(U_2)$ .** By denoting  $p_{N, \mathcal{S}_m}^\perp$  the orthogonal projection from  $\mathbb{L}^2(A, f_T(x) dx)$  onto  $\mathcal{S}_m$  with respect to the empirical scalar product  $\langle \cdot, \cdot \rangle_N$ ,

$$(10) \quad \|\widehat{b}_m - b_A\|_N^2 = \|\widehat{b}_m - p_{N, \mathcal{S}_m}^\perp(b_A)\|_N^2 + \inf_{\tau \in \mathcal{S}_m} \|b_A - \tau\|_N^2.$$

As in the proof of Comte and Genon-Catalot [12], Proposition 2.1, on  $\Omega(m)$ ,

$$\|\widehat{b}_m - p_{N, \mathcal{S}_m}^\perp(b_A)\|_N^2 = \mathbf{e}(m)^* \widehat{\Phi}(m)^{-1} \mathbf{e}(m) \leq 2\mathbf{e}(m)^* \Phi(m)^{-1} \mathbf{e}(m).$$

So,

$$\begin{aligned} \mathbb{E}(\|\widehat{b}_m - p_{N, \mathcal{S}_m}^\perp(b_A)\|_N^2 \mathbf{1}_{\Omega(m) \cap \widehat{\Lambda}(m)}) &\leq 2\mathbb{E} \left( \sum_{j,k=0}^{m-1} \mathbf{e}(m)_j \mathbf{e}(m)_k \Phi(m)_{j,k}^{-1} \right) \\ &= \frac{2\sigma^2}{NT^2} \sum_{j,k=0}^{m-1} \Phi(m)_{j,k}^{-1} \\ &\quad \times \mathbb{E} \left( \left( \int_0^T \varphi_j(X(s)) \delta B(s) \right) \left( \int_0^T \varphi_k(X(s)) \delta B(s) \right) \right) \\ &= \frac{2}{NT} \text{trace}(\Phi(m)^{-1} \Phi(m, \sigma)). \end{aligned}$$

Therefore, by Inequality (10),

$$\mathbb{E}(U_2) \leq \inf_{\tau \in \mathcal{S}_m} \|b_A - \tau\|_{f_T}^2 + \frac{2}{NT} \text{trace}(\Phi(m)^{-1} \Phi(m, \sigma)).$$

- **Bound for  $\mathbb{E}(U_3)$ .** On the one hand,

$$\|\widehat{b}_m - p_{N, \mathcal{S}_m}^\perp(b_A)\|_N^2 = \mathbf{e}(m)^* \widehat{\Phi}(m)^{-1} \mathbf{e}(m).$$

On the other hand, on  $\widehat{\Lambda}(m)$ , for  $N, T$  large enough,

$$\begin{aligned} (L(m) + R(m)) \|\widehat{\Phi}(m)^{-1}\|_{\text{op}} &\leq L(m) (\|\widehat{\Phi}(m)^{-1}\|_{\text{op}} \vee 1) + \rho L(m)^\kappa (\|\widehat{\Phi}(m)^{-1}\|_{\text{op}} \vee 1)^\kappa \\ &\leq (1 + \rho) \mathbf{c}_{\kappa, T} \left( \frac{NT}{\log(NT)} \right)^\kappa. \end{aligned}$$

Then, by Lemmas 6.1 and 6.2, there exists a deterministic constant  $\mathbf{c}_2 > 0$ , not depending on  $m$  (satisfying  $m \leq NT$ ),  $N$  and  $T$ , such that

$$\begin{aligned} \mathbb{E}(U_3) &\leq \mathbb{E}(\|\widehat{b}_m - p_{N, \mathcal{S}_m}^\perp(b_A)\|_N^2 + \|b_A\|_N^2) \mathbf{1}_{\widehat{\Lambda}(m) \cap \Omega(m)^c} \\ &\leq \left[ \frac{(1 + \rho) \mathbf{c}_{\kappa, T}}{L(m) + R(m)} \left( \frac{NT}{\log(NT)} \right)^\kappa \mathbb{E}(|\mathbf{e}(m)^* \mathbf{e}(m)|^2)^{1/2} + \mathbb{E}(\|b_A\|_N^4)^{1/2} \right] \mathbb{P}(\Omega(m)^c)^{1/2} \\ &\leq \mathbf{c}_2 \frac{\overline{\mathfrak{m}}_{2, H, M}(T)}{NT}. \end{aligned}$$

These bounds together with Inequality (9) allow to conclude.

