



**HAL**  
open science

# Stochastic processes associated to multidimensional parabolic transmission problems in divergence form

Pierre Etoré, Miguel Martinez

► **To cite this version:**

Pierre Etoré, Miguel Martinez. Stochastic processes associated to multidimensional parabolic transmission problems in divergence form. 2020. hal-02531467v1

**HAL Id: hal-02531467**

**<https://hal.science/hal-02531467v1>**

Preprint submitted on 3 Apr 2020 (v1), last revised 20 Feb 2023 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Stochastic processes associated to multidimensional parabolic transmission problems in divergence form\*

Pierre Etoré<sup>†</sup>      Miguel Martinez<sup>‡</sup>

April 6, 2020

**Abstract** In this note we define and study the stochastic process  $X$  in link with a parabolic transmission operator  $(A, \mathcal{D}(A))$  in divergence form. The transmission operator involves a diffraction condition along a transmission boundary. To that aim we gather and clarify some results coming from the theory of Dirichlet forms as exposed in [6] and [14] for general divergence form operators. We show that  $X$  is a semimartingale and that it is solution of a stochastic differential equation involving partial reflections in the co-normal directions along the transmission boundary.

## 1 Introduction

In this note we aim at tying - with all the necessary rigor - various theoretical results that come from different approaches concerning the probabilistic study of divergence form operators. We also provide a probabilistic representation of the underlying process when the divergence operator is a transmission operator involving a transmission condition across some smooth interface : in this particular case, we show that the process is solution of a stochastic differential equation involving partial reflections in the co-normal directions along the transmission boundary. All of these results are natural but we could not find them in the existing literature and would like to record them in print with a proof. Besides, these results ground the foundations of a research project pointing at probabilistic numerical methods for transmission problems (see [4]).

Let  $a : \mathbb{R}^d \rightarrow \mathcal{M}_d(\mathbb{R})$  a measurable symmetric matrix valued coefficient satisfying the following ellipticity and boundedness condition **(E – B)** :

**Assumption 1.1. (E-B)** : *There exists  $\lambda, \Lambda \in (0, \infty)$  such that*

$$\forall x \in \mathbb{R}^d, \quad \forall \xi \in \mathbb{R}^d, \quad \lambda |\xi|^2 \leq \xi^* a(x) \xi \leq \Lambda |\xi|^2. \quad (1)$$

Let us associate to the coefficient  $a$  the following unbounded operator  $A : \mathcal{D}(A) \subset L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  defined by

$$\mathcal{D}(A) = \left\{ u \in H^1(\mathbb{R}^d) \text{ with } \sum_{i,j=1}^d D_i(a_{ij} D_j u) \in L^2(\mathbb{R}^d) \right\}$$

and

$$\forall u \in \mathcal{D}(A), \quad Au = \sum_{i,j=1}^d D_i(a_{ij} D_j u).$$

There exists a closed symmetric Dirichlet form  $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$  and its corresponding semigroup  $(T_t)$  on  $L^2(\mathbb{R}^d)$  that are naturally in link with  $(A, \mathcal{D}(A))$ .

We define rigorously these objects and study their relations in Section 2. Using the spectral resolution of the identity associated to  $(A, \mathcal{D}(A))$ , we study the regularity in  $t$  of  $\mathcal{E}(T_t f, g)$ ,  $f \in L^2(\mathbb{R}^d)$ ,  $g \in \mathcal{D}[\mathcal{E}]$  (Subsection

---

\*Supported by the Labex Bézout.

<sup>†</sup>Université Grenoble-Alpes, LJK. ETORE@UNIV-GRENOBLE-ALPES.FR

<sup>‡</sup>Université Gustave Eiffel, LAMA France. MIGUEL.MARTINEZ@U-PEM.FR

2.1). This permits to establish rigorously the connection with the results of D.W. Stroock in [14] (Subsection 2.2) which are exposed in the  $C_b(\mathbb{R}^d)$  setting (Feller semigroup) and to assert the validity of Aronson's estimates for the transition function of  $(T_t)$  (see [1], [14], [2]).

Finally, in Section 3 we focus on the case where  $(A, \mathcal{D}(A))$  is a transmission operator and provide a Skorokhod representation of the Hunt process  $X$  associated to  $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$  in this particular case (Subsection 3.2).

## 2 Dirichlet form and Markovian semigroup associated to general elliptic divergence form operators

### 2.1 Definitions and first properties

To the coefficient matrix  $a$ , we may associate a closed symmetric Dirichlet form  $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$  defined on  $L^2(\mathbb{R}^d)$  by

$$\begin{cases} \mathcal{D}[\mathcal{E}] &= H^1(\mathbb{R}^d), \\ \mathcal{E}(u, v) &= \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij} D_j u D_i v, \quad u, v \in \mathcal{D}[\mathcal{E}] \end{cases}$$

(see [6], p111). This closed symmetric Dirichlet form is the starting point of our construction.

On the underlying Hilbert space  $L^2(\mathbb{R}^d)$ , we denote within this subsection by  $(A, \mathcal{D}(A))$  the (unique) self-adjoint operator associated to  $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$  and characterized by

$$\begin{cases} \mathcal{D}(A) \subset \mathcal{D}[\mathcal{E}], \\ \mathcal{E}(u, v) = -\langle Au, v \rangle_{L^2(\mathbb{R}^d)}, \quad u \in \mathcal{D}(A), v \in \mathcal{D}[\mathcal{E}] \end{cases}$$

([6], Theorem 1.3.1 and Corollary 1.3.1 p.21).

We aim at identifying this operator - as expected it will turn out that  $(A, \mathcal{D}(A))$  is nothing else than the operator defined in the Introduction, therefore the common notation.

By the very definition of  $(A, \mathcal{D}(A))$ , we have for any  $f \in \mathcal{D}(A)$  and any  $g \in C_c^\infty(\mathbb{R}^d)$

$$-\langle Af, g \rangle_{L^2(\mathbb{R}^d)} = \mathcal{E}(f, g) = \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij} D_j f D_i g = -\left\langle \sum_{i,j=1}^d D_i(a_{ij} D_j f), g \right\rangle_{H^{-1}(\mathbb{R}^d), H^1(\mathbb{R}^d)}$$

where  $\sum_{i,j=1}^d D_i(a_{ij} D_j f)$  is understood in the distributional sense as an element of  $H^{-1}(\mathbb{R}^d)$ . But as  $Af \in L^2(\mathbb{R}^d)$  by the definition of  $\mathcal{D}(A)$  the above equality shows that  $\sum_{i,j=1}^d D_i(a_{ij} D_j f) \in L^2(\mathbb{R}^d)$  (for any  $f \in \mathcal{D}(A)$ ).

Thus, it is proved that  $\mathcal{D}(A) \subseteq \{f \in H^1(\mathbb{R}^d) \text{ with } \sum_{i,j=1}^d D_i(a_{ij} D_j f) \in L^2(\mathbb{R}^d)\}$ .

