

On the Search Efficiency of Parallel Lévy Walks on \mathbb{Z}^2

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Abstract

Motivated by the *Lévy flight foraging hypothesis* – the premise that the movement of various animal species searching for food resembles a *Lévy walk* – we study the search efficiency of parallel Lévy walks on the infinite 2-dimensional grid. We assume that k independent identical discrete-time Lévy walks, with exponent parameter $\alpha \in (1, +\infty)$, start simultaneously at the origin, and we are interested in the time $h_{\alpha, k, \ell}$ until some walk visits a given target node at distance ℓ from the origin. First, we observe that the total work, i.e., the product $k \cdot h_{\alpha, k, \ell}$, is at least $\Omega(\ell^2)$, for any combination of the parameters α, k, ℓ . Then we provide a comprehensive analysis of the time and work, for the complete range of these parameters. Our main finding is that for any α , there is a specific choice of k that achieves optimal work, $\tilde{O}(\ell^2)$, whereas all other choices of k result in sub-optimal work. In particular, in the interesting super-diffusive regime of $2 < \alpha < 3$, the optimal value for k is $\tilde{\Theta}(\ell^{1-(\alpha-2)})$. Our results should be contrasted with several previous works showing that the exponent $\alpha = 2$ is optimal for a wide range of related search problems on the plane. On the contrary, in our setting of multiple walks which measures efficiency in terms of the natural notion of work, no single exponent is optimal: for each α (and ℓ) there is a specific choice of k that yields optimal efficiency.

1 Introduction

In this work, we present a general analytical study that proposes the discrete version of Lévy walks, a famous movement model [Lé54], as a natural parallel algorithm for searching a fixed target on the 2-dimensional grid.

We can informally define a *Lévy walk* as the following searching process. A single agent continuously moves on the infinite real plane \mathbb{R}^2 , in continuous time. It chooses a way-point u.a.r. among those at distance d , where d is chosen according to a power-law distribution with tail $\mathbb{P}(d > t) = \Omega(\frac{1}{d^{1-\tau}})$, and then moves towards the way-point at a certain speed. Once the way-point is reached, the agent repeats the procedure.

In this work, we propose and study a natural discretization of the Lévy walk process on the infinite grid \mathbb{Z}^2 , which we call *Pareto walk*. For any fixed real $\alpha > 1$, at time $t = 0$, the agent, which is placed on some node u , chooses a distance $d \in \mathbb{N}$ with probability distribution $\frac{c}{(1+d)^\alpha}$, where c is a normalization constant. The agent then chooses a way-point w u.a.r. among those at distance d , and moves towards it by following a path which closely approximates the direction $\vec{v} = \vec{w} - \vec{u}$ (see Definitions 2 and 3 in Section 4 for a rigorous description). Once w is reached, the agent repeats the procedure by sampling a new distance, a new direction (that is, a new way-point) independently from the previous ones.

The reason behind the name “Pareto walk” lies in the name of the jump length distribution we choose, which is a common variant of the Pareto distribution that decays as a power law [Arn08]. The scenario we address is the following: in the infinite two-dimensional grid \mathbb{Z}^2 , there are k independent probabilistic agents performing α -Pareto walks initially placed at the origin, which start moving. The goal, for each of them, is to *hit* a unique node (from now on, the *treasure*) placed at (unknown) Manhattan distance ℓ from the origin. We denote the first time any agent hits the treasure as the *hitting time* of the considered parallel process.

The above mathematical framework allows us to provide the *first rigorous, general analysis of the hitting time and the search efficiency of parallel Lévy walks in a discrete setting*, where the efficiency is here measured in terms of *work*, i.e., the total number of time steps performed by all agents. In particular, we give almost-tight results for the following algorithmic question: for any fixed constant $\alpha > 1$, find a suitable range of k such that the treasure is found with minimum work, *with high probability*¹ in ℓ . Our contribution entails the analysis of novel (discrete) random walk processes and issues about sums of power-law random variables which are of independent interest (see also Section 3.5.1 for more detail on this aspect).

It is reasonable to state that Lévy walks constitute nowadays the main mobility model in biology [Rey18], at least among models with comparable mathematical simplicity and elegance [VLRS11]. In the next paragraph, we concisely summarize the history of the study of Lévy walks in order to provide the main context for the present work. For a rich monograph on the topic, we defer the reader to [VLRS11].

Lévy walks and Lévy flights. Lévy flights were originally investigated by Paul Lévy in his 1937 treatise [Lé54],² and named after him by his student Benoît Mandelbrot which investigated their fractal properties [Man82]. [SK86], which investigates the relation between Lévy flights and Lévy walks, was among the first studies to discuss the relevance of Lévy walks and Lévy flights as a biological mobility model. Many works, among which a 1996 Nature paper by Viswanathan et al., supported such hypothesis on the basis of analyses of empirical data [VAB+96]. Statistical flaws were later pointed out in several of these works [EPW+07], including [VAB+96]; however, subsequent works which avoided previous methodological issues continued to corroborate what has been known as *Lévy flight foraging hypothesis*:

¹As usual, we say an event E , depending on a parameter n , holds with high probability (w.h.p. in short) w.r.t. n if $\mathbb{P}(E) \underset{n \rightarrow \infty}{\rightarrow} 1$.

²The referenced citation refers to the second edition of the work.

Since Lévy flights and walks can optimize search efficiencies, therefore natural selection should have led to adaptations for Lévy flight foraging. - [VRdL08]

The surge of interest on Lévy walks motivated theoretical investigations. We have to point out here that, despite our efforts, *we failed to identify rigorous proofs for a large part of the main results* in the Lévy walks literature.³

The main mathematical models on which Lévy walks have been investigated in the literature involves a single agent continuously moving on the infinite real space \mathbb{R}^n , in continuous time. Here, targets are randomly located in the plane according to a fixed density ρ . If, at any time, any target happens to be within some distance radius r from the agent, the target is *found*.⁴

We remark that the above setting is quite different from the one we consider. In particular, the optimality concept in the above framework is based on the *expected discovery rate*, that is the expected number of targets found per distance travelled, while our efficiency measure (the work) involves the number of agents k and the *hitting time* of one single target.

It is well-known that the range of exponent α which controls the tail distribution of Lévy walks can be partitioned into three main regimes [VRdL08, VLRS11]:

- the *diffusive* regime ($\alpha \geq 3$),
- the *super-diffusive* regime ($\alpha \in (2, 3)$) and
- the *ballistic* regime ($\alpha \in (1, 2]$).

In the continuous time-space setting, the diffusive and ballistic regimes have been observed to be qualitatively equivalent to two different processes [VLRS11]: the Wiener process and the *ballistic walk* (in which an agent moves along a fixed direction), respectively. The threshold value between the ballistic and the super-diffusive regime (i.e., when $\alpha = 2$) plays a crucial role in the literature. In the seminal paper [VBH⁺99],⁵ it is showed that a Pareto walk (in the language and notation of the present work) with exponent $\alpha \approx 2 - (\frac{1}{\log \frac{1}{2r^2}})^2$ optimizes the expected number of discovered targets per distance travelled for any dimension $n \geq 1$ of the underlying space; they thus suggest that $\alpha = 2$ can be considered an optimal choice when the quantity $r^2\rho$ is small but not exactly known. Subsequently, [PCM14] showed that the choice $\alpha = 2$ does not have *universal* value: this choice is in fact rather sensible to further aspects of the underlying model. In particular, they show that, when in the one-dimensional case \mathbb{R} of the above framework, an additional *bias* factor is considered (i.e., an *external drift* term in the jump distribution), the optimal choice for α varies in $(2, 3)$.⁶ More recently, in [LTBV20] the authors have argued that the results of [VBH⁺99] do not hold when the underlying space dimension is higher than one, by showing that the choice of the exponent $\alpha = 2$ is not always the best one, contesting the technical validity of some analytical expression derived in [VBH⁺99].

Performance analysis of random walks. The rigorous study of the *hitting time* of parallel random walks has been first introduced on *finite* graphs in [ER09], following the investigation of the *cover time* in [AAK⁺11]. In this setting, standard techniques for bounding the hitting time involve the closely-related notion of *mixing time* of a single random walk [ES11, KMTS19, STM06]. We notice that in the infinite setting the mixing time is not defined, as a single random walk does not have a stationary distribution. Further works are discussed in Section 3.3.

³In fact, none of the works we mention in the following is organized in standard mathematical form involving clearly marked statements and proofs.

⁴Our high level description is omitting several details; e.g., in [VBH⁺99], when a target happens to be within radius r from the agent, the agent start moving towards such target until reaching its exact location.

⁵We emphasize that we failed to reconstruct a rigours proof of the claimed result, as mentioned above.

⁶We remark that in several works the exponent corresponds to the exponent of the survival function $\Pr(X > t)$ where X is the random distance chosen by the agent, rather than being the exponent of the density $\Pr(X = t)$; in this work, we always adopt the latter convention.

α -Pareto walk	Equivalence	Optimal number of agents	Hitting time	Total work
$\alpha \geq 3$ (Theorems 3 and 6)	Simple random walk (Theorem 7)	$\log^{\Theta(1)}(\ell)$	$\tilde{\Theta}(\ell^2)$	$\tilde{\Theta}(\ell^2)$
$\alpha \in (2, 3)$ (Theorem 2)		$\tilde{\Theta}(\ell^{1-(\alpha-2)})$	$\tilde{\Theta}(\ell^{1+(\alpha-2)})$	$\tilde{\Theta}(\ell^2)$
$\alpha \in (1, 2]$ (Theorem 4)	Ballistic walk (Theorem 5)	$\tilde{\Theta}(\ell)$	$\tilde{\Theta}(\ell)$	$\tilde{\Theta}(\ell^2)$

Table 1: Optimal settings. The values of the hitting time and of the work hold w.h.p.

1.1 Our results

In [FK17], the authors investigate the hitting time of parallel search algorithms for a fixed treasure on \mathbb{Z}^2 . In particular, their analysis imply that, for k agents performing any search algorithm which has no information on the treasure location (even allowing communication among agents), $\Omega(\ell^2)$ work is necessary with constant probability and, thus, in expectation (see Lemma 2 in Section 4 for more details).

We present a set of probabilistic bounds on the hitting time of Pareto walks that globally show that, *for any value of α , there is a unique value of k such that the work efficiency is optimal (up to polylogarithms)*. We summarize these results in the following theorem.

Theorem 1 (Work efficiency of Pareto walks - informal statement). *Assume that the treasure is located in some node of the infinite grid at distance $\ell > 0$ from the origin. Let k agents perform mutually independent Pareto walks with parameter $\alpha > 1$, starting simultaneously at the origin. Then, optimal work $\tilde{\Theta}(\ell^2)$ is achieved, w.h.p., by setting k as follows.*

- (i) For $\alpha \geq 3$, $k = \log^{\Theta(1)} \ell$; the hitting time is $\tilde{\Theta}(\ell^2)$.
- (ii) For $2 < \alpha < 3$, $k = \tilde{\Theta}(\ell^{1-(\alpha-2)})$; the hitting time is $\tilde{\Theta}(\ell^{1+(\alpha-2)})$.
- (iii) For $1 < \alpha \leq 2$, $k = \tilde{\Theta}(\ell)$; the hitting time is $\tilde{\Theta}(\ell)$.

Furthermore, any polynomial variation of k in each of the three cases increases the work by at least a polynomial value, w.h.p.

Conversely, for any $0 < c \leq 1$ and for any $k = \tilde{\Theta}(\ell^c)$, Theorem 1 allows us to derive a suitable α such that the hitting time and the work are optimal up to a $\mathcal{O}(\ell^{-c})$ factor, for an arbitrarily small $\delta > 0$. Indeed, if we set $\alpha = 3 + \delta - c$, then the hitting time is $\tilde{\mathcal{O}}(\ell^{2+c})$ and the work is $\tilde{\mathcal{O}}(\ell^{2+c})$, w.h.p.