6.3.2. *Proof of Lemma 6.1.* On the one hand, for any bounded and measurable function  $\tau : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$(11) \quad \int_0^T \mathbb{E}(\tau(X(s))) ds = \int_{-\infty}^{\infty} \int_0^T \tau(x) p_s(x_0, x) ds dx = T \int_{-\infty}^{\infty} \tau(x) f_T(x) dx$$

and, by Jensen's inequality, for every  $p > 1/H$ ,

$$\begin{aligned} \int_0^T \mathbb{E}(\tau(X(s)))^{1/(pH)} ds &= T \int_0^T \left( \int_{-\infty}^{\infty} \tau(x) p_s(x_0, x) dx \right)^{1/(pH)} \frac{ds}{T} \\ &\leq T \left( \int_{-\infty}^{\infty} \int_0^T \tau(x) p_s(x_0, x) \frac{ds}{T} dx \right)^{1/(pH)} \\ (12) \quad &= T \left( \int_{-\infty}^{\infty} \tau(x) f_T(x) dx \right)^{1/(pH)}. \end{aligned}$$

On the other hand, by Jensen's inequality, since  $(B_1, X_1), \dots, (B_N, X_N)$  are i.i.d and by Equality (2),

$$\begin{aligned} \mathbb{E}(|\mathbf{e}(m)^* \mathbf{e}(m)|^2) &= \mathbb{E} \left( \left| \sum_{j=0}^{m-1} \mathbf{e}(m)_j^2 \right|^2 \right) \\ &\leq \frac{\sigma^4 m}{(NT)^4} \sum_{j=0}^{m-1} \mathbb{E} \left[ \left( \sum_{i=1}^N \int_0^T \varphi_j(X^i(s)) \delta B^i(s) \right)^4 \right] \\ &\leq \frac{\sigma^4 m}{N^3 T^4} \sum_{j=0}^{m-1} \mathbb{E}(Y_j^4) + \frac{\sigma^4 m}{N^2 T^4} \sum_{j=0}^{m-1} \mathbb{E}(Y_j^2)^2 \end{aligned}$$

where

$$Y_j := \int_0^T \varphi_j(X(s)) \delta B(s) ; \forall j \in \{0, \dots, m-1\}.$$

Moreover, by Theorem 2.9, Equality (11) and Inequality (12), for  $j = 0, \dots, m-1$ ,

$$\begin{aligned} \mathbb{E}(Y_j^4) &\leq \mathbf{c}_{4,H,\sigma} \bar{\mathbf{m}}_{4,H,M}(T) \left[ \left( \int_0^T \mathbb{E}(|\varphi_j(X(s))|^{1/H}) ds \right)^{4H} + \left( \int_0^T \mathbb{E}(|\varphi_j'(X(s))|^4)^{1/(4H)} ds \right)^{4H} \right] \\ &\leq \mathbf{c}_{4,H,\sigma} \bar{\mathbf{m}}_{4,H,M}(T) T^{4H} \left[ \left( \int_{-\infty}^{\infty} |\varphi_j(x)|^{1/H} f_T(x) dx \right)^{4H} + \int_{-\infty}^{\infty} |\varphi_j'(x)|^4 f_T(x) dx \right] \\ &\leq \mathbf{c}_{4,H,\sigma} \bar{\mathbf{m}}_{4,H,M}(T) T^{4H} \sup_{x \in A} \{|\varphi_j(x)|^4 + |\varphi_j'(x)|^4\}. \end{aligned}$$

Therefore, there exists a deterministic constant  $\mathbf{c}_e > 0$ , not depending on  $m$ ,  $N$  and  $T$ , such that

$$\mathbb{E}(|\mathbf{e}(m)^* \mathbf{e}(m)|^2) \leq \mathbf{c}_e \sigma^4 \frac{\bar{\mathbf{m}}_{4,H,M}(T)}{N^2 T^{4-4H}} m(L(m)^2 + R(m)^2).$$

6.3.3. *Proof of Lemma 6.2.* The proof of Lemma 6.2 is close to the proof of Comte and Genon-Catalot [12], Lemma 3.