In turn (by the density of  $C_c^\infty(\mathbb{R}^d)$  in  $L^2(\mathbb{R}^d)$ ) the equality permits to identify for any  $f \in \mathcal{D}(A)$ ,

$$Af = \sum_{i,j=1}^d D_i(a_{ij} D_j f).$$

Let us now prove the reverse inclusion  $\{f \in H^1(\mathbb{R}^d) \text{ with } \sum_{i,j=1}^d D_i(a_{ij} D_j f) \in L^2(\mathbb{R}^d)\} \subseteq \mathcal{D}(A)$ .

Let  $f \in \{f \in H^1(\mathbb{R}^d) \text{ with } \sum_{i,j=1}^d D_i(a_{ij} D_j f) \in L^2(\mathbb{R}^d)\}$ . By the symmetry of the coefficient matrix  $a$  and integration by parts, it is not hard to prove that for any  $v \in \mathcal{D}(A)$ ,

$$\langle Av, f \rangle_{L^2(\mathbb{R}^d)} = -\mathcal{E}(v, f) = -\sum_{j,i=1}^d \int_{\mathbb{R}^d} a_{ji} D_i f D_j v = \left\langle \sum_{j,i} D_j(a_{ji} D_i f), v \right\rangle_{L^2(\mathbb{R}^d)}$$

and in particular  $f \in \mathcal{D}(A^*) \stackrel{\text{def}}{=} \{g \in L^2(\mathbb{R}^d) \mid \exists h_g \in L^2(\mathbb{R}^d) \text{ s.t. } \langle Av, g \rangle = \langle v, h_g \rangle, \forall v \in \mathcal{D}(A)\}$  (see [9]). So that we get the reverse inclusion

$$\{f \in H^1(\mathbb{R}^d) \text{ with } \sum_{i,j=1}^d D_i(a_{ij} D_j f) \in L^2(\mathbb{R}^d)\} \subseteq \mathcal{D}(A^*) = \mathcal{D}(A)$$

where the equality comes from the fact that  $(A, \mathcal{D}(A))$  is self-adjoint. Finally, we have proved

$$\mathcal{D}(A) = \{f \in H^1(\mathbb{R}^d) \text{ with } \sum_{i,j=1}^d D_i(a_{ij}D_j f) \in L^2(\mathbb{R}^d)\} \quad (2)$$

and  $(A, \mathcal{D}(A))$  is fully identified as being the same operator of the Introduction 1.

Note that since  $a$  is only assumed to be measurable,  $C_c^\infty(\mathbb{R}^d)$  - which is a core for the Dirichlet form  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  - is not even a subset of  $\mathcal{D}(A)$ .

We now turn to the study of the spectral resolution and the semigroup associated to  $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$  and  $(A, \mathcal{D}(A))$ . For the sake of conciseness we denote  $(\cdot, \cdot) = \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^d)}$  and  $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^d)}$  till the end of the section.

Since  $(-A, \mathcal{D}(A))$  is a self-adjoint operator on the Hilbert space  $L^2(\mathbb{R}^d)$  that is non-negative definite, it admits a spectral resolution of the identity  $\{E_\gamma : \gamma \in [0, \infty)\}$ . For any  $\gamma \geq 0$  the operator  $E_\gamma : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  is a self-adjoint projection operator with  $(E_\gamma f, f) \geq 0$ ,  $f \in L^2(\mathbb{R}^d)$ , and the  $E_\gamma$ 's form a spectral family with in particular  $E_\mu E_\gamma = E_{\mu \wedge \gamma}$ , (see [6] p18 for a list of properties). The link with  $(-A, \mathcal{D}(A))$  is through

$$(-Af, g) = \int_{[0, \infty)} \gamma d(E_\gamma f, g) \quad \forall f \in \mathcal{D}(A), g \in L^2(\mathbb{R}^d)$$

and  $\mathcal{D}(A) = \left\{f \in L^2(\mathbb{R}^d) : \int_{[0, \infty)} \gamma^2 d(E_\gamma f, f) < \infty\right\}$  (see [6] paragraph 1.3.4 p.18).

Consequently, the family of operators  $\{T_t \stackrel{\text{def}}{=} e^{tA} : t > 0\}$  is a strongly continuous semigroup of self-adjoint contractions acting on  $L^2(\mathbb{R}^d)$  ([6] Lemma 1.3.2 p.19) and

$$(T_t f, g) = \int_{[0, \infty)} e^{-\gamma t} d(E_\gamma f, g) \quad \forall f \in L^2(\mathbb{R}^d), g \in L^2(\mathbb{R}^d).$$

Note that for any  $\gamma \geq 0$ ,  $t > 0$ , and any functions  $f \in L^2(\mathbb{R}^d)$  and  $g \in L^2(\mathbb{R}^d)$ , we have the commutation property

$$\begin{aligned} (T_t E_\gamma f, g) &= (E_\gamma f, T_t g) \\ &= \int_{[0, \infty)} e^{-\xi t} d_\xi(E_\gamma f, E_\xi g) = \int_{[0, \gamma]} e^{-\xi t} d_\xi(E_\xi E_\gamma f, g) + \int_{[\gamma, \infty)} e^{-\xi t} d_\xi(E_\xi E_\gamma f, g) \\ &= \int_{[0, \gamma]} e^{-\xi t} d_\xi(E_\xi f, g) \\ &= \int_{[0, \infty)} e^{-\xi t} d_\xi(E_\xi f, E_\gamma g) \\ &= (T_t f, E_\gamma g) = (E_\gamma T_t f, g). \end{aligned}$$

Note also that for any  $f \in L^2(\mathbb{R}^d)$  and any  $t > 0$ ,

$$\begin{aligned} \int_{[0, \infty)} \gamma^2 d(E_\gamma T_t f, T_t f) &= \int_{[0, \infty)} \gamma^2 d_\gamma \left( \int_{[0, \infty)} e^{-\xi t} d_\xi(E_\gamma E_\xi f, T_t f) \right) \\ &= \int_{[0, \infty)} \gamma^2 d_\gamma \left( \int_{[0, \infty)} e^{-\xi t} d_\xi \left( \int_{[0, \infty)} e^{-\theta t} d_\theta(E_\gamma E_\xi f, E_\theta f) \right) \right) \\ &= \int_{[0, \infty)} \gamma^2 e^{-2\gamma t} d_\gamma(E_\gamma f, f) \\ &\leq \frac{4}{t^2} e^{-2} \int_{[0, \infty)} e^{-\gamma t} d_\gamma(E_\gamma f, f) = \frac{4}{t^2} e^{-2} (T_t f, f) \leq \frac{4}{t^2} e^{-2} \|f\|^2 < +\infty, \end{aligned}$$

where we have used the spectral family property, the associativity of the Stieltjes integral and the inequality  $\gamma^2 e^{-\gamma t} \leq 4e^{-2}/t^2$ . The above inequality ensures that  $T_t f \in \mathcal{D}(A)$  for any  $t > 0$ .

