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Search via Parallel Lévy Walks on \mathbb{Z}^2

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Abstract

Motivated by the *Lévy foraging hypothesis* – the premise that various animal species have adapted to follow *Lévy walks* to optimize their search efficiency – we study the parallel hitting time of Lévy walks on the infinite two-dimensional grid. We consider k independent discrete-time Lévy walks, with the same exponent $\alpha \in (1, \infty)$, that start from the same node, and analyze the number of steps until the first walk visits a given target at distance ℓ . We show that for any choice of k and ℓ from a large range, there is a unique optimal exponent $\alpha_{k,\ell} \in (2, 3)$, for which the hitting time is $\tilde{O}(\ell^2/k)$ w.h.p., while modifying the exponent by an ϵ term increases the hitting time by a polynomial factor, or the walks fail to hit the target almost surely. Based on that, we propose a surprisingly simple and effective parallel search strategy, for the setting where k and ℓ are unknown: The exponent of each Lévy walk is just chosen independently and uniformly at random from the interval $(2, 3)$. This strategy achieves optimal search time (modulo polylogarithmic factors) among all possible algorithms (even centralized ones that know k). Our results should be contrasted with a line of previous work showing that the exponent $\alpha = 2$ is optimal for various search problems. In our setting of k parallel walks, we show that the optimal exponent depends on k and ℓ , and that randomizing the choice of the exponents works simultaneously for all k and ℓ .

1 Introduction

1.1 Background

A *Lévy walk* is a random walk process in which jump lengths are drawn from a power-law distribution. Thus, the walk consists of a mix of long trajectories and short, random movements. Over the last two decades, Lévy walks have attracted significant attention, as a result of increasing empirical evidence that the movement patterns of various animal species resemble Lévy walks. Examples of such species range from snails [31], bees [33], and albatross birds [37], to sharks [20, 35], deers [16], and humans [3, 29], among others [32]. Nowadays, Lévy walks are the most prominent movement model in biology [32], at least among models with comparable mathematical simplicity and elegance [39].

The *Lévy foraging hypothesis*, put forward by Viswanathan et al. [38, 40], stipulates that the observed Lévy walk movement patterns in animals must have been induced by natural selection, due

to the optimality of Lévy walks in searching for food. Indeed, it has been shown that Lévy walks achieve (near) optimal search time in certain settings. In particular, Lévy walks with exponent parameter $\alpha = 2$ are optimal for searching sparse randomly distributed revisitable targets [38]. However, these results were formally shown just for one-dimensional spaces [4], and do not carry over to higher-dimensions [26]. Currently, there is no strong analytical evidence supporting the optimality of Lévy walks in d -dimensional spaces for $d > 1$.

In this paper we focus on Lévy walks on two-dimensional spaces. Concretely, we assume the infinite lattice \mathbb{Z}^2 , and consider the Manhattan distance as the underlying metric. To determine each jump of the Lévy walk, an integer distance d is chosen independently at random such that the probability of $d = i$ is inversely proportional to i^α , where $\alpha \in (1, \infty)$ is the exponent parameter of the walk. A destination v is then chosen uniformly at random among all nodes at distance d from the current node u , and in the next d steps, the process moves from u to v along a shortest lattice path approximating the straight line segment \overline{uv} .

We evaluate the search efficiency of Lévy walks on \mathbb{Z}^2 , by analysing the parallel hitting time of multiple walks originating at the same node. Precisely, we assume that $k \geq 1$ independent Lévy walks start simultaneously from the origin $(0, 0)$ of the lattice. Then the parallel hitting time for any given target node u^* is the first step when some walk visits u^* . This very basic setting can be viewed as a model of natural cooperative foraging behavior, such as the behavior of ants around their nest. In fact, our setting is as a special instance of the more general ANTS problem introduced by Feinerman and Korman [14]. The ANTS problem asks for a search strategy for k independent agents that minimizes the parallel hitting time for an unknown target, subject to limited communication before the search starts.

1.2 Our Results

1.2.1 Hitting Time Bounds for a Single Lévy Walk

A main technical contribution of our work is an analysis of the hitting time $\tau_\alpha(u^*)$ of a single Lévy walk with exponent $\alpha \in (1, \infty)$, for an arbitrary target node u^* . We show the following bounds on $\tau_\alpha(u^*)$, assuming the Lévy walk starts at the origin $(0, 0)$, and u^* 's distance to the origin is $\ell = \|u^*\|_1$.

Consider first the *super-diffusive* regime, where $\alpha \in (2, 3)$. In this regime, jump lengths have bounded mean and unbounded variance. Roughly speaking, we show that in the first $t_\ell = \Theta(\ell^{\alpha-1})$ steps,¹ the walk stays inside a ball of radius $t_\ell \cdot \text{polylog } \ell$ with significant probability, while only a constant fraction of those steps are inside the smaller ball of radius ℓ . We also show a monotonicity property, which roughly implies that the probability of visiting a node decreases as the node's distance from the origin increases. Therefore, a constant fraction of the t_ℓ steps visits nodes at distances between ℓ and $\ell \cdot \text{polylog } \ell$, and the visit probability of each of these nodes is upper bounded by that of node u^* . We thus obtain that the probability of visiting u^* within t_ℓ steps is $\Omega(t_\ell/\ell^2 \text{polylog } \ell)$.

If we consider a smaller number of steps, $t = \mathcal{O}(t_\ell/\text{polylog } \ell)$, then it is very likely that the walk stays in a ball of radius smaller than ℓ , and we show a simple bound of $\mathcal{O}((t/t_\ell)^2 \cdot t_\ell/\ell^2)$ for the probability of $\tau_\alpha(u^*) \leq t$, i.e., ignoring $\text{polylog } \ell$ factors, the probability decreases by a factor of $\mathcal{O}((t/t_\ell)^2)$.

On the other hand, if we consider a larger number of steps (even if $t \rightarrow \infty$), the probability that u^* is hit does not increase significantly, just by at most a $\text{polylog } \ell$ factor.

¹Note that t_ℓ is of the same order as the expected number of steps before the first jump of length greater than ℓ .

Therefore, in regime $\alpha \in (2, 3)$, $\Theta(\ell^{\alpha-1})$ steps suffice to maximize the hitting probability (within polylog ℓ factors), while reducing this time reduces the probability super-linearly.

The *diffusive* regime, $\alpha \in (3, \infty)$, is similar to the case of a simple random walk, as jump lengths have bounded mean and bounded variance. We show that $\mathcal{O}(\ell^2 \text{polylog } \ell)$ steps suffice to hit the target with probability $\Omega(1/\text{polylog } \ell)$, while for a smaller number of steps t , the probability decreases by a factor of $\mathcal{O}((t/\ell^2)^2)$. The behavior is similar also in the threshold case of $\alpha = 3$, even though the variance of the jump length is unbounded in this case.

Finally, in the *ballistic* regime, $\alpha \in (1, 2]$, where jump lengths have unbounded mean and unbounded variance, the behavior is similar to that of a straight walk along a random direction. We show that the target is hit with probability $\Omega(1/\ell \text{polylog } \ell)$ in the first $\Theta(\ell)$ steps, while increasing the number of steps does not increase this probability significantly.

Below we give formal statements of these results, for the case where α is independent of ℓ , as $\ell \rightarrow \infty$. More refined statements and their proofs are given in Sections 4 to 6.

Theorem 1.1. *Let α be any real constant in $(2, 3)$ and u^* any node in \mathbb{Z}^2 with $\ell = \|u^*\|_1$. Then:*

- (a) $\mathbb{P}(\tau_\alpha(u^*) = \mathcal{O}(\ell^{\alpha-1})) = \Omega(1/\ell^{3-\alpha} \log^2 \ell)$;
- (b) $\mathbb{P}(\tau_\alpha(u^*) \leq t) = \mathcal{O}(t^2/\ell^{\alpha+1})$, for any step $\ell \leq t = \mathcal{O}(\ell^{\alpha-1})$;
- (c) $\mathbb{P}(\tau_\alpha(u^*) < \infty) = \mathcal{O}(\log \ell / \ell^{3-\alpha})$.

Theorem 1.2. *Let α be any real constant in $[3, \infty)$ and u^* any node in \mathbb{Z}^2 with $\ell = \|u^*\|_1$. Then:*

- (a) $\mathbb{P}(\tau_\alpha(u^*) = \mathcal{O}(\ell^2 \log^2 \ell)) = \Omega(1/\log^4 \ell)$;
- (b) $\mathbb{P}(\tau_\alpha(u^*) \leq t) = \mathcal{O}(t^2 \log \ell / \ell^4)$, for any step t with $\ell \leq t = \mathcal{O}(\ell^2)$.

Theorem 1.3. *Let α be any real constant in $(1, 2]$ and u^* any node in \mathbb{Z}^2 with $\ell = \|u^*\|_1$. Then:*

- (a) $\mathbb{P}(\tau_\alpha(u^*) = \mathcal{O}(\ell)) = \Omega(1/\ell \log \ell)$;
- (b) $\mathbb{P}(\tau_\alpha(u^*) < \infty) = \mathcal{O}(\log^2 \ell / \ell)$.

1.2.2 Parallel Lévy Walks with Common Exponent

Consider $k \geq 1$ independent identical Lévy walks with exponent $\alpha \in (1, \infty)$, that start simultaneously at the origin. Let $\tau_\alpha^k(u^*)$ denote the parallel hitting time for node u^* , i.e., the first step when some walk visits u^* . It is straightforward to derive upper and lower bounds on $\tau_\alpha^k(u^*)$ from the corresponding bounds on the hitting time of a single Lévy walk. For example, the next statement is a direct corollary of Theorem 1.1(a).

Corollary 1.4. *Let α be any real constant in $(2, 3)$ and u^* any node in \mathbb{Z}^2 with $\ell = \|u^*\|_1$. Then*

$$\mathbb{P}\left(\tau_\alpha^k(u^*) = \mathcal{O}(\ell^{\alpha-1})\right) = 1 - e^{-\Omega(k/\ell^{3-\alpha} \log^2 \ell)}.$$

From the bounds we obtain for τ_α^k , it follows² that, for each pair of k and $\ell = \|u^*\|_1$ with $\text{polylog } \ell \leq k \leq \ell \text{polylog } \ell$, there is a unique optimal exponent $\alpha = 3 - \frac{\log k}{\log \ell} + \mathcal{O}\left(\frac{\log \log \ell}{\log \ell}\right)$, which minimizes $\tau_\alpha^k(u^*)$, w.h.p. Moreover, increasing or decreasing this exponent by an arbitrarily small

²In fact, we use more refined versions of Theorems 1.1 to 1.3, to obtain bounds on τ_α^k which allow α to be a function of ℓ and k .

constant term, respectively increases the hitting time by a $\text{poly}(\ell)$ factor, or the walks never hit u^* with probability $1 - o(1)$. For the case of $k \leq \text{polylog } \ell$ or $k \geq \ell \text{ polylog } \ell$, all exponents $\alpha \in [3, \infty)$ or $\alpha \in (1, 2]$, respectively, achieve the same optimal value of $\tau_\alpha^k(u^*)$ (within $\text{polylog } \ell$ factors). Formal statements of these results are given in Sections 4 to 6. The theorem below bounds the parallel hitting time for (near) optimal choices of α .

Theorem 1.5. *Let u^* be any node in \mathbb{Z}^2 , and $\ell = \|u^*\|_1$.*

- (a) *If $\log^6 \ell \leq k \leq \ell \log^4 \ell$, then for $\alpha = 3 - \frac{\log k}{\log \ell} + 5 \frac{\log \log \ell}{\log \ell}$, $\mathbb{P}\left(\tau_\alpha^k(u^*) = \mathcal{O}\left(\frac{\ell^2 \log^6 \ell}{k}\right)\right) = 1 - e^{-\omega(\log \ell)}$;*
- (b) *If $k = \omega(\log^5 \ell)$, then $\mathbb{P}\left(\tau_3^k(u^*) = \mathcal{O}(\ell^2)\right) = 1 - e^{-\omega(\log \ell)}$;*
- (c) *If $k = \omega(\ell \log^2 \ell)$, then $\mathbb{P}\left(\tau_2^k(u^*) = \mathcal{O}(\ell)\right) = 1 - e^{-\omega(\log \ell)}$.*

Observe that for any given k, ℓ with $k = \omega(\log^5 \ell)$, if we choose the exponent α as in Theorem 1.5, then

$$\mathbb{P}\left(\tau_\alpha^k(u^*) = \mathcal{O}\left((\ell^2/k) \log^6 \ell + \ell\right)\right) = 1 - e^{-\omega(\log \ell)}. \quad (1)$$

1.2.3 Parallel Lévy Walks with Random Exponents

The right choice of α , according to Theorem 1.5, requires knowledge of the values of k and ℓ (at least within polylogarithmic factors). We propose a very simple randomized strategy for choosing the exponents of the k Lévy walks, which almost matches the parallel hitting time bounds of Theorem 1.5, for all distances ℓ simultaneously! The strategy does not require knowledge of ℓ , and works as long as $k \geq \text{polylog } \ell$. Interestingly, it does not require knowledge of k either. The strategy is the following:

The exponent of each walk is sampled independently and uniformly at random from the real interval $(2, 3)$.

The next theorem bounds the resulting parallel hitting time $\tau_{rand}^k(u^*)$, for an arbitrary node u^* . Its proof is given in Section 7.

Theorem 1.6. *Let u^* be any node in \mathbb{Z}^2 , $\ell = \|u^*\|_1$, and $k \geq \log^8 \ell$. Then*

$$\mathbb{P}\left(\tau_{rand}^k(u^*) = \mathcal{O}\left((\ell^2/k) \log^7 \ell + \ell \log^3 \ell\right)\right) = 1 - e^{-\omega(\log \ell)}. \quad (2)$$

By comparing Eq. (1) and Eq. (2), we observe that indeed the hitting time of the randomized strategy is only by a $\text{polylog } \ell$ factor worse than that of the deterministic strategy based on Theorem 1.5, which knows ℓ and k . Moreover, this hitting time is optimal within a $\text{polylog } \ell$ factor among *all* possible search strategies (deterministic or randomized) that do not know ℓ within a constant factor, since a universal lower bound of $\Omega\left(\ell^2/k + \ell\right)$ with constant probability applies to all such strategies, as observed in [14].

1.2.4 Implications on Lévy Hypothesis and Distributed Search

As already mentioned, our setting of k independent walks starting from the same location, aiming to hit an unknown target, can be viewed as a basic model of animals' foraging behavior around a central location, such as a nest, a food storage area, or a sheltered environment. The assumption that walks are independent is approximately true for certain animal species such as ants *Cataglyphis*, which

lack pheromone-based marking mechanisms [30]. Our results suggest that if the typical or maximum distance ℓ of the food (target) from the nest (source) is fixed, then a group of animals executing parallel Lévy walks with the same exponent can optimize search efficiency by tuning the exponent value and/or the number k of animals participating in the foraging. In that setting, no universally optimal exponent value exists, as the optimal exponent depends on k and ℓ . An alternative, novel approach suggested by our last result is that each animal performs a Lévy walk with a randomly chosen exponent. This strategy, which surprisingly achieves near optimal search efficiency for all distance scales, implies that different members of the same group follow different search patterns. The existence of such variation in the search patterns among individuals of the same species requires empirical validation.

In the context of the related ANTS problem [14], our result on parallel Lévy walks with randomly selected exponents directly implies a uniform solution to the problem (i.e., independent of k and ℓ), which is extremely simple and natural, and is optimal within polylog ℓ factors, w.h.p.

2 Related Work

Lévy walks (also referred to as Lévy flights) have been studied mostly by physicists, and mainly in continuous spaces [32, 42]. The idea that biological organisms could perform Lévy walks was first suggested in the mid 80s [34], as a potentially more efficient search strategy compared to Brownian motion. Lévy walks attracted significant attention after experimental work in the mid 90s showed that albatross birds follow Lévy walk-like trajectories [37], a pattern that was subsequently observed for various other organisms as well [3, 16, 20, 29, 31, 33, 35]. Even though statistical and methodological flaws were later pointed out in several of these works [9], there is currently ample evidence that many animals do exhibit Lévy walk movements [21, 39].

A possible explanation for this phenomenon is the *Lévy foraging hypothesis* [38, 40]: “According to the optimal foraging theory [41], natural selection drives species to adopt the most economically advantageous foraging pattern. Thus species must have adapted to follow Lévy walks because Lévy walks optimize search efficiency.” The main theoretical argument in support of this hypothesis was provided in [38], stating that a Lévy walk with exponent $\alpha = 2$ (known as Cauchy walk) maximizes the number of visits to targets, when targets are sparse and uniformly distributed. This result has been formally shown for one-dimensional spaces [4], but is not true for higher-dimensional spaces [26], at least not without additional assumptions [5, 27].

Very recently, a new argument was provided in [18] supporting the optimality of Lévy walks with $\alpha = 2$. In the considered setting, the space is a square torus of area n , and the walk must find a single, randomly selected target. Two critical model assumptions are that the target may have an arbitrary diameter D , and that the Lévy walk is “intermittent,” i.e., cannot detect the target during a jump, only at the end of the jump. Under these assumptions, the Cauchy walk was shown to achieve a (near) optimal search time of $\tilde{O}(n/D)$, whereas exponents $\alpha \neq 2$ are suboptimal.³

Our results add a new perspective to the Lévy foraging hypothesis. Unlike [38] and [18], we consider a *collective* search setting, where k individuals start from the same source and move independently. The space is two-dimensional as in [18] (but discrete and unbounded), and there is a single target (of unit size). If rough information about the target’s distance ℓ to the source is known then letting all individuals execute identical Lévy walks with a specific exponent, which depends on k and ℓ , achieves (near) optimal search time. If no information on ℓ is available, then using a random exponent for each walk, sampled independently from the super-diffusive range

³Note that if the target has a fixed size $D = 1$ or the walk is not intermittent, then all exponents $\alpha \geq 2$ or $\alpha \leq 2$, respectively, are optimal as well.

(2, 3), still achieves near optimal search time, for all distances ℓ .

In our analysis, we derive upper and lower bounds on the hitting time of a Lévy walk on \mathbb{Z}^2 . Bounds on the hitting time and related quantities for Lévy walks on the (one-dimensional) real line are given in [28]. Bounds for general random walks on \mathbb{Z}^d , for $d \geq 1$, in the case where the walk has bounded second (or higher) moments can be found in [36]. Recall that Lévy walks have unbounded second moment when $\alpha \leq 3$.

In the Ants-Nearby-Treasure-Search (ANTS) problem [14], k identical (probabilistic) agents starting from the same location, search for an unknown target on \mathbb{Z}^2 . Agents do not know k , and cannot communicate (or see each other). However, before the search begins, each agent receives a b -bit *advice* from an oracle. In [14], matching upper and lower bounds are shown for the trade-off between the expected time until the target is found, and the size b of the advice. The proposed optimal algorithms repeatedly execute the following steps: walk to a random location in a ball of a certain radius (chosen according to the algorithm specifics), perform a spiral movement of the same radius as the ball's, then return to the origin. Our results suggest a very simple algorithm for the setting where no advice is given ($b = 0$): Each agent performs a Lévy walk with a uniformly random exponent sampled from (2, 3). The algorithm is Monte Carlo, and finds the target w.h.p. in time that is larger than the optimal by at most a polylogarithmic factor.

Variants of the ANTS problem have been studied, where agents are (synchronous or asynchronous) finite state machines, which can communicate during the execution whenever they meet [7, 12, 13, 25]. Another variant, involving parallel search on the line by k non-communicating agents, is considered in [17].

In [2, 19], tight bounds were shown for the cover time on the cycle of a random walk with k different jump lengths. The optimal walk in this case is one that approximates (using k levels) a Lévy walk with exponent $\alpha = 2$.

When $\alpha \in (3, \infty)$, a Lévy walk on \mathbb{Z}^d behaves similarly to a simple random walk, as the variance of the jump length is bounded. In particular, as $\alpha \rightarrow \infty$, the Lévy walk jump converges in distribution to that of a simple random walk. Parallel independent simple random walks have been studied extensively on finite graphs, under various assumptions for their starting positions [1, 10, 11, 22, 23]. A main objective of that line of work has been to quantify the “speedup” achieved by k parallel walks on the cover time, hitting times, and other related quantities, compared to a single walk.

The following basic network model has been proposed by Kleinberg to study the small world phenomenon [24]. A square (or, more generally, d -dimensional) finite lattice is augmented by adding one “long-range” edge from each node u , to a uniformly random node v among all nodes at lattice distance k , where distance k is chosen independently for each u , from a power-law distribution with exponent α . That is, the distribution of long-range edges is the same as the jump distribution of a Lévy walk with the same exponent. It was shown that (distributed) greedy routing is optimized when $\alpha = 1$, whereas for $\alpha \neq 1$ the expected routing time is slowed down by polynomial factors [24].⁴ This result is of similar nature as our result for the hitting time of k identical Lévy walks, where exactly one exponent is optimal. However, in our case, this exponent depends on the target distance. In Kleinberg’s network, exponent $\alpha = 1$ ensures that the lengths of long-range links are uniformly distributed over all *distance scales*, which facilitates fast routing. In our randomized strategy, availability of a sufficient number of walks with the right exponent is achieved by choosing the exponents uniformly at random over the interval (2, 3).

⁴In the paper, the exponent considered is that of choosing the endpoint of u ’s long-range link to be a given node v at distance k , which is proportional to $1/k^\beta$, where $\beta = \alpha + d - 1$. Thus the optimal exponent is $\beta = 2$ for the square lattice, and $\beta = d$ for the d -dimensional lattice.