On the one hand, by the beginning of Comte and Genon-Catalot [12], Subsection 6.1 which remains true for  $H > 1/2$  without additional arguments,

$$\Omega(m) = \left\{ \|\Phi(m)^{-1/2} \widehat{\Phi}(m) \Phi(m)^{-1/2} - \mathbf{I}_m\|_{\text{op}} \leq \frac{1}{2} \right\}.$$

Then, as in the proof of Comte and Genon-Catalot [11], Proposition 3, by Cohen et al. [9], Theorem 1,

$$\mathbb{P}(\Omega(m)^c) \leq 2m \exp\left(-\mathbf{c}_1 \frac{NT}{L(m)(\|\Phi(m)^{-1}\|_{\text{op}} \vee 1)}\right)$$

with  $\mathbf{c}_1 := 1/2(3 \log(3/2) - 1)$ . Under Assumption 3.3,

$$L(m)(\|\Phi(m)^{-1}\|_{\text{op}} \vee 1) \leq \frac{\mathbf{c}_{\kappa,T}}{2} \cdot \frac{NT}{\log(NT)}.$$

Then, there exists a deterministic constant  $\mathbf{c}_\Omega > 0$ , not depending on  $m$ ,  $N$  and  $T$ , such that

$$\mathbb{P}(\Omega(m)^c) \leq \frac{\mathbf{c}_\Omega}{(NT)^{6+\kappa}}.$$

On the other hand, as in the proof of Comte and Genon-Catalot [11], Lemma 5, under Assumption 3.3 and on  $\widehat{\Lambda}(m)^c$ ,

$$L(m) \|\Phi(m)^{-1}\|_{\text{op}} \leq \frac{\mathbf{c}_{\kappa,T}}{2} \cdot \frac{NT}{\log(NT)}$$

and

$$L(m) \|\widehat{\Phi}(m)^{-1}\|_{\text{op}} > \mathbf{c}_{\kappa,T} \frac{NT}{\log(NT)},$$

and then

$$\begin{aligned} \widehat{\Lambda}(m)^c &\subset \left\{ L(m) \|\widehat{\Phi}(m)^{-1} - \Phi(m)^{-1}\|_{\text{op}} > \frac{\mathbf{c}_{\kappa,T}}{2} \cdot \frac{NT}{\log(NT)} \right\} \\ &\subset \{ \|\widehat{\Phi}(m)^{-1} - \Phi(m)^{-1}\|_{\text{op}} > \|\Phi(m)^{-1}\|_{\text{op}} \}. \end{aligned}$$

Finally, by Comte and Genon-Catalot [11], Proposition 4.(ii) which remains true for  $H > 1/2$  without additional arguments,

$$\{ \|\widehat{\Phi}(m)^{-1} - \Phi(m)^{-1}\|_{\text{op}} > \|\Phi(m)^{-1}\|_{\text{op}} \} \subset \Omega(m)^c.$$

Therefore,

$$\mathbb{P}(\widehat{\Lambda}(m)^c) \leq \mathbb{P}(\Omega(m)^c) \leq \frac{\mathbf{c}_\Omega}{(NT)^{6+\kappa}}.$$

6.4. **Proof of Corollary 3.5.** First of all,

$$(13) \quad \mathbb{E}(\|\widehat{b}_m - b_A\|_{f_T}^2) = \mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3$$

where

$$\begin{aligned} \mathbf{m}_1 &:= \mathbb{E}(\|b_A\|_{f_T}^2 \mathbf{1}_{\widehat{\Lambda}(m)^c}), \\ \mathbf{m}_2 &:= \mathbb{E}(\|\widehat{b}_m - b_A\|_{f_T}^2 \mathbf{1}_{\widehat{\Lambda}(m) \cap \Omega(m)}) \text{ and} \\ \mathbf{m}_3 &:= \mathbb{E}(\|\widehat{b}_m - b_A\|_{f_T}^2 \mathbf{1}_{\widehat{\Lambda}(m) \cap \Omega(m)^c}). \end{aligned}$$