Furthermore, we prove that Pareto walks with parameter $\alpha \geq 3$ achieve the same performance as the corresponding discrete processes of the Wiener process (simple random walks), and that Pareto walks with parameter $\alpha \in (1, 2]$ have the same performance as ballistic walks (a natural discrete version of the ballistic process), with respect to the hitting time and the total work. Finally, as for the range $\alpha \in (2, 3)$, the Pareto walk exhibits a peculiar search efficiency which clearly distinguishes it from the previous ranges. Table 1 shows, for each regime of α , these equivalences together with the related performance bounds.

Our work should be contrasted with evidence provided by classical results on Lévy walks we discussed above, which suggested a key role for the value $\alpha = 2$ in biological applications. Our finding that the optimal exponent α in fact depends on the considered setting and on the specific optimization problem, supports the thesis of [PCM14].

A key technical contribution of our work is the proof of a *monotonicity property* of the agent's spatial distributions yielded by a wide class of movement models. This distribution plays a crucial role in several aspects of these models since, for any node v of the grid and for any time step t , it gives the probability the agent lies in v at time t . Informally speaking,

consider *any random walk process* which moves from one node u to any other node v of the grid, according to a probability distribution $\pi_{u,v}$ which is *radial* and non-increasing with respect to the distance between u and v . We say that a distribution is *radial* if it only depends on the distance between u and v (see Definition 8 in Section 4.1). Then, we prove that the corresponding agent's spatial distribution satisfies a monotonicity property which is essentially equivalent to that of $\pi_{u,v}$, despite the fact that it may be non-radial with respect to the origin (see Lemma 3 in Section 4.1).

Algorithmic significance. This work aims at providing analytical tools for the design of new natural distributed algorithms by studying a discrete version of Lévy walks, a process which has been subject to a lot of interest in mathematical biology because of its *search efficiency*. Our main result provides the optimal choice of k given α as a function of ℓ . When the parameter ℓ is unknown, our analysis can be exploited to design a search algorithm which has the same work efficiency given by our analysis, up to a constant multiplicative factor. Indeed, consider the following protocol, which makes use of agents moving according to a Pareto walk for a fixed α . Let ℓ_i be the *guess* of the algorithm on the value of ℓ during *phase i* , and let k_i be the corresponding value of k suggested by Theorem 1. At the onset (phase 1), set ℓ_1 equal to some constant and let $k_1 = \mathcal{O}(1)$ random walks start at the origin and search for the treasure until some time t_1 such that $\tilde{\mathcal{O}}(\ell_1^2)$ work has been performed. At time $t_1 + 1$, start phase 2: set $\ell_2 = 2\ell_1$ and let $k_2 - k_1$ new agents start at the origin and search for the treasure, until some time t_2 such that the new random walks have performed work $\tilde{\mathcal{O}}(\ell_2^2)$. Repeat the previous process by doubling ℓ_i of phase i and generating $k_{i+1} - k_i$ new agents after work $\tilde{\mathcal{O}}(\ell_i^2)$ has been performed, until the treasure is found.

As for further algorithmic aspects of our study, in Section 3.2 we compare our results to those by Feinerman and Korman for the ANTS Problem [FK17].

1.2 Road-map of the paper

In Section 2 we present an outline of the analysis in the super-diffusive regime (i.e., $\alpha \in (2, 3)$) which allows to prove Claim (ii) of Theorem 1. This outline well represents the line of reasoning and the kind of technical arguments we essentially adopt in the whole analysis of Pareto walks. Further previous works are discussed in Section 3, while Section 4 provides some preliminary notions we use throughout the paper. In Section 5 we derive the efficiency bounds of Pareto walks in the super-diffusive regime (i.e., $\alpha \in (2, 3)$)⁷ that are formalized in Claim (i) (for $\alpha = 3$) and Claim (ii) (for $\alpha \in (2, 3)$) of Theorem 1. In Section 6 we prove the efficiency bounds of Pareto walks for the ballistic regime (i.e., $\alpha \in (1, 2]$). In particular, we prove Claim (iii) of Theorem 1 and, moreover, we show that the search efficiency of the (discrete) ballistic walk model is equivalent to that of the Pareto walk model in the ballistic regime [VLS11]. In Section 7 we prove the efficiency bounds of Pareto walks in the normal diffusive regime (i.e., $\alpha \in (3, +\infty)$) which are stated in Claim (i) of Theorem 1. We also show that the search efficiency of the simple random walk model is equivalent to that of the Pareto walk model in the normal diffusive regime [VLS11].

2 Analysis of the Super-Diffusive Regime

A main technical contribution of this work is the analysis of the super-diffusive regime of Pareto walks, i.e., the range $\alpha \in (2, 3)$, which represents the most intriguing technical scenario. In this section we give an outline of the proof of the upper bound in Claim (ii) of Theorem 1. Notice

⁷We remark that the Pareto walk model for $\alpha = 3$ lies in the normal diffusive regime but, for technical reasons, it is analyzed together with the super-diffusive regime.

that we here also include the case $\alpha = 3$ since its analysis is similar. More details on the proof of Claim (ii) of Theorem 1, together with the tightness of the result, are given in Section 5.

The Pareto walk process $\{\mathbf{P}_t\}_{t \in \mathbb{N}}$, where \mathbf{P}_t represents the coordinates of a Pareto walk at round t , is not a Markov chain over the state space \mathbb{Z}^2 : \mathbf{P}_t indeed depends on \mathbf{P}_{t-1} , but also on the state of the agent (i.e., if it is performing a jump or not), and its previous jump decision. We also remark that it is hard to recover which nodes are actually visited by the agent during its trip to the next destination (see Definition 4). To address the issues above we consider two simpler movement models, the Pareto flight (the discrete version of the well-known Lévy flight, which also has a per-se interest) and provide bounds on the performances of such a model. Then, using coupling arguments, we show how such bounds can be exploited to get similar bounds for the original Pareto walk model.

Definition 1 (α -Pareto flight). *Let $\alpha > 1$ be a real constant. At each round, the agent chooses a distance d with distribution $\frac{c}{(1+d)^\alpha}$, where c is a normalization constant, and chooses u.a.r. one node u among the $4d$ nodes of the grid at distance d from its current position. Then, in one step/unit time, the agent reaches u . Once reached u , the agent repeats the procedure above, and so on. If the chosen distance d is equal to zero, the agent keeps still for one time unit and then it repeats the procedure.*

By defining the two-dimensional random variable \mathbf{P}_t^f as the coordinates of the node the Pareto flight visits at time t , we can easily observe that the process $\{\mathbf{P}_t^f\}_{t \in \mathbb{N}}$ is a Markov chain over the state space \mathbb{Z}^2 . This important property will be exploited to prove the following result.

Proposition 1 (Hitting time of Pareto flight - case $\alpha \in (2, 3]$). *Consider a single agent that performs t steps of the Pareto flight with parameter $\alpha \in (2, 3]$. For some $t = \Theta(\ell^{1+(\alpha-2)})$, conditional on the event that the lengths of all performed jumps are less than $(t \log t)^{\frac{1}{1-\alpha}}$, the agent finds the treasure within the t -th step, with probability*

$$(i) \quad \Omega\left(\left(\ell^{1+(\alpha-2)}(\log \ell)^{\frac{2}{1-\alpha}}\right)^{-1}\right) \text{ if } \alpha \neq 3;$$

$$(ii) \quad \Omega((\log^4 \ell)^{-1}) \text{ if } \alpha = 3.$$

The next proposition shows a useful coupling between the Pareto flight process and the Pareto walk one.

Proposition 2 (Coupling between Pareto flight and Pareto walk - case $\alpha \in (2, 3]$). *Suppose an agent performing the Pareto flight with any $\alpha \in (2, 3]$ finds the treasure within t steps with probability $p = p(t) > 0$, conditional on the event that all the performed jump lengths are less than $(t \log t)^{\frac{1}{1-\alpha}}$. Then, another agent that performs the Pareto walk, with the same parameter α , finds the treasure within $\Theta(t)$ steps with probability at least $[1 - \mathcal{O}(1/\log t)] \cdot [p(t) - \exp(-t^{\Theta(1)})]$, without any conditional event.*

Informally, thanks to the above coupling, we can transform the upper bound in Proposition 1 on the Pareto flight hitting time into an (unconditional) upper bound on the Pareto walk hitting time, and we get that an agent performing a Pareto walk with parameter $\alpha \in (2, 3]$ finds the treasure within time $t = \Theta(\ell^{1+(\alpha-2)})$ with probability $\Omega\left(\left(\ell^{1+(\alpha-2)}(\log \ell)^{\frac{2}{1-\alpha}}\right)^{-1}\right)$. Then, $k = \Theta\left(\ell^{1+(\alpha-2)}(\log \ell)^{\frac{2}{1-\alpha}}\right)$ agents are sufficient to find the treasure within time $\mathcal{O}(\ell^{1+(\alpha-2)})$, w.h.p., with $\mathcal{O}\left(\ell^2(\log \ell)^{\frac{2}{1-\alpha}}\right)$ work.

In the next subsection, we present an outline of the proof of Proposition 1, while the proof of Proposition 2 is deferred to Section 5.4.

2.1 Outline of the proof of Proposition 1

We consider the following sequence of random variables

$$S_i = \text{length of the } i\text{-th jump of the agent, } i = 1, \dots, t.$$

For each $i = 1, \dots, t$, we define the events

$$E_i = \{S_i < (t \log t)^{-\frac{1}{\alpha}}\} \text{ and } E(t) = \bigcap_{i=1}^t E_i.$$

For brevity's sake, we write $E(t) = E$ when the dependency from t is clear from context. Then, for any node $u = (u_x, u_y)$ of the grid, we also define the random variable

$$Z_u(t) = \text{number of agent's visits at node } u \text{ within } t \text{ steps.}$$

In order to bound the probability that the node u has been visited at least once at time t , namely $\mathbb{P}(Z_u(t) > 0 \mid E)$, we define the following agent's spatial distribution

$$p_{u;i} = \mathbb{P}(\text{the agent is in node } u \text{ at step } i \mid E),$$

and we note that

$$\mathbb{E}[Z_u(t) \mid E] = \sum_{i=0}^t p_{u;i}. \quad (1)$$

The key idea of our approach is to estimate the expected number of visits on the treasure the agent does within the first t steps of the Pareto flight process. We recall that the treasure is an arbitrary node p of the infinite grid located at distance $\ell > 0$ from the origin.

We begin by considering a suitable partition of the grid in three concentric regions. The first one, named \mathcal{A}_1 , consists of all the nodes having distance from the origin roughly smaller than ℓ , i.e.,

$$\mathcal{A}_1 = Q(\ell) = \{(x, y) : \max(|x|, |y|) \leq \ell\}.$$

Informally speaking, the second region, named \mathcal{A}_2 , consists of all nodes whose distance from the origin ranges from ℓ and at most a logarithmic factor further. Actually, its formal definition depends on the current time step t the parallel process is running on. In detail, we wait until the parallel process performs $\Omega(\ell^{-1})$ steps, and, for any fixed $t = \Theta(\ell^{-1})$ (recall that α is fixed), we define

$$\mathcal{A}_2 = \{v \in \mathbb{Z}^2 : |v|_1 \leq 2(t \log t)^{-\frac{1}{\alpha}} \text{ if } \alpha \in (2, 3), \quad |v|_1 \leq 2\sqrt{t} \log t \text{ if } \alpha = 3\} \setminus \mathcal{A}_1.$$

Finally, the third region, which consists of all other (further) nodes, is defined as follows: for any $t = \Theta(\ell^{-1})$,

$$\mathcal{A}_3 = \{v \in \mathbb{Z}^2 : |v|_1 > 2(t \log t)^{-\frac{1}{\alpha}} \text{ if } \alpha \in (2, 3), \quad |v|_1 > 2\sqrt{t} \log t \text{ if } \alpha = 3\}.$$

Our analysis then proceeds along the following steps.