From the fact that  $|\frac{d}{dt}e^{-\gamma t}| \leq \gamma$  is integrable w.r.t.  $d(E_\gamma h, g)$  whenever  $h \in \mathcal{D}(A)$ , we deduce from the commutation property that for any  $f, g \in L^2(\mathbb{R}^d)$  and for any  $s > 0$

$$-\frac{d}{dt}(T_t f, T_s g) = \int_{[0, \infty)} \gamma e^{-\gamma t} d(E_\gamma f, T_s g) \xrightarrow{t \searrow 0^+} \int_{[0, \infty)} \gamma d(E_\gamma T_s f, g) = (-AT_s f, g)$$

where the limit exists and is well defined (since we have shown that  $T_s f \in \mathcal{D}(A)$ ).

If moreover  $g \in \mathcal{D}[\mathcal{E}]$  then

$$\begin{aligned} -\frac{d}{ds}(T_s f, g) &= -\frac{d}{dt}(T_{s+t} f, g)|_{t=0^+} = -\frac{d}{dt}(T_t f, T_s g)|_{t=0^+} \\ &= (-AT_s f, g) = \mathcal{E}(T_s f, g). \end{aligned} \quad (3)$$

And since  $-\frac{d}{ds}(T_s f, g) = -\frac{d}{ds}(T_s g, f)$  by the symmetry property of  $T_s$ , we deduce that  $\mathcal{E}(T_s f, g) = \mathcal{E}(f, T_s g)$  for any  $f, g \in \mathcal{D}[\mathcal{E}]$ .

Consequently, for any  $f \in \mathcal{D}[\mathcal{E}]$  and using the ellipticity of the coefficient matrix  $a$ ,

$$\begin{aligned} \lambda \|\nabla T_s f\|^2 &\leq \mathcal{E}(T_s f, T_s f) = \mathcal{E}(T_{2s} f, f) = (-AT_{2s} f, f) \\ &= \int_{[0, \infty)} \gamma e^{-2\gamma s} d(E_\gamma f, f) \leq \frac{e^{-1}}{s} \int_{[0, \infty)} e^{-\gamma s} d(E_\gamma f, f) = \frac{e^{-1}}{s} (T_s f, f) \leq \frac{\|f\|^2}{s}, \end{aligned}$$

from which we deduce the fundamental estimate

$$\|\nabla T_s f\| \leq \frac{\|f\|}{\sqrt{\lambda s}}, \quad \forall s > 0. \quad (4)$$

In turn this estimate implies that for any  $f \in L^2(\mathbb{R}^d)$ ,  $g \in \mathcal{D}[\mathcal{E}]$ , the function

$$s \mapsto \mathcal{E}(T_s f, g) \text{ is integrable on } (0, t],$$

and from (3) and the right continuity of  $s \mapsto T_s f$  at time  $s = 0^+$  (one may extend  $T_0 f = f$  as long as no differentiation of  $s \mapsto T_s f$  is implied at  $s = 0^+$  when  $f \notin \mathcal{D}(A)$ ), we deduce the integrated version of (3) namely  $\forall f \in L^2(\mathbb{R}^d)$ ,  $\forall g \in \mathcal{D}[\mathcal{E}]$ ,

$$(T_t f, g) - (f, g) = -\int_0^t \sum_{i,j=1}^d (a_{ij} D_j T_s f, D_i g) ds = -\int_0^t \mathcal{E}(T_s f, g) ds, \quad t \in (0, \infty). \quad (5)$$

## 2.2 Link with the results of D.W. Stroock [14]

In his celebrated article *Diffusion semigroups corresponding to uniformly elliptic divergence form operators* D.W. Stroock constructs via a regularization procedure a Feller continuous semigroup  $\{P_t : t > 0\}$  associated to  $a$  with the properties that (with our notations)

1. the map  $t \in [0, \infty) \mapsto P_t \phi \in H^1(\mathbb{R}^d)$  is a weakly continuous map for each  $\phi \in C_c^\infty(\mathbb{R}^d)$ .
2.  $\forall \phi, \psi \in C_c^\infty(\mathbb{R}^d)$ ,

$$(P_t \phi, \psi) - (\phi, \psi) = -\int_0^t (a \nabla P_s \phi, \nabla \psi) ds = -\int_0^t \mathcal{E}(P_s \phi, \psi) ds, \quad t \in (0, \infty). \quad (6)$$

(Nota : please note that there is a sign error in the original version of [14]).

In fact,  $\{P_t : t > 0\}$  determines a unique strongly continuous semigroup  $\{\bar{P}_t : t > 0\}$  of self-adjoint contractions on  $L^2(\mathbb{R}^d)$ .

The aim of this subsection is to prove the following equality, which clarifies the relationship between the results obtained in [14] and the those provided by the theory of Dirichlet forms [6].

**Proposition 2.1.**

$$\{\bar{P}_t : t > 0\} = \{T_t : t > 0\} \quad \text{on } L^2(\mathbb{R}^d). \quad (7)$$

*Proof.* The semigroup  $\{\bar{P}_t : t > 0\}$  is strongly continuous on  $H^1(\mathbb{R}^d)$ . Moreover, for each  $t > 0$ ,  $\bar{P}_t$  maps  $L^2(\mathbb{R}^d)$  into  $H^1(\mathbb{R}^d)$  and for each  $f \in H^1(\mathbb{R}^d) = \mathcal{D}[\mathcal{E}]$ , we have the fundamental estimate

$$\|\nabla \bar{P}_s f\| \leq \frac{1}{\sqrt{\lambda}} \left( \frac{\|f\|}{\sqrt{s}} \right) \wedge \|\nabla f\|, \quad \forall s > 0. \quad (8)$$

(See [14] Theorem II.3.1. p.341).

This estimate implies that for for each  $f, g \in \mathcal{D}[\mathcal{E}]$  and any  $t, s > 0$ ,

$$|\mathcal{E}(\bar{P}_t f, g) - \mathcal{E}(\bar{P}_s f, g)| \leq \Lambda \|\nabla g\| \frac{\|\bar{P}_{t \vee s - t \wedge s} f - f\|}{\sqrt{\lambda(t \wedge s)}} \xrightarrow{s \rightarrow t} 0,$$

which ensures the continuity of  $s \mapsto \mathcal{E}(\bar{P}_s \phi, \psi)$  for any  $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$ . Since  $(\bar{P}_t)_{t>0}$  and  $(P_t)_{t>0}$  coincide on  $C_c^\infty(\mathbb{R}^d)$ , we may differentiate in (6) (as long as  $t > 0$ ) to find that

$$\frac{d}{dt}(\bar{P}_t \phi, \psi) = -\mathcal{E}(\bar{P}_t \phi, \psi), \quad t \in (0, \infty). \quad (9)$$

This has to be compared to (3).