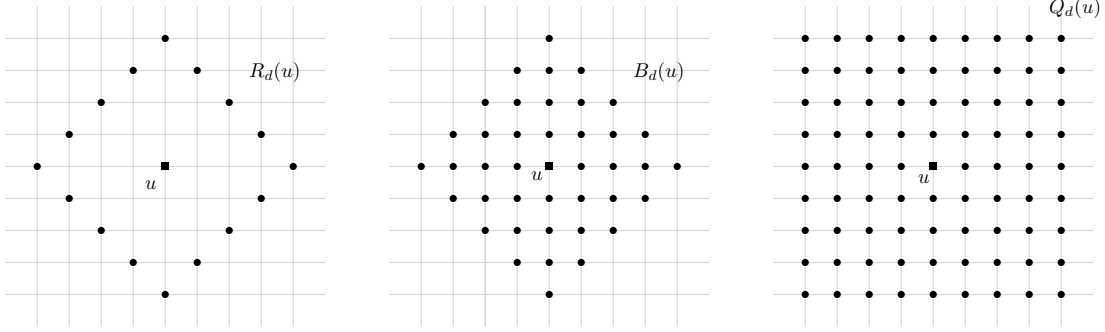


Figure 1: Illustrations of $R_d(u)$, $B_d(u)$, and $Q_d(u)$, for $d = 4$.

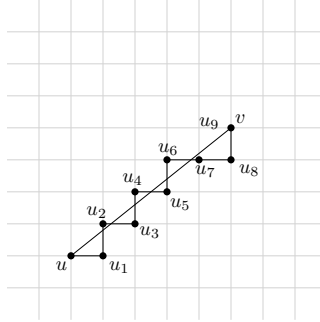


Figure 2: Example of a line segment \overline{uv} and the direct-path between u and v .

Roadmap

The paper is organized as follows. In Section 3, we give some basic definitions and facts, and investigate a monotonicity property that plays a key role in our analysis. In Section 4, we provide the analysis of the regime $\alpha \in (2, 3]$ while in Sections 5 and 6, we investigate the regimes $\alpha \in (1, 2]$ and $\alpha \in (3, \infty)$, respectively. Finally, in Section 7, we use results from Section 4 to analyze the efficiency of our simple distributed search algorithm.

3 Preliminaries

3.1 Main definitions and notation

For each point $s = (x, y) \in \mathbb{R}^2$, we write $\|s\|_p$ to denote its p -norm $(|x|^p + |y|^p)^{1/p}$. The p -norm distance between points $s = (x, y)$ and $s' = (x', y')$ is $\|s - s'\|_p = (|x - x'|^p + |y - y'|^p)^{1/p}$.

We consider the infinite grid graph $G = (\mathbb{Z}^2, E)$, where $E = \{\{u, v\} : \|u - v\|_1 = 1\}$. The shortest-path distance between two nodes $u, v \in \mathbb{Z}^2$ in G equals $\|u - v\|_1$. In the following, we will say just *distance* to refer to the shortest-path distance.

We denote by $R_d(u)$ the set of all nodes $v \in \mathbb{Z}^2$ that are at distance exactly d from u , i.e., $R_d(u) = \{v \in \mathbb{Z}^2 : \|u - v\|_1 = d\}$. We also define $B_d(u) = \{v \in \mathbb{Z}^2 : \|u - v\|_1 \leq d\}$ and $Q_d(u) = \{v \in \mathbb{Z}^2 : \|u - v\|_\infty \leq d\}$. See Fig. 1 for an illustration.

By \overline{uv} we denote the straight-line segment on the real plane \mathbb{R}^2 between nodes u and v . A *direct-path* between u and v in G is a shortest path that “closely follows” the real segment \overline{uv} . See Fig. 2 for an illustration.

Definition 3.1. (Direct-path) A *direct-path* from node u to v is a shortest path $u, u_1, \dots, u_k = v$, where $k = \|u - v\|_1$, and for each $1 \leq i < k$, $u_i \in R_i(u)$ and $\|u_i - w_i\|_2 = \min_{v' \in R_i(u)} \|v' - w_i\|_2$, where w_i is the (unique) point w in the real segment \overline{uv} with $\|u - w\|_1 = i$.

It is not hard to verify that u, u_1, \dots, u_k is indeed a path of G . Also, unlike point w_i , node u_i is not necessarily unique, since there may be two different nodes in $R_i(u)$ that are closest to w_i . The next simple fact about direct-paths is proved in Appendix B.1.

Lemma 3.2. Let $u \in \mathbb{Z}^2$ and $d \geq 1$ be an integer. Suppose we sample a node v uniformly at random from set $R_d(u)$, and then sample a direct-path $u, u_1, \dots, u_d = v$ from u to v uniformly at random among all such paths. Then, for every $1 \leq i < d$ and $w \in R_i(u)$,

$$\frac{(i/d) \cdot \lfloor d/i \rfloor}{4i} \leq \mathbb{P}(u_i = w) \leq \frac{(i/d) \cdot \lceil d/i \rceil}{4i}.$$

A (discrete-time) *jump process* on \mathbb{Z}^2 is just an infinite sequence of random variables $(J_t)_{t \geq 0}$, where $J_t \in \mathbb{Z}^2$ for each integer $t \geq 0$. We say that the process *visits* node $v \in \mathbb{Z}^2$ at step $t \geq 0$ if $J_t = v$. Our analysis will focus on the following two jump processes.

Definition 3.3 (Lévy flight). A jump process $L^f = (L_t^f)_{t \geq 0}$ on \mathbb{Z}^2 is a Lévy flight with exponent parameter $\alpha \in (1, \infty)$, and start node $s \in \mathbb{Z}^2$, if $L_0^f = s$, and for each $t \geq 0$, if $L_t^f = u$ then $L_{t+1}^f = v$, where node $v \in \mathbb{Z}^2$ is selected as follows: First a jump distance d is chosen independently at random such that

$$\mathbb{P}(d = 0) = 1/2, \text{ and } \mathbb{P}(d = i) = c_\alpha / i^\alpha \text{ for } i \geq 1, \quad (3)$$

where c_α is a normalizing factor. Then, node v is chosen independently and uniformly at random among all nodes in $R_d(u)$ (i.e., all nodes at distance d from u).

Definition 3.4 (Lévy walk). A jump process $L^w = (L_t^w)_{t \geq 0}$ on \mathbb{Z}^2 is a Lévy walk with exponent parameter $\alpha \in (1, \infty)$, and start node $s \in \mathbb{Z}^2$, if $L_0^w = s$, and the process consists of a infinite sequence of *jump-phases*, where each jump-phase is defined as follows: Suppose that the jump-phase begins at step $t + 1$ (the first jump-phase begins at the first step), and suppose also that $L_t^w = u$. First a distance d and a node v at distance d from u are chosen, in exactly the same way as in the Lévy flight. If $d = 0$ then the jump-phase has length 1, and $L_{t+1}^w = u$, i.e., the process stays put for one step. If $d \neq 0$ then the jump-phase has length d , and in the next d steps the process follows a path $u, u_1, \dots, u_d = v$ chosen uniformly at random among all direct-paths from u to v , i.e., $L_{t+i}^w = u_i$, for all $1 \leq i \leq d$.⁵

We observe that a Lévy flight is a Markov chain, while a Lévy walk is not.

Remark 3.5. Throughout the analysis, we assume α to be a (not necessarily constant) real value greater than $1 + \epsilon$ for some arbitrarily small constant $\epsilon > 0$.

We will often use the following bound on the tail distribution of the jump length d chosen according to (3):

$$\mathbb{P}(d \geq i) = \Theta(1/i^{\alpha-1}) \quad (4)$$

The next statement follows immediately from Lemma 3.2.

Corollary 3.6. Let $u, v \in \mathbb{Z}^2$, and $d = \|u - v\|_1 > 0$. If a Lévy walk is at node u at the beginning of a jump-phase, then the probability it visits v during the jump-phase is $\Theta(1/d^\alpha)$.

⁵Our analysis works also if an arbitrary direct-path between u and v is selected, instead of a random one.

Definition 3.7 (Hitting Time). The *hitting time* for node $u^* \in \mathbb{Z}^2$ of a jump process is the first step $t \geq 0$ when the process visits u . For a set of k independent jump processes that run in parallel, their *parallel hitting time* for u^* is the first step in which some (at least one) of the k processes visits u^* .

We will denote by $\tau_\alpha(u^*)$ the hitting time for u^* of a single Lévy walk processes with exponent α starting from the origin $\mathbf{0} = (0, 0)$; and by $\tau_\alpha^k(u^*)$ the parallel hitting time for u^* of k independent copies of the above Lévy walk. Unless stated otherwise, we will always assume that the starting node of a jump processes is the origin $\mathbf{0} = (0, 0)$.

For a Lévy flight $L^f = (L_t^f)_{t \geq 0}$, we denote by $Z_u^f(t)$ the number of times the process visits node $u \in \mathbb{Z}^2$ until step t , i.e., $Z_u^f(t) = |\{i: L_i^f = u\} \cap \{1, \dots, t\}|$. We define $Z_u^w(t)$ similarly for a Lévy walk.

3.2 Bounds via Monotonicity

We will now describe an intuitive monotonicity property that applies to a family of jump processes that includes Lévy flights (but not Lévy walks). We then use this property, and the similarity between Lévy flights and walks, to show upper bounds on the probability that a Lévy walk visits a given target. We start by defining the family of *monotone radial* jump processes.

Definition 3.8 (Monotone radial process). A jump process $(J_t)_{t \geq 0}$ is *monotone radial* if, for any pair of nodes $u, v \in \mathbb{Z}^2$, and any $t \geq 0$, $\mathbb{P}(J_{t+1} = v \mid J_t = u) = \rho(\|u - v\|_1)$, for some non-increasing function ρ .

Clearly, Lévy flights are monotone radial processes. For all such processes, we use geometric arguments to prove the following property. The proof is deferred to Appendix B.2.

Lemma 3.9 (Monotonicity property). *Let $(J_i)_{i \geq 0}$ be any monotone radial jump process. For every pair $u, v \in \mathbb{Z}^2$ and any $t \geq 0$, if $\|v\|_\infty \geq \|u\|_1$ then $\mathbb{P}(J_t = u) \geq \mathbb{P}(J_t = v)$.*

Next, we use Lemma 3.9 to upper bound the probability that a target node is visited during a given jump-phase of a Lévy walk, and then bound the probability that the target is visited during *at least one* jump-phase.

Lemma 3.10. *Let u^* be an arbitrary node with $\ell = \|u^*\|_1$. Let $\epsilon > 0$ be an arbitrarily small constant. For any $\alpha \geq 1 + \epsilon$, the probability that a Lévy walk visits u^* during its i -th jump-phase is $\mathcal{O}(\mu \cdot (\ell^{-2} + \ell^{-\alpha}))$ if $\alpha \neq 2$, where $\mu = \min\{\log \ell, \lfloor \frac{1}{2-\alpha} \rfloor\}$, and $\mathcal{O}(\log \ell / \ell^2)$ if $\alpha = 2$.*

Proof. For any $v \in B_{\ell/4}(u^*)$, the probability that the i -th jump starts in v is at most $\mathcal{O}(1/\ell^2)$ due to Lemma 3.9, since the process restricted only to the jumps endpoints is a Lévy flight. Moreover, for any $1 \leq d \leq \ell/4$, there are at most $4d$ nodes in $B_{\ell/4}(u^*)$ located at distance d from u^* . Then, from the chain rule and Corollary 3.6, the probability that the i -th jump starts from $B_{\ell/4}(u^*)$ and the agent visits the target during the jump-phase is bounded by

$$\mathcal{O}\left(\frac{1}{\ell^2}\right) \sum_{d=1}^{\ell/4} 4d \cdot \mathcal{O}\left(\frac{1}{d^\alpha}\right) + \mathcal{O}\left(\frac{1}{\ell^2}\right),$$

where, in the first expression, the last term $\mathcal{O}(1/\ell^2)$ is the contribution of u^* itself. The latter equals $\mathcal{O}(\mu \ell^{2-\alpha} / \ell^2)$ if $\alpha \neq 2$, and $\mathcal{O}(\log \ell / \ell^2)$ if $\alpha = 2$. Notice that, for any fixed node $v \notin B_{\ell/4}(u^*)$, the probability that, during a jump-phase starting in v , the agent visits the target is at most $\mathcal{O}(1/\ell^\alpha)$.

Let F_i be the above event, and V_i be the event that the starting point of the i -th jump is in $B_{\ell/4}(u^*)$. Then,

$$\mathbb{P}(F_i) \leq \mathbb{P}(F_i | V_i) \mathbb{P}(V_i) + \mathbb{P}(F_i | V_i^C) \leq \mathbb{P}(F_1 | V_i) + \mathcal{O}\left(\frac{1}{\ell^\alpha}\right).$$

Then, if $\alpha > 2$ we get

$$\mathbb{P}(F_i) = \mathcal{O}\left(\frac{\mu}{\ell^2} + \frac{1}{\ell^\alpha}\right) = \mathcal{O}\left(\frac{\mu}{\ell^2}\right).$$

If $\alpha = 2$ we get $\mathbb{P}(F_i) = \mathcal{O}(\log \ell / \ell^2)$. If $1 < \alpha < 2$ we get

$$\mathbb{P}(F_i) = \mathcal{O}\left(\frac{\mu}{\ell^\alpha} + \frac{1}{\ell^\alpha}\right) = \mathcal{O}\left(\frac{\mu}{\ell^\alpha}\right),$$

which completes the proof. \square

Lemma 3.11. *Let u^* be an arbitrary node with $\ell = \|u^*\|_1$. Let $\epsilon > 0$ be an arbitrarily small constant. The probability that a Lévy walk with exponent $1 + \epsilon \leq \alpha < 3$ visits u^* at least once (at any step t) is $\mathcal{O}(\mu \log \ell (\ell^{-1} + \ell^{3-\alpha}))$ if $\alpha \neq 2$, where $\mu = \min\{\log \ell, \lfloor \frac{1}{2-\alpha} \rfloor\}$, and $\mathcal{O}\left(\frac{\log^2 \ell}{\ell}\right)$ if $\alpha = 2$.*

Proof. For each $i \geq 0$, consider the first time t_i the agent is at distance at least $\lambda_i = 2^i \ell$ from the origin. Define, for $i \geq 1$, the values $\tau_i = 2\lambda_i^{\alpha-1} \log \lambda_i$. The probability that $t_i \geq n\tau_i$ is bounded from above by the probability that at least one between the first $n\tau_i$ jumps has length at least λ_i . Since the jump lengths are mutually independent, from Eq. (4) we get

$$\begin{aligned} \mathbb{P}(t_i \geq n\tau_i) &= \left[1 - \mathcal{O}\left(\frac{1}{\lambda_i^{\alpha-1}}\right)\right]^{2n\lambda_i^{\alpha-1} \log \lambda_i} \leq \exp(-\mathcal{O}(2n \log \lambda_i)) \\ &= \mathcal{O}\left(\frac{1}{\lambda_i^{2n}}\right) = \mathcal{O}\left(\frac{1}{2^{2ni} \ell^{2n}}\right), \end{aligned}$$

where we have used the well known inequality $1 - x \leq e^{-x}$ that holds for every real x . We next bound the expected number of visits to the target node from time t_i to time t_{i+1} as follows:

$$\begin{aligned} &\mathbb{E}[Z_{u^*}^w(t_{i+1}) - Z_{u^*}^w(t_i)] \\ &\leq \mathbb{E}[Z_{u^*}^w(t_{i+1}) - Z_{u^*}^w(t_i) | t_{i+1} \leq \tau_{i+1}] \mathbb{P}(t_{i+1} \leq \tau_{i+1}) \\ &+ \sum_{n \geq 1} \mathbb{E}[Z_{u^*}^w(t_{i+1}) - Z_{u^*}^w(t_i) | n\tau_{i+1} < t_{i+1} \leq (n+1)\tau_{i+1}] \mathbb{P}(t_{i+1} \geq n\tau_i). \end{aligned}$$

We now proceed by analysing three different ranges for the exponent α and use Lemma 3.10. Notice that the agent starts at distance $\Omega(\lambda_i)$ from the target.

If $\alpha > 2$, the latter expression is equal to

$$\mathcal{O}\left(\frac{\mu\tau_{i+1}}{\lambda_i^2}\right) + \sum_{n \geq 1} \mathcal{O}\left(\frac{\mu(n+1)\tau_{i+1}}{2^{2ni} \ell^{2n} \lambda_i^2}\right) = \mathcal{O}\left(\frac{\mu\tau_i}{\lambda_i^2}\right)$$

since the sum $\sum_{n \geq 1} (n+1) (2^i \ell)^{-2n}$ is less than a constant.

If $\alpha = 2$, we get

$$\mathcal{O}\left(\frac{\tau_{i+1} \log \lambda_i}{\lambda_i^2}\right) + \sum_{n \geq 1} \mathcal{O}\left(\frac{(n+1)\tau_{i+1} \log \lambda_i}{2^{2ni} \ell^{2n} \lambda_i^2}\right) = \mathcal{O}\left(\frac{\tau_i \log \lambda_i}{\lambda_i^2}\right).$$

Finally, if $1 < \alpha < 2$, we obtain

$$\mathcal{O}\left(\frac{\mu \tau_{i+1}}{(2-\alpha)\lambda_i^\alpha}\right) + \sum_{n \geq 1} \mathcal{O}\left(\frac{\mu(n+1)\tau_{i+1}}{2^{2ni} \ell^{2n} \lambda_i^\alpha}\right) = \mathcal{O}\left(\frac{\mu \tau_i}{\lambda_i^\alpha}\right).$$

Moreover, the same bounds with $i = 1$ hold for the expected number of visits to the target until time t_1 . From the facts above, for $\alpha > 2$, the expected total number of visits to the target can be bounded as follows:

$$\begin{aligned} & \mathcal{O}\left(\frac{\mu \tau_1}{\lambda_1^2}\right) + \sum_{i \geq 1} \mathcal{O}\left(\frac{\mu \tau_i}{\lambda_i^2}\right) \\ &= \mathcal{O}\left(\frac{\mu \log \ell}{\ell^{3-\alpha}}\right) + \sum_{i \geq 1} \mathcal{O}\left(\mu \cdot \frac{\log(2^i) + \log \ell}{(2^{2i(3-\alpha)} \ell^{3-\alpha})}\right) \\ &= \mathcal{O}\left(\frac{\mu \log \ell}{\ell^{3-\alpha}}\right). \end{aligned}$$

Similarly, for $\alpha = 2$, we obtain $\mathcal{O}(\log^2 \ell / \ell^{3-\alpha})$, while, for $1 < \alpha < 2$, we get

$$\begin{aligned} & \mathcal{O}\left(\frac{\mu \log \ell}{\ell}\right) + \sum_{i \geq 1} \mathcal{O}\left(\mu \cdot \frac{\log(2^i) + \log \ell}{(2^{2i(3-\alpha)} \ell)}\right) \\ &= \mathcal{O}\left(\frac{\mu \log \ell}{\ell}\right). \end{aligned}$$

Finally, we bound the probability the agent visits the target at least once using the Markov's inequality. This concludes the proof. \square

4 The Case $\alpha \in (2, 3]$

In this section, we analyze the hitting time of Lévy walks when the exponent parameter α belongs to the range $(2, 3]$. In this case, the jump length has bounded mean and unbounded variance.

Recall that $\tau_\alpha(u^*)$ is the hitting time for target u^* of a single Lévy walk with exponent α , and $\tau_\alpha^k(u^*)$ is the parallel hitting time for u^* of k independent copies of the above Lévy walk. All walks start from the origin.

We will prove the following bounds on the hitting time τ_α .

Theorem 4.1. *Let $\alpha \in (2, 3)$, $u^* \in \mathbb{Z}^2$, and $\ell = \|u^*\|_1$. Let $\mu = \min\{\log \ell, \frac{1}{\alpha-2}\}$, $\nu = \min\{\log \ell, \frac{1}{3-\alpha}\}$, and $\gamma = \frac{(\log \ell)^{\frac{2}{\alpha-1}}}{(3-\alpha)^2}$. Then:*

- (a) $\mathbb{P}(\tau_\alpha(u^*) = \mathcal{O}(\mu \cdot \ell^{\alpha-1})) = \Omega(1/(\gamma \cdot \ell^{3-\alpha}))$, if $3 - \alpha = \omega(1/\log \ell)$;
- (b) $\mathbb{P}(\tau_\alpha(u^*) \leq t) = \mathcal{O}(\mu \nu \cdot t^2 / \ell^{\alpha+1})$, for any step $\ell \leq t = \mathcal{O}(\ell^{\alpha-1} / \nu)$;

$$(c) \mathbb{P}(\tau_\alpha(u^*) < \infty) = \mathcal{O}(\mu \cdot \log \ell / \ell^{3-\alpha}).$$

Using Theorem 4.1, we can easily obtain the following bounds on the parallel hitting time τ_α^k . The proof is given in Section 4.5.

Corollary 4.2. *Let $u^* \in \mathbb{Z}^2$ and $\ell = \|u^*\|_1$, and let k be any integer such that $\log^6 \ell \leq k \leq \ell \log^4 \ell$. Let $\alpha^* = 3 - \frac{\log k}{\log \ell}$, and $\bar{\alpha} = \max\{2, \alpha^* - 4 \frac{\log \log \ell}{\log \ell}\}$. Then:*

- (a) For $\alpha = \alpha^* + 5 \frac{\log \log \ell}{\log \ell}$, $\mathbb{P}(\tau_\alpha^k(u^*) = \mathcal{O}(\ell^2 \log^6 \ell / k)) = 1 - e^{-\omega(\log \ell)}$;
- (b) For $\bar{\alpha} < \alpha < 3$, $\mathbb{P}(\tau_\alpha^k(u^*) \leq (\ell^2/k) \cdot \ell^{(\alpha-\alpha^*)/2} / \log^4 \ell) = o(1)$;
- (c) For $2 < \alpha \leq \alpha^*$, $\mathbb{P}(\tau_\alpha^k(u^*) < \infty) = \mathcal{O}(\log^2 \ell / \ell^{\alpha^*-\alpha})$.