1. **Upper bound for the number of visits in \mathcal{A}_1 .** For a suitable $t = \Theta(\ell^{-1})$, we first get a simple linear upper bound on the average number of visits to region \mathcal{A}_1 : $\mathbb{E}\left[\sum_{v \in \mathcal{A}_1} Z_v(t) \mid E\right] \leq ct$, where $c \in (0, 1)$ is a sufficiently large constant.
2. **Monotonicity of $p_{u;t}$.** We show that the agent's spatial distribution $p_{u;t}$ has the following property. For any $u = (u_x, u_y)$, define $d_u = |u_x| + |u_y|$. Then, it holds that $p_{u;t} \geq p_{v;t}$ for all nodes v lying outside the square $Q(d_u) = \{(x^\theta, y^\theta) : \max(|x^\theta|, |y^\theta|) \leq d_u\}$.

3. **Upper bound for the number of visits in \mathcal{A}_2 .** Thanks to Eq. (1) and Item 2 above, we get that, for a node u , $\mathbb{E}[Z_u(t) | E] \geq \mathbb{E}[Z_v(t) | E]$ for all nodes v outside $Q(d_u)$. So, by taking $u = p$ and by observing that each $v \in \mathcal{A}_2$ lies outside $Q(\ell)$, we get that the average number of visits in \mathcal{A}_2 is at most the expected number of visits on the treasure p (i.e. $\mathbb{E}[Z_p(t) | E]$) times (any upper bound of) the size of \mathcal{A}_2 : in formula, it is upper bounded by $\mathbb{E}[Z_p(t) | E] \cdot 4(t \log t)^{\frac{2}{\alpha-1}}$ if $\alpha \in (2, 3)$, and by $\mathbb{E}[Z_p(t) | E] \cdot 4t \log^2 t$ if $\alpha = 3$.
4. **Upper bound for the number of visits in \mathcal{A}_3 .** Using Chebyshev Inequality, we get the following upper bound on the average number of visits in region \mathcal{A}_3 :

$$\sum_{\substack{v=(x,y): \\ jx+jy \leq 2(t \log t)^{\frac{1}{\alpha-1}}}} \mathbb{E}[Z_v(t) | E] = \mathcal{O}\left(\frac{t}{\log t}\right) \quad \text{if } \alpha \in (2, 3);$$

$$\sum_{\substack{v=(x,y): \\ jx+jy \leq 2^{\frac{1}{\alpha-1}} t \log t}} \mathbb{E}[Z_v(t) | E] = \mathcal{O}\left(\frac{t}{\log t}\right) \quad \text{if } \alpha = 3;$$

5. **Lower bound on the number of visits in p .** By combining the upper bounds in Items 1, 3 and 4, for some $t = \Theta(\ell^{-1})$, we obtain:

$$ct + \mathbb{E}[Z_p(t) | E] \cdot 4(t \log t)^{\frac{2}{\alpha-1}} + \mathcal{O}\left(\frac{t}{\log t}\right) \geq t \quad \text{if } \alpha \in (2, 3);$$

$$ct + \mathbb{E}[Z_p(t) | E] \cdot 4t \log^2 t + \mathcal{O}\left(\frac{t}{\log t}\right) \geq t \quad \text{if } \alpha = 3.$$

From the inequalities above, we get the lower bounds on the expected number of visits on the treasure, until time t :

$$\mathbb{E}[Z_p(t) | E] = \Omega\left(\frac{1}{t^{\frac{3}{\alpha-1}} [\log(t)]^{\frac{2}{\alpha-1}}}\right) \quad \text{if } \alpha \in (2, 3);$$

$$\mathbb{E}[Z_p(t) | E] = \Omega\left(\frac{1}{\log^2 t}\right) \quad \text{if } \alpha = 3.$$

6. **Upper bounds on the number of visits on the origin.** Using Item 3, we prove that the expected number of visits in the origin $\mathbb{E}[Z_o(t) | E]$ is a positive constant w.r.t. t , i.e., $\mathbb{E}[Z_o(t) | E] = a_t(\alpha) = \Theta(1)$, for any choice of $\alpha \in (2, 3)$, while this expectation grows at most as a poly-logarithmic function for the case $\alpha = 3$, i.e. $a_t(3) = \mathcal{O}(\log^2 t)$.
7. **Hitting probability via expected number of visits.** By simple calculations we link the probability to visit a node u within time t with the average number of visits to that node (within time t) and $a_t(\alpha)$:

$$\mathbb{E}[Z_u(t) | E] \geq \mathbb{P}(Z_u(t) > 0 | E) \geq \mathbb{E}[Z_u(t) | E] / a_t(\alpha).$$

8. **Wrap-up.** From Items 5 to 7, for some $t = \Theta(\ell^{-1})$, we get that

$$\mathbb{P}(Z_p(t) > 0 | E) = \Omega\left(\frac{1}{t^{\frac{3}{\alpha-1}} [\log(t)]^{\frac{2}{\alpha-1}}}\right) \quad \text{if } \alpha \in (2, 3), \text{ and}$$

$$\mathbb{P}(Z_p(t) > 0 \mid E) = \Omega\left(\frac{1}{\log^4 t}\right) \text{ if } \alpha = 3.$$

Finally, the claim of Proposition 1 follows from the above inequalities by fixing $t = \Theta(\ell^{-1})$.

3 Related Work

Random searching strategies performed by simple agents is a topic that attracted strong attention of the researchers from several scientific fields. A huge amount of models, questions, analytical and experimental results are now available. We thus restrict our comparison to previous results that are more related to our setting and that we have not already mentioned in the introduction.

3.1 Further recent works on Levy walks

Besides the works mentioned in Section 1.1, a recent work [GK20] analyzes a single *intermittent* Lévy walk as a search strategy for one target of any size and shape in the real two dimensional torus of area n and in continuous time, where an intermittent Lévy walk is a Lévy walk where the walker can look for the target only in-between steps. In particular, if D is the diameter of the target, they prove an unconditional lower bound of $\Omega(n/D)$ on the expected time to find the target, and an upper bound $\tilde{O}(n/D)$ which holds for the intermittent Lévy walk with exponent $\alpha = 2$, thus emphasizing the optimal role of such choice for the exponent in some specific setting. We observe that the result for constant diameter can be obtained in the discrete two dimensional grid of area n and in discrete time by a single simple random walk, which has expected time $\tilde{O}(n)$ to find a single node [DY08].

We remark that the above setting is different from ours. Indeed, in [GK20] the walker lives in a limited two dimensional continuous space (the torus) and in continuous time, and can look for the target only in between steps. The analysis is based on mean-field equations as well as the optimality concept. The above framework results to be similar to that in [VBH⁺99], in which targets are randomly distributed in \mathbb{R}^2 according to a fixed distribution, and the optimality measure to maximize is the expected discovery rate, which corresponds to minimizing the expected time to first hit any target. On the contrary, we study k parallel processes in the discrete infinite two dimensional grid \mathbb{Z}^2 and in discrete time, in the presence of one single target of unitary size (which corresponds to a precise node of the infinite grid), and our goal is to minimize an efficiency measure which involves the number of agents and the hitting time (namely, the work).

3.2 Comparison with the ANTS Problem

With the aim to investigate the key computational aspects of food-searching processes in biological agent systems, in their seminal paper [FK17] Feinerman and Korman introduced an effective framework called the *Ants Nearby Treasure Search* Problem (for short, the *ANTS* Problem). Here, k probabilistic agents, initially placed at the origin of the infinite two-dimensional grid \mathbb{Z}^2 , can move at each discrete time step on one of the neighboring nodes.⁸ The goal is to minimize the hitting time, i.e. the time needed for any agent to find the treasure.

The analysis in [FK17] provides the first known bounds on the trade-off between the agent memory size and the time complexity (i.e. the number of time steps) of the parallel searching algorithm adopted by the agents. They consider the case where agents are allowed, upon initialization (i.e. before leaving the origin), to make use of their local memory in order to coordinate their action. Such initial information is provided by a centralized oracle and, for

⁸More precisely, the problem was first introduced in the conference version of their paper [FKLS12].

instance, may consist of an approximation of the parameter k . They provide tight bounds on the memory/time trade-off. More in detail, it can be easily shown that $\Omega(\ell + \ell^2/k)$ time steps are necessary for any algorithm, both with constant probability and in expectation. Then, in order to achieve such a lower bound, they proved that $\log \log k + \Theta(1)$ bits of local memory are necessary and sufficient, while larger time bounds are obtained for smaller memory size.

As for their optimal algorithms, the authors remark that they are too complex to be considered realistic strategies for the biological agent systems the paper is inspired from.

In [FK17], the authors also consider a simple, more natural strategy, the *Harmonic Search algorithm*, and prove that it finds the treasure within time and work which is not too large compared to the optimal algorithms. In the Harmonic algorithm, agents are independent, perform the same strategy and do not use any initial information given by the oracle: it is thus more realistic w.r.t. biological systems. In detail, let $\alpha > 1$ be an arbitrarily small constant, denote with d_U the ℓ_1 -distance of a node u from the origin, and let c be a normalization constant such that $p(u) = c / d_U^{1+\alpha}$ turns out to be a probability distribution over \mathbb{Z}^2 . Then, the Harmonic algorithms let each agent perform the following three phases: i) independently choose a node u with probability $p(u)$ and reach that node, ii) perform a local *spiral* search around u , and iii) go back to the nest. For details on the spiral trajectory, see [FK17].

The authors of [FK17] analyze the Harmonic algorithm for each given $\alpha > 1$, and find the range of k such that k agents have almost optimal hitting time and, thus, work. In particular, let δ and ϵ be arbitrarily small positive constants and let $\alpha = 1 + \delta$. For $k = \Omega(\ell)$, the Harmonic Search algorithm finds the treasure within time $\mathcal{O}(\ell + \ell^{2+\alpha}/k)$ with probability $1 - \epsilon$. According to our efficiency measure, the algorithm takes $\mathcal{O}(\ell k + \ell^{2+\alpha})$ work, with probability $1 - \epsilon$. We remark that the Harmonic Search algorithm takes a work complexity which is up to an $\mathcal{O}(\ell)$ from the general lower bound $\Omega(\ell^2)$ for $k \in [\Theta(\ell), \Theta(\ell^{1+\delta})]$, where δ is an arbitrarily small positive constant.

We can now compare the performance of Pareto walks we derived in this work with those of the Harmonic Search algorithm [FK17]: Pareto walks can achieve optimal work up to some polylogarithmic factor, for any fixed $\alpha > 1$, while the Harmonic Search algorithm always has a polynomial gap $\mathcal{O}(\ell)$ from the optimal work, for an arbitrary small $\delta > 0$. Moreover, for any constant $0 < c \leq 1$, Theorem 1 implies that there is some value $k = \tilde{\Theta}(\ell^c)$ such that the hitting time of k Pareto walks is $\tilde{\Theta}(\ell^{2-c})$ w.h.p., which turns out to be optimal thanks to the general lower bound $\Omega(\ell + \ell^2/k)$ [FK17].

Conversely, as we discussed in Section 1.1, for any $k = \tilde{\Theta}(\ell^c)$, it is possible to set a suitable value of α for which Pareto walks reach a work which is up to a $\mathcal{O}(\ell)$ polynomial factor from the optimal bound, for any $\delta > 0$ arbitrarily small, w.h.p. So, we get, according to the known upper bound in [FK17], the same efficiency (up to polylogarithms) as the Harmonic algorithm: in fact, for any $\delta > 0$ arbitrary small, for any constant $\delta < c \leq 1 + \delta$, and for any $k = \tilde{\Theta}(\ell^c)$, the Harmonic algorithm has hitting time $\mathcal{O}(\ell^{2-c})$ and work $\mathcal{O}(\ell^{2+\alpha})$ with probability $1 - \epsilon$, where $\epsilon > 0$ is an arbitrary small constant.

Finally, we emphasize that all upper bounds in our analysis hold w.h.p., while the bounds provided for the Harmonic Search algorithm in [FK17] hold with probability $1 - \epsilon$, where $\epsilon > 0$ is an arbitrarily small constant.