Let us now justify rigorously that for any  $t > 0$ ,  $s \in (0, t)$  and  $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$ ,

$$\frac{d}{ds}(T_s \phi, \bar{P}_{t-s} \psi) = \frac{d}{du}(T_u \phi, \bar{P}_{t-s} \psi)|_{u=s} - \frac{d}{du}(T_s \phi, \bar{P}_{t-u} \psi)|_{u=s}. \quad (10)$$

We have for sufficiently small  $0 \leq h < t - s$  and using the strong continuity of  $(\bar{P}_t)_{t>0}$

$$\begin{aligned} |(T_{s+h} \phi - T_s \phi, \bar{P}_{t-s+h} \psi - \bar{P}_{t-s} \psi)| &\leq \|\bar{P}_{t-s+h} \psi - \bar{P}_{t-s} \psi\| \|T_{s+h} \phi - T_s \phi\| \\ &\leq \varepsilon_\psi(h) \left( \int_{[0, \infty)} e^{-2\gamma s} (e^{-\gamma h} - 1)^2 d(E_\gamma \phi, \phi) \right)^{1/2} \\ &\leq \varepsilon_\psi(h) \left( \int_{[0, \infty)} e^{-2\gamma s} (\gamma h)^2 d(E_\gamma \phi, \phi) \right)^{1/2} \\ &\leq h \varepsilon_\psi(h) \left( \int_{[0, \infty)} e^{-\gamma s} (\gamma^2 e^{-\gamma s}) d(E_\gamma \phi, \phi) \right)^{1/2} \\ &\leq h \varepsilon_\psi(h) \frac{2e^{-1}}{s} \|\phi\|, \end{aligned}$$

where as usual  $\varepsilon_\psi(\cdot)$  denotes some positive continuous function vanishing at zero. We deduce that

$$\frac{1}{h} |(T_{s+h} \phi - T_s \phi, \bar{P}_{t-s+h} \psi - \bar{P}_{t-s} \psi)| \xrightarrow{h \rightarrow 0} 0,$$

implying (10).

Hence, from (10) and applying (3) and (9), we have that

$$\frac{d}{ds}(T_s \phi, \bar{P}_{t-s} \psi) = 0, \quad s \in (0, t). \quad (11)$$

Integrating the identity (11) on  $(0, t)$  and using the time continuity of both semigroups  $(T_t)$  and  $(\bar{P}_t)$  up to time  $s = 0+$  gives

$$(T_t \phi, \psi) = (\phi, \bar{P}_t \psi) = (\bar{P}_t \phi, \psi) \quad (12)$$

which holds for any  $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$ . Since  $C_c^\infty(\mathbb{R}^d)$  is dense in  $L^2(\mathbb{R}^d)$ , using the strong continuity of both semigroups  $(T_t)$  and  $(\bar{P}_t)$ , we finally deduce from (12) the identification (7).  $\square$

Consequently, all results in [14] that are valid for  $\{\bar{P}_t : t > 0\}$  are true for  $\{T_t : t > 0\}$ . For example, identifying abusively  $\{T_t : t > 0\}$  with its Feller restriction  $\{P_t : t > 0\}$  on  $C_c^\infty(\mathbb{R}^d)$ , we deduce that there is a  $p \in C((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d)$  such that

$$[T_t \phi](x) = \int_{\mathbb{R}^d} \phi(y) p(t, x, y) dy, \quad \ell(dx) - \text{a.e.}, \quad \phi \in C_c^\infty(\mathbb{R}^d). \quad (13)$$

Moreover, the fundamental function  $p$  satisfies the well-known Aronson's estimates for the fundamental solutions of elliptic divergence form operators, namely there exists a constant  $M(\lambda, \Lambda, d) \in [1, \infty)$  such that

$$\frac{1}{Mt^{d/2}} \exp(-M|x-y|^2/t) \leq p(t, x, y) \leq \frac{M}{t^{d/2}} \exp(-|x-y|^2/Mt). \quad (14)$$

Finally, we have the convergence result of [14] (Theorem II.3.1. p.341) that we state roughly without introducing the necessary notations (see [14] for details) : if  $\{a_n\}_1^\infty \subset \mathcal{A}(\lambda, \Lambda)$  and  $a_n \rightarrow a$  almost everywhere, then  $p^n(t, x, y) \rightarrow p(t, x, y)$  uniformly on compacts (in  $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ ) and for each  $t \in [0, \infty)$  and  $\phi \in C_c^\infty(\mathbb{R}^d)$ ,  $T_t^n \phi \rightarrow T_t \phi$  in  $H^1(\mathbb{R}^d)$ .

### 3 Stochastic representation of transmission operators in divergence form

#### 3.1 An application of the Revuz correspondence for additive functionals

Since  $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$  is a regular Dirichlet form (with the space  $\mathcal{D}[\mathcal{E}] \cap C_c(\mathbb{R}^d)$  or  $C_c^\infty(\mathbb{R}^d)$  as a special standard core, see e.g. Exercice 1.4.1 in [6]), we are in position to apply Theorem 7.2.1 p. 380 of [6].

We may associate to  $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$  and its corresponding semigroup  $(T_t)$  a Hunt process, symmetric w.r.t the Lebesgue measure  $\ell(dx)$  on  $\mathbb{R}^d$ . We shall denote by  $\mathbb{M} = (\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, (X_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in \mathbb{R}^d})$  this Hunt process, with  $X = (X^1, \dots, X^d)$ . The correspondence with  $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$  and  $(T_t)$  is through

$$\mathbb{E}^x[f(X_t)] = T_t f(x), \quad \forall f \in L^2(\mathbb{R}^d), \quad \forall t \geq 0, \quad \forall x \in \mathbb{R}^d \quad (15)$$

(see the discussion p160 in [6], at the beginning of Section 4.2).

Let us also denote by  $\{R_\alpha : \alpha > 0\}$  the Markovian resolvent kernel of the Markovian transition function  $\{\hat{p}(t, x, dy) := p(t, x, y) dy : t > 0\}$ . Then, for any  $\alpha > 0$ ,  $f \in \mathcal{B}_b(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$ ,  $R_\alpha f(x) = \int_{\mathbb{R}^d} r_\alpha(x, y) f(y) dy$  with  $r_\alpha(x, y) = \int_0^\infty e^{-\alpha t} p(t, x, y) dt$ .

Denote by  $S$  the set of positive Radon measures on  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ . For  $\mu \in S$  define  $R_1 \mu(x) = \int_{\mathbb{R}^d} r_1(x, y) \mu(dy)$  ( $x \in \mathbb{R}^d$ ) and introduce the subset of *finite energy measures*

$$\begin{aligned} S_0 &:= \left\{ \mu \in S : \exists C > 0, \quad \forall v \in \mathcal{D}[\mathcal{E}] \cap C_c(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} |v(x)| \mu(dx) \leq C (\mathcal{E}(v, v) + (v, v))^{1/2} \right\} \\ &= \left\{ \mu \in S : \sup_{v \in \mathcal{D}[\mathcal{E}] \cap C_c(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{|v(x)|}{\|v\|_{\mathcal{E}_1}} \mu(dx) < \infty \right\}, \end{aligned}$$

(where we follow the notations of [6]).

Finally, introduce

$$S_{00} := \{ \mu \in S_0 : \mu(\mathbb{R}^d) < \infty, \quad \|R_1 \mu(\cdot)\|_\infty < \infty \}.$$

Let us denote respectively by  $\mathbf{A}_c^+$  and  $\mathbf{A}_{c,1}^+$  the families of all Positive Continuous Additive Functionals (PCAF in short) (resp. the family of all PCAF in the strict sense) associated to  $\mathbb{M}$  (for the distinction between  $\mathbf{A}_c^+$  and  $\mathbf{A}_{c,1}^+$ , see [6] the introduction of Section 5.1).