Corollary 4.2(a) states that $\tau_\alpha^k(u^*) \leq (\ell^2/k) \cdot \text{polylog } \ell$, w.h.p., when $\alpha \approx \alpha^* = 3 - \log k / \log \ell$. On the other hand, Corollary 4.2(b) says that the lower bound $\tau_\alpha^k(u^*) \geq (\ell^2/k) \cdot \ell^{(\alpha-\alpha^*)/2} / \text{polylog } \ell$ holds with probability $1 - o(1)$ for any $\alpha \geq \alpha^* - \mathcal{O}(\log \log \ell / \log \ell)$, and Corollary 4.2(c) says that the target is never hit with probability at least $1 - \text{polylog } \ell / \ell^{\alpha^*-\alpha}$, if $\alpha \leq \alpha^*$. Therefore, the optimal hitting time of ℓ^2/k is achieved (modulo $\text{polylog } \ell$ factors) only for values of α very close to α^* , precisely only if $|\alpha - \alpha^*| = \mathcal{O}(\log \log \ell / \log \ell)$.

For the threshold case $\alpha = 3$, similar bounds to those of Theorem 4.1 apply, modulo some $\text{polylog } \ell$ factors, as stated in the next theorem.

Theorem 4.3. *Let $u^* \in \mathbb{Z}^2$ and $\ell = \|u^*\|_1$. Then:*

- (a) $\mathbb{P}(\tau_3(u^*) = \mathcal{O}(\ell^2)) = \Omega(1/\log^4 \ell)$;
- (b) $\mathbb{P}(\tau_3(u^*) \leq t) = \mathcal{O}(t^2 \log \ell / \ell^4)$, for any step t with $\ell \leq t = \mathcal{O}(\ell^2 / \log \ell)$.

The next corollary gives bounds on the parallel hitting time τ_3^k . The proof is similar to that of Corollary 4.2.

Corollary 4.4. *Let $u^* \in \mathbb{Z}^*$ and $\ell = \|u^*\|_1 = \ell$, and let k be any integer such that $\omega(\log^5 \ell) \leq k \leq \ell^2 / \log^2 \ell$. Then:*

- (a) $\mathbb{P}(\tau_3^k(u^*) = \mathcal{O}(\ell^2)) = 1 - e^{-\omega(\log \ell)}$.
- (b) $\mathbb{P}(\tau_3^k(u^*) \leq \ell^2 / \sqrt{k}) = o(1)$.

Corollary 4.4(a) says that $\tau_3^k(u^*) = \mathcal{O}(\ell^2)$, w.h.p., for any $k \geq \text{polylog } \ell$, and Corollary 4.4(b) provides a very crude lower bound indicating that increasing k beyond $\text{polylog } \ell$, can only result in sublinear improvement.

4.1 Proof of Theorems 4.1 and 4.3

We will use the following key lemma, which is shown in Section 4.2. The lemma provides an upper bound on the hitting time of a Lévy flight, assuming the maximum jump length is capped to some appropriate value.

Lemma 4.5 (Lévy flight with $\alpha \in (2, 3]$). *Let h_f be the hitting time of a Lévy flight for target u^* with $\|u^*\|_1 = \ell$. Let \mathcal{E}_t be the event that each of the first t jumps has length less than $(t \log t)^{1/(\alpha-1)}$. Then, there is a $t = \Theta(\ell^{\alpha-1})$ such that:*

(a) $\mathbb{P}(h_f \leq t \mid \mathcal{E}_t) = \Omega(1/(\gamma \ell^{3-\alpha}))$, if $2 < \alpha \leq 3 - \omega(1/\log \ell)$, where $\gamma = \frac{(\log \ell)^{\frac{2}{\alpha-1}}}{(3-\alpha)^2}$;

(b) $\mathbb{P}(h_f \leq t \mid \mathcal{E}_t) = \Omega(1/\log^4 \ell)$, if $\alpha = 3$.

The second lemma we need is an upper bound on the hitting time of a Lévy walk in terms of the hitting time of the capped Lévy flight considered above. The proof proceeds by coupling the two processes, and is given in Section 4.3.

Lemma 4.6. *Let h_f and \mathcal{E}_t be defined as in Lemma 4.5, and let $\tau_\alpha(u^*)$ be the hitting time of a Lévy walk with the same exponent α , for the same target u^* . Then, for any $2 < \alpha \leq 3$ and step t , and for $\mu = \min\{\log \ell, \frac{1}{\alpha-2}\}$,*

$$\mathbb{P}(\tau_\alpha(u^*) = \mathcal{O}(\mu t)) \geq (1 - \mathcal{O}(1/\log t)) \cdot \left(\mathbb{P}(h_f \leq t \mid \mathcal{E}_t) - e^{-t^{\Theta(1)}} \right).$$

The last lemma we need is the following lower bound on the hitting time of a Lévy walk, proved in Section 4.4.

Lemma 4.7. *Let $u^* \in \mathbb{Z}^2$ and $\ell = \|u^*\|_1$. For any step $t \geq \ell$,*

(a) $\mathbb{P}(\tau_\alpha(u^*) \leq t) = \mathcal{O}\left(\frac{\nu \mu t^2}{\ell^{\alpha+1}}\right)$ if $\alpha \neq 3$ and $t = \mathcal{O}(\ell^{\alpha-1}/\nu)$, where $\nu = \min\{\log \ell, \frac{1}{3-\alpha}\}$ and $\mu = \min\{\log \ell, \frac{1}{\alpha-2}\}$;

(b) $\mathbb{P}(\tau_\alpha(u^*) \leq t) = \mathcal{O}\left(\frac{t^2 \log \ell}{\ell^{\alpha+1}}\right)$ if $\alpha = 3$ and $t = \mathcal{O}(\ell^2/\log \ell)$.

Proof of Theorem 4.1. Lemmas 4.5 and 4.6 imply that, for some $t = \Theta(\ell^{\alpha-1})$,

$$\mathbb{P}(\tau_\alpha(u^*) = \mathcal{O}(\mu t)) = \Omega\left(\frac{(3-\alpha)^2}{\ell^{3-\alpha} \log^{\frac{2}{\alpha-1}} \ell}\right),$$

obtaining (a). Parts (b) and (c) follow from Lemma 4.7 and Lemma 3.11, respectively. \square

Proof of Theorem 4.3. The proof proceeds in exactly the same way as the proof of Theorem 4.1, using Lemmas 4.5 and 4.6 to show (a), and Lemma 3.11 to show (b). \square

4.2 Proof of Lemma 4.5

We first define some notation. Then we give an overview of the analysis, before we provide the detailed proof.

Recall that $(L_i^f)_{i \geq 0}$ denotes the Lévy flight process, and $Z_u^f(i) = |\{j: L_j^f = u\} \cap \{1, \dots, i\}|$ is the number of visits to node u in the first i steps. Let $t = \Theta(\ell^{\alpha-1})$ be a step to be fixed later. For each $i \geq 1$, let S_i be the length of the i -th jump of the Lévy flight, i.e.,

$$S_i = \|L_i^f - L_{i-1}^f\|_1.$$

Define also the events

$$E_i = \{S_i \leq (t \log t)^{1/(\alpha-1)}\},$$

and let $\mathcal{E}_i = \bigcap_{j=1}^i E_j$. For each node u and $i \geq 0$, let

$$p_{u,i} = \mathbb{P}(L_i^f = u \mid \mathcal{E}_i),$$

and note that $\mathbb{E} \left[Z_u^f(i) \mid \mathcal{E}_i \right] = \sum_{j=0}^i p_{u,j}$. We partition the set of nodes into disjoint sets $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ defined as follows:

$$\begin{aligned} \mathcal{A}_1 &= \{v: \|v\|_\infty \leq \ell\} \\ \mathcal{A}_2 &= \begin{cases} \{v: \|v\|_1 \leq 2(t \log t)^{1/(\alpha-1)}\} \setminus \mathcal{A}_1 & \text{if } \alpha \in (2, 3) \\ \{v: \|v\|_1 \leq 2\sqrt{t} \log t\} \setminus \mathcal{A}_1 & \text{if } \alpha = 3 \end{cases} \\ \mathcal{A}_3 &= \mathbb{Z}^2 \setminus (\mathcal{A}_1 \cup \mathcal{A}_2). \end{aligned}$$

4.2.1 Proof Overview

Here, we discuss just the case of $\alpha \in (2, 3)$; the case of $\alpha = 3$ is similar. We assume all probability and expectation quantities below are conditional on the event \mathcal{E}_t , and we omit writing this conditioning explicitly. First, we show a simple upper bound on the mean number of visits to \mathcal{A}_1 until step $t = \Theta(\ell^{\alpha-1})$, namely,

$$\sum_{v \in \mathcal{A}_1} \mathbb{E} \left[Z_v^f(t) \right] \leq ct,$$

for a constant $c < 1$. To bound the mean number of visits to \mathcal{A}_2 , we use the monotonicity property from Section 3.2 and the fact that $\|v\|_\infty \geq \ell = \|u^*\|_1$ for all $v \in \mathcal{A}_2$, to obtain

$$\sum_{v \in \mathcal{A}_2} \mathbb{E} \left[Z_v^f(t) \right] \leq |\mathcal{A}_2| \cdot \mathbb{E} \left[Z_{u^*}^f(t) \right] \leq 4(t \log t)^{1/(\alpha-1)} \cdot \mathbb{E} \left[Z_{u^*}^f(t) \right].$$

For the number of visits to \mathcal{A}_3 we obtain the following bound using Chebyshev's inequality,

$$\sum_{v \in \mathcal{A}_3} \mathbb{E} \left[Z_v^f(t) \right] \leq c't / ((3 - \alpha) \log t),$$

where c' is a constant. From the above results, and the fact that the total number of visits to all three sets is t , we get

$$ct + 4(t \log t)^{1/(\alpha-1)} \cdot \mathbb{E} \left[Z_{u^*}^f(t) \right] + c't / ((3 - \alpha) \log t) \geq t,$$

which implies

$$\mathbb{E} \left[Z_{u^*}^f(t) \right] = \Omega \left(t^{\frac{\alpha-3}{\alpha-1}} \cdot (\log t)^{-\frac{2}{\alpha-1}} \right)$$

if $3 - \alpha = \omega(1/\log t)$. We can express the probability of $h_f \leq t$ in terms of the above mean as

$$\mathbb{P}(h_f \leq t) = \mathbb{P} \left(Z_{u^*}^f(t) > 0 \right) = \mathbb{E} \left[Z_{u^*}^f(t) \right] / \mathbb{E} \left[Z_{u^*}^f(t) \mid Z_{u^*}^f(t) > 0 \right],$$

and we have

$$\mathbb{E} \left[Z_{u^*}^f(t) \mid Z_{u^*}^f(t) > 0 \right] \leq \mathbb{E} \left[Z_{\mathbf{0}}^f(t) \right] + 1.$$

We also compute

$$\mathbb{E} \left[Z_{\mathbf{0}}^f(t) \right] = \mathcal{O}(1/(3 - \alpha)^2).$$

Combining the last four equations yields $\mathbb{P}(h_f \leq t) = \Omega \left(t^{\frac{\alpha-3}{\alpha-1}} \cdot (\log t)^{-\frac{2}{\alpha-1}} \cdot (3 - \alpha)^2 \right)$, and substituting $t = \Theta(\ell^{\alpha-1})$ completes the proof.

4.2.2 Detailed Proof

We give now the details of the analysis. Throughout the section $2 < a \leq 3 - \omega(1/\log \ell)$.

Lemma 4.8 (Bound on the visits to \mathcal{A}_1). *For any $t = \Theta(\ell^{\alpha-1})$ large enough, there is a constant $c \in (0, 1)$ such that*

$$\sum_{v \in Q_\ell(\mathbf{0})} \mathbb{E} \left[Z_v^f(t) \mid \mathcal{E}_t \right] \leq ct.$$

Proof. We bound the probability the walk has moved to distance $\frac{5}{2}\ell$ at least once, within time $t = \Theta(\ell^{\alpha-1})$, by the probability that at least one of the performed jumps is no less than 5ℓ (we denote this latter event by H). Indeed, if there is a jump of length at least 5ℓ , the walk moves necessarily to distance no less than $\frac{5}{2}\ell$. Then,

$$\begin{aligned} \mathbb{P} \left(S_j \geq 5\ell \mid S_j \leq (t \log t)^{\frac{1}{\alpha-1}} \right) &= \sum_{k=5\ell}^{(t \log t)^{\frac{1}{\alpha-1}}} \frac{c_\alpha}{k^\alpha} \\ &\stackrel{(*)}{\geq} \frac{c_\alpha}{\alpha-1} \left(\frac{1}{(5\ell)^{\alpha-1}} - \frac{1}{t \log t} \right) \\ &\stackrel{(\star)}{\geq} \frac{c_\alpha}{2(\alpha-1)(5\ell)^{\alpha-1}}, \end{aligned}$$

where $(*)$ follows for the integral test (Lemma A.4), while (\star) easily holds for a large enough ℓ since $t = \Theta(\ell^{\alpha-1})$. Thanks to the mutual independence among the random destinations chosen by the agent, the probability of the event “the desired jump takes place within time $c' \cdot 2(\alpha-1)(5\ell)^{\alpha-1}/c_\alpha$ ” is bounded by

$$1 - \left[1 - \frac{c_\alpha}{2(\alpha-1)(5\ell)^{\alpha-1}} \right]^{c' \frac{2(\alpha-1)(5\ell)^{\alpha-1}}{c_\alpha}} \geq \frac{3}{4},$$

for some constant $c' > 0$ and for ℓ large enough. Hence, by choosing $t \geq 4c' \cdot 2(\alpha-1)(5\ell)^{\alpha-1}/c_\alpha$, the desired jump takes place with probability $\frac{3}{4}$, within time $\frac{t}{4}$. Once reached such a distance (conditional on the previous event), Fig. 3 shows there are at least other 3 mutually disjoint regions which are at least as equally likely as $Q_\ell(\mathbf{0})$ to be visited at any future time.

Thus, the probability to visit $Q_\ell(\mathbf{0})$ at any future time step is at most $\frac{1}{4}$. Observe that

$$\begin{aligned} \mathbb{E} \left[\sum_{v \in Q_\ell(\mathbf{0})} Z_v^f(t) \mid \mathcal{E}_t \right] &= \mathbb{E} \left[\sum_{v \in Q_\ell(\mathbf{0})} Z_v^f(t) \mid H, \mathcal{E}_t \right] \mathbb{P}(H \mid \mathcal{E}_t) + \mathbb{E} \left[\sum_{v \in Q_\ell(\mathbf{0})} Z_v^f(t) \mid H^C, \mathcal{E}_t \right] \mathbb{P}(H^C \mid \mathcal{E}_t) \\ &\leq \left(\frac{1}{4}t + \frac{1}{4} \cdot \frac{3}{4}t \right) \frac{3}{4} + t \cdot \frac{1}{4} \\ &= \frac{t}{4} \left(1 + \frac{3}{4} + \frac{9}{16} \right) = \frac{37}{64}t, \end{aligned}$$

and the proof is completed. \square

For the rest of Section 4.2, let $t = \Theta(\ell^{\alpha-1})$ as in Lemma 4.13

Remark 4.9. The monotonicity property (Lemma 3.9) holds despite the conditional event \mathcal{E}_t . The proof is exactly the same.

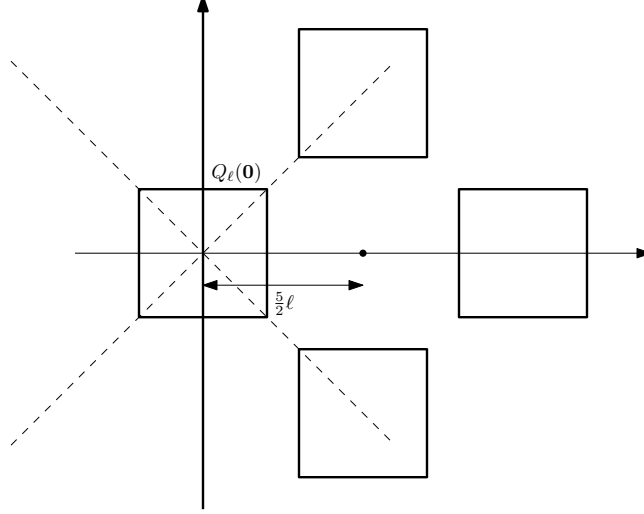


Figure 3: The disjoint zones at least as equally likely as $Q_\ell(\mathbf{0})$ to be visited.

Notice that, from $\mathbb{E} \left[Z_v^f(t) \mid \mathcal{E}_t \right] = \sum_{i=0}^t p_{v,i}$ and the monotonicity property, we easily get the following bound.

Corollary 4.10. $\mathbb{E} \left[Z_u^f(t) \mid \mathcal{E}_t \right] \geq \mathbb{E} \left[Z_v^f(t) \mid \mathcal{E}_t \right]$ for all $v \notin Q_{d_u}(\mathbf{0})$.

Namely, the more the node is “far” (according to the sequence of squares $\{Q_d(\mathbf{0})\}_{d \in \mathbb{N}}$) from the origin, the less it is visited on average. Thus, each node is visited at most as many times as the origin, on average. This easily gives an upper bound on the total number of visits to \mathcal{A}_2 until time t , namely, by taking $u = u^*$ and by observing that each $v \in \mathcal{A}_2$ lies outside $Q_\ell(\mathbf{0})$, we get that the average number of visits to \mathcal{A}_2 is at most the expected number of visits to the target u^* (i.e. $\mathbb{E} \left[Z_{u^*}^f(t) \mid \mathcal{E}_t \right]$) times (any upper bound of) the size of \mathcal{A}_2 : in formula, it is upper bounded by $\mathbb{E} \left[Z_{u^*}^f(t) \mid \mathcal{E}_t \right] \cdot 4(t \log t)^{\frac{2}{\alpha-1}}$ if $\alpha \in (2, 3)$, and by $\mathbb{E} \left[Z_{u^*}^f(t) \mid \mathcal{E}_t \right] \cdot 4t \log^2 t$ if $\alpha = 3$.

The next lemma considers \mathcal{A}_3 .

Lemma 4.11 (Bound on visits to \mathcal{A}_3). *It holds that*

$$\sum_{\substack{v=(x,y): \\ |x|+|y| \geq 2(t \log t)^{\frac{1}{\alpha-1}}}} \mathbb{E} \left[Z_v^f(t) \mid \mathcal{E}_t \right] = \mathcal{O} \left(\frac{t}{(3-\alpha) \log t} \right) \quad \text{if } \alpha \in (2, 3); \quad (5)$$

$$\sum_{\substack{v=(x,y): \\ |x|+|y| \geq 2\sqrt{t} \log t}} \mathbb{E} \left[Z_v^f(t) \mid \mathcal{E}_t \right] = \mathcal{O} \left(\frac{t}{\log t} \right) \quad \text{if } \alpha = 3. \quad (6)$$

Proof. Let $L_{t'}^f$ be the two dimensional random variable representing the coordinates of the node the agent performing the Lévy flight is located in at time t' . Consider the projection of the Lévy flight on the x -axis, namely the random variable $X_{t'}$ such that $L_{t'}^f = (X_{t'}, Y_{t'})$. The random variable $X_{t'}$ can be expressed as the sum of t' random variables S_j^x , $j = 1, \dots, t'$, representing the jumps (with sign) that the projection of the walk takes at each of the t' rounds. The partial distribution

of the jumps along the x -axis, conditional on the event \mathcal{E}_t , can be derived as follows.⁶ For any $0 \leq d \leq (t \log t)^{\frac{1}{\alpha-1}}$,

$$\begin{aligned} & \mathbb{P}\left(S_j^x = \pm d \mid S_j \leq (t \log t)^{\frac{1}{\alpha-1}}\right) \\ &= \left[\frac{1}{2} + \sum_{k=1}^{(t \log t)^{\frac{1}{\alpha-1}}} \frac{c_\alpha}{2k^{\alpha+1}} \right] \mathbb{1}_{d=0} + \left[\frac{c_\alpha}{2d^{\alpha+1}} + \sum_{k=1+d}^{(t \log t)^{\frac{1}{\alpha-1}}} \frac{c_\alpha}{k^{\alpha+1}} \right] \mathbb{1}_{d \neq 0}, \end{aligned} \quad (7)$$

where: $\mathbb{1}_{d \in A}$ returns 1 if $d \in A$ and 0 otherwise, the term

$$\frac{\mathbb{1}_{d=0}}{2} + \frac{c_\alpha}{2d^{\alpha+1}} \mathbb{1}_{d \neq 0}$$

is the probability that the original jump lies along the horizontal axis and has “length” exactly d (there are two such jumps if $d > 0$), and, for $k \geq 1 + d$, the terms

$$\frac{c_\alpha}{2k^{\alpha+1}} \mathbb{1}_{d=0} + \frac{c_\alpha}{k^{\alpha+1}} \mathbb{1}_{d \neq 0}$$

are the probability that the original jump has “length” exactly k and its projection on the horizontal axis has “length” d (there are two such jumps if $d = 0$, and four such jumps if $d > 0$). Observe that (7) is of the order of

$$\Theta \left(\frac{1}{d^{\alpha+1}} + \sum_{k=1+d}^{(t \log t)^{\frac{1}{\alpha-1}}} \frac{1}{k^{\alpha+1}} \right).$$

By the integral test (Lemma A.4 in A), we know that this probability is

$$\mathbb{P}(S_j^x = \pm d \mid \mathcal{E}_j) = \Theta \left(\frac{1}{d^\alpha} \right).$$

Due to symmetry, it is easy to see that $\mathbb{E}[X_{t'} \mid \mathcal{E}_t] = 0$ for each time t' , while

$$\text{Var}(X_{t'} \mid \mathcal{E}_t) = \sum_{i=1}^{t'} \text{Var}(S_j^x \mid \mathcal{E}_j) = t' \text{Var}(S_1^x \mid E_1)$$

since $S_1^x, \dots, S_{t'}^x$ are i.i.d.