3.3 Related work on finite graphs

In [BGK⁺18], Boczkowski et al. analyze a single-agent searching strategy, called *m-intermittent* search, over the *ring topology* of n nodes. They provide some bounds on the cover time (and, thus, on the hitting time) as a function of the number m of different step lengths $\{L_i\}_{i \in [m]}$ an agent randomly selects for its jumps on every step, where L_i is chosen with probability p_i . They show that an expected hitting time which is linear in the size n of the ring can be achieved by choosing a distribution $\{p_i\}_{i \in [m]}$ on the step lengths according to a *Weierstrassian*

random walk [HSM81]: the latter can be regarded as an efficient approximation of the ballistic regime of our Pareto walk (that is, when α is approaching 1). More in detail, a Weierstrassian random walk is obtained by fixing some integer $B > 1$ and choosing $m = \log n$, $L_i = B^i$ and $p_i \propto B^{-i}$. This setting results in an heavy tail w.r.t. n , formally $\mathbb{P}(\text{step length} > t) \geq \frac{1}{2t}$ for $t < \frac{n}{2}$. In [GK19], Guinard and Korman refined the results of [BGK⁺18] on the ring topology by providing bounds on the hitting time that can be achieved by considering small values of m . In particular, they show that by setting $B = n^{\frac{1}{m}}$ in the Weierstrassian random walk described above, the expected hitting time on the ring is $\tilde{\Theta}\left(n^{1+\frac{1}{m}}\right)$.

3.3.1 Mobility models

Agent’s movement models are fundamental tools in the study of mobile communication networks [LV06, Roy11] (in the corresponding research community, such models are called *mobility models*). Analytical results in this area are essentially available for the two most popular classes of mobility models, the random walk and the random way-point models, and some generalizations of them. In more detail, for the continuous space-time setting, stationary agents’ spatial distributions over finite support spaces have been derived in [LV06] for the general model known as *random trip model*. As for the discrete setting, similar results have been derived in [CMS11, CST14] for the general *Markovian trace model* under the assumption that the set of nodes of the (possibly infinite) support graph that admit feasible agent’s mobility paths has finite size.

3.3.2 Other related works in Computer Science

Our setting is reminiscent of the *Parallel Search without Coordination Problem* investigated in [FKR16], in which an infinite list of *boxes* is given, with a *treasure* hidden in one of them, where the boxes’ order reflects the importance of finding the treasure in a given box. At each time step, a search protocol executed by a searcher can *probe* one box, and see whether it contains the treasure. The author study the best running time achievable by *non-coordinating* algorithms, motivated by *robustness* requirements. Crucially, in their setting the searching agent are provided an approximate knowledge of the importance ordering according to which the treasure is present in a given box.

Finally, some readers may find some connection between our work and some research on the Small World phenomenon, which we discuss in Section 3.5.

3.4 Biological relevance of our analysis

The present work rigorously addresses the mathematical problem of investigating the efficiency of parallel, independent Pareto walks as a natural solution for the ANTS Problem. As for the general question of the biological relevance of Lévy walks as a realistic mobility model, we defer the reader to the discussion in [VLR11, Chapter 8]. Assuming that Lévy walks are a relevant mobility model, the main issue with respect to the biological relevance of our results is the absence of coordination among the agents. We argue here that, in some biological scenarios, it appears that the stochastic dependencies among the searching agents do not play a relevant role for the efficiency of the process. Experiments by Fourcassié et al. in [VBVT03] on a population of ants of the species *Messor sancta* consider groups of ants of different sizes and observe several aspects of their dispersion movements over a large arena. Their statistical results show that, while the geometry of an ant’s path appears to be *mechanically* affected by random *collisions* with other ants, the size of the area explored by each ant moving in a group and the average number of interactions among them is not significantly different from the explored area and number of encounters that would be observed if ants were moving on the area independently of

each other. This fact leads the authors to conclude that there is no coordination (i.e. explicit information exchange) among the ants moving on the arena.

3.5 Power laws and stable distributions

In Lévy walks, the jump lengths are chosen according to a Pareto distribution, which naturally relates their analysis to the vast literature on *stable distribution* [Nol07]. [Nol19] provides an extensive updated bibliography on such research area. In Section 3.5.1, we briefly argue that it does not appear to be possible to obtain our results using standard tools from such literature.

Another famous research endeavour evoked by the power-law jump distribution of Pareto walks is the research on the small-world phenomenon in social networks, in particular Kleinberg’s analysis of his generalized Watts-Strogatz model [Kle00b, Kle00a, EK10]. In its simplest form, Kleinberg’s model is a two-dimensional grid where, to each node u , a *long-range* edge is added by connecting it to another node v chosen according to a Pareto distribution w.r.t. the ℓ_1 distance $\text{dist}(u, v)$ between u and v . Kleinberg considers the simple routing algorithm according to which an agent greedily moves, at each step, across the edge which minimizes its distance from the target, and shows that, given a target node v , it achieves optimal performance ($\mathcal{O}(\log^2(\text{dist}(u, v)))$) when long-range connections of length ℓ are chosen with probability proportional to $\frac{1}{\ell}$. Notice that, besides the similarities given by the common presence of a Pareto distribution, our problem is of a different nature as our parallel Pareto walks do not know the location of the target.

3.5.1 Relation with the continuous setting and stable distributions

In this section we discuss how our analysis in the discrete setting relates to known results in the continuous setting. Essentially, we argue that, to the best of our knowledge, our results cannot be derived from known results for the classical, continuous version of Lévy walk.

The sum of i.i.d. random variables with Pareto distribution converges to a stable distribution⁹ for which there is no-known closed-form expression in terms of elementary function, except for few special cases (Gaussian, Cauchy and Lévy distributions) [Sam94, Nol07]. Hence, although it is a classical result that the *normalized* sum of random variables whose tail distribution decays as $x^{-\alpha}$ converges to a distribution with the same tail [GK54, BB08], it does not seem possible to avoid *ad hoc* arguments in order to estimate the distribution of several steps of Pareto walks. More precisely, we failed to work directly with the distribution of the sum of the jumps by employing asymptotic bounds [BB08] in order to estimate the probability for the random walk process to be located on the target node, as we ended up with the need to evaluate the aforementioned distribution on constant values rather than having to estimate its tail.

As for the hitting time of Lévy walks or Lévy flights, a recent study focuses on the one-dimensional continuous setting [PBL⁺19]. Here, the authors estimate the distribution of the hitting time of a single Lévy walk (and Lévy flight), as time $t \rightarrow \infty$ and for fixed, finite values of ℓ . They also provide the distribution of the hitting time of a single Lévy flight for finite t and $\ell \rightarrow \infty$. Our analysis cannot be derived from such estimations since we focus on the setting where time t is some bounded function of ℓ .

4 Preliminaries

In this work we analyze and compare the search efficiency of some mobility models in the following setting. We have an infinite grid \mathbb{Z}^2 and a special node p of the grid called the treasure, which is at distance $\ell > 0$ from the origin $o = (0, 0)$. Two nodes of the grid, (x_1, y_1) and (x_2, y_2) , are connected by an edge if their Manhattan distance is one, i.e. if $|x_1 - x_2| + |y_1 - y_2| = 1$.

⁹We remark how this fact is sometimes subject to misunderstandings in the literature [Lem08].

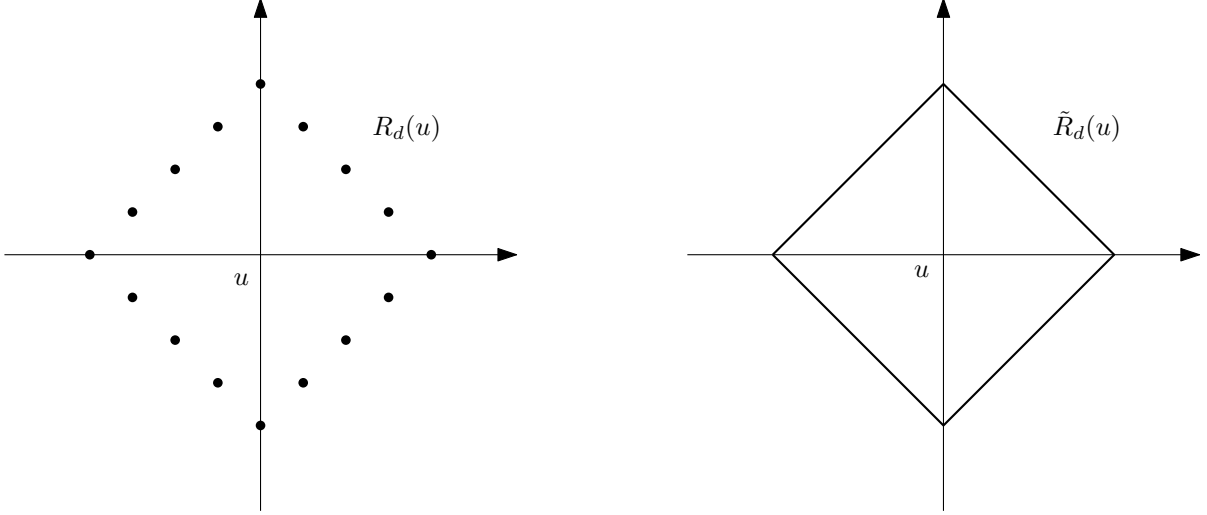


Figure 1: Examples of $R_d(u)$ and $\tilde{R}_d(u)$.

For any node $u = (u_x, u_y) \in \mathbb{Z}^2$, we write $|u|_1 = |u_x| + |u_y|$ for its Manhattan distance from the origin. At the same time, we write $|u|_2 = \sqrt{u_x^2 + u_y^2}$ to denote its Euclidean distance from the origin. With an abuse of notation, for a given set S , we denote as $|S|$ its cardinality. Let u and w be two nodes and define $\vec{v} := \vec{w} - \vec{u}$; we refer to the set of point $\{u + \lambda\vec{v} \mid \lambda > 0\}$ as a *ray* from u to w , or alternatively as a ray starting from u with direction \vec{v} .

Time is discrete and is marked by a global clock. Let $k \in \mathbb{N}$ be any positive integer. At time $t = 0$, k agents are positioned in the origin o and start moving, independently one from each other, over the edges of the grid, to search the treasure. We say an agent finds the treasure at time $t \geq 0$ if, at that time, the agent is located on the treasure (so we assume here that the agent's detection radius is 0). We call step a move that takes one unit time.

To introduce the considered mobility model over the infinite grid, we need the notion of *r-approximating path*, which approximates the unique ray r identified by a unit vector \vec{v} (the *direction*) applied to some node u (the starting node).

As usual, the length of a path is defined as the number of edges it consists of, and the distance between two nodes is the length of any of the shortest paths between them. Consider an agent in some node u of the grid that wants to walk along a ray r . The agent chooses a path *approximating* r according to the following definition.

Definition 2 (*r-approximating path*). *Let r be the unique ray identified by some unit vector \vec{v} applied to some node u . Consider, for $d \geq 0$, the rhombus centered at u*

$$R_d(u) = \{v \in \mathbb{Z}^2 : |u - v|_1 = d\}.$$

For each $d \geq 0$, consider the “natural immersion” of the rhombus in the continuous plane, namely

$$\tilde{R}_d(u) = \{(x, y) \in \mathbb{R}^2 : (y + x + 1)(y + x - 1)(y - x + 1)(y - x - 1) = 0, |x| \leq d, |y| \leq d\},$$

*as in Figure 1. Let v_d the intersection between r and $\tilde{R}_d(u)$. An *r-approximating path* is a simple path which starts at u and whose d -th node, for any d , is the node $w_d \in R_d(u)$ that minimize the distance $\min_{w \in R_d(u)} |w - v_d|_2$. Ties are broken uniformly at random.*

A geometric description of such a path is given in Figure 2, while the well-posedness of Definition 2 is discussed in Appendix B. We now give a definition of what we mean when we say that an agent chooses a direction \vec{v} and moves along it.