The *Revuz correspondence* asserts that there is a one-to-one correspondence (up to equivalence of processes) between  $\mathbf{A}_c^+$  and  $S$ . This correspondence permits to construct for any  $\mu \in S_{00}$  a unique PCAF in the strict sense  $A \in \mathbf{A}_{c,1}^+$  such that

$$\forall x \in \mathbb{R}^d, \quad \mathbb{E}^x \int_0^\infty e^{-t} dA_t = R_1 \mu(x). \quad (16)$$

(see for e.g. Theorem 5.1.4 in [6]).

In order to get a bijective map, introduce a new subset  $S_1$  of  $S$  defined by  $\mu \in S_1$  if there exists a sequence  $(E_n)_{n \geq 0}$  of Borel finely open sets increasing to  $\mathbb{R}^d$  satisfying that  $\mathbb{I}_{E_n} \cdot \mu \in S_{00}$  for each  $n$ . Then, there is a one-to-one correspondence between  $S_1$  and  $\mathbf{A}_{c,1}^+$  (up to equivalence) which is given by relation (16) whenever  $\mu \in S_{00}$ . The set of measures  $S_1$  is called the set of *smooth measures (in the strict sense)*.

Let us introduce  $\mathcal{D}[\mathcal{E}]_b$  (resp.  $\mathcal{D}[\mathcal{E}]_{b,\text{loc}}$ ) the space of essentially bounded functions belonging to  $\mathcal{D}[\mathcal{E}]$  (resp. locally to  $\mathcal{D}[\mathcal{E}]$ ). A function  $u$  is in  $\mathcal{D}[\mathcal{E}]_{b,\text{loc}}$  if for any compact set  $G$ , there exists a bounded function  $\omega$  such that  $u = \omega$ ,  $\ell(dx)$ -a.e. on  $G$ .

For  $u \in \mathcal{D}[\mathcal{E}]_b$ , we may associate a unique positive Radon measure  $\mu_{\langle u \rangle} \in S$ , satisfying

$$\int_{\mathbb{R}^d} f(x) \mu_{\langle u \rangle}(dx) = 2\mathcal{E}(uf, u) - \mathcal{E}(u^2, f), \quad \forall f \in \mathcal{D}[\mathcal{E}] \cap C_c(\mathbb{R}^d). \quad (17)$$

If  $u \in \mathcal{D}[\mathcal{E}]_{b,\text{loc}}$ , we may construct  $\mu_{\langle u \rangle} \in S$  with the help of a sequence  $(G_n)_{n \geq 0}$  of relatively compact open sets such that  $\overline{G_n} \subset G_{n+1}$  and  $\bigcup_{n \geq 0} G_n = \mathbb{R}^d$ . Let  $(u_n)_{n \geq 0}$  a sequence of functions in  $\mathcal{D}[\mathcal{E}]_b$  satisfying  $u_n = u$  on  $G_n$ . There is no ambiguity in defining  $\mu_{\langle u \rangle} = \mu_{\langle u_n \rangle}$  on  $G_n$  because the construction is consistent (since  $\mu_{\langle u_n \rangle} = \mu_{\langle u_{n+1} \rangle}$  on  $G_n$ ). For an account on the above assertions, please refer to [6] Section 3.2.

Note that obviously  $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$  is strong local, so we may apply Theorem 5.5.5 in [6].

Suppose that a function  $u$  satisfies the following conditions :

1.  $u \in \mathcal{D}[\mathcal{E}]_{b,\text{loc}}$ ,  $u$  is finely continuous on  $\mathbb{R}^d$ .
2.  $\mathbb{I}_G \cdot \mu_{\langle u \rangle} \in S_{00}$  for any relatively compact open set  $G$ .
3.  $\exists \varrho = \varrho^{(1)} - \varrho^{(2)}$  with  $\mathbb{I}_G \cdot \varrho^{(1)}, \mathbb{I}_G \cdot \varrho^{(2)} \in S_{00}$  for any relatively compact open set  $G$  and

$$\mathcal{E}(u, v) = (\varrho, v), \quad \forall v \in C_c^\infty(\mathbb{R}^d).$$

(Note that even though  $u$  is not formally in  $\mathcal{D}[\mathcal{E}]$ , the quantity  $\mathcal{E}(u, v)$  is well-defined because  $v$  has compact support and  $u \in \mathcal{D}[\mathcal{E}]_{b,\text{loc}}$ ).

Let  $A^{(1)}$ ,  $A^{(2)}$ , and  $B$  be PCAF's in the strict sense with Revuz measures  $\varrho^{(1)}$ ,  $\varrho^{(2)}$ , and  $\mu_{\langle u \rangle}$  respectively. Then, Theorem 5.5.5 in [6] asserts that

$$u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}, \quad \mathbb{P}^x - a.s., \quad \forall x \in \mathbb{R}^d. \quad (18)$$

Here,

$$N^{[u]} = -A^{(1)} + A^{(2)}, \quad \mathbb{P}^x - a.s., \quad \forall x \in \mathbb{R}^d \quad (19)$$

and  $M^{[u]}$  is a local Additive Functional in the strict sense such that for any relatively compact set  $G$ ,

$$\mathbb{E}^x M_{t \wedge \tau_G}^{[u]} = 0, \quad \forall x \in G$$

and

$$\mathbb{E}^x \left[ (M_{t \wedge \tau_G}^{[u]})^2 \right] = \mathbb{E}^x B_{t \wedge \tau_G}, \quad \forall x \in G,$$

where  $\tau_G = \inf\{s > 0 : X_s \notin G\}$  stands for the first leaving time from  $G$  (with the convention  $\inf \emptyset = \infty$ ) and  $B$  denotes the PCAF in the strict sense with Revuz measure  $\mu_{\langle u \rangle}$ .

### 3.2 Skorokhod representation of the Hunt process associated to a transmission operator in divergence form

Consider  $\mathbb{R}^d = \bar{D}_+ \cup D_-$  with  $D_+$  and  $D_-$  two open connected subdomains separated by a transmission boundary  $\Gamma$  that is to say

$$\Gamma = \bar{D}_+ \cap \bar{D}_-.$$

We denote

$$D = D_+ \cup D_- = \mathbb{R}^d \setminus \Gamma \subset \mathbb{R}^d.$$



For a point  $x \in \Gamma$  we denote by  $\nu(x) \in \mathbb{R}^d$  the unit normal to  $\Gamma$  at point  $x$ , pointing to  $D_+$ .

In the following, " $f \in C^p(\bar{D}_+; \mathbb{R}) \cap C^p(\bar{D}_-; \mathbb{R})$ " means that the restriction  $f_+$  of the real valued function  $f$  to  $D_+$  (and the restriction  $f_-$  of  $f$  to  $D_-$ ) coincides on  $D_+$  (resp.  $D_-$ ) with a function  $\tilde{f}_+$  of class  $C^p(\mathbb{R}^d)$  (resp.  $\tilde{f}_-$ ).