As for the case $\alpha \in (2, 3)$, the variance of S_1^x conditioned to the event $E_1 = \left\{ S_1 \leq (t \log t)^{\frac{1}{\alpha-1}} \right\}$, can be bounded as follows

$$\begin{aligned} \text{Var}(S_1^x \mid E_1) &\leq \sum_{k=1}^{(t \log t)^{\frac{1}{\alpha-1}}} \mathcal{O} \left(\frac{k^2}{k^\alpha} \right) \\ &\stackrel{(*)}{=} \mathcal{O} \left(\frac{1}{3-\alpha} \left[(t \log t)^{\frac{3-\alpha}{\alpha-1}} - 1 \right] \right) \\ &= \mathcal{O} \left(\frac{(t \log t)^{\frac{3-\alpha}{\alpha-1}}}{3-\alpha} \right), \end{aligned}$$

⁶We remark that in Appendix C we estimate the unconditional distribution of the jump projection length on the x -axis (Lemma C.1) for any $\alpha > 1$. Nevertheless, in this case we are conditioning on the event the original two dimensional jump is bounded, and thus we cannot make use of Lemma C.1.

where, in (*), we used the integral test (Lemma A.4). Observe that the event $\mathcal{E}_t = \bigcap_{i=1}^t E_i$ has probability

$$\mathbb{P}(\mathcal{E}_t) = 1 - \mathcal{O}\left(\frac{1}{\log t}\right).$$

Then, for each $t' \leq t$, from the Chebyshev's inequality and the fact that $\mathbb{E}[X_{t'} | \mathcal{E}_t] = 0$,

$$\mathbb{P}\left(|X_{t'}| \geq (t \log t)^{\frac{1}{\alpha-1}} \mid \mathcal{E}_t\right) \leq \frac{t' \text{Var}(S_1^x \mid E_1)}{(t \log t)^{\frac{2}{\alpha-1}}} \leq \frac{t \text{Var}(S_1^x \mid E_1)}{(t \log t)^{\frac{2}{\alpha-1}}} = \mathcal{O}\left(\frac{1}{(3-\alpha) \log t}\right),$$

which implies that

$$\mathbb{P}\left(|X_{t'}| \geq (t \log t)^{\frac{1}{\alpha-1}}\right) \leq \mathbb{P}\left(|X_{t'}| \geq (t \log t)^{\frac{1}{\alpha-1}} \mid \mathcal{E}_t\right) + \mathbb{P}(\mathcal{E}_t^C) = \mathcal{O}\left(\frac{1}{(3-\alpha) \log t}\right).$$

Then, the probability that both $X_{t'}$ and $Y_{t'}$ are less than $(t \log t)^{\frac{1}{\alpha-1}}$ (call the corresponding events $A_{x,t'}$ and $A_{y,t'}$, respectively) is

$$\mathbb{P}(A_{x,t'} \cap A_{y,t'}) = \mathbb{P}(A_{x,t'}) + \mathbb{P}(A_{y,t'}) - \mathbb{P}(A_{x,t'} \cup A_{y,t'}) \geq 1 - \mathcal{O}\left(\frac{1}{(3-\alpha) \log t}\right),$$

for any $t' \leq t$. Then, let $Z'(t)$ be the random variable indicating the number of times the Lévy flight visits the set of nodes whose coordinates are both no less than $(t \log t)^{\frac{1}{\alpha-1}}$, until time t . Then,

$$\mathbb{E}[Z'(t) \mid \mathcal{E}_t] \leq \sum_{\substack{v=(x,y) \\ |x|+|y| \geq 2(t \log t)^{\frac{1}{\alpha-1}}}} \mathbb{E}\left[Z_v^f(t) \mid \mathcal{E}_t\right],$$

and

$$\begin{aligned} \mathbb{E}[Z'(t) \mid \mathcal{E}_t] &= \sum_{i=0}^t \mathbb{E}[Z'(i) \mid A_{x,i} \cap A_{y,i}, \mathcal{E}_t] \mathbb{P}(A_{x,i} \cap A_{y,i} \mid \mathcal{E}_t) \\ &\quad + \sum_{i=0}^t \mathbb{E}[Z'(i) \mid (A_{x,i} \cap A_{y,i})^C, \mathcal{E}_t] \mathbb{P}((A_{x,i} \cap A_{y,i})^C \mid \mathcal{E}_t) \\ &= \sum_{i=0}^t \mathbb{E}[Z'(i) \mid (A_{x,i} \cap A_{y,i})^C, \mathcal{E}_t] \mathbb{P}((A_{x,i} \cap A_{y,i})^C \mid \mathcal{E}_t) \\ &\leq t \cdot \mathcal{O}\left(\frac{1}{(3-\alpha) \log t}\right) = \mathcal{O}\left(\frac{t}{(3-\alpha) \log t}\right), \end{aligned}$$

which proves Eq. (5).

As for the case $\alpha = 3$, the variance of S_1^x conditional on E_1 is $\mathcal{O}(\log(t \log t))$. Then, we look at the probability that $|X_{t'}|$ is at least $\sqrt{t} \cdot \log t$ conditional on \mathcal{E}_t , which is, again, $\mathcal{O}(1/\log t)$. Finally, the proof proceeds in exactly the same way of the previous case, obtaining Eq. (6). \square

Lemma 4.12. For $t = \Theta(\ell^{\alpha-1})$,

$$ct + \mathbb{E}\left[Z_{u^*}^f(t) \mid \mathcal{E}_t\right] \cdot 4(t \log t)^{\frac{2}{\alpha-1}} + \mathcal{O}\left(\frac{t}{(3-\alpha) \log t}\right) \geq t \quad \text{if } \alpha \in (2, 3); \quad (8)$$

$$ct + \mathbb{E}\left[Z_{u^*}^f(t) \mid \mathcal{E}_t\right] \cdot 4t \log^2 t + \mathcal{O}\left(\frac{t}{\log t}\right) \geq t \quad \text{if } \alpha = 3. \quad (9)$$

Proof. Suppose the agent has made t jumps for some $t = \Theta(\ell^{\alpha-1})$ (the same t of Lemma 4.8), thus visiting exactly t nodes. Then,

$$\mathbb{E} \left[\sum_{v \in \mathbb{Z}^2} Z_v^f(t) \mid \mathcal{E}_t \right] = t.$$

As for Eq. (8), we observe that, from Lemma 4.8, the number of visits to $\mathcal{A}_1 = Q_\ell(\mathbf{0})$ until time t is at most ct , for some constant $c \in (0, 1)$. From Lemma 4.11, the number of visits to \mathcal{A}_3 is at most $\mathcal{O}(t / ((3 - \alpha) \log t))$. Thanks to Corollary 4.10, each of the remaining nodes, i.e., the nodes in \mathcal{A}_2 (whose size is at most $4(t \log t)^{\frac{2}{\alpha-1}}$), is visited by the agent at most $\mathbb{E} \left[Z_{u^*}^f(t) \mid \mathcal{E}_t \right]$ times. It follows that

$$ct + \mathbb{E} \left[Z_{u^*}^f(t) \mid \mathcal{E}_t \right] \cdot 4(t \log t)^{\frac{2}{\alpha-1}} + \mathcal{O} \left(\frac{t}{(3 - \alpha) \log t} \right) \geq t.$$

As for Eq. (9), we proceed as for the first case above, by noticing that the number of visits to \mathcal{A}_2 is at most $\mathbb{E} \left[Z_{u^*}^f(t) \mid \mathcal{E}_t \right] \cdot (4t \log^2 t)$. This gives Eq. (9). \square

The next two lemmas provide a clean relationship between the probability to hit a node u within time t to the average number of visits to the origin and to the average number of visits to u itself. In particular, the first lemma estimate the average number of visits to the origin. For any $t \geq 0$ and $\alpha \in (2, 3]$, let $\mathbb{E} \left[Z_{\mathbf{0}}^f(t) \mid \mathcal{E}_t \right] = a_t(\alpha)$.

Lemma 4.13 (Visits to the origin).

- (a) If $\alpha \in (2, 3)$, then $a_t(\alpha) = \mathcal{O}(1/(3 - \alpha)^2)$.
- (b) If $\alpha = 3$, then $a_t(3) = \mathcal{O}(\log^2 t)$.

Proof. For the case $\alpha \in (2, 3)$, we proceed as follows. Since $\mathbb{E} \left[Z_{\mathbf{0}}^f(t) \mid \mathcal{E}_t \right] = \sum_{k=1}^t p_{\mathbf{0},k}$, it suffices to accurately bound the probability $p_{\mathbf{0},k}$ for each $k = 1, \dots, t$. Let us make a partition of the natural numbers in the following way

$$\mathbb{N} = \bigcup_{t'=1}^{\infty} \left[\mathbb{N} \cap [2t' \log t', 2(t'+1) \log(t'+1)) \right].$$

For each $k \in \mathbb{N}$, there exists t' such that $k \in [2t' \log t', 2(t'+1) \log(t'+1))$. Then, within $2t' \log t'$ steps, we claim that the walk has moved to distance $\lambda = \frac{(t')^{\frac{1}{\alpha-1}}}{2}$ at least once, with probability $\Omega\left(\frac{1}{(t')^2}\right)$. Indeed, if there is one jump of length at least 2λ , then the walk has necessarily moved to a distance at least λ from the origin. We now bound the probability that one jump is at least 2λ . For the integral test and for $\lambda > 0$, we get

$$\begin{aligned} \mathbb{P} \left(S_j \geq 2\lambda \mid S_j \leq (t \log t)^{\frac{1}{\alpha-1}} \right) &\geq \frac{1}{\mathbb{P} \left(S_j \leq (t \log t)^{\frac{1}{\alpha-1}} \right)} \left[\int_{2\lambda}^{(t \log t)^{\frac{1}{\alpha-1}}} \frac{c_\alpha}{s^\alpha} ds \right] \\ &\geq \frac{c_\alpha}{\alpha - 1} \left(\frac{1}{t'} - \frac{1}{t \log t} \right) \\ &\geq \frac{c_\alpha}{\alpha - 1} \left(\frac{1 - \frac{t'}{t \log t}}{t'} \right) \end{aligned}$$

$$\begin{aligned}
&\geq \frac{c_\alpha}{\alpha-1} \left(\frac{1 - \frac{1}{2 \log(t') \log t}}{t'} \right) \\
&= \Omega\left(\frac{1}{t'}\right),
\end{aligned}$$

where the last inequality holds since $2t' \log t' \leq t$. Thus, the probability that the first $2t' \log t'$ jumps are less than 2λ is

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{j=1}^{2t' \log t'} \{S_j < 2\lambda\} \mid \mathcal{E}_t\right) &\stackrel{(*)}{=} \left[1 - \mathbb{P}\left(S_1 < 2\lambda \mid S_1 \leq (t \log t)^{\frac{1}{\alpha-1}}\right)\right]^{2t' \log t'} \\
&\geq \left[1 - \Omega\left(\frac{1}{t'}\right)\right]^{2t' \log t'} \\
&= \mathcal{O}\left(\frac{1}{(t')^2}\right),
\end{aligned}$$

where in (*) we used the independence among the agent's jumps. Once the agent reaches such a distance, Lemma 3.9 implies that there are at least $\lambda^2 = \Omega\left((t')^{\frac{2}{\alpha-1}}\right)$ different nodes that are at least as equally likely as $\mathbf{0}$ to be visited at any given future time. Thus, the probability to reach the origin at any future time is at most $\mathcal{O}\left(\frac{1}{(t')^{\frac{2}{\alpha-1}}}\right) = \mathcal{O}\left(\frac{1}{(t')^{1+\epsilon}}\right)$ with $\epsilon = (3-\alpha)/(\alpha-1) > 0$: in particular the bound holds for $p_{\mathbf{0},k}$. Observe that in an interval $[2t' \log t', 2(t'+1) \log(t'+1))$ there are

$$2(t'+1) \log(t'+1) - 2t' \log t' = 2t' \left[\log\left(1 + \frac{1}{t'}\right) \right] + 2 \log(t'+1) = \mathcal{O}(\log t')$$

integers. Let L_t^f be the two-dimensional random variable denoting the node visited at time t by an agent which started from the origin, and let $H_{t'}$ be the event $\bigcup_{j=1}^{2t' \log t'} \{S_j \geq 2\lambda\}$. Observe that, by the law of total probability,

$$p_{\mathbf{0},k} = \mathbb{P}\left(L_t^f = \mathbf{0} \mid H_{t'}, \mathcal{E}_t\right) \mathbb{P}(H_{t'} \mid \mathcal{E}_t) + \mathbb{P}\left(L_t^f = \mathbf{0} \mid H_{t'}^C, \mathcal{E}_t\right) \mathbb{P}(H_{t'}^C \mid \mathcal{E}_t).$$

Thus, if $I_{t'} = [2t' \log t', 2(t'+1) \log(t'+1))$, we get

$$\begin{aligned}
\sum_{k=1}^t p_{\mathbf{0},k} &\leq \sum_{t'=1}^t \sum_{k \in I_{t'}} p_{\mathbf{0},k} \\
&\leq \sum_{t'=1}^t \left[\mathbb{P}\left(L_t^f = \mathbf{0} \mid H_{t'}, \mathcal{E}_t\right) \mathbb{P}(H_{t'} \mid \mathcal{E}_t) + \mathbb{P}\left(L_t^f = \mathbf{0} \mid H_{t'}^C, \mathcal{E}_t\right) \mathbb{P}(H_{t'}^C \mid \mathcal{E}_t) \right] \mathcal{O}(\log t') \\
&\leq \sum_{t'=1}^t \left[\mathcal{O}\left(\frac{1}{(t')^{1+\epsilon}}\right) + \mathcal{O}\left(\frac{1}{(t')^2}\right) \right] \mathcal{O}(\log t') \\
&= \sum_{t'=1}^t \mathcal{O}\left(\frac{\log t'}{(t')^{1+\epsilon}}\right) \stackrel{(*)}{=} \mathcal{O}\left(\frac{1}{\epsilon^2}\right) = \mathcal{O}\left(\frac{1}{(3-\alpha)^2}\right),
\end{aligned}$$

where for \star we used the integral test and partial integration. In particular, it holds that

$$\int_1^t \frac{\log(x)}{x^{1+\epsilon}} dx = \frac{1}{\epsilon^2} \left[\frac{\epsilon(1 - \log x)}{x^\epsilon} - \frac{\epsilon+1}{x^\epsilon} \right]_1^t.$$

For the case $\alpha = 3$, we can consider the same argument above for the previous case where we fix $\lambda = \sqrt{t}$. Then the proof proceeds as in the previous case by observing that the average number of visits until time t is, now, of magnitude $\mathcal{O}(\log^2 t)$. \square

Lemma 4.14. *Let $u \in \mathbb{Z}^2$ be any node. Then,*

$$(i) \mathbb{E} \left[Z_u^f(t) \mid \mathcal{E}_t \right] \leq a_t(\alpha),$$

$$(ii) 1 \leq \mathbb{E} \left[Z_u^f(t) \mid Z_u^f(t) > 0, \mathcal{E}_t \right] \leq a_t(\alpha),$$

$$(iii) \mathbb{E} \left[Z_u^f(t) \mid \mathcal{E}_t \right] / a_t(\alpha) \leq \mathbb{P} \left(Z_u^f(t) > 0 \mid \mathcal{E}_t \right) \leq \mathbb{E} \left[Z_u^f(t) \mid \mathcal{E}_t \right].$$

Proof. Claim (i) is a direct consequence of (ii), since $\mathbb{E} \left[Z_u^f(t) \mid Z_u^f(t) > 0, \mathcal{E}_t \right] \geq \mathbb{E} \left[Z_u^f(t) \mid \mathcal{E}_t \right]$. As for Claim (ii), let τ be the first time the agent visits u . Then, conditional on $Z_u^f(t) > 0$, τ is at most t , and

$$\mathbb{E} \left[Z_u^f(t) \mid Z_u^f(t) > 0, \mathcal{E}_t \right] = \mathbb{E} \left[Z_{\mathbf{0}}^f(t - \tau) \mid \tau \leq t, \mathcal{E}_t \right] \leq \mathbb{E} \left[Z_{\mathbf{0}}^f(t) \mid \mathcal{E}_t \right] = a_t(\alpha).$$

Notice that this expectation is at least 1 since we have the conditional event. As for Claim (iii), let us explicitly write the term $\mathbb{E} \left[Z_u^f(t) \mid Z_u^f(t) > 0, \mathcal{E}_t \right] \cdot \mathbb{P} \left(Z_u^f(t) > 0 \mid \mathcal{E}_t \right)$:

$$\begin{aligned} & \sum_{i=1}^t i \mathbb{P} \left(Z_u^f(t) = i \mid Z_u^f(t) > 0, \mathcal{E}_t \right) \cdot \mathbb{P} \left(Z_u^f(t) > 0 \mid \mathcal{E}_t \right) \\ &= \sum_{i=1}^t i \frac{\mathbb{P} \left(Z_u^f(t) = i, Z_u^f(t) > 0, \mathcal{E}_t \right)}{\mathbb{P} \left(Z_u^f(t) > 0, \mathcal{E}_t \right)} \cdot \frac{\mathbb{P} \left(Z_u^f(t) > 0, \mathcal{E}_t \right)}{\mathbb{P}(\mathcal{E}_t)} \\ &= \sum_{i=1}^t i \frac{\mathbb{P} \left(Z_u^f(t) = i, Z_u^f(t) > 0, \mathcal{E}_t \right)}{\mathbb{P}(\mathcal{E}_t)} \\ &= \sum_{i=1}^t i \mathbb{P} \left(Z_u^f(t) = i \mid \mathcal{E}_t \right) \\ &= \mathbb{E} \left[Z_u^f(t) \mid \mathcal{E}_t \right]. \end{aligned}$$

Then,

$$\mathbb{E} \left[Z_u^f(t) \mid \mathcal{E}_t \right] \geq \mathbb{P} \left(Z_u^f(t) > 0 \mid \mathcal{E}_t \right) = \frac{\mathbb{E} \left[Z_u^f(t) \mid \mathcal{E}_t \right]}{\mathbb{E} \left[Z_u^f(t) \mid Z_u^f(t) > 0, \mathcal{E}_t \right]} \geq \frac{\mathbb{E} \left[Z_u^f(t) \mid \mathcal{E}_t \right]}{a_t(\alpha)},$$

since, from Claim (ii), $\mathbb{E} \left[Z_u^f(t) \mid Z_u^f(t) > 0, \mathcal{E}_t \right] \leq a_t(\alpha)$. \square

We can now complete the proof of Lemma 4.5, as follows. From Lemma 4.12 we have that $\mathbb{E} \left[Z_{u^*}^f(t) \mid \mathcal{E}_t \right] = \Omega \left((3 - \alpha)^2 / (t^{(3-\alpha)/(\alpha-1)} (\log t)^{2/(\alpha-1)}) \right)$ if $\alpha \in (2, 3)$ and $3 - \alpha = \omega(1/\log t)$ and $\mathbb{E} \left[Z_{u^*}^f(t) \mid \mathcal{E}_t \right] = \Omega \left(1/(\log t)^2 \right)$ if $\alpha = 3$. Then, Lemma 4.13 and claim (iii) of Lemma 4.14 give the results by substituting $t = \Theta(\ell^{\alpha-1})$.

4.3 Proof of Lemma 4.6

Let S_j be the random variable denoting the j -th jump-length. From Eq. (4), we get

$$\mathbb{P}\left(S_j > (t \log t)^{\frac{1}{\alpha-1}}\right) = \Theta\left(\frac{1}{t \log t}\right).$$

Let E_j be the event $\left\{S_j \leq (t \log t)^{\frac{1}{\alpha-1}}\right\}$, and let \mathcal{E}_t be the intersection of E_j for $j = 1, \dots, t$. Notice that, by the union bound, the probability of \mathcal{E}_t is $1 - \mathcal{O}(1/\log t)$. We next apply the multiplicative form of the Chernoff bound to the sum of S_j , conditional on the event \mathcal{E}_t . This is possible since the variable $S_j/(t \log t)^{\frac{1}{\alpha-1}}$ takes values in $[0, 1]$. To this aim, we first bound the expectation of the sum of the random variables S_j , for $j = 1, \dots, t$ conditional on \mathcal{E}_t .

$$\begin{aligned} \mathbb{E}\left[\sum_{j=1}^t S_j \mid \mathcal{E}_t\right] &= \sum_{j=1}^t \mathbb{E}[S_j \mid \mathcal{E}_t] = \Theta(t) + t \frac{c_\alpha}{\mathbb{P}(\mathcal{E}_t)} \sum_{d=1}^{(t \log t)^{\frac{1}{\alpha-1}}} \frac{d}{d^\alpha} \\ &\leq \Theta(t) + 2c_\alpha t \sum_{d=1}^{(t \log t)^{\frac{1}{\alpha-1}}} \frac{1}{d^{\alpha-1}} \\ &\stackrel{(a)}{\leq} \mathcal{O}(\mu t), \end{aligned}$$

where in (a) we have $\mu = \min\{\log \ell, \frac{1}{\alpha-2}\}$ for the integral test (Lemma A.4). We now use the Chernoff bound (Theorem A.2) on the normalized sum of all jumps, to show that such a sum is at most linear in $\mathcal{O}(\mu t)$ with probability $1 - \exp(-t^{\Theta(1)})$, conditional on \mathcal{E}_t . In formula,

$$\begin{aligned} \mathbb{P}\left(\sum_{j=1}^t S_j \geq \Theta(\mu t) \mid \mathcal{E}_t\right) &= \mathbb{P}\left(\frac{\sum_{j=1}^t S_j}{(t \log t)^{\frac{1}{\alpha-1}}} \geq \frac{2\Theta(\mu t)}{(t \log t)^{\frac{1}{\alpha-1}}} \mid \mathcal{E}_t\right) \\ &\leq \exp\left(-\frac{2\Theta(\mu t)}{3\left((t \log t)^{\frac{1}{\alpha-1}}\right)}\right) \\ &\leq \exp\left(-\Theta\left(\frac{\mu t^{\frac{\alpha-2}{\alpha-1}}}{(\log t)^{\frac{1}{\alpha-1}}}\right)\right) \leq \exp\left(-\Theta\left(t^{\frac{\alpha-2}{2(\alpha-1)}}\right)\right). \end{aligned}$$

Then, define

$$\begin{aligned} A &= \{\text{the Lévy walk finds the target within time } \Theta(\mu t)\}, \\ A_1 &= \left\{\sum_{j=1}^t S_j = \Theta(\mu t)\right\}, \text{ and} \\ A_2 &= \{\text{the Lévy flight finds the target within } t \text{ jumps}\}. \end{aligned}$$

Observe that the event $A_1 \cap A_2$ implies that the Lévy walk finds the target within t jumps, which, in turn, implies the event A . Indeed, $A_1 \cap A_2$ implies the target is found in one of the t jump endpoints, and the overall amount of “travel” time is $\Theta(\mu t)$. Let $p(t) = \mathbb{P}(h_f \leq t \mid \mathcal{E}_t)$. Then

$$\mathbb{P}(A) \geq \mathbb{P}(A_1, A_2) \geq \mathbb{P}(A_1, A_2, \mathcal{E}_t)$$

$$\begin{aligned}
&= \mathbb{P}(\mathcal{E}_t) [\mathbb{P}(A_1 | \mathcal{E}_t) + \mathbb{P}(A_2 | \mathcal{E}_t) - \mathbb{P}(A_1 \cup A_2 | \mathcal{E}_t)] \\
&\geq \left(1 - \mathcal{O}\left(\frac{1}{\log t}\right)\right) \left[1 - \exp(-t^{\Theta(1)}) + p(t) - 1\right] \\
&= \left(1 - \mathcal{O}\left(\frac{1}{\log t}\right)\right) \left(p(t) - \exp(-t^{\Theta(1)})\right),
\end{aligned}$$

where in the second line we used the definition of conditional probability and the inclusion-exclusion principle, and in the third line we used that $\mathbb{P}(\mathcal{E}_t) = (1 - \mathcal{O}(1/\log t))$, $\mathbb{P}(A_1 | \mathcal{E}_t) \geq 1 - \exp(-t^{\Theta(1)})$, and $\mathbb{P}(A_1 \cup A_2 | \mathcal{E}_t) \leq 1$.