On the one hand, by Lemma 6.2, there exists a deterministic constant  $\mathbf{c}_1 > 0$ , not depending on  $m$ ,  $N$  and  $T$ , such that

$$(14) \quad \mathbf{m}_1 \leq \frac{\mathbf{c}_1}{NT},$$

and as in the proof of Comte and Genon-Catalot [12], Proposition 1, Theorem 3.4 allows to get

$$(15) \quad \mathbf{m}_2 \leq \left(1 + \frac{4\mathbf{c}_{\kappa,T}}{\log(NT)}\right) \inf_{\tau \in \mathcal{S}_m} \|\tau - b_A\|_{f_T}^2 + \frac{8}{NT} \text{trace}(\Phi(m)^{-1} \Phi(m, \sigma)).$$

On the other hand, by Lemma 6.1,

$$\begin{aligned} \mathbb{E}(|\widehat{\mathbf{x}}(m)^* \widehat{\mathbf{x}}(m)|^2) &\leq 4m \sum_{j=0}^{m-1} \mathbb{E}(\langle \varphi_j, b \rangle_N^4) + 4\mathbb{E}(|\mathbf{e}(m)^* \mathbf{e}(m)|^2) \\ &\leq 4mL(m)^2 \int_{-\infty}^{\infty} b_A(x)^4 f_T(x) dx \\ &\quad + 4\mathbf{c}_e \sigma^4 \frac{\overline{\mathbf{m}}_{4,H,M}(T)}{N^2 T^{4-4H}} m(L(m)^2 + R(m)^2) \\ &\leq \mathbf{c}_2 \left( mL(m)^2 + \frac{\overline{\mathbf{m}}_{4,H,M}(T)}{N^2 T^{4-4H}} m(L(m)^2 + \rho^2 L(m)^{2\kappa}) \right) \end{aligned}$$

where

$$\mathbf{c}_2 := \left( 4 \int_{-\infty}^{\infty} b_A(x)^4 f_T(x) dx \right) \vee (4\mathbf{c}_e \sigma^4).$$

As in the proof of Comte and Genon-Catalot [12], Proposition 1,  $\|\Phi(m)\|_{\text{op}} \leq L(m)$ , and by the definition of  $\widehat{\Lambda}(m)$ ,

$$\begin{aligned} \mathbb{E}(\|\widehat{b}_m\|_{f_T}^2 \mathbf{1}_{\widehat{\Lambda}(m) \cap \Omega(m)^c}) &\leq \|\Phi(m)\|_{\text{op}} \mathbb{E}(\|\widehat{\Phi}(m)^{-1}\|_{\text{op}}^2 |\widehat{\mathbf{x}}(m)^* \widehat{\mathbf{x}}(m)| \mathbf{1}_{\widehat{\Lambda}(m) \cap \Omega(m)^c}) \\ &\leq \frac{\mathbf{c}_{\kappa,T}^2}{L(m)} \left( \frac{NT}{\log(NT)} \right)^2 \mathbb{P}(\Omega(m)^c)^{1/2} \mathbb{E}(|\widehat{\mathbf{x}}(m)^* \widehat{\mathbf{x}}(m)|^2)^{1/2} \\ &\leq \frac{\mathbf{c}_2^{1/2} \mathbf{c}_{\kappa,T}^2}{L(m)} \left( \frac{NT}{\log(NT)} \right)^2 \frac{\mathbf{c}_{\Omega}^{1/2}}{(NT)^{3+\kappa/2}} \\ &\quad \times \left( m^{1/2} L(m) + \frac{\overline{\mathbf{m}}_{2,H,M}(T)}{NT^{2-2H}} m^{1/2} (L(m) + \rho L(m)^{\kappa}) \right). \end{aligned}$$

Then, there exists a deterministic constant  $\mathbf{c}_3 > 0$ , not depending on  $m$ ,  $N$  and  $T$ , such that

$$(16) \quad \mathbf{m}_3 \leq 2\mathbb{E}(\|\widehat{b}_m\|_{f_T}^2 \mathbf{1}_{\widehat{\Lambda}(m) \cap \Omega(m)^c}) + 2\|b_A\|_{f_T}^2 \mathbb{P}(\Omega(m)^c) \leq \mathbf{c}_3 \frac{\overline{\mathbf{m}}_{2,H,M}(T)}{NT}.$$

Inequalities (14), (15) and (16) together with Inequality (13) allow to conclude.