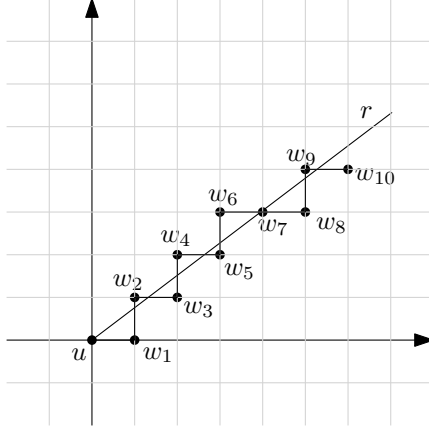


Figure 2: An r -approximating path example.

Definition 3 (Direction choice procedure). *An agent at some node u chooses a direction \vec{v} in the following way: it samples uniformly at random one node w of $\tilde{R}_1(u)$ and takes $\vec{v} = \vec{w} - \vec{u}$ as the unit vector from u to w .*

Observe that an approximating path is unique except for the choices of \vec{v} such that, for some d , there are v_d which are equidistant from two nodes of $R_d(u)$. However, by choosing uniformly at random a direction \vec{v} according to the above procedure, the event that the approximating path is not unique has probability equal to zero, since there is just a countable number of choices for \vec{v} which give rise to this kind of ambiguity (see Appendix B for a formal argument). We remark that there may be different possible procedures an agent may follow to approximate a direction which are simpler and more lightweight. We believe that any procedure that makes an agent choose a direction and a path which crosses a node at distance d with probability $\mathcal{O}(1/d)$ leads to the same results. Our choice, which is an attempt to minimize the distance of the grid path from the ray, simplifies the analysis. Indeed, the following lemma holds, whose proof is deferred to Appendix B.

Lemma 1. *Let u be any node of \mathbb{Z}^2 , $d \geq 1$, and $v \in R_d(u)$. Suppose an agent is on u and chooses a direction according to the procedure in Definition 3. Then, there is probability $1/(4d)$ that the corresponding approximating path crosses v .*

We now introduce the discrete version of Lévy walk that we name Pareto walk.

Definition 4 (α -Pareto walk). *Let $\alpha > 1$ be a real constant. At time $t = 0$, each agent chooses a distance $d \in \mathbb{N}$ with probability distribution $\frac{c}{(1+d)^\alpha}$, where c is a normalization constant. Then, it chooses a direction according to the direction-choice procedure (Definition 3), and walks along the corresponding approximating path (Definition 2) for d steps, reaching some node v . Once reached v , the agent repeats the procedure from the beginning, sampling a new distance and a new direction independently from the previous ones. If the chosen distance d is equal to zero, the agent keeps still for one time unit and then it repeats the procedure.*

As discussed in the introduction, by varying α in the range $(1, +\infty)$, the Pareto walk “simulates” the behaviours of some popular movement models: the simple random walk (for $\alpha \geq 3$) and the ballistic walk (for $\alpha \in (1, 2]$), which we will define later. We are going to compare the search efficiency of each of these models as, essentially, a function of the number k of agents, the first hitting time, and the important notion of work we define below.

Definition 5 (Work). *Let k be any positive integer. Suppose k agents move independently on the grid performing t_1, t_2, \dots, t_k steps, respectively. Then the (overall) work made by the agents is $\sum_{i=1}^k t_i$.*

Notice that if k agents move on the grid for t steps, then the work is equal to $k \cdot t$. Informally, one crucial optimization aspect we consider here is to derive upper and lower bound on the best trade-off between the number k of agents and the hitting time of the corresponding parallel Pareto walk process. We first present a simple extension of the lower bound of [FK17] to the total work made by the agents performing *any* search algorithm. We emphasize that the above lower bound applies independently from the number of agents k , and even assuming that agents share all the information available to them at all time steps.

Lemma 2 (Lower bound on the work). *Let ℓ be any integer such that $\ell \geq 1$, and locate the treasure u.a.r. in one node of the infinite grid among those having distance at most ℓ from the origin. Then, for any $k \geq 1$, and for any search algorithm \mathcal{A} adopted by the k agents, the total work required to find the treasure is $\Omega(\ell^2)$ both with constant probability and in expectation.*

Proof of Lemma 2. Suppose we have k agents looking for the treasure, and let H be the random variable denoting the first hitting time of the treasure. Let $t = \ell^2/(4k)$. Within time $2t$, the agents can cover at most $2kt = \ell^2/2$ nodes. Since the treasure is located u.a.r. on one out of ℓ^2 nodes, there is probability at least $1/2$ that it is not found within $2t$ time steps. Then,

$$\mathbb{E}[kH] = k \sum_{i=1}^{\infty} i \mathbb{P}(H = i) \geq k \sum_{i>2t} i \mathbb{P}(H = i) > 2kt \mathbb{P}(H > 2t) \geq \ell^2/4.$$

□

Our analysis shows that this lower bound can be achieved up to polylogarithmic factor, w.h.p. Namely, if the treasure is located in some node \mathfrak{p} at distance $d_{\mathfrak{p}} = \ell$ from the origin, we show that the Pareto walk in some optimal setting, and the different mobility models we are going to define, find the treasure making a total work of $\tilde{\Theta}(\ell^2)$, w.h.p. (Theorems 2 to 7).

The simple random walk is a well known process, but we are going to present a definition as well.

Definition 6 (Simple random walk). *At each round, the agent, located on node u , chooses one (grid) neighbor v of u u.a.r., and takes a step toward it.*

The simple random walk is a reliable process to search for any specific node (i.e. the treasure) of the grid since it guarantees that with probability one the treasure will be found. On the other hand, this property is achieved at a very-high cost in terms of hitting time since it has high redundancy: the expected number of times the agent visits the same nodes is very large. As we will see in the next sections, the Pareto walk obviates this problem in the regime $\alpha \in (1, 3)$, having a smaller redundancy. Last, we also consider the ballistic walk which is the basic mobility model having redundancy zero.

Definition 7 (Ballistic walk). *At the first round, the agent chooses a direction according to the procedure in Definition 3, and walks along the corresponding approximating path forever, moving over one edge at each step.*

We remark that the Ballistic walk process is just intended to challenge the search efficiency of the other processes, because an agent moving according to it finds the treasure in time exactly ℓ with probability $\Theta(1/\ell)$. If it does not find the treasure within that time, then it will never find it. As discussed in the introduction, the above three processes can be seen as particular cases of the Pareto walk process, for some specific choices of $\alpha > 1$.

Informally, if we set $\alpha \geq 3$, the Pareto walk behaves like a simple random walk; if instead $\alpha \in (1, 2]$, the ballistic walk is a good approximation of the Pareto walk. To provide concrete argument for the above claims we need some preliminaries. We name S_j the random variable

denoting the j -th jump-length of a Pareto walk. Then, for the expected value of a jump-length, the following bound holds:

$$\mathbb{E}[S_j] = \sum_{d=0}^{\ell} \frac{c d}{(1+d)} = \Theta\left(\sum_{d=0}^{\ell} \frac{1}{(1+d)^{\alpha-1}}\right). \quad (2)$$

Notice that the latter bound is finite if $\alpha > 2$, and is infinite if $1 < \alpha \leq 2$. We can also calculate the variance of a jump length (which is well-defined only if the mean is finite) as follows:

$$\text{Var}(S_j) = \sum_{d=0}^{\ell} \frac{c d^2}{(1+d)} - \left[\sum_{d=0}^{\ell} \frac{c d}{(1+d)}\right]^2 = \Theta\left(\sum_{d=0}^{\ell} \frac{1}{(1+d)^{\alpha-2}}\right), \quad (3)$$

which takes finite values if $\alpha > 3$, and is infinite if $2 < \alpha \leq 3$. Looking at this threshold behaviour of the expectation and the variance is the key to show the equivalence between the various Pareto walks and the mobility models we focus on. In detail, if $1 < \alpha \leq 2$, within $\Theta(t)$ steps, the Pareto walk has moved to a distance $\Theta(t)$ from the origin, in average, since the expected step-length is infinite, like the ballistic walk. If $\alpha > 3$, the variance of the jump-length is finite, so the random jump-length is concentrated around its expectation: this fact makes this case of Pareto walks similar to a simple random walk with a longer step-size. Another intuition on the ‘‘equivalence’’ in this case is that, since the variance of a jump-length is finite, the distribution of the Pareto walk at the i -th jump is a two-dimensional Gaussian distribution, due to the Central Limit Theorem, similarly to the case of simple random walk. The case $\alpha = 3$ is a threshold case in which we observe the transition from a super-diffusive regime to the diffusive one, yielding roughly the same results as the simple random walk. Such equivalences will be formally discussed in Section 6 (case $\alpha \in (1, 2]$) and Section 7 (case $\alpha \in [3, +\infty)$), showing that the work done by the Pareto walk and that of the corresponding mobility model (according to the choice of α) are essentially the same.

A useful fact we use several times throughout our analysis is the following. A single agent, performing a Pareto walk, chooses a jump of length at least $d \geq 0$ with probability

$$\mathbb{P}(S_1 \geq d) = \sum_{h=d}^{\ell} \frac{c}{(1+h)} = \Theta\left(\frac{1}{(1+d)^{\alpha-1}}\right), \quad (4)$$

where the last equation follows by applying the standard integral test (Fact 1 in the Appendix A).

We remark that most of the asymptotic bounds we will obtain in this paper hold *with high probability* (in short, *w.h.p.*) with respect to the parameter ℓ : as usual, considering increasing values of ℓ , we say that an event E holds w.h.p. if $\mathbb{P}(E) \geq 1 - 1/\ell^{\Theta(1)}$. Furthermore, each event whose probability is dominated by a factor $1/\ell^{\Theta(1)}$ is said to hold with *negligible* probability, while any other event is said to hold with *non-negligible* probability. Furthermore, we recall that the notation $f(n) = \tilde{O}(g(n))$ means that there exists some $k \in \mathbb{Z}$ such that $f(n) = \mathcal{O}(g(n) \log^k n)$. The same holds for the notations $\tilde{\Omega}(g(n))$ and $\tilde{\Theta}(g(n))$ in an analogous way.

Finally, one remark on the meaning of the statements of the theorems we are going to present.

Remark 1. We emphasize that with the statement ‘‘if $k = \tilde{\Theta}(f(\ell))$, then k agents find the treasure in time $\tilde{\Theta}(g(\ell))$, w.h.p.’’, we mean that there exists at least one k in the family $\tilde{\Theta}(f(\ell))$ such that the agents find the treasure in time $\tilde{O}(g(\ell))$, w.h.p., and for all k in the above family, the treasure is found in time $\tilde{\Omega}(g(\ell))$ with probability at least $1 - o(1)$. For Theorems 4 to 7, the lower bound actually holds w.h.p. Furthermore, from the respective analyses, it will be clear that

- (i) if we increase the optimal k by multiplying by polylogarithmic factors, the upper bound on the hitting time still holds w.h.p.;

- (ii) if we decrease the optimal k by dividing by polylogarithmic factors, the upper bound on the hitting time holds with non-negligible probability.

4.1 Further properties of Pareto models

The Pareto walk process $\{\mathbf{P}_t\}_{t \in \mathbb{N}}$, where \mathbf{P}_t represents the coordinates of a Pareto walk at round t , is not a Markov chain over the state space \mathbb{Z}^2 : \mathbf{P}_t indeed depends on \mathbf{P}_{t-1} , but, also, on the state of the agent, and its last choice of a direction. Moreover, it is hard to recover which nodes are actually visited by the agent during its trip to the next destination (see Definition 4).

To address the issues above we consider two simpler movement models, the Pareto flight (already introduced in Section 2), which is the discrete version of the well-known Lévy flight and which also has a per-se interest, and the Pareto run, and provide bounds on the performances of such models. Then, using coupling arguments, we show how such bounds can be exploited to get similar bounds for the original Pareto walk model.