4.4 Proof of Lemma 4.7

Let X_i be the x -coordinate of the agent at the end of the i -th jump. For any $i \leq t$, we bound the probability that $X_i > \ell/4$. The probability that there is a jump whose length is at least ℓ among the first i jumps is $\Theta(i/\ell^{\alpha-1})$. We first consider the case $\alpha \in (2, 3)$. Conditional on the event that the first i jump-lengths are all smaller than ℓ (event C_i), the expectation of X_i is zero and its variance is

$$\Theta(1) + i \cdot \sum_{d=1}^{\ell/4} \Theta\left(\frac{d^2}{d^\alpha}\right) = \Theta(i \cdot \nu \ell^{3-\alpha}),$$

for the integral test (Lemma A.4), where $\nu = \min\{\log \ell, \frac{1}{3-\alpha}\}$. Chebyshev's inequality implies that

$$\mathbb{P}(|X_i| \geq \ell/4 | C_i) \leq \frac{\Theta(i \cdot \nu \ell^{3-\alpha})}{\Theta(\ell^2)} = \Theta\left(\frac{i\nu}{\ell^{\alpha-1}}\right).$$

Since the conditional event has probability $1 - \Theta(i/\ell^{\alpha-1})$, then the ‘‘unconditional’’ probability of the event $|X_i| \leq \ell/4$ is

$$\left[1 - \Theta\left(\frac{i}{\ell^{\alpha-1}}\right)\right] \left[1 - \mathcal{O}\left(\frac{i\nu}{\ell^{\alpha-1}}\right)\right] = 1 - \mathcal{O}\left(\frac{\nu t}{\ell^{\alpha-1}}\right),$$

since $i \leq t$, for t which is some $\mathcal{O}(\ell^{\alpha-1}/\nu)$. The same result holds analogously for Y_i (the y -coordinate of the agent after the i -th jump), thus obtaining $|X_i| + |Y_i| \leq \ell/2$, with probability $1 - \mathcal{O}(\nu t/\ell^{\alpha-1})$ by the union bound.

During the first jump-phase, thanks to Corollary 3.6, the probability the agents visits the target is $\mathcal{O}(1/\ell^\alpha)$. Let $2 \leq i \leq t$. We want to estimate the probability that during i -th jump-phase the agents visits the target, having the additional information that $t = \mathcal{O}(\ell^{\alpha-1})$. As in the proof of Lemma 3.10, we consider the node u^* where the target is located on, and the rhombus centered in u^* that contains the nodes within distance $\ell/4$ from u^* , namely $B_{\ell/4}(u^*)$. Let F_i be the event that during the i -th jump-phase the agent visits the target; let V_{i-1} be the event that the $(i-1)$ -th jump ends in $B_{\ell/4}(u^*)$, and let W_{i-1} be the event that the $(i-1)$ -th jump ends at distance farther than $\ell/2$ from the origin. Finally, let L_i^f be the two-dimensional random variable denoting the coordinates of the node the agent is located on at the end of the i -th jump-phase. Then,

$$\begin{aligned}
\mathbb{P}(F_i | V_{i-1}) \mathbb{P}(V_{i-1} | W_{i-1}) &= \sum_{v \in B_{\ell/4}(u^*)} \mathbb{P}(F_i | L_i^f = v) \mathbb{P}(L_i^f = v | W_{i-1}) \\
&\leq \mathcal{O}\left(\frac{1}{\ell^2}\right) \sum_{v \in B_{\ell/4}(u^*)} \mathbb{P}(F_i | L_i^f = v),
\end{aligned}$$

where in the above inequalities we used the monotonicity property (Lemma 3.9, which holds since the process restricted to the jump endpoints is a Lévy flight), and the fact that, for each $v \in B_{\ell/4}(u^*)$, there are at least $\Theta(\ell^2)$ nodes at distance at least $\ell/2$ from the origin which are more likely to be the destination of the i -th jump than v . Then, we proceed as in the proof of Lemma 3.10 and obtain

$$\mathbb{P}(F_i | V_{i-1}) \mathbb{P}(V_{i-1} | W_{i-1}) = \mathcal{O}\left(\frac{\mu}{\ell^2}\right), \quad (10)$$

where $\mu = \min\{\log \ell, \frac{1}{\alpha-2}\}$. By the law of total probabilities, we get

$$\begin{aligned} \mathbb{P}(F_i) &= \mathbb{P}(F_i | W_{i-1}) \mathbb{P}(W_{i-1}) + \mathbb{P}(F_i | W_{i-1}^C) \mathbb{P}(W_{i-1}^C) \\ &= [\mathbb{P}(F_i | W_{i-1}, V_{i-1}) \mathbb{P}(V_{i-1} | W_{i-1}) + \mathbb{P}(F_i | W_{i-1}, V_{i-1}^C) \mathbb{P}(V_{i-1}^C | W_{i-1})] \mathbb{P}(W_{i-1}) \\ &\quad + \mathbb{P}(F_i | W_{i-1}^C) \mathbb{P}(W_{i-1}^C) \\ &\leq [\mathbb{P}(F_i | V_{i-1}) \mathbb{P}(V_{i-1} | W_{i-1}) + \mathbb{P}(F_i | W_{i-1}, V_{i-1}^C)] \mathbb{P}(W_{i-1}) + \mathbb{P}(F_i | W_{i-1}^C) \mathbb{P}(W_{i-1}^C) \\ &\leq \left[\mathcal{O}\left(\frac{\mu}{\ell^2}\right) + \mathcal{O}\left(\frac{1}{\ell^\alpha}\right) \right] \mathcal{O}\left(\frac{\nu t}{\ell^{\alpha-1}}\right) + \mathcal{O}\left(\frac{1}{\ell^\alpha}\right) = \mathcal{O}\left(\frac{\nu \mu t}{\ell^{\alpha+1}}\right), \end{aligned} \quad (11)$$

where for second-last inequality we used that $V_{i-1} \subset W_{i-1}$ and that $\mathbb{P}(V_{i-1}^C | W_{i-1}) \leq 1$, while for the last inequality we used Eq. (10), and that $\mathbb{P}(F_i | W_{i-1}, V_{i-1}^C) = \mathcal{O}(1/\ell^\alpha)$, which is true because the jump starts in a node whose distance from the target is $\Omega(\ell)$, and that $\mathbb{P}(F_i | W_{i-1}^C) = \mathcal{O}(1/\ell^\alpha)$, which is true for the same reason.

Thus, by the union bound and by Eq. (11), the probability that during at least one between the t jump-phases, the agent finds the target is

$$\mathcal{O}\left(\frac{1}{\ell^\alpha}\right) + (t-1) \mathcal{O}\left(\frac{\nu \mu t}{\ell^{\alpha+1}}\right) = \mathcal{O}\left(\frac{\nu \mu t^2}{\ell^{\alpha+1}}\right)$$

since $t \geq \ell$, which gives the first claim of the lemma by observing that within time t at most t jumps can be performed.

Consider now the case $\alpha = 3$. The proof proceeds exactly as in the first case, with the only key difference that the variance of X_i is $\Theta(i \log \ell)$. This means that the probability that $|X_i|$ is at least $\ell/4$ conditional to C_i is $\mathcal{O}(\log \ell / \ell^2)$, and the “unconditional” probability that $|X_i|$ is less than $\ell/4$ is $1 - \mathcal{O}(t \log \ell / \ell^2)$. It thus follows that

$$\mathbb{P}(F_i) = \mathcal{O}(t \log \ell / \ell^4).$$

Then we get the second claimed bound of the lemma: $\mathcal{O}(t^2 \log \ell / \ell^4)$.

4.5 Proof of Corollary 4.2

From Theorem 4.1(a) and the independence among the agents, we get that

$$\mathbb{P}\left(\tau_\alpha^k(u^*) = \mathcal{O}(\mu \ell^{\alpha-1})\right) = 1 - \left[1 - \Omega\left(\frac{1}{\gamma \ell^{3-\alpha}}\right)\right]^k \geq 1 - e^{-\Omega\left(\frac{k}{\gamma \ell^{3-\alpha}}\right)},$$

where we have used the inequality $1 - x \leq e^{-x}$ for all $x \in \mathbb{R}$. Then, if $\alpha = \alpha^* + 5 \frac{\log \log \ell}{\log \ell}$,

$$\mathbb{P}\left(\tau_\alpha^k(u^*) = o\left(\frac{\ell^2 \log^6 \ell}{k}\right)\right) = 1 - e^{-\omega(\log \ell)},$$

since $\mu \leq \frac{1}{\alpha-2} \leq \frac{\log \ell}{\log \log \ell}$, $\ell^{\alpha-1} = \frac{\ell^2 \log^5 \ell}{k}$, $\gamma = o(\log^4 \ell)$ and $\ell^{3-\alpha} = \frac{k}{\log^5 \ell}$, thus giving Claim (a). From Theorem 4.1.(b) and the independence among the agents, we get

$$\mathbb{P}\left(\tau_\alpha^k(u^*) > t\right) = \left[1 - \mathcal{O}\left(\frac{\mu \nu t^2}{\ell^{\alpha+1}}\right)\right]^k,$$

for $\ell \leq t = o(\ell^{\alpha-1}/\nu)$. Let $t = \frac{\ell^2 \cdot \ell^{\frac{\alpha-\alpha^*}{2}}}{k \log^4 \ell}$ which is a function in $o(\ell^{\alpha-1}/\nu)$ since $\alpha > \bar{\alpha}$. If $t \geq \ell$, we get

$$\mathbb{P}\left(\tau_\alpha^k(u^*) > t\right) = \left[1 - \mathcal{O}\left(\frac{\mu \nu \ell^4}{k^2 \ell^{\alpha^*+1} \log^8 \ell}\right)\right]^k \geq e^{-\mathcal{O}\left(\frac{1}{\log^6 \ell}\right)} = 1 - \mathcal{O}\left(\frac{1}{\log^6 \ell}\right),$$

since $\mu \nu \leq \log^2 \ell$, $\ell^{\alpha^*+1} = \ell^4/k$. Notice that, in the inequality we have used that $1 - x \geq e^{-\frac{x}{1-x}}$ if $x < 1$, and in the last equality we have used the Taylor's expansion of the exponential function. If $t < \ell$, we get $\mathbb{P}(\tau_\alpha^k(u^*) > t) = 1$ (at least ℓ steps are needed to reach the target). Therefore, Claim (b) follows. Finally, from Theorem 4.1.(c) and the independence among the agents, we get

$$\mathbb{P}\left(\tau_\alpha^k(u^*) = \infty\right) = \left[1 - \mathcal{O}\left(\frac{\mu \log \ell}{\ell^{3-\alpha}}\right)\right]^k = \left[1 - \mathcal{O}\left(\frac{\log^2 \ell}{k \ell^{\alpha^*-\alpha}}\right)\right]^k \geq \exp\left(-\mathcal{O}\left(\frac{\log^2 \ell}{\ell^{\alpha^*-\alpha}}\right)\right),$$

since $\mu \leq \log^2 \ell$, $\ell^{3-\alpha} = \ell^{3-\alpha^*} \cdot \ell^{\alpha^*-\alpha} = k \ell^{\alpha^*-\alpha}$. Notice that in the last inequality we have used that $1 - x \geq e^{-\frac{x}{1-x}}$ for $x < 1$, which is our case since $k \geq \log^6 \ell$. Hence, we have Claim (c).

5 The Case $\alpha \in (1, 2]$

We now analyze the hitting time of Lévy walks with parameter $\alpha \in (1, 2]$, which is the exponent range for which the jump length has unbounded mean and unbounded variance. We show the following theorems.

Theorem 5.1. *Let $\alpha \in [1 + \epsilon, 2)$, where $\epsilon > 0$ is an arbitrarily small constant. Let $u^* \in \mathbb{Z}^2$, and $\ell = \|u^*\|_1$. Let $\mu = \min\{\log \ell, \frac{1}{2-\alpha}\}$. Then:*

- (a) $\mathbb{P}(\tau_\alpha(u^*) = \mathcal{O}(\ell)) = \Omega(1/\mu\ell)$;
- (b) $\mathbb{P}(\tau_\alpha(u^*) < \infty) = \mathcal{O}(\mu \log \ell / \ell)$.

Theorem 5.2. *Let $u^* \in \mathbb{Z}^2$ and $\ell = \|u^*\|_1$. Then:*

- (a) $\mathbb{P}(\tau_2(u^*) = \mathcal{O}(\ell)) = \Omega(1/\ell \log \ell)$;
- (b) $\mathbb{P}(\tau_2(u^*) < \infty) = \mathcal{O}(\log^2 \ell / \ell)$.

The above theorems imply the following bounds on the parallel hitting time.

Corollary 5.3. *Let $\alpha \in [1 + \epsilon, 2]$, where $\epsilon > 0$ is an arbitrarily small constant. Let $u^* \in \mathbb{Z}^2$ and $\ell = \|u^*\|_1$. Let $\mu = \min\{\log \ell, \frac{1}{2-\alpha}\}$. Then:*

- (a) $\mathbb{P}(\tau_\alpha^k(u^*) = \mathcal{O}(\ell)) = 1 - e^{-\omega(\log \ell)}$, if $k = \omega(\ell \log^2 \ell)$;
- (b) $\mathbb{P}(\tau_\alpha^k(u^*) < \infty) = o(1)$, if $k = o(\ell / \log^2 \ell)$.

Recall also that the trivial lower bound $\tau_\alpha^k(u^*) \geq \ell$ holds.

5.1 Proof of Theorems 5.1 and 5.2

We first show the following lemma, which bounds the hitting time for a single Lévy walk.

Lemma 5.4. *Let $\alpha \in (1, 2]$ and $u^* \in \mathbb{Z}^2$ with $\|u^*\|_1 = \ell$. Then,*

- (a) $\mathbb{P}(\tau_\alpha(u^*) = \mathcal{O}(\ell)) = \Omega\left(\frac{1}{\mu\ell}\right)$, if $\alpha \in [1 + \epsilon, 2)$ for an arbitrarily small constant $\epsilon > 0$, where $\mu = \min\{\frac{1}{\alpha-2}, \log \ell\}$.
- (b) $\mathbb{P}(\tau_\alpha(u^*) = \mathcal{O}(\ell)) = \Omega\left(\frac{1}{\ell \log \ell}\right)$, if $\alpha = 2$.

Proof. Consider a single agent moving according the Lévy walk with parameter $\alpha \in (1, 2]$. By Equation (4) in Section 3, the probability the agent chooses a jump of length at least ℓ is of the order of $\Theta(1/d^{\alpha-1})$. Let c be some constant to be fixed later, and let μ_α be equal to $\min\{\frac{1}{\alpha-2}, \log \ell\}$ if $\alpha < 2$, and to $\log \ell$ if $\alpha = 2$. Then, the probability that all the first $c\ell^{\alpha-1}/\mu_\alpha$ jumps have length less than ℓ is

$$\left(1 - \Theta\left(\frac{1}{\ell^{\alpha-1}}\right)\right)^{\frac{c\ell^{\alpha-1}}{\mu_\alpha}}$$

which is greater than positive constant strictly less than 1 thanks to the inequality $\exp(-x/(1-x)) < (1-x)$ for $x < 1$. Let E_i be the event that the i -th jump-length is less than ℓ and $\mathcal{E}_i = \cap_{1 \leq j \leq i} E_j$. By what has been said before, we have

$$\mathbb{P}(\mathcal{E}_i) \geq \Theta(1) \quad \text{for all } i \leq c\ell^{\alpha-1}/\mu_\alpha.$$

Conditional on \mathcal{E}_i , the sum of the first i jumps is at most $3\ell/4$ with constant probability. Indeed, if $j < i$, the expected value of S_j is, for the integral test (Lemma A.4)

$$\mathbb{E}[S_j | \mathcal{E}_i] = \mathcal{O}(1) + \sum_{d=1}^{\ell-1} \frac{c_\alpha d}{d^\alpha} = \mathcal{O}(\mu_\alpha \ell^{2-\alpha}).$$

Thus,

$$\mathbb{E}\left[\sum_{j=1}^i S_j | \mathcal{E}_i\right] \leq \sum_{j=1}^{c\ell^{\alpha-1}/\mu_\alpha} \mathbb{E}[S_j | \mathcal{E}_i] = \mathcal{O}(c\ell).$$

We choose c small enough so that this expression is less than $\ell/2$. Conditional on \mathcal{E}_i , the $\{S_j\}_{j \leq i}$ random variables are non negative and we can use the Markov's inequality to get that their sum is bounded by $3\ell/4$ with constant probability. Indeed

$$\mathbb{P}\left(\sum_{j=1}^i S_j \geq \frac{3\ell}{4} \mid \mathcal{E}_i\right) \leq \mathbb{P}\left(\sum_{j=1}^i S_j \geq \frac{3\mathbb{E}\left[\sum_{j=1}^i S_j \mid \mathcal{E}_i\right]}{2} \mid \mathcal{E}_i\right) \leq \frac{2}{3}.$$

The latter implies there is at least constant probability the agent has displacement at most $3\ell/4$ from the origin during the first $c\ell^{\alpha-1}/\mu_\alpha$ jumps and in time $\mathcal{O}(\ell)$ (since the sum of all jumps is at most linear), without any conditional event:

$$\mathbb{P}\left(\sum_{j \leq i} S_j \leq 3\ell/4\right) \geq \mathbb{P}\left(\sum_{j \leq i} S_j \leq 3\ell/4 \mid \mathcal{E}_i\right) \mathbb{P}(\mathcal{E}_i)$$

$$\geq \Theta(1)$$

for each $i \leq c\ell^{\alpha-1}/\mu_\alpha$.

Define the event $W_i = \{\sum_{j \leq i} S_j \leq 3\ell/4\}$. We now compute the probability that, given $i \leq c\ell^{\alpha-1}/\mu_\alpha$, in the first $i-1$ jumps the displacement has been at most $3\ell/4$ and during the i -th jump-phase the agent finds the target. Let F_i be such the latter event. Since

$$\mathbb{P}(F_i, W_{i-1}) = \mathbb{P}(F_i \mid W_{i-1}) \mathbb{P}(W_{i-1}),$$

we estimate $\mathbb{P}(F_i \mid W_{i-1})$. Let L_t^w be the two-dimensional random variable representing the coordinates of the nodes the Lévy walk visits at time t . If t_i is the time the agent ends the i -th jump-phase, we have

$$\mathbb{P}(F_i \mid W_{i-1}) \geq \sum_{v \in Q_{3\ell/4}(\mathbf{0})} \mathbb{P}\left(F_i \mid L_{t_{i-1}}^w = v, W_{i-1}\right) \mathbb{P}\left(L_{t_{i-1}}^w = v \mid W_{i-1}\right).$$

By Corollary 3.6, the term $\mathbb{P}\left(F_i \mid L_{t_{i-1}}^w = v, W_{i-1}\right)$ is $\Theta(1/\ell^\alpha)$, and, since $L_{t_{i-1}}^w \in Q_{3\ell/4}(\mathbf{0})$ is implied by W_{i-1} , we have

$$\sum_{v \in Q_{3\ell/4}(\mathbf{0})} \mathbb{P}\left(F_i \mid L_{t_{i-1}}^w = v, W_{i-1}\right) \mathbb{P}\left(L_{t_{i-1}}^w = v \mid W_{i-1}\right) \geq \Theta\left(\frac{1}{\ell^\alpha}\right) \cdot \mathbb{P}\left(L_{t_{i-1}}^w \in Q_{3\ell/4}(\mathbf{0}) \mid W_{i-1}\right) = \Theta\left(\frac{1}{\ell^\alpha}\right),$$

implying $\mathbb{P}(F_i, W_{i-1}) = \Omega\left(\frac{1}{\ell^\alpha}\right)$ for all $i \leq c\ell^{\alpha-1}/(1 + \log \ell \cdot \mathbf{1}_{[\alpha=2]})$. Then, for the chain rule, the probability that none of the events $F_i \cap W_{i-1}$ holds for each $i \leq \ell^{\alpha-1}/\log(c\ell)$ is

$$\begin{aligned} \mathbb{P}\left(\bigcup_{i \leq \frac{c\ell^{\alpha-1}}{\mu_\alpha}} (F_i \cap W_{i-1})\right) &= 1 - \mathbb{P}\left(\bigcap_{i \leq \frac{c\ell^{\alpha-1}}{\mu_\alpha}} (F_i^C \cup W_{i-1}^C)\right) \\ &= 1 - \prod_{i \leq \frac{c\ell^{\alpha-1}}{\mu_\alpha}} \mathbb{P}\left(F_i^C \cup W_{i-1}^C \mid \bigcap_{j \leq i-1} (J_j^C \cup W_{j-1}^C)\right) \\ &= 1 - \prod_{i \leq \frac{c\ell^{\alpha-1}}{\mu_\alpha}} \left(1 - \mathbb{P}\left(F_i \cap W_{i-1} \mid \bigcap_{j \leq i-1} (J_j^C \cup W_{j-1}^C)\right)\right) \\ &\stackrel{(*)}{\geq} 1 - \prod_{i \leq \frac{c\ell^{\alpha-1}}{\mu_\alpha}} \left(1 - \mathbb{P}\left(F_i \cap W_{i-1}, \bigcap_{j \leq i-1} (J_j^C \cup W_{j-1}^C)\right)\right) \\ &\stackrel{(\star)}{=} 1 - \prod_{i \leq \frac{c\ell^{\alpha-1}}{\mu_\alpha}} (1 - \mathbb{P}(F_i \cap W_{i-1})) \\ &= 1 - \left(1 - \Omega\left(\frac{1}{\ell^\alpha}\right)\right)^{\frac{c\ell^{\alpha-1}}{\mu_\alpha}} \\ &\geq 1 - e^{-\Omega\left(\frac{c}{\mu_\alpha \ell}\right)} = \Omega\left(\frac{c}{\mu_\alpha \ell}\right), \end{aligned}$$

where, $(*)$ holds since $\mathbb{P}(A | B) \geq \mathbb{P}(A, B)$, (\star) holds since $W_{i-1} \subseteq (W_{j-1}^C \cup J_j^C)$ for $j \leq i-1$, and the last equality holds by the inequality $e^{-x} \geq 1 - x$ for all x , and by the Taylor's expansion of $f(x) = e^x$. Then, there is probability at least $\Omega(c/(\mu_\alpha \ell))$ to find the target within time $\mathcal{O}(\ell)$. \square

Lemma 5.4 gives part (a) of Theorems 5.1 and 5.2, while part (b) comes from Lemma 3.11.