**6.5. Proof of Proposition 3.7.** Let us denote by  $\Phi(m)^{-1/2}$  a symmetric square root of  $\Phi(m)^{-1}$ , and write

$$(17) \quad \begin{aligned} \text{trace}(\Phi(m)^{-1}\Phi(m, \sigma)) &= \text{trace}(\Phi(m)^{-1/2}\Phi(m, \sigma)\Phi(m)^{-1/2}) \\ &\leq m \|\Phi(m)^{-1/2}\Phi(m, \sigma)\Phi(m)^{-1/2}\|_{\text{op}} \\ &= m \sup_{\|x\|_{2,m}=1} x^* \Phi(m)^{-1/2}\Phi(m, \sigma)\Phi(m)^{-1/2}x = m \sup_{\|\Phi(m)^{1/2}y\|_{2,m}=1} y^* \Phi(m, \sigma)y. \end{aligned}$$

For any  $y \in \mathbb{R}^m$  such that  $\|\Phi(m)^{1/2}y\|_{2,m} = 1$ ,

$$\begin{aligned} y^* \Phi(m, \sigma)y &= \frac{\sigma^2}{T} \sum_{j,k=0}^{m-1} y^j y^k \mathbb{E} \left( \left( \int_0^T \varphi_j(X(s)) \delta B(s) \right) \left( \int_0^T \varphi_k(X(s)) \delta B(s) \right) \right) \\ &= \frac{\sigma^2}{T} \mathbb{E} \left[ \left( \int_0^T \tau(X(s)) \delta B(s) \right)^2 \right] \quad \text{with } \tau := \sum_{j=0}^{m-1} y^j \varphi_j. \end{aligned}$$

By Theorem 2.9, Equality (11) and Inequality (12),

$$(18) \quad \begin{aligned} \mathbb{E} \left[ \left( \int_0^T \tau(X(s)) \delta B(s) \right)^2 \right] &\leq \mathbf{c}_{2,H,\sigma} \bar{\mathbf{m}}_{2,H,M}(T) \left[ \left( \int_0^T \mathbb{E}(|\tau(X(s))|^{1/H}) ds \right)^{2H} \right. \\ &\quad \left. + \left( \int_0^T \mathbb{E}(|\tau'(X(s))|^2)^{1/(2H)} ds \right)^{2H} \right] \\ &\leq \mathbf{c}_{2,H,\sigma} \bar{\mathbf{m}}_{2,H,M}(T) T^{2H} \left( \int_{-\infty}^{\infty} \tau(x)^2 f_T(x) dx + \int_{-\infty}^{\infty} \tau'(x)^2 f_T(x) dx \right). \end{aligned}$$

Since

$$\int_{-\infty}^{\infty} \tau(x)^2 f_T(x) dx = y^* \Phi(m)y = 1,$$

by Inequalities (17) and (18),

$$\text{trace}(\Phi(m)^{-1}\Phi(m, \sigma)) \leq \mathbf{c}_{2,H,\sigma} \sigma^2 \frac{\bar{\mathbf{m}}_{2,H,M}(T)}{T^{1-2H}} m (1 + \|\Phi(m)^{-1/2}\Psi(m, \sigma)\Phi(m)^{-1/2}\|_{\text{op}})$$

where

$$\Psi(m, \sigma) := \left( \int_{-\infty}^{\infty} \varphi'_j(x) \varphi'_k(x) f_T(x) dx \right)_{j,k=0,\dots,m-1}.$$

Moreover,

$$\|\Phi(m)^{-1/2}\Psi(m, \sigma)\Phi(m)^{-1/2}\|_{\text{op}} \leq \|\Phi(m)^{-1/2}\|_{\text{op}}^2 R(m).$$

Therefore,

$$\text{trace}(\Phi(m)^{-1}\Phi(m, \sigma)) \leq \mathbf{c}_{2,H,\sigma} \sigma^2 \frac{\bar{\mathbf{m}}_{2,H,M}(T)}{T^{1-2H}} m (1 + \|\Phi(m)^{-1}\|_{\text{op}} R(m)).$$

This concludes the proof.