Assume the  $a_{ij}$ 's satisfy  $(a_{\pm})_{ij} \in C(\bar{D}_{\pm}; \mathbb{R})$ . We may define then the co-normal vector fields  $\gamma_+(x) := a_+(x)\nu(x)$  and  $\gamma_-(x) := -a_-(x)\nu(x)$ , for  $x \in \Gamma$ .

We shall consider restricted operators and bilinear forms in the following sense. We define  $A_+ : H^1(D_+) \rightarrow H^{-1}(D_+)$  by

$$\forall v \in H^1(D_+), \quad A_+v = \sum_{i,j=1}^d D_i((a_+)_{ij}D_jv).$$

We define  $A_- : H^1(D_-) \rightarrow H^{-1}(D_-)$  in the same manner (note that we do not specify here any domain  $\mathcal{D}(A_{\pm})$ ). Further, we define

$$\mathcal{E}_{\pm}(u, v) = \sum_{i,j=1}^d \int_{D_{\pm}} (a_{\pm})_{ij} D_j u D_i v, \quad \forall u, v \in H^1(D_{\pm}).$$

We have, for  $u_{\pm} \in H^1(D_{\pm})$  with  $A_{\pm}u_{\pm} \in L^2(D_{\pm})$ ,

$$\mathcal{E}_{\pm}(u_{\pm}, v) = \int_{D_{\pm}} (-A_{\pm}u_{\pm})v, \quad \forall v \in H_0^1(D_{\pm}). \quad (20)$$

Imagine now that in (20) we wish to take the test function in  $H^1(D_{\pm})$  instead of  $H_0^1(D_{\pm})$ . There will still be a link between  $A_{\pm}$  and  $\mathcal{E}_{\pm}$ , but through Green type identities, involving conormal derivatives and boundary integrals.

We introduce a specific notation for the one-sided conormal derivatives on  $\Gamma$  of  $u \in L^2(\mathbb{R}^d)$  with  $u_{\pm} \in H^2(D_{\pm})$ . Provided the  $(a_{\pm})_{ij}$  are in  $C_b^1(\bar{D}_{\pm}; \mathbb{R})$  and  $\Gamma$  is bounded and Lipschitz we set

$$\mathcal{B}_{\nu}^{\pm} u = \nu^* \text{Tr}^{\pm}(a_{\pm} \nabla u_{\pm}) = \sum_{i=1}^d \sum_{j=1}^d \nu_i \text{Tr}^{\pm}((a_{\pm})_{ij} D_j u_{\pm}) \quad \text{on } \Gamma \quad (21)$$

where  $\text{Tr}^{\pm} : H^1(D_{\pm}) \rightarrow H^{1/2}(\Gamma)$  stand for the usual trace operators on  $\Gamma$ .

For  $g \in H^{-\frac{1}{2}}(\Gamma)$  and  $f \in H^{\frac{1}{2}}(\Gamma)$  we denote by  $(g, f)_{\Gamma}$  the action of  $g$  on  $f$ . If both  $f, g$  are in  $H^{\frac{1}{2}}(\Gamma)$  the quantity  $(g, f)_{\Gamma}$  coincides with the surface integral  $\int_{\Gamma} g f d\varsigma$ .

Let us recall the version of the Green identity that is used in the sequel.

**Proposition 3.1** (First Green identity, first version; [8], Lemma 4.1). *Assume  $\Gamma$  is bounded and  $C^2$ . Let  $u \in L^2(\mathbb{R}^d)$  with  $u_+ \in H^2(D_+)$  and  $u_- \in H^2(D_-)$ . Assume that the coefficients  $(a_{\pm})_{ij}$  are in  $C_b^1(\bar{D}_{\pm}; \mathbb{R})$ . Then*

$$\mathcal{E}_+(u_+, v) = \int_{D_+} (-A_+u_+)v - \left( \mathcal{B}_{\nu}^+ u, \text{Tr}^+(v) \right)_{\Gamma}, \quad \forall v \in H^1(D_+)$$

and

$$\mathcal{E}_-(u_-, v) = \int_{D_-} (-A_-u_-)v + \left( \mathcal{B}_{\nu}^- u, \text{Tr}^-(v) \right)_{\Gamma}, \quad \forall v \in H^1(D_-).$$

We have the following result.

**Theorem 3.2.** *Assume  $\Gamma$  is bounded and  $C^2$ . Assume that Assumption **(E – B)** is fulfilled and that for all  $1 \leq i, j \leq d$   $a_{ij} \in C_b^1(\bar{D}_+; \mathbb{R}) \cap C_b^1(\bar{D}_-; \mathbb{R})$  with  $a_{ij}$  possessing a possible discontinuity on  $\Gamma$ . Then, the Hunt process  $\mathbb{M}$  associated to  $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$  is a diffusion which possesses the following Skorokhod decomposition : for any  $k \in \{1, \dots, d\}$ ,*

$$\begin{aligned} X_t^k &= x_k + \int_0^t \sum_{j=1}^d \sigma_{kj}(X_s) dW_s^j + \int_0^t \sum_{j=1}^d \partial_j a_{kj}(X_s) \mathbb{1}_{X_s \in D} ds \\ &\quad - \frac{1}{2} \int_0^t \gamma_{+,k}(X_s) dK_s + \frac{1}{2} \int_0^t \gamma_{-,k}(X_s) dK_s, \quad t \geq 0, \mathbb{P}^x - a.s., \forall x = (x_1, \dots, x_d) \in \mathbb{R}^d. \end{aligned} \quad (22)$$

In the above equality  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  denotes the positive square-root of coefficient  $2a$  i.e. the positive matrix real valued coefficient satisfying

$$\sigma \sigma^*(x) = 2a(x), \quad \forall x \in D.$$

(Note that this coefficient exists because  $a(x)$  is non-negative definite for all  $x \in D$ ). The process  $W = (W^1, \dots, W^d)$  is a  $d$ -dimensional standard Brownian motion starting from zero and  $(K_t)_{t \geq 0}$  denotes the unique PCAF associated to the surface measure  $\zeta(d\xi) \in S$  on  $\Gamma$  through the Revuz correspondence. The process  $(K_t)$  increases only at times where  $X$  lies on  $\Gamma$ ,

$$\int_0^t \mathbb{1}_{X_s \in \Gamma} dK_s = K_t, \quad t \geq 0.$$

*Proof.* We apply the results of Theorem 5.5.5 in [6] in this context for the coordinate functions

$$p_k(x_1, \dots, x_d) := x_k \quad (k \in \{1, \dots, d\}).$$

and follow the ideas of [16] Theorem 5.2. Of course  $p_k \in \mathcal{D}[\mathcal{E}]_{b, \text{loc}}$  and  $p_k$  is finely continuous on  $\mathbb{R}^d$ . Let  $G$  a relatively compact open set containing  $\Gamma$  and a function  $f_k \in \mathcal{D}[\mathcal{E}]_b$  such that  $p_k = f_k$  on  $G$ . Let  $\langle M^{[f_k]} \rangle$  the square bracket of  $M^{[f_k]}$ . Then, an easy computation from (17) shows that the energy measure of  $M^{[f_k]}$  (the Revuz measure of  $\langle M^{[f_k]} \rangle$ ) is