5.2 Proof of Corollary 5.3

First, suppose $\alpha \in [1 + \epsilon, 2)$. From Corollary 5.3(a) and the independence between agents, we get that

$$\mathbb{P}\left(\tau_\alpha^k(u^*) = \mathcal{O}(\ell)\right) = 1 - \left[\Omega\left(\frac{1}{\mu\ell}\right)\right]^k \geq 1 - e^{-\Omega\left(\frac{k}{\ell \log \ell}\right)},$$

where we used the inequality $1 - x \leq e^{-x}$ for every real x and the bound $\mu \leq \log \ell$. From Corollary 5.3(b) and the independence between agents, we get that

$$\mathbb{P}\left(\tau_\alpha^k(u^*) = \infty\right) = \left[1 - \mathcal{O}\left(\frac{\mu \log \ell}{\ell}\right)\right]^k \geq \exp\left(-\mathcal{O}\left(\frac{k \log^2 \ell}{\ell}\right)\right),$$

where we used again $\mu \leq \log \ell$ and the inequality $1 - x \geq e^{-\frac{x}{1-x}}$ for every real x . Thus, if $k = o(\ell/\log^2 \ell)$, for the Taylor's expansion of the exponential function, we get hitting time ∞ with probability $1 - o(1)$. For $\alpha = 2$ the proof proceeds exactly in the same way.

6 The Case $\alpha \in (3, \infty)$

We analyze now the hitting time of Lévy walks with parameter $\alpha \in (3, \infty)$, which is the exponent range for which the jump length has bounded mean and bounded variance.

Theorem 6.1. *Let $\alpha \in (3, \infty)$ and $u^* \in \mathbb{Z}^2$ with $\ell = \|u^*\|_1$. Let $\nu = \min\{\log \ell, \frac{1}{\alpha-3}\}$, and $\gamma = \frac{\alpha^2}{(\alpha-3)^2}$. Then:*

- (a) $\mathbb{P}(\tau_\alpha(u^*) = \mathcal{O}(\ell^2 \log^2 \ell)) = \Omega(1/(\gamma \log^4 \ell))$, if $\alpha \geq 3 + \omega(\log \log \ell / \log \ell)$;
- (b) $\mathbb{P}(\tau_\alpha(u^*) \leq t) = \mathcal{O}(\nu \cdot t^2 / \ell^{\alpha+1})$, for any step $\ell \leq t = \mathcal{O}(\ell^2 / \nu)$.

From the above result, we easily obtain the following bounds on the parallel hitting time.

Corollary 6.2. *Let $\alpha \in (3, \infty)$, $u^* \in \mathbb{Z}^2$, $\ell = \|u^*\|_1$, and $1 \leq k \leq o(\ell^2)$. Let $\nu = \min\{\log \ell, \frac{1}{\alpha-3}\}$ and $\gamma = \frac{\alpha^2}{(\alpha-3)^2}$. Then:*

- (a) $\mathbb{P}(\tau_\alpha^k(u^*) = \mathcal{O}(\ell^2 \log^2 \ell)) = 1 - e^{-\omega(\log \ell)}$, if $k \geq \Omega(\log^6 \ell)$ and $\alpha \geq 3 + \omega\left(\frac{\log \log \ell}{\log \ell}\right)$;
- (b) $\mathbb{P}\left(\tau_\alpha^k(u^*) \leq \ell^2 / \sqrt{k}\right) = 1 - o(1)$.

Corollary 6.2(a) says that $\tau_\alpha^k(u^*) = \mathcal{O}(\ell^2 \log^2 \ell)$, w.h.p., for $k \geq \text{polylog } \ell$ and $\alpha \geq 3 + \omega\left(\frac{\log \log \ell}{\log \ell}\right)$, and Corollary 6.2(b) provides a crude lower bound indicating that increasing k beyond $\text{polylog } \ell$, can only result in sublinear improvement.

6.1 Proof of Theorem 6.1

The structure of the proof is similar to that for Theorems 4.1 and 4.3. We will use the next three lemmas, which are analogous to Lemmas 4.5 to 4.7, respectively

Lemma 6.3 (Lévy flight with $\alpha \in (3, \infty)$). *Let h_f be the hitting time of a Lévy flight for target $u^* \in \mathbb{Z}^2$, and let $\ell = \|u^*\|_1$. If $\alpha - 3 = \omega(\log \log \ell / \log \ell)$, then*

$$\mathbb{P}(h_f = \mathcal{O}(\ell^2 \log^2 \ell)) = \Omega\left(\frac{(\alpha - 3)^3}{\alpha^2 \log^4 \ell}\right).$$

Lemma 6.4. *Let h_f be defined as in Lemma 6.3, and let $\tau_\alpha(u^*)$ be the hitting time of a Lévy walk with the same $\alpha \in (3, \infty)$, for the same target. Then, for every step t ,*

$$\mathbb{P}(\tau_\alpha(u^*) = \mathcal{O}(t)) \geq \mathbb{P}(h_f \leq t) - \mathcal{O}\left(\frac{\alpha}{(\alpha - 3)t}\right).$$

Lemma 6.5. *Let $\alpha \in (3, \infty)$, $u^* \in \mathbb{Z}^2$, and $\ell = \|u^*\|_1$. For any step t such that $\ell \leq t = \mathcal{O}(\ell^2/\nu)$,*

$$\mathbb{P}(\tau_\alpha(u^*) \leq t) = \mathcal{O}(\nu t^2 / \ell^{\alpha+1}),$$

where $\nu = \min\{\log \ell, \frac{1}{\alpha-3}\}$.

The proofs of the above lemmas are given in Sections 6.2 to 6.4, respectively. Using these lemmas can now prove our main result as follows. From Lemmas 6.3 and 6.4, by substituting $t = \Theta(\ell^2 \log^2 \ell)$, we get

$$\mathbb{P}(\tau_\alpha(u^*) = \mathcal{O}(\ell^2 \log^2 \ell)) = \Omega\left(\frac{(\alpha - 3)^2}{\alpha^2 \log^4 \ell}\right) - \mathcal{O}\left(\frac{\alpha}{(\alpha - 3)\ell^2 \log^2 \ell}\right) = \Omega\left(\frac{(\alpha - 3)^2}{\alpha^2 \log^4 \ell}\right)$$

if $3 + \omega(\log \log \ell / \log \ell) \leq \alpha$, which is part (a) of Theorem 6.1. Also, by applying Lemma 6.5, we get part (b) of Theorem 6.1.

6.2 Proof of Lemma 6.3

The proof is similar to that of Lemma 4.5, in Section 4.2. We reuse some of the notation defined there. Namely, $(L_i^f)_{i \geq 0}$ denotes the Lévy flight process, S_i is the length of the i -th jump, and $Z_u^f(i)$ is the number of visits to u in the first i steps. We also use the following modified definitions: For each node u and $i \geq 0$, we let

$$p_{u,i} = \mathbb{P}(L_i^f = u),$$

thus $\mathbb{E}[Z_u^f(i)] = \sum_{j=0}^i p_{u,j}$. We partition \mathbb{Z}^2 into sets $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ as follows. Let $\delta > 0$ be some value to be fixed later. Then,

$$\begin{aligned} \mathcal{A}_1 &= \{v: \|v\|_\infty \leq \ell\} \\ \mathcal{A}_2 &= \{v: \|v\|_1 \leq 4\sqrt{2(1+\delta)t \log t}\} \setminus \mathcal{A}_1 \\ \mathcal{A}_3 &= \mathbb{Z}^2 \setminus (\mathcal{A}_1 \cup \mathcal{A}_2). \end{aligned}$$

6.2.1 Proof Overview

First, we bound the mean number of visits to \mathcal{A}_1 until a given step t . We show that

$$\mathbb{E} \left[Z_{\mathbf{0}}^f(t) \right] = \mathcal{O}(\log^2 t).$$

The monotonicity property from Section 3.2 implies that

$$\sum_{v \in \mathcal{A}_1} \mathbb{E} \left[Z_v^f(t) \right] \leq |\mathcal{A}_1| \cdot \mathbb{E} \left[Z_{\mathbf{0}}^f(t) \right] \leq c(3 - \alpha)(\ell \log t)^2 / \alpha,$$

where c is a constant. To bound the mean number of visits to \mathcal{A}_2 , as before, we use the monotonicity property again to obtain

$$\sum_{v \in \mathcal{A}_2} \mathbb{E} \left[Z_v^f(t) \right] \leq |\mathcal{A}_2| \cdot \mathbb{E} \left[Z_{u^*}^f(t) \right] \leq 32(1 + \delta)t \log^2 t \cdot \mathbb{E} \left[Z_{u^*}^f(t) \right].$$

For the number of visits to \mathcal{A}_3 , using a Chernoff-Hoeffding bound we show that

$$\sum_{v \in \mathcal{A}_3} \mathbb{E} \left[Z_v^f(t) \right] \leq c' \left(t^{1 - (\alpha - 3)/2} + 1 \right),$$

for some constant c' and for $\delta = \Theta(1/(\alpha - 3)^2)$ large enough. Combining the above we obtain

$$c\ell^2 \log^2 t + 32(1 + \delta)t \log^2 t \cdot \mathbb{E} \left[Z_{u^*}^f(t) \right] + c' \left(t^{1 - (\alpha - 3)/2} + 1 \right) \geq t.$$

By choosing $t = \Theta(\ell^2 \log^2 \ell)$ and $\alpha - 3 = \omega(\log \log \ell / \log \ell)$, the above inequality implies

$$\mathbb{E} \left[Z_{u^*}^f(t) \right] = \Omega \left(\frac{1}{(1 + \delta) \log^2 \ell} \right).$$

Since

$$\mathbb{P}(h_f \leq t) = \mathbb{P} \left(Z_{u^*}^f(t) > 0 \right) = \mathbb{E} \left[Z_{u^*}^f(t) \right] / \mathbb{E} \left[Z_{u^*}^f(t) \mid Z_{u^*}^f(t) > 0 \right],$$

and $\mathbb{E} \left[Z_{u^*}^f(t) \mid Z_{u^*}^f(t) > 0 \right] \leq \mathbb{E} \left[Z_{\mathbf{0}}^f(t) \right] + 1 = \mathcal{O}(\log^2 t)$, we obtain $\mathbb{P}(h_f \leq t) = \Omega((\alpha - 3)^2 / (\alpha^2 \log^4 \ell))$.

6.2.2 Detailed Proof

Lemma 6.6. *For any $t \geq 0$, $\mathbb{E} \left[Z_{\mathbf{0}}^f(t) \right] = b_t = \mathcal{O}(\log^2 t)$.*

Proof. First, we show the following. Let $L_{t'}^f$ be the two dimensional random variable representing the coordinates of the agent performing the Lévy flight at time t' . Let $L_{t'}^f = (X_{t'}, Y_{t'})$ and consider the projection of the Lévy flight on the x -axis $X_{t'}$: it can be expressed as the sum of t' random variables S_j^x , $j = 1, \dots, t'$, representing the projection of the jumps (with sign) of the agent on the x -axis at times $j = 1, \dots, t'$. The partial distribution of the jumps along the x -axis is given by Lemma C.1 in Appendix C, and states that, for any given $d \geq 1$, we have

$$\mathbb{P}(S_j^x = \pm d) = \Theta \left(\frac{1}{d^\alpha} \right).$$

Since $\mathbb{E} \left[Z_{\mathbf{0}}^f(t) \right] = \sum_{k=1}^t p_{\mathbf{0},k}$, it suffices to accurately bound the probability $p_{\mathbf{0},k}$ for each $k = 1, \dots, t$. Let us partition the natural numbers in the following way

$$\mathbb{N} = \bigcup_{t'=1}^{\infty} \left[\mathbb{N} \cap [2t' \log t', 2(t'+1) \log(t'+1)) \right].$$

For each $k \in \mathbb{N}$, there exists t' such that $k \in [2t' \log t', 2(t'+1) \log(t'+1))$. Then, within $2t' \log t'$ steps the walk has moved to distance $\Theta(\sqrt{t'})$ at least once, with probability $\Omega\left(\frac{1}{(t')^2}\right)$. Indeed, the sequence $\{S_j^x\}_{1 \leq j \leq t'}$ consists of i.i.d. r.v.s with zero mean and finite variance $\Omega(1)$. Thus, the central limit theorem (Theorem A.1 in Appendix A) implies that the variable

$$\frac{S_1^x + \dots + S_{t'}^x}{\sigma \sqrt{t'}}$$

converges in distribution to a standard normal random variable Z , with $\sigma = \Omega(1)$. Let $\epsilon > 0$ be a small enough constant, then there exists a \bar{t}' large enough, such that for all $t' \geq \bar{t}'$ it holds that

$$\mathbb{P} \left(S_1^x + \dots + S_{t'}^x \geq \sigma \sqrt{t'} \right) \geq \mathbb{P}(Z \geq 1) - \epsilon = \frac{c}{2} > 0,$$

for some suitable constant $c \in (0, 1)$. The symmetrical results in which the normalized sum is less than $-\sigma \sqrt{t'}$ holds analogously. Thus, for all $t' \geq \bar{t}'$, we have that $\left| \sum_{j=1}^{t'} S_j^x \right| \geq \sigma \sqrt{t'}$ with constant probability $c > 0$. In $2t' \log t'$ jumps, we have $2 \log t'$ sets of t' consequent i.i.d. jumps. For independence, the probability that at least in one round before round $2t' \log t'$ the Lévy flight has displacement $\Theta(\sqrt{t'})$ from the origin is at least

$$1 - (1 - c)^{2 \log t'} = 1 - \mathcal{O}\left(\frac{1}{(t')^2}\right).$$

Once reached such a distance, there are at least $\lambda^2 = \Theta(t')$ different nodes that are at least as equally likely as $\mathbf{0}$ to be visited at any given future time for the monotonicity property (Lemma 3.9). Thus, the probability to reach the origin at any future time is at most $\mathcal{O}(1/t')$. Let $H_{t'}$ be the event that in any instant before time $2t' \log t'$ the Lévy flight has displacement at least $\Theta(\sqrt{t'})$. Observe that

$$p_{\mathbf{0},k} = \mathbb{P} \left(L_t^f = \mathbf{0} \mid H_{t'} \right) \mathbb{P}(H_{t'}) + \mathbb{P} \left(L_t^f = \mathbf{0} \mid H_{t'}^C \right) \mathbb{P}(H_{t'}^C),$$

by the law of total probability. We remark that in an interval $[2t' \log t', 2(t'+1) \log(t'+1))$ there are

$$2(t'+1) \log(t'+1) - 2t' \log t' = 2t' \left[\log \left(1 + \frac{1}{t'} \right) \right] + 2 \log(t'+1) = \mathcal{O}(\log t')$$

integers. Thus, if $I_{t'} = [2t' \log t', 2(t'+1) \log(t'+1))$, we have

$$\begin{aligned} \sum_{k=1}^t p_{\mathbf{0},k} &\leq \sum_{t'=1}^t \sum_{k \in I_{t'}} p_{\mathbf{0},k} \\ &\leq \sum_{t'=1}^t \left[\mathbb{P} \left(L_t^f = \mathbf{0} \mid H_{t'} \right) \mathbb{P}(H_{t'}) + \mathbb{P} \left(L_t^f = \mathbf{0} \mid H_{t'}^C \right) \mathbb{P}(H_{t'}^C) \right] \mathcal{O}(\log t') \\ &\leq \bar{t}' + \sum_{t'=\bar{t}'}^t \left[\mathcal{O}\left(\frac{1}{t'}\right) + \mathcal{O}\left(\frac{1}{(t')^2}\right) \right] \mathcal{O}(\log t') = \mathcal{O}(\log^2 t), \end{aligned}$$

since \bar{t}' is a constant. □

We have also the following.

Lemma 6.7. *For any node $u \in \mathbb{Z}^2$, it holds that*

- (i) $\mathbb{E} \left[Z_u^f(t) \right] \leq b_t$;
- (ii) $1 \leq \mathbb{E} \left[Z_u^f(t) \mid Z_u^f(t) > 0 \right] \leq b_t$;
- (iii) $\mathbb{E} \left[Z_u^f(t) \right] / b_t \leq \mathbb{P} \left(Z_u^f(t) > 0 \right) \leq \mathbb{E} \left[Z_u^f(t) \right]$.

Proof. The proof is exactly as that of Lemma 4.14. □

Thus, the total number of visits to \mathcal{A}_1 is upper bounded by $m_{u^*} b_t$, where $m_{u^*} = |Q_\ell(\mathbf{0})|$. Furthermore, from the monotonicity property (Lemma 3.9), the following holds.

Corollary 6.8. *For any node u in \mathbb{Z}^2 , we have $\mathbb{E} \left[Z_u^f(t) \right] \geq \mathbb{E} \left[Z_v^f(t) \right]$ for all $v \notin Q_{d_u}(\mathbf{0})$.*

Namely, almost all the nodes that are “further” than u from the origin are less likely to be visited at any given future time. This easily gives an upper bound on the total number of visits to \mathcal{A}_2 until time t , namely, by taking $u = u^*$ and by observing that each $v \in \mathcal{A}_2$ lies outside $Q_\ell(\mathbf{0})$, we get that the average number of visits to \mathcal{A}_2 is at most the expected number of visits to the target u^* (i.e. $\mathbb{E} \left[Z_{u^*}^f(t) \right]$) times (any upper bound of) the size of \mathcal{A}_2 : in formula, it is upper bounded by $\mathbb{E} \left[Z_{u^*}^f(t) \right] \cdot 32(1 + \delta)t \log^2 t$.

We also give a bound to the average number of visits to nodes that are further roughly $\sqrt{t} \cdot \log t$ from the origin.