Definition 1 (α -Pareto flight). *Let $\alpha > 1$ be a real constant. At each round, the agent chooses a distance d with distribution $\frac{c}{(1+d)^\alpha}$, where c is a normalization constant, and chooses u.a.r. one node u among the $4d$ nodes of the grid at distance d from its current position. Then, in one step/unit time, the agent reaches u . Once reached u , the agent repeats the procedure above, and so on. If the chosen distance d is equal to zero, the agent keeps still for one time unit and then it repeats the procedure.*

By defining the two-dimensional random variable \mathbf{P}_t^f as the coordinates of the node the Pareto flight visits at time t , we can easily observe that the process $\{\mathbf{P}_t^f\}_{t \in \mathbb{N}}$ is a Markov chain over the state space \mathbb{Z}^2 . As for the coupling linking this process to the former Pareto walk, the result for the regime $\alpha \in (2, 3]$ has already been given in Section 5.4, while in Section 7 the regime $\alpha \in (3, +\infty)$ is discussed. The analysis of Pareto walks in the remaining case, $\alpha \in (1, 2]$, does not need to make use of the Pareto flight.

We show a monotonicity property which holds for any mobility model such that the step-length distribution is non-increasing and does not depend on the direction (including the Pareto flight model). More formally, we say that a mobility model is *radially monotone* if it satisfies the following property.

Definition 8 (Radially monotone mobility model). *We say that a mobility model is radially monotone if it exists a non-increasing distribution function $\rho : \mathbb{N} \rightarrow \mathbb{R}$, such that, given any two nodes u and v , the probability to jump in just one step from u to v is $p(u, v) = \rho(|u - v|_1)$.*

This monotonicity is characterized by a geometric shape that can be roughly characterized as follows. For any node $u = (u_x, u_y)$, let $d_u = |u_x| + |u_y|$, and define the square

$$Q(d_u) = \{(x^\theta, y^\theta) \in \mathbb{Z}^2 : \max(|x^\theta|, |y^\theta|) \leq d_u\},$$

(See Fig. 3 below). Furthermore, for any node $u \in \mathbb{Z}^2$, let $p_{u,t}$ be the probability that the agent is located in u at time t . The following geometric property holds.

Lemma 3 (Monotonicity property). *Let $u \in \mathbb{Z}^2$ be an arbitrary node, and consider an agent which moves according to a radially monotone mobility model, starting at the origin. Then, for each node $v \notin Q(d_u)$ and each time t , it holds that $p_{u,t} \geq p_{v,t}$.*

Since the Pareto flight model is radially monotone, the next corollary holds trivially.

Lemma 4. *Consider an agent performing a Pareto flight for any parameter $\alpha > 1$ starting at the origin. Let $u \in \mathbb{Z}^2$ be an arbitrary node. Then, for each node $v \notin Q(d_u)$ and each step t , it holds that $p_{u,t} \geq p_{v,t}$, where*

$$Q(d_u) = \{(x, y) \in \mathbb{Z}^2 : \max(|x|, |y|) \leq d_u\}.$$

Proof of Lemma 4. The Pareto flight mobility model fulfill the hypothesis of Lemma 3 in Appendix C. The latter gives the desired result. \square

Fig. 3 gives an idea of the geometrical shape of the discussed *monotonicity* property.

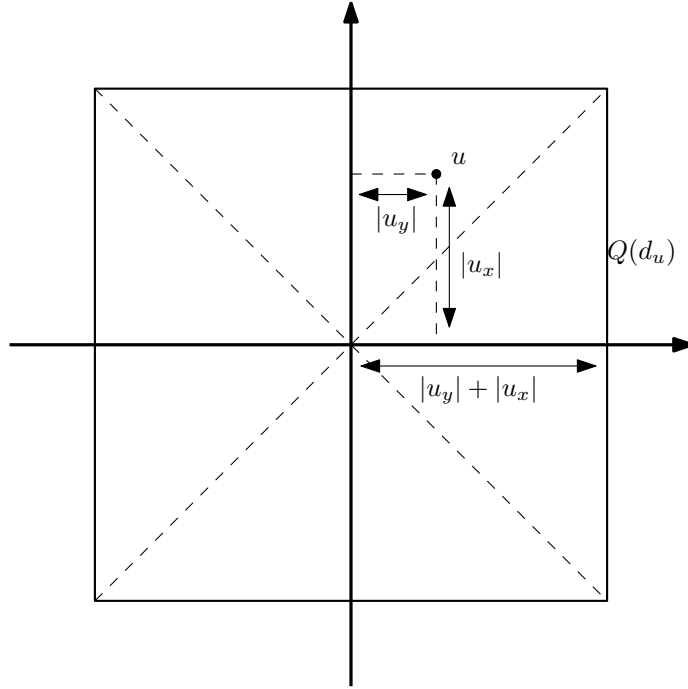


Figure 3: The set $Q(d_u)$.

We need an equivalent framework to derive a lower bound for the Pareto walk. To this aim, we define another similar process, the Pareto run, which we show to be at least as efficient as the Pareto walk.

Definition 9 (α -Pareto run). *Let $\alpha > 1$ be a real constant. At each round, the agent chooses a distance d with distribution $\frac{c}{(1+d)^\alpha}$, where c is a normalization constant, and chooses a direction according to the procedure in Definition 3. Then, it walks along the corresponding approximating path (visiting all the path nodes) in one step/unit time until it reaches the end-point v of the path at distance d . Once v is reached, the agent repeats all the procedure, and so on. If the chosen distance d is equal to zero, the agent keeps still for one time unit and then it repeats the procedure.*

Similarly to Pareto flight, the sequence of random variables denoting the coordinate of the Pareto run at the end of each jump define a Markov chain on the space \mathbb{Z}^2 . Furthermore, we know that at each iteration of the procedure in Definition 9, the agent takes just one time unit to visit all the nodes of the chosen path: this allows us to avoid dealing with the time needed to cover the path.

The next proposition links the Pareto run to the Pareto walk.

Proposition 3 (Coupling: Pareto run into Pareto walk). *Suppose an agent a_1 that moves according to the Pareto run with parameter $\alpha > 1$ finds the treasure within t steps with probability $p > 0$. Then, another agent a_2 that moves according to the Pareto walk with the same parameter α finds the treasure within $\Omega(t)$ steps with probability p . Furthermore, if a_1 never finds the treasure with probability q , then a_2 never finds the treasure with probability at least q .*

Proof of Proposition 3. The proof is trivial, since the Pareto walk behaves as a Pareto run, except for the time needed to perform a jump. \square

Furthermore, the following result will be used throughout the analysis, which is proved using Lemma 1 in the preliminaries (Section 4).

Lemma 5. *Consider an agent performing a Pareto walk (or, equivalently, a Pareto run) which is located at distance $d \geq 0$ from some node u . Then, the probability that it visits u during the next jump is $\Theta(1/d)$.*

Proof of Lemma 5. By Lemma 1, the probability to choose a direction leading to u is $\Theta(1/d)$. Independently, the probability to choose to walk for a distance at least d along the chosen direction is $\Theta(1/d^{-1})$ by Equation (4). Thus, the probability to eventually reach u is $\Theta(1/d)$. \square

5 The Pareto Walk Model: Super-Diffusive Regime

The aim of this section is to analyze the Pareto walk model for the parameter range $\alpha \in (2, 3]$ on the infinite grid and prove the following results.

Theorem 2 (Performances of Pareto walks - case $\alpha \in (2, 3)$). *Let $\alpha \in (2, 3)$ be a real constant and assume that the treasure is located in some node of the infinite grid at distance $\ell > 0$. Let k agents perform mutually independent Pareto walks with parameter α . If $k = \tilde{\Theta}(\ell^{1-(\alpha-2)})$, then the agents find the treasure in time $\tilde{\Theta}(\ell^{1+(\alpha-2)})$, making total work $\tilde{\Theta}(\ell^2)$, w.h.p.¹⁰ Furthermore, the result is almost-tight in a two-fold sense:*

- (i) *Let $k = \tilde{\Theta}(\ell^{1-(\alpha-2)-\epsilon})$ for an arbitrary constant $\epsilon \in (0, 1 - (\alpha - 2)]$. Then the treasure is never hit by the agents, w.h.p., letting the work to be infinite, w.h.p.;*
- (ii) *Let $k = \tilde{\Theta}(\ell^{1-(\alpha-2)+\min(+\frac{2}{2}, \frac{3}{2})-\epsilon})$ for an arbitrary constant $\epsilon > 0$. Then the treasure is found in time at least $\tilde{\Omega}(\ell^{1+(\alpha-2)})$, letting the work to be $\tilde{\Omega}(\ell^{2+\min(-\frac{2}{2}, \frac{3}{2})})$, w.h.p.*

The Pareto walk model for $\alpha = 3$ represents a “threshold” case since, as we will see in the next theorem, it shows the transition from the super-diffusive regime (for $\alpha \in (2, 3)$ the hitting time is sub-quadratic) to the diffusive one. Indeed, its hitting time has roughly the same distribution of that of the simple random walks. However, its analysis cannot rely on the same arguments we will use for the simple random walks since the variance of the jump length is infinite. Instead, we will exploit the same tools used for the case of $\alpha \in (2, 3)$ leading to slightly different results from that of the simple random walks. This technical closeness leads us to present the analysis of the two cases above in the same section.

Theorem 3 (Performances of Pareto walks - case $\alpha = 3$). *Assume that the treasure is located in some node of the infinite grid at distance $\ell > 0$. Let k agents perform mutually independent Pareto walks with parameter $\alpha = 3$. If $k = \log^{\alpha(1)} \ell$, then k agents find the treasure in time $t = \tilde{\Theta}(\ell^2)$, making a total work of $\tilde{\Theta}(\ell^2)$, w.h.p.¹¹ Furthermore, the result is almost-tight in the following sense: if $k = \tilde{\Theta}(\ell^{\min(+\frac{2}{2}, \frac{3}{2})-\epsilon})$ for any arbitrary constant $\epsilon > 0$, then k agents find the treasure in time at least $\tilde{\Omega}(\ell^2)$, letting the work to be $\tilde{\Omega}(\ell^{2+\min(\frac{1}{2}, \frac{3}{2})})$, w.h.p.*

Theorem 2 and Theorem 3 provide, respectively, Claim (ii) and Claim (i) (just for $\alpha = 3$) of Theorem 1 in Section 1.1. The reason why our analysis first focuses on the super-diffusive regime is due to the fact that no other mobility model is known to be equivalent to the Pareto walk model when $\alpha \in (2, 3)$, and we believe this is the most intriguing technical scenario.

¹⁰See Remark 1 in the preliminaries (Section 4) for some formal details.

¹¹See Remark 1 in the preliminaries (Section 4) for some formal details.

5.1 Proofs of Theorem 2 and Theorem 3: main tools and general scheme

As discussed in Section 2, we consider the Pareto flight process and we get the following results. The first one is on the hitting time and will be proved in Section 5.2.

Proposition 1 (Hitting time of Pareto flight - case $\alpha \in (2, 3]$). *Consider a single agent that performs t steps of the Pareto flight with parameter $\alpha \in (2, 3]$. For some $t = \Theta(\ell^{1+(\alpha-2)})$, conditional on the event that the lengths of all performed jumps are less than $(t \log t)^{\frac{1}{1-\alpha}}$, the agent finds the treasure within the t -th step, with probability*

- (i) $\Omega\left(\ell^{1+(\alpha-2)}(\log \ell)^{\frac{2}{1-\alpha}-1}\right)$ if $\alpha \neq 3$;
- (ii) $\Omega((\log^4 \ell)^{-1})$ if $\alpha = 3$.

The second one is a coupling with the former process, and will be proved in Section 5.4.