**6.6. Proof of Proposition 4.2.** Consider  $x \in \mathbb{R}$  and  $\varepsilon, t > 0$ . For every  $s \in [0, t]$ ,

$$\partial_x X_x(s) = 1 + \int_0^s b'(X_x(r)) \partial_x X_x(r) dr$$

and, by Taylor's formula,

$$X_{x+\varepsilon}(s) - X_x(s) = \varepsilon + \int_0^s (X_{x+\varepsilon}(r) - X_x(r)) \int_0^1 b'(X_x(r) + \theta(X_{x+\varepsilon}(r) - X_x(r))) d\theta dr.$$

So, for every  $(u, v) \in [0, t]^2$  such that  $v < u$ ,

$$\frac{\partial_x X_x(u)}{\partial_x X_x(v)} = \exp \left( \int_v^u b'(X_x(r)) dr \right)$$

and

$$\frac{X_{x+\varepsilon}(u) - X_x(u)}{X_{x+\varepsilon}(v) - X_x(v)} = \exp\left(\int_v^u \int_0^1 b'(X_x(r) + \theta(X_{x+\varepsilon}(r) - X_x(r)))d\theta dr\right).$$

For a given  $\varphi \in \text{Lip}_b^1(\mathbb{R})$ , by Comte and Marie [14], Proposition 2.7,

$$\Delta_\varphi^S(x, \varepsilon, t) \leq \alpha_H \sigma^2 \int_0^t \int_0^u |\varphi'(X_x(u))| \Delta_\varphi(x, \varepsilon, u, v) (u-v)^{2H-2} dv du,$$

where

$$\Delta_\varphi^S(x, \varepsilon, t) := \left| \int_0^t \varphi(X_x(u)) \delta X_x(u) - S_\varphi(x, \varepsilon, t) \right|$$

and, for every  $(u, v) \in [0, t]^2$  such that  $v < u$ ,

$$\Delta_\varphi(x, \varepsilon, u, v) := \left| \frac{\partial_x X_x(u)}{\partial_x X_x(v)} - \frac{X_{x+\varepsilon}(u) - X_x(u)}{X_{x+\varepsilon}(v) - X_x(v)} \right|.$$

Since  $b'(\mathbb{R}) \subset [m, M]$  and  $b$  is two times continuously differentiable,

$$\begin{aligned} \Delta_\varphi(x, \varepsilon, u, v) &= \left| \exp\left(\int_v^u b'(X_x(r))dr\right) \right. \\ &\quad \left. - \exp\left(\int_v^u \int_0^1 b'(X_x(r) + \theta(X_{x+\varepsilon}(r) - X_x(r)))d\theta dr\right) \right| \\ &\leq \sup_{z \in [m(u-v), M(u-v)]} e^z \\ &\quad \times \int_v^u \left| b'(X_x(r)) - \int_0^1 b'(X_x(r) + \theta(X_{x+\varepsilon}(r) - X_x(r)))d\theta \right| dr \\ &\leq (1 \vee e^{Mt}) \int_v^u \int_0^1 |b'(X_x(r)) - b'(X_x(r) + \theta(X_{x+\varepsilon}(r) - X_x(r)))| d\theta dr \\ &\leq \frac{\|b''\|_\infty}{2} (1 \vee e^{Mt}) \int_v^u |X_{x+\varepsilon}(r) - X_x(r)| dr. \end{aligned}$$

Consider  $s \in \mathbb{R}_+$ . By Equation (1),

$$\begin{aligned} (X_{x+\varepsilon}(s) - X_x(s))^2 &= \varepsilon^2 + 2 \int_0^s (X_{x+\varepsilon}(r) - X_x(r)) d(X_{x+\varepsilon} - X_x)(r) \\ &= \varepsilon^2 + 2 \int_0^s (X_{x+\varepsilon}(r) - X_x(r)) (b(X_{x+\varepsilon}(r)) - b(X_x(r))) dr. \end{aligned}$$