Lemma 6.9. *A sufficiently large positive real δ exists such that $\delta = \Theta \left(1/(\alpha - 3)^2 \right)$ and*

$$\sum_{\substack{v \in \mathbb{Z}^2 : \\ \|v\|_1 \geq 4\sqrt{2(1+\delta)t \log t}}} \mathbb{E} \left[Z_v^f(t) \right] = \mathcal{O} \left(t^{1 - \frac{\alpha-3}{2}} + 1 \right).$$

Proof. Since $\alpha > 3$, the expectation and the variance of a single jump-length are finite. By Equation (4) in the preliminaries (Section 3), the probability a jump length is at least \sqrt{t} is $\Theta \left(1/t^{\frac{\alpha-1}{2}} \right)$. Let us call A_j the event that the j -th jump-length is less than \sqrt{t} . Let us recall that L_j^f is the random variable denoting the coordinates of the nodes the corresponding Lévy flight visits at the end of the j -th jump. We can write $L_j^f = (X_j, Y_j)$, where X_j is x -coordinate of the Lévy walk after the j -th jump, and Y_j is the y -coordinate. Then, X_j can be seen as the sum $\sum_{i=1}^j S_i^x$ of j random variables representing the projections of the jumps along the x -axis. For symmetry, $\mathbb{E}[X_j] = 0$ for each j , while $\text{Var}(X_j) = j \text{Var}(S_1^x) = \mathcal{O}(j/(\alpha - 3) + j) = \mathcal{O}(\alpha j/(\alpha - 3))$ since S_1^x has finite variance $\mathcal{O}(1 + 1/(\alpha - 3))$. This comes by observing that $S_1^x \leq S_1$. Then, conditional on $A = \cap_{i=1}^t A_i$, we can apply the Chernoff bound (Theorem A.3) on the sum of the first j jumps, for $j \leq t$. We have

$$\begin{aligned} \mathbb{P} \left(|X_t| \geq 2\sqrt{2(1+\delta)t \log t} \mid A \right) &\leq 2 \exp \left(- \frac{8(1+\delta)t \cdot \log^2 t}{\mathcal{O} \left(\frac{\alpha t}{\alpha-3} \right) + \Theta \left(\sqrt{(1+\delta)t \cdot \log t} \right) \sqrt{t}} \right) \\ &\leq 2 \exp \left(-\Theta \left(\frac{\alpha-3}{\alpha} \sqrt{1+\delta} \cdot \log t \right) \right) \end{aligned}$$

$$\leq \frac{2}{t^{\Theta\left(\frac{\alpha-3}{\alpha}\sqrt{1+\delta}\right)}},$$

which is less than $1/t^2$ if we choose $\delta = \Theta(1/(\alpha-3)^2)$ large enough. The same result holds for the random variable X_j for each $j < t$, since the variance of X_j is smaller than the variance of X_t . Notice that

$$\begin{aligned} \mathbb{P}\left(\cap_{j=1}^t \{|X_j| < 2\sqrt{2(1+\delta)t \log t}\} \mid A\right) &= 1 - \mathbb{P}\left(\cup_{j=1}^t \{|X_t| \geq 2\sqrt{2(1+\delta)t \log t}\} \mid A\right) \\ &\geq 1 - \frac{t}{t^2} = 1 - \frac{1}{t}, \end{aligned}$$

and that

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^C) = 1 - \mathbb{P}\left(\cup_{j=1}^t A_j^C\right) \geq 1 - \mathcal{O}\left(\frac{t}{t^{\frac{\alpha-1}{2}}}\right) = 1 - \mathcal{O}\left(\frac{1}{t^{\frac{\alpha-3}{2}}}\right).$$

An analogous argument holds for the random variable Y_t conditioned to the event A . Then,

$$\begin{aligned} &\mathbb{P}\left(\cap_{j=1}^t \{\|X_j\|_1 < 2\sqrt{2(1+\delta)t \cdot \log t}\}, \cap_{j=1}^t \{\|Y_j\|_1 < 2\sqrt{2(1+\delta)t \cdot \log t}\}\right) \\ &\geq \mathbb{P}\left(\cap_{j=1}^t \{\|X_j\|_1 < 2\sqrt{2(1+\delta)t \cdot \log t}\}, \cap_{j=1}^t \{\|Y_j\|_1 < 2\sqrt{2(1+\delta)t \cdot \log t}\} \mid A\right) \mathbb{P}(A) \\ &\geq \left(2\mathbb{P}\left(\cap_{j=1}^t \{\|X_j\|_1 < 2\sqrt{2(1+\delta)t \cdot \log t}\} \mid A\right) - 1\right) \mathbb{P}(A) \\ &\stackrel{(*)}{\geq} \left[2\left(1 - \frac{1}{t}\right) - 1\right] \left(1 - \mathcal{O}\left(\frac{1}{t^{\frac{\alpha-3}{2}}}\right)\right) \\ &\geq 1 - \mathcal{O}\left(\frac{1}{t^{\frac{\alpha-3}{2}}} + \frac{1}{t}\right), \end{aligned}$$

where $(*)$ holds for symmetry (the distribution of Y_t is the same as the one of X_t) and for the union bound. Thus, in t jumps (which take at least time t), the walk has never reached distance $4\sqrt{2(1+\delta)t \cdot \log t}$, w.h.p. The average number of visits until time t to nodes at distance at least $4\sqrt{2(1+\delta)t \cdot \log t}$ is then less than $t \cdot \mathcal{O}\left(1/t^{\frac{\alpha-3}{2}} + 1/t\right) = \mathcal{O}\left(t^{1-\frac{\alpha-3}{2}} + 1\right)$. \square

The following puts together the previous estimations in order to get a lower bound on the average number of visits the target u^* . Let $\delta > 0$ be as given in Lemma 6.9 for the rest of the section.

Lemma 6.10. *For every node $u^* \in \mathbb{Z}^2$ and every time $t \geq 1$,*

$$m_{u^*} b_t + \mathbb{E}\left[Z_{u^*}^f(t)\right] \cdot 32(1+\delta)(t \log^2 t) + \mathcal{O}\left(t^{1-\frac{\alpha-3}{2}} + 1\right) \geq t.$$

Proof. Suppose the agent has made t jumps, thus visiting t nodes. Then,

$$\mathbb{E}\left[\sum_{v \in \mathbb{Z}^2} Z_v^f(t)\right] = t.$$

We divide the plane in different zones, and we bound the number of visits over each zone in expectation. From Lemma 6.7, the number of visits inside $\mathcal{A}_1 = Q_\ell(\mathbf{0})$ until time t is at most $m_{u^*} b_t$, where $m_{u^*} = |Q_\ell(\mathbf{0})| = 4\ell^2$. From Lemma 6.9, the number of visits \mathcal{A}_3 is at most $\mathcal{O}\left(t^{1-\frac{\alpha-3}{2}}\right)$.

Each of the remaining nodes, i.e. the nodes in \mathcal{A}_2 , which are at most $32(1 + \delta)(t \log^2 t)$ in total, is visited by the agent at most $\mathbb{E} \left[Z_u^f(t) \right]$ times, for Corollary 6.8. Then, we have that

$$m_{u^*} b_t + \mathbb{E} \left[Z_{u^*}^f(t) \right] \cdot 32(1 + \delta)(t \log^2 t) + \mathcal{O} \left(t^{1 - \frac{\alpha-3}{2}} + 1 \right) \geq t. \quad \square$$

We can now complete the proof of Lemma 6.3 as follows. Lemma 6.10 implies that

$$\mathbb{E} \left[Z_{u^*}^f(t) \right] = \Omega \left(\frac{t - t^{1 - \frac{\alpha-3}{2}} - 1 - m_{u^*} b_t}{(1 + \delta)t \log^2 t} \right),$$

while Lemma 6.7 implies that

$$\mathbb{P}(h_f \leq t) = \Omega \left(\frac{t - t^{1 - \frac{\alpha-3}{2}} - 1 - m_{u^*} b_t}{(1 + \delta)t \log^2 t \cdot b_t} \right).$$

Lemma 6.6 gives $b_t = \mathcal{O}(\log^2 t)$, while Lemma 6.9 gives $\delta = \Theta(1/(\alpha - 3)^2)$. If $t = \Theta(\ell^2 \log^2 \ell)$ is large enough and $\alpha - 3 = \omega(\log \log \ell / \log \ell)$, so that $t - t^{1 - \frac{\alpha-3}{2}} - m_{u^*} b_t = \Theta(t)$, we get the result.

6.3 Proof of Lemma 6.4

If S_i is the random variable yielding the i -th jump length, then it has expectation $\Theta(1)$ and variance. This means that the sum $\bar{S}_t = \sum_{i=1}^t S_i$ has expectation $\Theta(t)$ and variance $\mathcal{O}(t + t/(\alpha - 3)) = \mathcal{O}(\alpha t / (\alpha - 3))$. Then, from Chebyshev's inequality,

$$\mathbb{P}(\bar{S}_t \geq \Theta(t) + t) \leq \frac{\text{Var}(\bar{S}_t)}{t^2} = \mathcal{O} \left(\frac{\alpha}{(\alpha - 3)t} \right).$$

Hence,

$$\mathbb{P}(h_w = \mathcal{O}(t)) \geq \mathbb{P}(h_f \leq t, \bar{S}_t \leq \Theta(t) + t) = \mathbb{P}(h_f \leq t) - \mathcal{O} \left(\frac{\alpha}{(\alpha - 3)t} \right),$$

where the latter equality is obtained using the union bound.

6.4 Proof of Lemma 6.5

Let X_i be the x -coordinate of the agent at the end of the i -th jump-phase. For any $i \leq t$, we bound the probability that $X_i > \ell/4$. The probability that there is a jump whose length is at least ℓ among the first i jumps is $\mathcal{O}(i/\ell^{\alpha-1})$ for the union bound. Conditional on the event that the first i jump lengths are all smaller than ℓ (event \mathcal{E}_i), the expectation of X_i is zero and its variance is

$$i \cdot \sum_{d=1}^{\ell/4} \Theta(d^2/d^\alpha) = \mathcal{O}(i\nu\ell^{3-\alpha}),$$

for the integral test (Lemma A.4), where $\nu = \min\{\log \ell, \frac{1}{\alpha-3}\}$. Chebyshev's inequality implies that

$$\mathbb{P}(|X_i| \geq \ell/4 \mid \mathcal{E}_i) \leq \frac{\mathcal{O}(i\nu\ell^{3-\alpha})}{\Theta(\ell^2)} = \mathcal{O} \left(\frac{i\nu}{\ell^{\alpha-1}} \right),$$

Since the conditional event has probability $1 - \mathcal{O}(i/\ell^{\alpha-1})$, then the “unconditional” probability that of the event $|X_i| \leq \ell/4$ is

$$\left[1 - \mathcal{O}\left(\frac{i}{\ell^{\alpha-1}}\right)\right] \cdot \left[1 - \mathcal{O}\left(\frac{i\nu}{\ell^{\alpha-1}}\right)\right] = 1 - \mathcal{O}\left(\frac{\nu t}{\ell^{\alpha-1}}\right),$$

since $i \leq t$, with t which is some function in $\mathcal{O}(\ell^{\alpha-1}/\nu)$. The same result holds analogously for Y_i (the y -coordinate of the agent after the i -th jump), obtaining that $|X_i| + |Y_i| \leq \ell/2$ with probability $1 - \mathcal{O}(\nu t/\ell^{\alpha-1})$ by the union bound.

Consider the first jump-phase. The probability the agents visits the target during it is $\mathcal{O}(1/\ell^\alpha)$ for Corollary 3.6 (Section 3). Now, let $2 \leq i \leq t$. We want to estimate the probability the agent visits the target during the i -th jump-phase. We recall that $B_{\ell/4}(u^*)$ is the rhombus centered in u^* that contains the nodes at distance at most $\frac{\ell}{4}$ from u^* . We denote the event that the agent visits the target during the i -th jump-phase by F_i . Furthermore, let V_{i-1} be the event that the $(i-1)$ -th jump ends in $B_{\ell/4}(u^*)$, and W_{i-1} the event that $(i-1)$ -th jump ends at distance farther than $\ell/2$ from the origin. Then, by the law of total probabilities, we have

$$\begin{aligned} \mathbb{P}(F_i) &= \mathbb{P}(F_i | W_{i-1}) \mathbb{P}(W_{i-1}) + \mathbb{P}(F_i | W_{i-1}^C) \mathbb{P}(W_{i-1}^C) \\ &= \left[\mathbb{P}(F_i | W_{i-1}, V_{i-1}) \mathbb{P}(V_{i-1} | W_{i-1}) + \mathbb{P}(F_i | W_{i-1}, V_{i-1}^C) \mathbb{P}(V_{i-1}^C | W_{i-1}) \right] \mathbb{P}(W_{i-1}) \\ &\quad + \mathbb{P}(F_i | W_{i-1}^C) \mathbb{P}(W_{i-1}^C) \\ &\stackrel{(*)}{\leq} \left[\mathbb{P}(F_i | V_{i-1}) \mathbb{P}(V_{i-1} | W_{i-1}) + \mathbb{P}(F_i | W_{i-1}, V_{i-1}^C) \right] \mathbb{P}(W_{i-1}) + \mathbb{P}(F_i | W_{i-1}^C) \mathbb{P}(W_{i-1}^C) \\ &\stackrel{(\star)}{\leq} \left[\mathcal{O}\left(\frac{1}{\ell^2}\right) + \mathcal{O}\left(\frac{1}{\ell^\alpha}\right) \right] \mathcal{O}\left(\frac{\nu t}{\ell^{\alpha-1}}\right) + \mathcal{O}\left(\frac{1}{\ell^\alpha}\right) = \mathcal{O}\left(\frac{\nu t}{\ell^{\alpha+1}}\right) \end{aligned} \quad (12)$$

where in $(*)$ we used that $V_{i-1} \subset W_{i-1}$ and that $\mathbb{P}(V_{i-1}^C | W_{i-1}) \leq 1$, while in (\star) we used that

$$\mathbb{P}(F_i | V_{i-1}) \mathbb{P}(V_{i-1} | W_{i-1}) = \mathcal{O}\left(\frac{1}{\ell^2}\right), \text{ (the proof is below)}$$

that $\mathbb{P}(F_i | W_{i-1}, V_{i-1}^C) = \mathcal{O}(1/\ell^\alpha)$ because the jump starts in a node whose distance form the target is $\Omega(\ell)$, and that $\mathbb{P}(F_i | W_{i-1}^C) = \mathcal{O}(1/\ell^\alpha)$ for the same reason. As for the term $\mathbb{P}(F_i | V_{i-1}) \cdot \mathbb{P}(V_{i-1} | W_{i-1})$ we observe the following. Let t_i be the time at the end of the i -th jump phase. Then

$$\begin{aligned} \mathbb{P}(F_i | V_{i-1}) \mathbb{P}(V_{i-1} | W_{i-1}) &= \sum_{v \in B_{\ell/4}(u^*)} \mathbb{P}(F_i | L_{t_i}^w = v) \mathbb{P}(L_{t_i}^w = v | W_{i-1}) \\ &\leq \mathcal{O}\left(\frac{1}{\ell^2}\right) \sum_{v \in B_{\ell/4}(u^*)} \mathbb{P}(F_i | L_{t_i}^w = v), \end{aligned}$$

since Lemma 3.9 holds in a consequent way conditional on W_{i-1} , and since, for each $v \in B_{\ell/4}(u^*)$, there are at least $\Theta(\ell^2)$ nodes at distance at least $\ell/2$ from the origin which are more probable to be visited than v . Then, we proceed similarly to the proof of Lemma 3.10 to show that $\sum_{v \in B_{\ell/4}(u^*)} \mathbb{P}(F_i | L_{t_i}^w = v) = \mathcal{O}(1)$, and we obtain $\mathbb{P}(F_i | V_{i-1}) \mathbb{P}(V_{i-1} | W_{i-1}) = \mathcal{O}(1/\ell^2)$.

Thus, by the union bound and by the inequality (12), the probability that at least during one of the t jump-phases the agent finds the target is, for some $t = \mathcal{O}(\ell^2/\nu)$,

$$\frac{1}{\ell^\alpha} + \mathcal{O}\left(\frac{\nu t^2}{\ell^{\alpha+1}}\right) = \mathcal{O}\left(\frac{\nu t^2}{\ell^{\alpha+1}}\right),$$

since $t \geq \ell$.

6.5 Proof of Corollary 6.2

From Theorem 6.1.(a) and the independence between agents we get that

$$\mathbb{P}\left(\tau_\alpha^k(u^*) = \mathcal{O}(\ell^2 \log^2 \ell)\right) = 1 - \left[1 - \Omega\left(\frac{1}{\gamma \log^4 \ell}\right)\right]^k \geq 1 - e^{-\Omega\left(\frac{k}{\gamma \log^4 \ell}\right)},$$

where we have used the inequality $1 - x \leq e^{-x}$ for all x . Then, part (a) follows. Let $t = \ell^2/k \cdot \sqrt{k}/\log^2 \ell$; hence, we have $t \leq o(\ell^{\alpha-1}/\nu)$ since $k = o(\ell^2)$. If $t < \ell$, then $\mathbb{P}(\tau_\alpha^k(u^*) > t) = 1$, since ℓ steps are needed to reach distance ℓ . If $t \geq \ell$, from Theorem 6.1.(b) and the independence between agents, we get that

$$\mathbb{P}\left(\tau_\alpha^k(u^*) > t\right) = \left[1 - \mathcal{O}\left(\frac{\nu t^2}{\ell^{\alpha+1}}\right)\right]^k \geq \exp\left(-\mathcal{O}\left(\frac{kt^2 \log \ell}{\ell^{\alpha+1}}\right)\right),$$

where we have used the inequality $1 - x \geq e^{-\frac{x}{1-x}}$ for $x < 1$, that $\nu t^2/\ell^{\alpha+1} = o(1)$, and that $\nu \leq \log \ell$. Then, by substituting $t = \ell^2/k \cdot \sqrt{k}/\log^2 \ell$, and by the Taylor's expansion of the exponential function, we get

$$\exp\left(-\mathcal{O}\left(\frac{kt^2 \log \ell}{\ell^{\alpha+1}}\right)\right) = 1 - \mathcal{O}\left(\frac{1}{\log^3 \ell}\right).$$

Corollary 6.2.(b) follows by observing that $\ell^2/k \cdot \sqrt{k}/\log^2 \ell \leq \ell^2/\sqrt{k}$.

7 Distributed Search Algorithm

In this section, we prove the following theorem, which provides a simple distributed search algorithm, which allows k agents to find an arbitrary, unknown target on \mathbb{Z}^2 in optimal time (modulo polylogarithmic factors).

Theorem 7.1. *Consider k independent Lévy walks that start simultaneously from the origin, and the exponent of each walk is sampled independently and uniformly at random from the real interval $(2, 3)$. Let $\tau_{rand}^k(u^*)$ be the parallel hitting time for a given target u^* . If $k \geq \log^8 \ell$, and $\ell = \|u^*\|_1$ is large enough, then*

$$\mathbb{P}\left(\tau_{rand}^k(u^*) = \mathcal{O}\left((\ell^2/k) \cdot \log^7 \ell + \ell \log^3 \ell\right)\right) = 1 - e^{-\omega(\log \ell)}.$$

We need the next lemma, which is a slight generalization of Corollary 4.2(a) that bounds the hitting time of a collection of Lévy walks with different exponent values.

Lemma 7.2. *Consider k independent Lévy walks that start simultaneously from the origin, and the exponent of each walk is in $[\alpha_1, \alpha_2]$. Let h_{diff} be the parallel hitting time for a target u^* with $\|u^*\|_1 = \ell$. If $2 < \alpha_1 \leq \alpha_2 \leq 3 - \epsilon$ and $\epsilon = \omega(1/\log \ell)$, then*

$$\mathbb{P}\left(h_{diff} = \mathcal{O}\left(\frac{\ell^{\alpha_2-1}}{\alpha_1 - 2}\right)\right) = 1 - e^{-\Omega\left(\frac{(3-\alpha_2)^2 k}{\ell^{3-\alpha_1} \log^2 \ell}\right)}.$$

Proof. First, we recall that the k agents move independently from each other. Let $\tau_\alpha(u^*)$ be the hitting time of a single walk. If $\alpha \in [\alpha_1, \alpha_2]$, then from Lemmas 4.5 and 4.6,

$$\mathbb{P}\left(\tau_\alpha(u^*) = \mathcal{O}\left(\frac{\ell^{\alpha-1}}{\alpha - 2}\right)\right) = \Omega\left(\frac{(3-\alpha)^2}{\ell^{3-\alpha} \log^{\frac{2}{\alpha-1}} \ell}\right) = \Omega\left(\frac{(3-\alpha_2)^2}{\ell^{3-\alpha_1} \log^2 \ell}\right),$$

provided that $3 - \alpha_2 = \omega(1/\log \ell)$. Observe that $\mathbb{P}\left(\tau_\alpha(u^*) = \mathcal{O}\left(\frac{\ell^{\alpha_2-1}}{\alpha_1-2}\right)\right) \geq \mathbb{P}\left(\tau_\alpha(u^*) = \mathcal{O}\left(\frac{\ell^{\alpha-1}}{\alpha-2}\right)\right)$. Then,

$$\mathbb{P}\left(h_{diff} = \mathcal{O}\left(\frac{\ell^{\alpha_2-1}}{\alpha_1-2}\right)\right) = 1 - \left(1 - \Omega\left(\frac{(3-\alpha_2)^2}{\ell^{3-\alpha_1} \log^2 \ell}\right)\right)^k \leq 1 - e^{-\Omega\left(\frac{(3-\alpha_2)^2 k}{\ell^{3-\alpha_1} \log^2 \ell}\right)}. \quad \square$$

Proof of Theorem 7.1. Fix k, ℓ such that $k \geq \log^8 \ell$, and let $\epsilon = \log \log \ell / \log \ell$. Let $\alpha \in [2 + \epsilon, 3 - 2\epsilon]$, and let k_α be the number of Lévy walks whose exponent is in the interval $[\alpha, \alpha + \epsilon]$. Then $\mathbb{E}[k_\alpha] = \epsilon k$, and by the Chernoff bound Theorem A.2,

$$\mathbb{P}(k_\alpha \geq \epsilon k / 2) = 1 - e^{-\Omega(\epsilon k)}.$$

Clearly, the parallel hitting time of the k Lévy walks is upper bounded by the parallel hitting time of the k_α Lévy walks whose exponent is in $[\alpha, \alpha + \epsilon]$. Then, from Lemma 7.2, it follows that

$$\mathbb{P}\left(\tau_{rand}^k(u^*) = \mathcal{O}\left(\frac{\ell^{\alpha+\epsilon-1}}{\alpha-2}\right) \mid k_\alpha \geq \epsilon k / 2\right) = 1 - e^{-\Omega\left(\frac{(3-\alpha-\epsilon)^2 \epsilon k}{\ell^{3-\alpha} \log^2 \ell}\right)}.$$

Combining the last two equations, we obtain

$$\mathbb{P}\left(\tau_{rand}^k(u^*) = \mathcal{O}\left(\frac{\ell^{\alpha+\epsilon-1}}{\alpha-2}\right)\right) = 1 - e^{-\Omega\left(\frac{(3-\alpha-\epsilon)^2 \epsilon k}{\ell^{3-\alpha} \log^2 \ell}\right)}. \quad (13)$$

We distinguish the following two cases.