Proposition 2 (Coupling between Pareto flight and Pareto walk - case $\alpha \in (2, 3]$). *Suppose an agent performing the Pareto flight with any $\alpha \in (2, 3]$ finds the treasure within t steps with probability $p = p(t) > 0$, conditional on the event that all the performed jump lengths are less than $(t \log t)^{\frac{1}{1-\alpha}}$. Then, another agent that performs the Pareto walk, with the same parameter α , finds the treasure within $\Theta(t)$ steps with probability at least $[1 - \mathcal{O}(1/\log t)] \cdot [p(t) - \exp(-t^{\Theta(1)})]$, without any conditional event.*

Thanks to the above coupling, we can transform the upper bound in Proposition 1 on the Pareto flight hitting time into an (unconditional) upper bound on the Pareto walk hitting time.

Instead, for an agent performing a Pareto run, the following holds.

Proposition 4 (Hitting time of Pareto run - case $\alpha \in (2, 3]$). *Let a single agent perform a Pareto run with $\alpha \in (2, 3]$. The followings hold:*

- (i) *If $\alpha \neq 3$, the agent never finds the treasure with probability $1 - \mathcal{O}(\log \ell / \ell^{1+(\alpha-2)})$;*
- (ii) *Let $c \geq 0$ be any arbitrary constant, and let t be any function in $\Theta(\ell^{1+(\alpha-2)} / (\log^c \ell))$. Then, the probability the agent finds the treasure within time t is $\mathcal{O}(1 / (\ell^{1+(\alpha-2)} \log^c \ell))$;*
- (iii) *For an arbitrary constant $\epsilon > 0$, the agent finds the treasure within time $\Theta(\ell^{1+(\alpha-2)-\epsilon})$ with probability:*
 - (a) $\mathcal{O}(1 / \ell^{1+(\alpha-2)+\min(\alpha-2, 2)})$ if $\alpha \neq 3$;
 - (b) $\mathcal{O}(\log \ell / \ell^{\min(\alpha-1, 2)})$ if $\alpha = 3$.

Section 5.3 is devoted to the proof of the above proposition. Next proposition, already given in Section 4.1, links the Pareto run to the Pareto walk.

Proposition 3 (Coupling: Pareto run into Pareto walk). *Suppose an agent a_1 that moves according to the Pareto run with parameter $\alpha > 1$ finds the treasure within t steps with probability $p > 0$. Then, another agent a_2 that moves according to the Pareto walk with the same parameter α finds the treasure within $\Omega(t)$ steps with probability p . Furthermore, if a_1 never finds the treasure with probability q , then a_2 never finds the treasure with probability at least q .*

Thanks to the above results, we can now prove the main result on the Pareto walks for $\alpha \in (2, 3)$. The proof for the case $\alpha = 3$ follows in the successive subsection.

5.1.1 Wrap-up I: proof of Theorem 2

As for the claimed upper bound on the hitting time, the proof proceeds as follows. From Proposition 1 and Proposition 2, we get that a single agent, that moves according to the Pareto walk and perform $t = \Theta(\ell^{1+(\alpha-2)})$ steps, finds the treasure within the t -th step with probability at least $\Omega\left(\left(\ell^{1+(\alpha-2)}(\log \ell)^{-\frac{2}{1-\alpha}}\right)^{\frac{1}{\alpha}}\right)$. Then, if we take k mutually-independent Pareto walk agents, with

$$k = \Theta\left(\ell^{1+(\alpha-2)}(\log \ell)^{-\frac{2}{1-\alpha}} \log \ell\right) = \Theta\left(\ell^{1+(\alpha-2)}(\log \ell)^{-\frac{1}{1-\alpha}}\right),$$

the treasure will be found by at least one of such agents within time $t = \Theta(\ell^{1+(\alpha-2)})$, w.h.p., making a total work of $\tilde{\mathcal{O}}(\ell^2)$. Furthermore, if we increase the number of agents by multiplying by any polylogarithmic factor, the same upper bound on the hitting time holds, w.h.p., while if we decrease it by dividing by any polylogarithmic factor, the upper bound on the hitting time holds with non-negligible probability.

As for the lower bound, for Proposition 4 we have that the agent will not find the treasure within time $t = \Theta(\ell^{1+(\alpha-2)}/\log^{2c} \ell)$ with probability

$$\left[1 - \mathcal{O}\left(\frac{1}{\ell^{1+(\alpha-2)} \log^{2c} \ell}\right)\right]^{\Theta(\ell^{1+(\alpha-2)} \log^c \ell)} = \exp\left(-\mathcal{O}\left(\frac{1}{\log^c \ell}\right)\right) = 1 - \mathcal{O}\left(\frac{1}{\log^c \ell}\right),$$

for any $c \geq 0$, where the last equality comes from the Taylor's expansion. Then, if $k = \Theta(\ell^{1+(\alpha-2)} \log^c \ell)$ is the number of agents, a time of the order of $\tilde{\Omega}(\ell^{1+(\alpha-2)})$ is needed to find the treasure with probability $1 - o(1)$, making a total work of $\tilde{\Omega}(\ell^2)$. On the other hand, if we decrease the number of agents to be $k = \Theta(\ell^{1+(\alpha-2)}/\log^c \ell)$ for some $c > 0$, then the hitting time cannot improve and the same bound holds with probability $1 - o(1)$.

As for Claim (i), Propositions 3 and 4 tell us that an agent performing a Pareto walk has probability $\mathcal{O}(\log \ell / \ell^{1+(\alpha-2)})$ to eventually find the treasure. Let $\epsilon \in (0, 1 - (\alpha - 2)]$ be a constant, and $k = \tilde{\Theta}(\ell^{1+(\alpha-2)})$. Then, k agents never find the treasure with probability $\tilde{\mathcal{O}}(1/\ell)$ (thus the work is infinite, w.h.p.).

As for Claim (ii), let $\epsilon > 0$ be any positive constant, and $k = \tilde{\Theta}\left(\ell^{1+(\alpha-2)+\min(\epsilon, -\frac{2}{2-\alpha}; \frac{3}{2})}\right)$. An agent performing the Pareto walk has probability $\mathcal{O}(1/\ell^{1+(\alpha-2)+\min(\epsilon, (\alpha-2)/2)})$ to find the treasure within time $t = \Theta(\ell^{1+(\alpha-2)})$, for Propositions 3 and 4. Thus, $k = \tilde{\Theta}\left(\ell^{1+(\alpha-2)+\min(\epsilon, -\frac{2}{2-\alpha}; \frac{3}{2})}\right)$ agents performing mutually independent Pareto walks find the treasure in time $\tilde{\Omega}(\ell^{1+(\alpha-2)})$, w.h.p. for the union bound, letting the work to be $\Omega\left(\ell^{2+\min(\epsilon, -\frac{2}{2-\alpha}; \frac{3}{2})}\right)$.

5.1.2 Wrap-up II: proof of Theorem 3

From Proposition 1 and Proposition 2, we get that a single agent, that moves according to the Pareto walk and perform $t = \Theta(\ell^2)$ steps, finds the treasure within the t -th step with probability at least $\Omega\left((\log^4 \ell)^{-1}\right)$. Then, if we take k mutually-independent Pareto walk agents, with

$$k = \Theta(\log^5 \ell),$$

the treasure will be found by at least one of such agents within time $t = \Theta(\ell^2)$, w.h.p., making a total work of $\tilde{\mathcal{O}}(\ell^2)$. Furthermore, if we increase the number of agents by multiplying by any polylogarithmic factor, the same upper bound on the hitting time holds, w.h.p., while if we decrease it by dividing by any polylogarithmic factor, the upper bound on the hitting time holds with non-negligible probability.

As for the lower bound, for Proposition 4 we have that the agent will not find the treasure within time $t = \Theta(\ell^2 / \log^{2c} \ell)$ with probability

$$\left[1 - \mathcal{O}\left(\frac{1}{\log^{2c} \ell}\right)\right]^{\Theta(\log^c \ell)} = \exp\left(-\mathcal{O}\left(\frac{1}{\log^c \ell}\right)\right) = 1 - \mathcal{O}\left(\frac{1}{\log^c \ell}\right),$$

for any $c \geq 0$, where the last equality comes from the Taylor's expansion. Then, if $k = \Theta(\log^c \ell)$ is the number of agents, a time of the order of $\tilde{\Omega}(\ell^{1+(c-2)})$ is needed to find the treasure with probability $1 - o(1)$, making a total work of $\tilde{\Omega}(\ell^2)$.

As for the almost-tightness result, let $\epsilon > 0$ be any positive constant, and $k = \tilde{\Theta}\left(\ell^{\min(c + \frac{1}{2}, \frac{3}{2})}\right)$. An agent performing the Pareto walk has probability $\mathcal{O}(\log \ell / \ell^{\min(c + 1.2, 2)})$ to find the treasure within time $t = \Theta(\ell^2)$ for Propositions 3 and 4. Thus, $k = \tilde{\Theta}\left(\ell^{\min(c + \frac{1}{2}, \frac{3}{2})}\right)$ agents performing mutually independent Pareto walks find the treasure in time $\tilde{\Omega}(\ell^2)$, w.h.p. for the union bound, letting the work to be $\Omega\left(\ell^{2 + \min(\frac{1}{2}, \frac{3}{2})}\right)$.

5.2 The Pareto flight model with $\alpha \in (2, 3]$: proof of Proposition 1

In this subsection we give the analysis for Proposition 1 following the outline presented in Section 2.1. We recall we analyze the Pareto flight for any fixed $\alpha \in (2, 3]$, that represents the key-ingredient to derive the upper bounds on the performances of the Pareto walk model. We proceed as follows.

We consider the following sequence of random variables

$$S_i = \text{length of the } i\text{-th jump of the agent, } i = 1, \dots, t.$$

For each $i = 1, \dots, t$, we define the events

$$E_i = \{S_i < (t \log t)^{-\frac{1}{\alpha}}\} \text{ and } E(t) = \bigcap_{i=1}^t E_i.$$

For brevity's sake, we write $E(t) = E$ when the dependency from t is clear from context. Then, for any node $u = (u_x, u_y)$ of the grid, we also define the random variable

$$Z_u(t) = \text{number of agent's visits at node } u \text{ within } t \text{ steps.}$$

In order to bound the probability that the node u has been visited at least once at time t , namely $\mathbb{P}(Z_u(t) > 0 \mid E)$, we define the following agent's spatial distribution

$$p_{u,i} = \mathbb{P}(\text{the agent is in node } u \text{ at step } i \mid E),$$

and we note that

$$\mathbb{E}[Z_u(t) \mid E] = \sum_{i=0}^t p_{u,i}. \quad (5)$$

The road-map of the analysis has already been discussed in Section 2.1.

5.2.1 Full analysis

We recall the grid partition defined in Section 2.1, namely

$$\mathcal{A}_1 = Q(\ell) = \{(x, y) : \max(|x|, |y|) \leq \ell\},$$

$$\begin{aligned}\mathcal{A}_2 &= \{v \in \mathbb{Z}^2 : |v|_1 \leq 2(t \log t)^{\frac{1}{\alpha-1}} \text{ if } \alpha \in (2, 3), \quad |v|_1 \leq 2\sqrt{t} \log t \text{ if } \alpha = 3\} \setminus \mathcal{A}_1, \\ \mathcal{A}_3 &= \{v \in \mathbb{Z}^2 : |v|_1 > 2(t \log t)^{\frac{1}{\alpha-1}} \text{ if } \alpha \in (2, 3), \quad |v|_1 > 2\sqrt{t} \log t \text{ if } \alpha = 3\},\end{aligned}$$

where \mathcal{A}_2 and \mathcal{A}_3 are defined for any $t = \Theta(\ell^{-1})$. We now provide suitable upper bounds on the expected number of visits the agent makes in \mathcal{A}_1 until time t . Indeed, for the nodes inside $Q(\ell) = \{(x, y) \in \mathbb{Z}^2 : \max(|x|, |y|) \leq \ell\}$, the following holds (this corresponds to Item 1 in Section 2.1).