By the mean value theorem, there exists  $x_s \in \mathbb{R}$  such that

$$\begin{aligned} \partial_s (X_{x+\varepsilon}(s) - X_x(s))^2 &= 2(X_{x+\varepsilon}(s) - X_x(s))^2 \frac{b(X_{x+\varepsilon}(s)) - b(X_x(s))}{X_{x+\varepsilon}(s) - X_x(s)} \\ &= 2(X_{x+\varepsilon}(s) - X_x(s))^2 b'(x_s) \leq 2M(X_{x+\varepsilon}(s) - X_x(s))^2 \end{aligned}$$

and then,

$$|X_{x+\varepsilon}(s) - X_x(s)| \leq \varepsilon e^{Ms}.$$

Therefore,

$$\begin{aligned} \Delta_\varphi(x, \varepsilon, u, v) &\leq \frac{\|b''\|_\infty}{2} \varepsilon (1 \vee e^{Mt}) \int_v^u e^{Mr} dr \\ &\leq \frac{\|b''\|_\infty}{2} \varepsilon (1 \vee e^{Mt}) \mathbf{m}_M(u, v) \end{aligned}$$

where

$$\mathbf{m}_M(u, v) := -\frac{1}{M} e^{Mv} \mathbf{1}_{M < 0} + (u-v) \mathbf{1}_{M=0} + \frac{1}{M} e^{Mu} \mathbf{1}_{M > 0}.$$

Finally, using the above bounds, and in a second stage, the integration by parts formula, we get

$$\begin{aligned}
\Delta_\varphi^S(x, \varepsilon, t) &\leq \alpha_H \sigma^2 \frac{\|b''\|_\infty}{2} \varepsilon (1 \vee e^{Mt}) \int_0^t \int_0^u |\varphi'(X_x(u))| \mathbf{m}_M(u, v) (u-v)^{2H-2} dv du \\
&\leq \alpha_H \sigma^2 \frac{\|b''\|_\infty \|\varphi'\|_\infty}{2} \varepsilon (1 \vee e^{Mt}) \left( -\frac{1}{M} \mathbf{1}_{M < 0} \int_0^t e^{Mv} \int_v^t (u-v)^{2H-2} dudv \right. \\
&\quad \left. + \mathbf{1}_{M=0} \int_0^t \int_0^u (u-v)^{2H-2} dv du + \frac{1}{M} \mathbf{1}_{M > 0} \int_0^t e^{Mu} \int_0^u (u-v)^{2H-2} dv du \right) \\
&\leq \alpha_H \sigma^2 \frac{\|b''\|_\infty \|\varphi'\|_\infty}{2} \varepsilon (1 \vee e^{Mt}) \left( \frac{1}{M^2} \mathbf{1}_{M < 0} \left( \frac{t^{2H-1}}{2H-1} - \int_0^t e^{Mv} (t-v)^{2H-2} dv \right) \right. \\
&\quad \left. + \frac{1}{2H} \mathbf{1}_{M=0} \int_0^t u^{2H} du + \frac{1}{M^2} \mathbf{1}_{M > 0} \left( \frac{e^{Mt} t^{2H-1}}{2H-1} - \int_0^t e^{Mu} u^{2H-2} du \right) \right) \\
&\leq \alpha_H \sigma^2 \frac{\|b''\|_\infty \|\varphi'\|_\infty}{2} \varepsilon t^{2H-1} \mathbf{m}_{H,M}(t)
\end{aligned}$$

where

$$\mathbf{m}_{H,M}(t) = \frac{1}{M^2(2H-1)} \mathbf{1}_{M < 0} + \frac{t^2}{2H(2H+1)} \mathbf{1}_{M=0} + \frac{e^{2Mt}}{M^2(2H-1)} \mathbf{1}_{M > 0}.$$

**6.7. Proof of Corollary 4.4.** First of all,

$$\mathbb{E}(\|\tilde{b}_{m,\varepsilon} - b_A\|_{f_T}^2) \leq 2\mathbb{E}(\|\hat{b}_{m,\varepsilon} - \hat{b}_m\|_{f_T}^2 \mathbf{1}_{\hat{\Lambda}(m)}) + 2\mathbb{E}(\|\tilde{b}_m - b_A\|_{f_T}^2)$$

and

$$\begin{aligned}
\mathbb{E}(\|\tilde{b}_m - b_A\|_{f_T}^2) &\leq \left( 1 + 4 \frac{3 \log(3/2) - 1}{(7 + \kappa) \log(NT)} \right) \inf_{\tau \in S_m} \|\tau - b_A\|_{f_T}^2 \\
&\quad + \frac{8}{NT} \text{trace}(\mathbf{\Phi}(m)^{-1} \mathbf{\Phi}(m, \sigma)) + \bar{\mathfrak{c}}_{\rho, \kappa, \sigma, b} \frac{\bar{\mathfrak{m}}_{2,H,M}(T)}{NT}
\end{aligned}$$