Case $\log^7 \ell \leq k \leq \ell \log^3 \ell$. Choose some $\alpha \in [2 + \epsilon, 3 - 2\epsilon]$ such that

$$k = \frac{\ell^{3-\alpha} \cdot \log^2 \ell}{(3-\alpha-\epsilon)^2 \cdot \epsilon} \cdot \log \ell \cdot \log \log \ell.$$

Such an α exists because the values of function $f(\alpha) = \frac{\ell^{3-\alpha} \cdot \log^3 \ell \cdot \log \log \ell}{(3-\alpha-\epsilon)^2 \cdot \epsilon}$ at the extreme points of α are $f(2+\epsilon) \geq \ell \log^3 \ell \geq k$ and $f(3-2\epsilon) \leq \log^8 \ell \leq k$. Substituting the above value of α to Eq. (13), we obtain

$$\mathbb{P}\left(\tau_{rand}^k(u^*) = \mathcal{O}\left(\frac{\ell^{\alpha+\epsilon-1}}{\alpha-2}\right)\right) = 1 - e^{-\omega(\log \ell)}.$$

Thus, with probability $1 - e^{-\omega(\log \ell)}$,

$$k \cdot \tau_{rand}^k(u^*) = k \cdot \mathcal{O}\left(\frac{\ell^{\alpha+\epsilon-1}}{\alpha-2}\right) = \mathcal{O}\left(\frac{\ell^2 \cdot \log^5 \ell}{(3-\alpha-\epsilon)^2 (\alpha-2)}\right) = o(\ell^2 \cdot \log^7 \ell).$$

Case $k \geq \ell \log^3 \ell$. In this case, we set $\alpha = 2 + \epsilon$ and substitute this value of α to Eq. (13) to obtain $\mathbb{P}\left(\tau_{rand}^k(u^*) = \mathcal{O}\left(\ell^{1+2\epsilon}/\epsilon\right)\right) = 1 - e^{-\Omega\left(\frac{k \log \log \ell}{\ell \log^2 \ell}\right)} = 1 - e^{-\omega(\log \ell)}$. Thus, with probability $1 - e^{-\omega(\log \ell)}$,

$$\tau_{rand}^k(u^*) = \mathcal{O}\left(\ell^{1+2\epsilon}/\epsilon\right) = o(\ell \cdot \log^3 \ell).$$

Combining the two cases completes the proof. □

APPENDIX

A Tools

Below we list some standard concentration bounds we use, and prove some simple algebraic inequalities.

A.1 Concentration Bounds

Theorem A.1 (Central limit theorem [15, Chapter X]). *Let $\{X_k\}_{k \geq 1}$ be a sequence of i.i.d. random variables. Let $\mu = \mathbb{E}[X_1]$, $\sigma^2 = \text{Var}(X_1)$, and $S_n = \sum_{k=1}^n X_k$ for any $n \geq 1$. Let $\Phi : \mathbb{R} \rightarrow [0, 1]$ be the cumulative distribution function of a standard normal distribution. Then, for any $\beta \in \mathbb{R}$,*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{S_n - n\mu}{\sigma\sqrt{n}} < \beta \right) = \Phi(\beta).$$

Theorem A.2 (Multiplicative Chernoff bounds [8]). *Let X_1, X_2, \dots, X_n be independent random variables taking values in $[0, 1]$. Let $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[X]$. Then:*

(i) *For any $\delta > 0$ and $\mu \leq \mu_+ \leq n$,*

$$P(X \geq (1 + \delta)\mu_+) \leq e^{-\frac{1}{3}\delta^2\mu_+}, \quad (14)$$

(ii) *For any $\delta \in (0, 1)$ and $0 \leq \mu_- \leq \mu$,*

$$P(X \leq (1 - \delta)\mu_-) \leq e^{-\frac{1}{2}\delta^2\mu_-}. \quad (15)$$

Theorem A.3 (Additive Chernoff bound using variance [6, Theorem 3.4]). *Let X_1, \dots, X_n be independent random variables satisfying $X_i \leq \mathbb{E}[X_i] + M$ for some $M \geq 0$, for all $i = 1, \dots, n$. Let $X = \sum_{i=1}^n X_i$, $\mu = \mathbb{E}[X]$, and $\sigma^2 = \text{Var}(X)$. Then, for any $\lambda > 0$,*

$$\mathbb{P}(X \geq \mu + \lambda) \leq \exp \left(-\frac{\lambda^2}{\sigma^2 + \frac{M\lambda}{3}} \right). \quad (16)$$

A.2 Inequalities

Lemma A.4. *Let $0 < d < d_{max}$ be any integers. For any $\alpha > 1$,*

$$\frac{1}{(\alpha - 1)d^{\alpha-1}} \leq \sum_{k \geq d} \frac{1}{k^\alpha} \leq \frac{1}{(\alpha - 1)d^{\alpha-1}} + \frac{1}{d^\alpha}, \quad \text{and} \quad (17)$$

$$\frac{1}{(\alpha - 1)} \left(\frac{1}{d^{\alpha-1}} - \frac{1}{d_{max}^{\alpha-1}} \right) \leq \sum_{k=d}^{d_{max}} \frac{1}{k^\alpha} \leq \frac{1}{(\alpha - 1)} \left(\frac{1}{d^{\alpha-1}} - \frac{1}{d_{max}^{\alpha-1}} \right) + \frac{1}{d^\alpha}. \quad (18)$$

Also,

$$\log \left(\frac{d_{max}}{d} \right) \leq \sum_{k=d}^{d_{max}} \frac{1}{k} \leq \log \left(\frac{d_{max}}{d} \right) + \frac{1}{d}, \quad (19)$$

and for any $0 < \alpha < 1$,

$$\frac{(d_{max})^{1-\alpha} - d^{1-\alpha}}{1 - \alpha} \leq \sum_{k=d}^{d_{max}} \frac{1}{k^\alpha} \leq \frac{(d_{max})^{1-\alpha} - d^{1-\alpha}}{1 - \alpha} + \frac{1}{d^\alpha}. \quad (20)$$

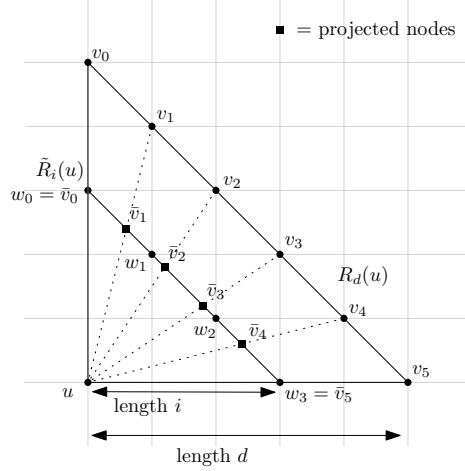


Figure 4: Projection from $R_d(u)$ to $\tilde{R}_i(u)$, with $d = 5$, $i = 3$.

Proof. By the integral test, it holds that

$$\int_d^{d_{\max}} \frac{1}{k^\alpha} dk \leq \sum_{k=d}^{d_{\max}} \frac{1}{k^\alpha} \leq \int_d^{d_{\max}} \frac{1}{k^\alpha} dk + \frac{1}{d^\alpha}.$$

Straightforward calculations give the result for Eqs. (18) to (20). As for Eq. (17), it comes from the integral test letting $d_{\max} \rightarrow \infty$. \square

B Proofs Omitted from Section 3

B.1 Proof of Lemma 3.2

For $1 \leq i < d$, consider the real rhombus $\tilde{R}_i(u)$ which is the set $\{v \in \mathbb{R}^2 : \|u - v\|_1 = i\}$. Project each element of $R_d(u)$ on $\tilde{R}_i(u)$ as in Fig. 4. We obtain $4d$ equidistant points $\bar{v}_1, \dots, \bar{v}_{4d}$ in Euclidean distance. Then, for each $w \in R_i(u)$, consider the set $C_w \subseteq \{\bar{v}_1, \dots, \bar{v}_{4d}\}$ of points that are closest to w than to any other node of $R_i(u)$ in Euclidean distance. Note that each \bar{v}_j belongs to either one or two sets C_w ; in the latter case we say that \bar{v}_j is *shared*. For the cardinality of set C_w , we have the following cases: (i) if $d \equiv 0 \pmod{i}$ then $|C_w| = d/i + 1$, and two of the elements of C_w are shared; (ii) if $d \not\equiv 0 \pmod{i}$ then either $|C_w| = \lfloor d/i \rfloor$ and no elements of C_w are shared, or $|C_w| = \lceil d/i \rceil$ and at most one element of C_w is shared.

Choose a node $v \in R_d(u)$ u.a.r. and look at a node $w \in R_i(u)$. The probability that w is on a direct-path chosen u.a.r. is exactly the probability that the projection \bar{v} of v on $\tilde{R}_i(u)$ belongs to C_w , where shared points contribute by 1/2. In all cases, this probability is between $\lfloor d/i \rfloor / (4d)$ and $\lceil d/i \rceil / (4d)$.

B.2 Proof of Lemma 3.9

For any node w , define $D(w)$ as the set $B_{\|w\|_1}(\mathbf{0}) \cup Q_{\|w\|_\infty}(\mathbf{0})$. Notice that $D(u) \subseteq Q_{\|u\|_1}(\mathbf{0})$. Then it suffice to prove that for all nodes $v \notin D(u)$ we have

$$\mathbb{P}(J_t = u) \geq \mathbb{P}(J_t = v).$$

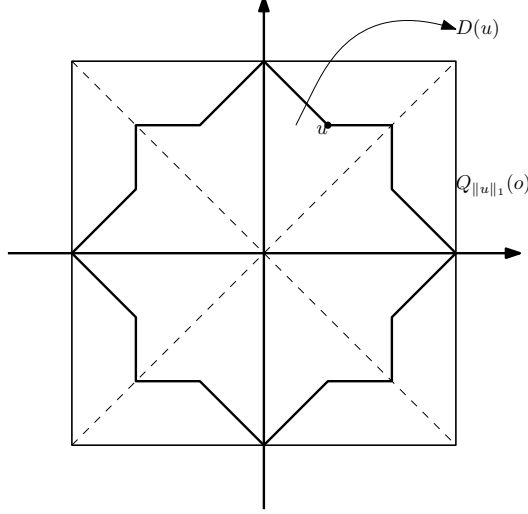


Figure 5: The set $D(u)$, consisting in all inner nodes of the “star”, and the square $Q_{d_u}(\mathbf{0})$.

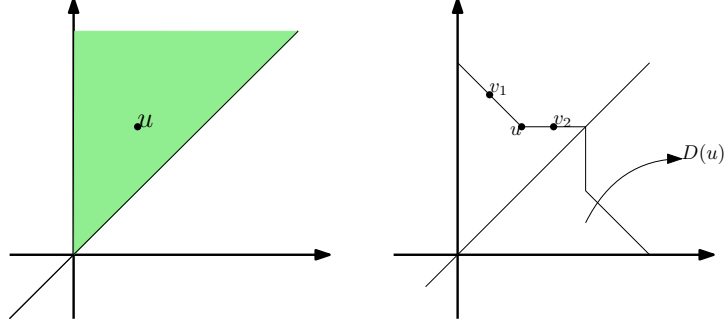


Figure 6: The “area” in which we take u , and the possible choices of v .

Let $u = (x_u, y_u)$ and, without loss of generality, suppose u is in the first quadrant and not below the main bisector, i.e. in the set $\{(x, y) \in \mathbb{Z}^2 : y \geq 0, x \geq y\}$ (Fig. 6). If we show that, for any v in $\{v_1 = (x_u - 1, y_u + 1), v_2 = (x_u + 1, y_u)\}$ (Fig. 6), we have $\mathbb{P}(J_t = u) \geq \mathbb{P}(J_t = v)$, then we have the statement. Indeed, for any $v \notin D(u)$ that “lives” in the highlighted area in Fig. 6, there exists a sequence of nodes $u = w_0, w_1, \dots, w_k = v$ from u to v such that w_{i+1} belongs to the set

$$\{(x_{w_i} - 1, y_{w_i} + 1), (x_{w_i} + 1, y_{w_i})\},$$

where $w_i = (x_{w_i}, y_{w_i})$, as Fig. 7 shows. Thus, if the thesis is true for $v \in \{v_1, v_2\}$, then it is true also for all $v \notin D(u)$ in the highlighted area in Fig. 6. At the same time, for any other $v \notin D(u)$, outside the highlighted area in Fig. 6, there exists a symmetrical argument explained in Fig. 8. Thus, if the thesis is true for all $v \notin D(u)$ in the highlighted area in Fig. 6, then it is also true for any $v \notin D(u)$. We now consider some geometric constructions which will be used in the proof, one for each choice of v . The following description is showed in Fig. 9 .

- (i) $v = (x_u - 1, y_u + 1)$: consider the strict line defined by $r : y = x + (y_u - x_u) + 1$ (i.e. the line in \mathbb{R}^2 which is the set of points that are equidistant from u and v in Euclidean distance). Call $V \subset \mathbb{Z}^2$ the set of nodes that are “above” this line, namely the ones that are closer to v than u . Define $U = \mathbb{Z}^2 \setminus (V \cup r)$ the complementary set without line r . Consider the injective function $f : V \rightarrow U$ such that $f(x, y) = (y - (y_u - x_u) - 1, x + (y_u - x_u) + 1)$, which is the

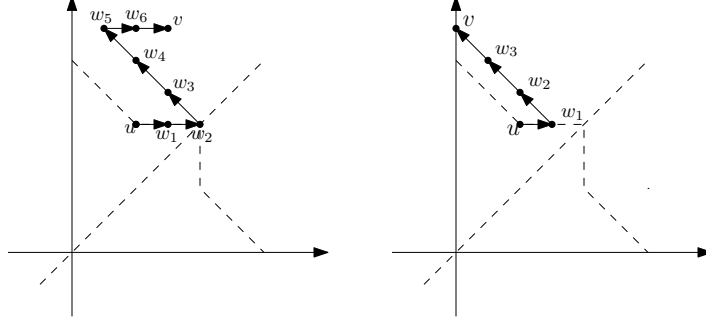


Figure 7: Two “path” examples.

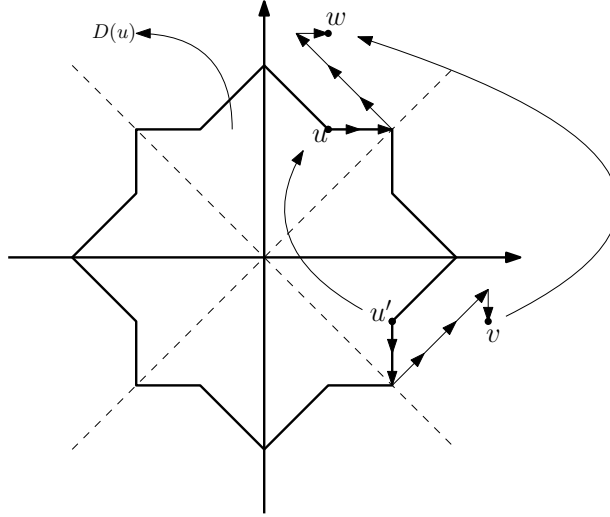


Figure 8: Symmetrical argument.

symmetry with respect to r . It trivially holds that for any $w \in V$, $\|w - v\|_1 = \|f(w) - u\|_1$ and $\|w - u\|_1 = \|f(w) - v\|_1$. Furthermore, it holds that for each $w \in V$, either $w \notin D(f(w))$, or w lies on the “border” of $D(f(w))$. All these properties are well-shown in Fig. 10.

- (ii) $v = (x_u + 1, y_u)$: the same construction can be done in this case. Indeed, the strict line will be $x = x_u + \frac{1}{2}$, and the injective function $f(x, y) = (2x_u + 1 - x, y)$. The same properties we have seen in the previous case hold here too.

Now we go for the proof. For any time i , and any two nodes $u', v' \in \mathbb{Z}^2$, define

$$p^i(u', v') = \mathbb{P}(\mathbf{X}_i = v' \mid \mathbf{X}_0 = u').$$

Let $v \in \{v_1, v_2\}$. We show that $p^t(\mathbf{0}, u) \geq p^t(\mathbf{0}, v)$ by induction on t . The base case is $t = 1$. From the monotonicity, we know that

$$p^1(\mathbf{0}, u) - p^1(\mathbf{0}, v) \geq 0$$

for any u and v in \mathbb{Z}^2 such that $\|u\|_1 \leq \|v\|_1$. We now suppose $t \geq 2$ and the thesis true for $t - 1$. Fix u and v as in Fig. 6; then, for the geometric construction we made above, it holds that

$$p^t(\mathbf{0}, u) - p^t(\mathbf{0}, v) = \sum_{w \in \mathbb{Z}^2} p^{t-1}(\mathbf{0}, w) (p^1(w, u) - p^1(w, v))$$

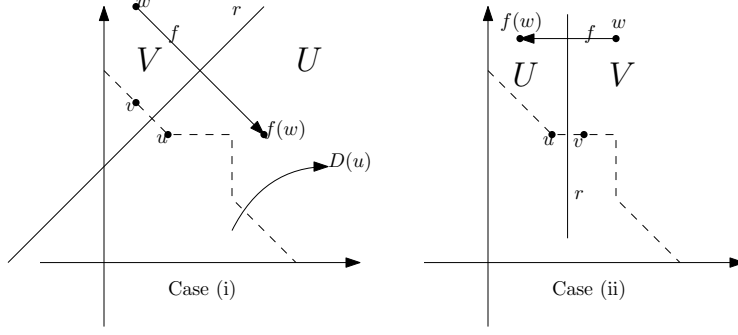


Figure 9: Geometric constructions in the two cases.

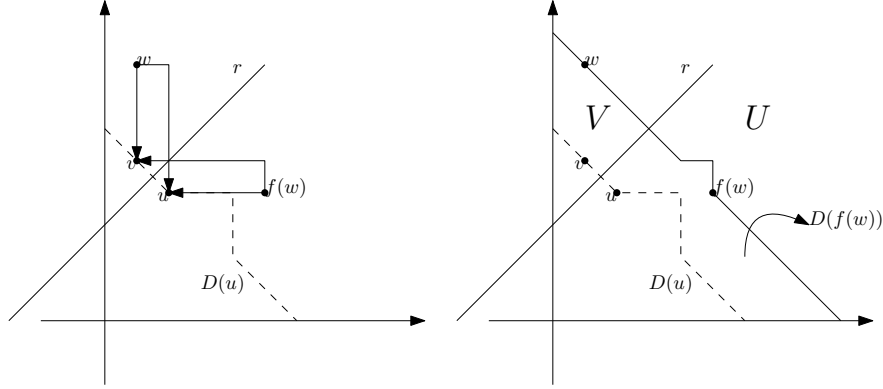


Figure 10: Left: equivalence between distances. Right: $w \notin D(f(w))$.

$$\geq \sum_{w \in U} p^{t-1}(\mathbf{0}, w) (p^1(w, u) - p^1(w, v)) + \sum_{w \in V} p^{t-1}(\mathbf{0}, w) (p^1(w, u) - p^1(w, v))$$

where last inequality is immediate for case (ii), indeed the line r does not contain elements of \mathbb{Z}^2 , while in case (i) the sum over nodes in line r is zero. Then, the previous value is equal to

$$\sum_{w \in V} p^{t-1}(\mathbf{0}, f(w)) (p^1(f(w), u) - p^1(f(w), v)) + \sum_{w \in V} p^{t-1}(\mathbf{0}, w) (p^1(w, u) - p^1(w, v))$$

because of the definition of $f : V \rightarrow U$, and, changing the sign of the second sum, we obtain

$$\sum_{w \in V} p^{t-1}(\mathbf{0}, f(w)) (p^1(f(w), u) - p^1(f(w), v)) - \sum_{w \in V} p^{t-1}(\mathbf{0}, w) (p^1(w, v) - p^1(w, u)).$$

Now, observe that the definition of f implies that for each $w \in V$, $\|w - v\|_1 = \|f(w) - u\|_1$ and $\|f(w) - v\|_1 = \|w - u\|_1$ (Fig. 10). Thus we can group out the term $p^1(f(w), u) - p^1(f(w), v) = p^1(w, v) - p^1(w, u)$, and we have

$$\sum_{w \in V} (p^{t-1}(\mathbf{0}, f(w)) - p^{t-1}(\mathbf{0}, w)) (p^1(f(w), u) - p^1(f(w), v)). \quad (21)$$

We observe that $p^{t-1}(\mathbf{0}, f(w)) - p^{t-1}(\mathbf{0}, w) \geq 0$ by the inductive hypothesis, since either $w \notin D(f(w))$ or w lies on the “border” of $D(f(w))$ (Fig. 10), and $p^1(f(w), u) - p^1(f(w), v) \geq 0$ by definition of f , since the distance between $f(w)$ and u is no more than the distance between $f(w)$ and v . It follows that (21) is non-negative, and, thus, the thesis.

C Projection of a Lévy Flight Jump

Let L_t^f be the two dimensional random variable representing the coordinates of an agent performing an α -Lévy flight at time t , for any $\alpha > 1$. Consider the projection of the Lévy flight on the x -axis, namely the random variable X_t such that $L_t^f = (X_t, Y_t)$. The random variable X_t can be expressed as the sum of t random variables S_j^x , $j = 1, \dots, t$, representing the projection of the jumps (with sign) of the agent on the x -axis at times $j = 1, \dots, t$. With the next lemma, we prove that the jump projection length has the same tail distribution as the original jump length.

Lemma C.1. *The probability that a jump S_j^x has length equal to d is $\Theta(1/d^\alpha)$.*

Proof. The partial distribution of the jumps along the x -axis is given by the following. For any $d \geq 0$,

$$\mathbb{P}(S_j^x = \pm d) = \left[\frac{1}{2} + \sum_{k=1}^{\infty} \frac{c_\alpha}{2k^{\alpha+1}} \right] \mathbb{1}_{d=0} + \left[\frac{c_\alpha}{2d^{\alpha+1}} + \sum_{k=1+d}^{\infty} \frac{c_\alpha}{k^{\alpha+1}} \right] \mathbb{1}_{d \neq 0}, \quad (22)$$

where $\mathbb{1}_{d \in A}$ returns 1 if $d \in A$ and 0 otherwise, the term

$$\frac{\mathbb{1}_{d=0}}{2} + \frac{c_\alpha}{2d^{\alpha+1}} \mathbb{1}_{d \neq 0}$$

is the probability that the original jump lies along the horizontal axis and has “length” exactly d (there are two such jumps if $d > 0$), and, for $k \geq 1 + d$, the terms

$$\frac{c_\alpha}{2k^{\alpha+1}} \mathbb{1}_{d=0} + \frac{c_\alpha}{k^{\alpha+1}} \mathbb{1}_{d \neq 0}$$

are the probability that the original jump has “length” exactly k and its projection on the horizontal axis has “length” d (there are two such jumps if $d = 0$, and four such jumps if $d > 0$). By the integral test (Lemma A.4) we know that quantity (22) is

$$\mathbb{P}(S_j^x = \pm d) = \Theta\left(\frac{1}{d^{\alpha+1}}\right).$$

□

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