$$\mu_{\langle f_k \rangle}(dy) = \mu_{\langle M^{[f_k]} \rangle}(dy) = \langle 2a(y) \nabla f_k(y), \nabla f_k(y) \rangle \ell(dy)$$

and we know that  $\mu_{\langle f_k \rangle} = \mu_{\langle p_k \rangle}$  on  $G$ . It is easy to show that  $\mathbb{1}_G \cdot \mu_{\langle p_k \rangle}$  is a finite Radon measure belonging to  $S_{00}$  and that  $\mu_{\langle p_k \rangle}$  is a smooth measure. Then, an easy computation from (16) shows that

$$\langle M^{[f_k]} \rangle_t = \int_0^t \langle 2a(X_s) \nabla f_k(X_s), \nabla f_k(X_s) \rangle ds, \quad k \in \{1, \dots, d\}$$

and by the well-known results on stochastic representation of martingales, there exists a  $d$  dimensional Brownian motion  $W = (W^1, \dots, W^d)$  such that

$$M_t^{[f_k]} = \int_0^t [\sigma(X_s) \nabla f_k(X_s)]^* dW_s, \quad \mathbb{P}^x - \text{a.s.} \quad \forall x \in \mathbb{R}^d, \quad k \in \{1, \dots, d\}$$

(see for e.g. [11] Chapter V Theorem 3.9 and the remark following its proof).

Moreover, for any  $v \in C_c^\infty(\mathbb{R}^d)$ , using the Green Identities of Proposition 3.1 and taking into account that  $v$  is of compact support, we have :

$$\begin{aligned} \mathcal{E}(f_k, v) &= \mathcal{E}_+(f_{k,+}, v) + \mathcal{E}_-(f_{k,-}, v) \\ &= \int_{D_+} (-A_+ f_{k,+}) v - \left( \mathcal{B}_\nu^+ f_{k,+}, \text{Tr}^+(v) \right)_\Gamma + \int_{D_-} (-A_- f_{k,-}) v + \left( \mathcal{B}_\nu^- f_{k,-}, \text{Tr}^-(v) \right)_\Gamma \\ &= \int_D \sum_{i,j=1}^d D_i (a_{ij}(y) D_j f_k(y)) v(y) \mathbb{1}_{y \in D} \ell(dy) \\ &\quad - \int_\Gamma \nu^* [\text{Tr}^+(a_+ \nabla f_{k,+}) - \text{Tr}^-(a_- \nabla f_{k,-})] \gamma(v) d\zeta \\ &= \int_D \sum_{j=1}^d \partial_j a_{kj}(y) v(y) \mathbb{1}_{y \in D} \ell(dy) - \int_\Gamma [\text{Tr}^+((a_+ \nu)_k) - \text{Tr}^-((a_- \nu)_k)] v d\zeta \\ &= \int_D \sum_{j=1}^d \partial_j a_{kj}(y) v(y) \mathbb{1}_{y \in D} \ell(dy) - \int_\Gamma [(\tilde{a}_+ \nu)_k - (\tilde{a}_- \nu)_k] v d\zeta \\ &= (\varrho_k^+, v) - (\varrho_k^-, v) \end{aligned}$$

with

$$\varrho_k^\pm(dy) := \sum_{j=1}^d [\partial_j a_{kj}(y)]^\pm \mathbb{1}_{y \in D} \ell(dy) + [(\gamma_-)_k - (\gamma_+)_k]^\pm(y) \mathbb{1}_{y \in \Gamma} \zeta(dy).$$

(here, the notation  $[a]^+$  (resp.  $[a]^-$ ) stands for the positive (resp. negative) part of some real number  $a$ ).

Let us now proceed to show that the measures  $\mathbb{I}_G \cdot \varrho_k^\pm$  belong to  $S_{00}$ .

Note that  $\|\partial_j a_{kj} \mathbb{I}_{D \cap G}\|_\infty < \infty$  and from the definition of  $S_{00}$  and the Revuz correspondence (16), it is not difficult to prove that the measures  $[\partial_j a_{kj}]_\pm(y) \mathbb{I}_{y \in D} \ell(dy)$  are smooth with their corresponding additive functional writing as  $\left( \int_0^t [\partial_j a_{kj}]_\pm(X_s) \mathbb{I}_{X_s \in D} ds \right)_{t \geq 0}$ .

We now turn to the surface measures  $\zeta_k^\pm(dy) := [(\gamma_-)_k - (\gamma_+)_k]^\pm(y) \mathbb{I}_{y \in \Gamma} \varsigma(dy)$ . It is well-known (see e.g. [5] p.134 3. (\*\*), (\*\*\*)) that there exists a universal constant  $C_0 > 0$ , depending only on the Lipschitz domain  $D_+$ , such that for all  $h \in C^1(\overline{D}_+)$ ,

$$\int_\Gamma |h(y)| \varsigma(dy) \leq C_0 \int_{D_+} (|\nabla h(x)| + |h(x)|) \ell(dx).$$

Thus, for all  $h \in \mathcal{D}[\mathcal{E}] \cap C_c(\mathbb{R}^d)$ , we have

$$\begin{aligned} \int_\Gamma |h(y)| \varsigma(dy) &\leq C_0 \int_{D_+} (|\nabla h(x)| + |h(x)|) \ell(dx) \\ &\leq C_0 \ell(D_+)^{1/2} \left( \int_{D_+} (|\nabla h(x)| + |h(x)|)^2 \ell(dx) \right)^{1/2} \\ &\leq C_0 (2\ell(D_+))^{1/2} \left( \int_{\mathbb{R}^d} (|\nabla h(x)|^2 + |h(x)|^2) \ell(dx) \right)^{1/2} \\ &\leq C_0 \sqrt{\frac{(2\ell(D_+))}{\lambda}} (\mathcal{E}(h, h) + (h, h))^{1/2} \end{aligned}$$

so that the surface measure  $\varsigma(dy)$  belongs to  $S_0$ . Since

$$\forall y \in \Gamma, \quad |[(\gamma_-)_k - (\gamma_+)_k]^\pm(y)| \leq 2|\tilde{a}_\pm(y)\nu(y)| \leq 2\Lambda,$$

the surface measures  $\zeta_k^\pm(dy) := [(\gamma_-)_k - (\gamma_+)_k]^\pm(y) \mathbb{I}_{y \in \Gamma} \varsigma(dy)$  belong also to  $S_0$ .

Note that from Aronson's estimates (14) we retrieve the following estimations

$$r_1(x, y) \leq C|x - y|^{-(d-2)} \quad \text{if } d > 2; \quad r_1(x, y) \leq C(\ln(1/|x - y|) \vee 1) \quad \text{if } d = 2.$$

Then, using the same arguments as in [6] (Example 5.2.2 p.255), we can assert that the measures  $\zeta_k^\pm(dy)$  belong to  $S_{00}$ . Moreover, let  $(K_t)_{t \geq 0}$  denote the PCAF associated to  $\varsigma(dy)$ ; in regard of the results stated in the original article of D. Revuz (cf. [10] p.507) we may assert that  $\left( \int_0^t [(\gamma_-)_k - (\gamma_+)_k]^\pm(X_s) \mathbb{I}_{X_s \in \Gamma} dK_s \right)_{t \geq 0}$  is the PCAF associated to  $\zeta_k^\pm(dy)$  via the Revuz correspondence.