by Corollary 3.5. So, let us find a suitable control for  $\mathbb{E}(\|\hat{b}_{m,\varepsilon} - \hat{b}_m\|_{f_T}^2 \mathbf{1}_{\hat{\Lambda}(m)})$ . On the one hand,

$$\begin{aligned}
\|\hat{b}_{m,\varepsilon} - \hat{b}_m\|_{f_T}^2 &= \|\langle \hat{\theta}(m, \varepsilon) - \hat{\theta}(m), (\varphi_0, \dots, \varphi_{m-1})(\cdot) \rangle_{2,m}\|_{f_T}^2 \\
&\leq \|\hat{\theta}(m, \varepsilon) - \hat{\theta}(m)\|_{2,m}^2 \int_A \|(\varphi_0, \dots, \varphi_{m-1})(x)\|_{2,m}^2 f_T(x) dx \\
&\leq \|\hat{\theta}(m, \varepsilon) - \hat{\theta}(m)\|_{2,m}^2 L(m).
\end{aligned}$$

On the other hand, by Proposition 4.2, for every  $i \in \{1, \dots, N\}$  and  $j \in \{0, \dots, m-1\}$ ,

$$\left| S_{\varphi_j}^i(x_0, \varepsilon, T) - \int_0^T \varphi_j(X_{x_0}^i(s)) \delta X_{x_0}^i(s) \right| \leq \alpha_H \sigma^2 \frac{\|b''\|_\infty \|\varphi'\|_\infty}{2} \varepsilon T^{2H-1} \mathbf{m}_{H,M}(T).$$

Then,

$$\begin{aligned}
\|\hat{\theta}(m, \varepsilon) - \hat{\theta}(m)\|_{2,m}^2 &\leq \|\hat{\mathbf{\Phi}}(m)^{-1}\|_{\text{op}}^2 \|\hat{\mathbf{x}}(m, \varepsilon) - \hat{\mathbf{x}}(m)\|_{2,m}^2 \\
&= \|\hat{\mathbf{\Phi}}(m)^{-1}\|_{\text{op}}^2 \sum_{j=0}^{m-1} \left| \frac{1}{NT} \sum_{i=1}^N \left( S_{\varphi_j}^i(x_0, \varepsilon, T) - \int_0^T \varphi_j(X_{x_0}^i(s)) \delta X_{x_0}^i(s) \right) \right|^2 \\
&\leq \mathbf{c}_1 \|\hat{\mathbf{\Phi}}(m)^{-1}\|_{\text{op}}^2 \varepsilon^2 T^{4H-4} \mathbf{m}_{H,M}(T)^2 R(m)
\end{aligned}$$

with

$$\mathbf{c}_1 := \alpha_H^2 \sigma^4 \frac{\|b''\|_\infty^2}{4}.$$

Therefore, by the definition of  $\widehat{\Lambda}(m)$ , there exists a deterministic constant  $\mathbf{c}_{\rho, \kappa, \sigma, H, \bar{\mathbf{b}}} > 0$ , depending only on  $\rho, \kappa, \sigma, H$  and  $\bar{\mathbf{b}}$ , such that

$$\begin{aligned} 2\mathbb{E}(\|\widehat{\mathbf{b}}_{m, \varepsilon} - \widehat{\mathbf{b}}_m\|_{f_T}^2 \mathbf{1}_{\widehat{\Lambda}(m)}) &\leq 2\mathbf{c}_1 \mathbf{c}_{\kappa, T}^2 \left( \frac{NT}{\log(NT)} \right)^2 L(m)^{-1} \varepsilon^2 T^{4H-4} \mathbf{m}_{H, M}(T)^2 R(m) \\ &\leq \mathbf{c}_{\rho, \kappa, \sigma, H, \bar{\mathbf{b}}} L(m)^{\kappa-1} \left( \frac{NT}{\log(NT)} \right)^2 \frac{\mathbf{m}_{H, M}(T)^2}{T^{4-4H}} \varepsilon^2. \end{aligned}$$

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