By application of Theorem 5.5.5 in [6] and since all the necessary hypothesis are fulfilled, we get the decomposition (22) on the set  $\{t \geq 0 : t \leq \tau_{G_q}\}$  where  $G_q := \{x \in \mathbb{R}^d : |x| < q\}$ . The identification of the process for all times follows by letting  $q$  tend to infinity.  $\square$

Let  $u_0 \in \mathcal{D}(A)$ . From the Hille-Yosida theorem ([3] Theorems VII.4 and VII.5) we can prove that there exists a unique function

$$u \in C^1([0, T]; L^2(\mathbb{R}^d)) \cap C([0, T]; \mathcal{D}(A))$$

satisfying

$$\frac{du}{dt} = Au, \quad u(0) = u_0. \tag{23}$$

where the first equality in (23) has to be understood in the weak sense.

Under the hypothesis of Theorem 3.2, we deduce the following Corollary.

**Corollary 3.3.** *Let  $0 < T < \infty$ . Under the conditions of Theorem 3.2, for any  $u_0 \in \mathcal{D}(A)$ , we have*

$$\mathbb{E}^x[u_0(X_t)] = u(t, x), \quad \forall t \in [0, T], \quad \forall x \in \mathbb{R}^d, \tag{24}$$

where  $X$  is the diffusion considered in Theorem 3.2 and  $u$  is the solution of (23).

In particular, the following transmission condition

$$\langle a_+ \nabla_x u_+(t, y) - a_- \nabla_x u_-(t, y), \nu(y) \rangle = 0, \text{ for a.e. } (t, y) \in (0, T] \times \Gamma \quad (\star) \quad (25)$$

is satisfied.

*Proof.* On the one hand we have  $\mathbb{E}^x[u_0(X_t)] = T_t u_0(x)$  thanks to (15). On the other hand we have  $\frac{d}{dt} T_t u_0 = A T_t u_0$  (see [9] Thm 2.4-c), i.e.  $T_t u_0$  solves (23) whose solution is unique. Thus  $T_t u_0$  and  $u$  are equal in the space  $C^1([0, T]; L^2(\mathbb{R}^d)) \cap C([0, T]; \mathcal{D}(A))$ , and finally  $T_t u_0(x) = u(t, x)$  for any  $t, x$  (where we have used the fact that  $\mathcal{D}(A) \subset H^1(\mathbb{R}^d)$  and elements of  $H^1(\mathbb{R}^d)$  are identified with their continuous versions). For a proof of (25) see [4].  $\square$

In the light of (24) and in order to compute an approximate value of  $u(t, x)$ , one could think of producing a Monte Carlo method. Our preprint [4] is an attempt to tackle this issue.

## References

- [1] ARONSON, D. G., (1967). Bounds for the fundamental solution of a parabolic equation, *Bull. Amer. Math. Soc.*, Vol. 73, p. 890–896, MR0217444, DOI 10.1090/S0002-9904-1967-11830-5, <https://doi.org/10.1090/S0002-9904-1967-11830-5>.
- [2] BASS, RICHARD F., (1998). *Diffusions and elliptic operators*, Probability and its Applications (New York), Springer-Verlag, New York, p. xiv+232, MR1483890.
- [3] BREZIS, HAÏM, *Analyse fonctionnelle*, (1983). Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree], Masson Paris, p. xiv+234, MR697382 (85a:46001).
- [4] ETORÉ, P. AND MARTINEZ, M., A transformed Euler scheme for multidimensional transmission PDE, (2020). *submitted*.
- [5] EVANS, LAWRENCE C. AND GARIEPY, RONALD F., Measure theory and fine properties of functions, (2015). *Textbooks in Mathematics*, p. xiv+299, MR3409135.
- [6] MASATOSHI AND OSHIMA, YOICHI AND TAKEDA, MASAYOSHI, *Dirichlet forms and symmetric Markov processes*, (2011). De Gruyter Studies in Mathematics, Vol. 19, Walter de Gruyter & Co., Berlin, p. x+489, MR2778606.
- [7] FUKUSHIMA, MASATOSHI, *Dirichlet forms and Markov processes*, (1980). North-Holland Mathematical Library, Vol. 23, North-Holland Publishing Co., Amsterdam-New York; Kodansha, Ltd., Tokyo, p. x+196, MR569058.
- [8] MCLEAN, W.C.H., *Strongly Elliptic Systems and Boundary Integral Equations*, (2000). Cambridge University Press, <https://books.google.ca/books?id=RILqjEeMfK0C>.
- [9] PAZY, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, (1992). Applied Mathematical Sciences, <https://books.google.fr/books?id=sIAyOgM4R3kC>.
- [10] REVUZ, D., Mesures associées aux fonctionnelles additives de Markov. I, (1970). *Trans. Amer. Math. Soc.*, Vol. 148, p. 501–531, MR0279890, DOI 10.2307/1995386, <https://doi.org/10.2307/1995386>.
- [11] REVUZ, DANIEL AND YOR, MARC, *Continuous martingales and Brownian motion*, (1999). Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 293, p.xiv+602, Third edition, Springer-Verlag, Berlin, MR1725357, DOI 10.1007/978-3-662-06400-9, <https://dx-doi-org.fennec.u-pem.fr/10.1007/978-3-662-06400-9>.
- [12] ROZKOSZ, ANDRZEJ, (1996). Stochastic representation of diffusions corresponding to divergence form operators, *Stochastic Process. Appl.*, Vol. 63, 1, p. 11–33, MR1411187, DOI 10.1016/0304-4149(96)00059-2, [http://dx.doi.org/10.1016/0304-4149\(96\)00059-2](http://dx.doi.org/10.1016/0304-4149(96)00059-2).

- [13] ROZKOSZ, ANDRZEJ, (1996). Weak convergence of diffusions corresponding to divergence form operators, *Stochastics Stochastics Rep.*, Vol. 57, p. 129–157, MR1407951.
- [14] STROOCK, DANIEL W., Diffusion semigroups corresponding to uniformly elliptic divergence form operators, (1988). *Séminaire de Probabilités, XXII*, Lecture Notes in Math., Vol. 1321, p. 316–347, Springer, Berlin, MR960535, 10.1007/BFb0084145, <http://dx.doi.org/10.1007/BFb0084145>.
- [15] STROOCK, DANIEL W. AND ZHENG, WEIAN, Markov chain approximations to symmetric diffusions, (1997). *Annales de l'I.H.P. Probabilités et statistiques*, Vol. 33, 5, p. 619-649, <http://eudml.org/doc/77584>.
- [16] TRUTNAU, GERALD, Multidimensional skew reflected diffusions, (2005). *Stochastic analysis: classical and quantum*, p. 228–244, World Sci. Publ., Hackensack, NJ, MR2233163.