Lemma 6 (Bound on the visits in \mathcal{A}_1). *Let $\alpha \in (2, 3]$. Then, for some $t = \Theta(\ell^{-1})$, a constant $c \in (0, 1)$ exists such that*

$$\sum_{v \in 2Q(\cdot)} \mathbb{E}[Z_v(t) \mid E] \leq ct.$$

Proof of Lemma 6. We bound the probability the walk has moved to distance $\frac{5}{2}\ell$ at least once, within time $t = \Theta(\ell^{-1})$, by the probability that at least one of the performed jumps is no less than 5ℓ (we denote this latter event as H). Indeed, if there is a jump of length at least 5ℓ , the walk moves necessarily to distance no less than $\frac{5}{2}\ell$. Then,

$$\begin{aligned}\mathbb{P}\left(S_j \geq 5\ell \mid S_j < (t \log t)^{\frac{1}{\alpha-1}}\right) &= \sum_{k=5}^{(t \log t)^{\frac{1}{\alpha-1}} - 1} \frac{c}{(1+k)} \\ &\stackrel{(a)}{\geq} \frac{c}{\alpha-1} \left(\frac{1}{(5\ell+1)^{\alpha-1}} - \frac{1}{t \log t} \right) \\ &\stackrel{(b)}{\geq} \frac{c}{2(\alpha-1)(5\ell)^{\alpha-1}},\end{aligned}$$

where (a) follows for the integral test (Fact 1 in Appendix A), while (b) easily holds for a large enough ℓ since $t = \Theta(\ell^{-1})$. Thanks to the mutual independence among the random destinations chosen by the agent, the probability of the event “the desired jump takes place within time $c^\ell \cdot 2(\alpha-1)(5\ell)^{\alpha-1}/c$ ” is bounded by

$$1 - \left[1 - \frac{c}{2(\alpha-1)(5\ell)^{\alpha-1}}\right]^{c^\ell \frac{2(\alpha-1)(5\ell)^{\alpha-1}}{c}} \geq \frac{3}{4},$$

for some constant $c^\ell > 0$. Hence, by choosing $t \geq 4c^\ell \cdot 2(\alpha-1)(5\ell)^{\alpha-1}/c$, the desired jump takes place with probability $\frac{3}{4}$, within time $\frac{t}{4}$. Once reached such a distance (conditional on the previous event), Fig. 4 shows there are at least other 3 mutually disjoint regions which are at least as equally likely as $Q(\ell)$ to be visited at any future time.

Thus, the probability to visit $Q(\ell)$ at any future time step is at most $\frac{1}{4}$. Observe that

$$\begin{aligned}\mathbb{E}\left[\sum_{v \in 2Q(\cdot)} Z_v(t) \mid E\right] &= \mathbb{E}\left[\sum_{v \in 2Q(\cdot)} Z_v(t) \mid H, E\right] \mathbb{P}(H \mid E) + \mathbb{E}\left[\sum_{v \in 2Q(\cdot)} Z_v(t) \mid H^C, E\right] \mathbb{P}(H^C \mid E) \\ &\leq \left(\frac{1}{4}t + \frac{1}{4} \cdot \frac{3}{4}t\right) \frac{3}{4} + t \cdot \frac{1}{4} \\ &= \frac{t}{4} \left(1 + \frac{3}{4} + \frac{9}{16}\right) = \frac{37}{64}t,\end{aligned}$$

and the proof is completed. \square

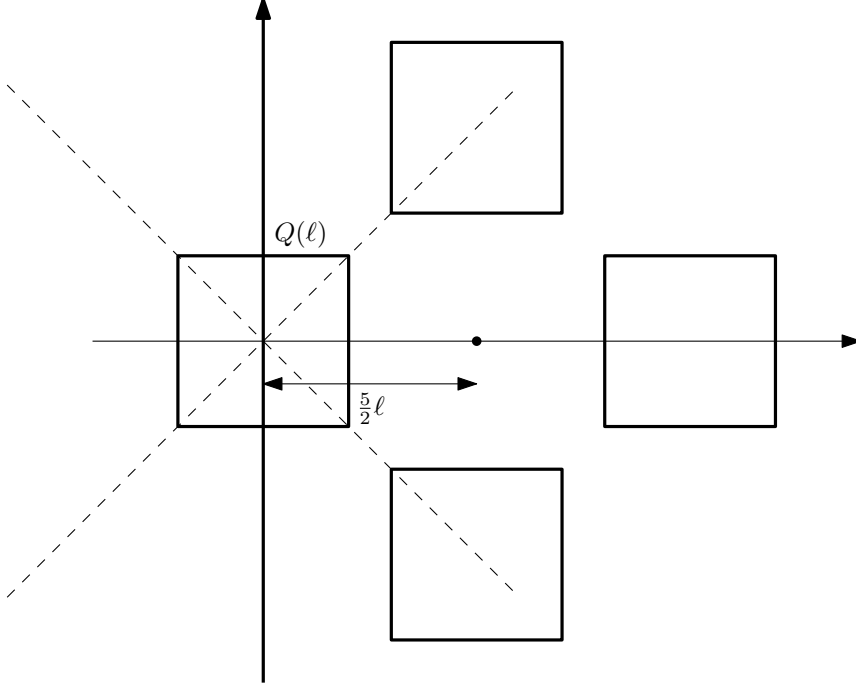


Figure 4: The disjoint zones at least as equally likely as $Q(\ell)$ to be visited.

The spatial distribution yielded by the Pareto flight process has a useful geometric shape that can be roughly characterized as follows. For any node $u = (u_x, u_y)$, we let $d_u = |u_x| + |u_y|$ and consider the square

$$Q(d_u) = \{(x^\theta, y^\theta) \in \mathbb{Z}^2 : \max(|x^\theta|, |y^\theta|) \leq d_u\}$$

(See Fig. 5 in the proof of Lemma 7). Then, the following geometric property holds (described in Item 2 in Section 2.1).

Lemma 7 (Monotonicity property of $p_{u;t}$). *Let $u \in \mathbb{Z}^2$ be an arbitrary node. Then, for each node $v \notin Q(d_u)$ and each step t , it holds that $p_{u;t} \geq p_{v;t}$.*

Proof of Lemma 7. The proof follows from Lemma 3 in Appendix C, observing that the Pareto flight model satisfies the hypothesis of that lemma, and that the conditional event E does not interfere with the proof. \square

We remark that this result is slightly different from that of Lemma 4 because of the presence of the conditional event. Furthermore, notice that, from $\mathbb{E}[Z_v(t) | E] = \sum_{i=0}^t p_{v;i}$, we easily get the following bound (corresponding to Item 3 in Section 2.1).

Corollary 1. $\mathbb{E}[Z_u(t) | E] \geq \mathbb{E}[Z_v(t) | E]$ for all $v \notin Q(d_u)$.

Namely, the more the node is “far” (according to the sequence of squares $\{Q(d)\}_{d \in 2\mathbb{N}}$) from the origin, the less it is visited in average. Thus, each node is visited at most as many times as the origin, in average. This easily gives an upper bound on the total number of visits in \mathcal{A}_2 until time t , namely, by taking $u = \mathfrak{p}$ and by observing that each $v \in \mathcal{A}_2$ lies outside $Q(\ell)$, we get that the average number of visits in \mathcal{A}_2 is at most the expected number of visits on the treasure \mathfrak{p} (i.e. $\mathbb{E}[Z_{\mathfrak{p}}(t) | E]$) times (any upper bound of) the size of \mathcal{A}_2 : in formula, it is upper bounded by $\mathbb{E}[Z_{\mathfrak{p}}(t) | E] \cdot 4(t \log t)^{\frac{2}{\alpha-1}}$ if $\alpha \in (2, 3)$, and by $\mathbb{E}[Z_{\mathfrak{p}}(t) | E] \cdot 4t \log^2 t$ if $\alpha = 3$.

The next lemma considers \mathcal{A}_3 and corresponds to Item 4 in Section 2.1.

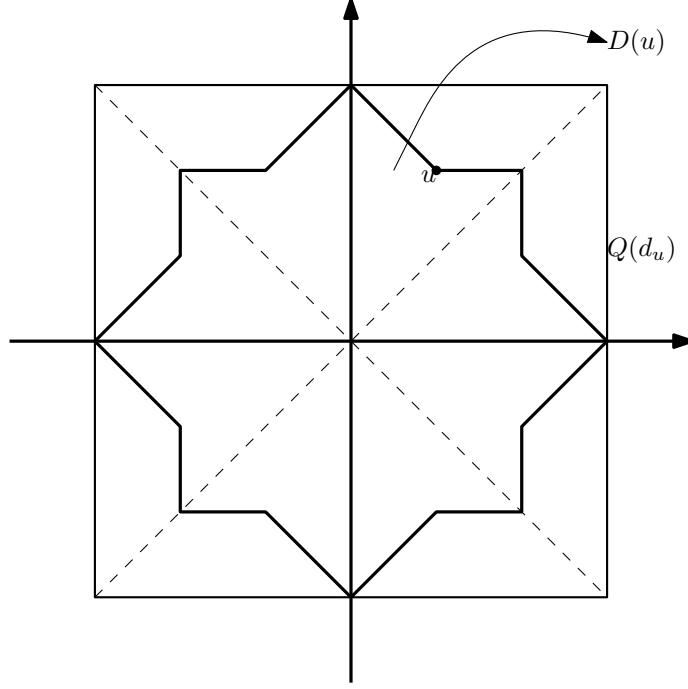


Figure 5: The set $D(u)$, consisting in all inner nodes of the “star”, and the square $Q(d_u)$.

Lemma 8 (Bound on visits in \mathcal{A}_3). *For $\alpha \in (2, 3]$, it holds that*

$$\sum_{\substack{v=(x,y): \\ jx+jy \leq 2(t \log t)^{\frac{1}{\alpha-1}}}} \mathbb{E}[Z_v(t) \mid E] = \mathcal{O}\left(\frac{t}{\log t}\right) \quad \text{if } \alpha \in (2, 3); \quad (6)$$

$$\sum_{\substack{v=(x,y): \\ jx+jy \leq 2^{\frac{1}{\alpha-1}} t \log t}} \mathbb{E}[Z_v(t) \mid E] = \mathcal{O}\left(\frac{t}{\log t}\right) \quad \text{if } \alpha = 3. \quad (7)$$

Proof of Lemma 8. Let $\mathbf{P}_{t^\ell}^f$ be the two dimensional random variable representing the coordinates of the node the agent performing the Pareto flight is located in at time t^ℓ . Consider the projection of the Pareto flight on the x -axis, namely the random variable X_{t^ℓ} such that $\mathbf{P}_{t^\ell}^f = (X_{t^\ell}, Y_{t^\ell})$. The random variable X_{t^ℓ} can be expressed as the sum of t^ℓ random variables S_j^x , $j = 1, \dots, t^\ell$, representing the jumps (with sign) the projection of the walk takes at each of the t^ℓ rounds. The partial distribution of the jumps along the x -axis, conditional on the event E , can be derived as follows¹². For any $0 \leq d \leq (t \log t)^{\frac{1}{\alpha-1}} - 1$,

$$\begin{aligned} & \mathbb{P}\left(S_j^x = \pm d \mid S_j < (t \log t)^{\frac{1}{\alpha-1}}\right) \\ &= \begin{cases} c + \sum_{k=1}^{(t \log t)^{\frac{1}{\alpha-1}} - 1} \frac{c}{2k(1+k)} & d=0 \\ \frac{c}{2d(1+d)} + \sum_{k=1+d}^{(t \log t)^{\frac{1}{\alpha-1}} - 1} \frac{c}{k(1+k)} & d \neq 0 \end{cases} \quad (8) \end{aligned}$$

where: d_{2A} returns 1 if $d \in A$ and 0 otherwise, the term

$$c \cdot d_{=0} + \frac{c}{2d(1+d)} \cdot d_{\neq 0}$$

¹²We remark that in Appendix D we estimate the unconditional distribution of the jump projection length on the x -axis (Lemma 32) for any $\alpha > 1$. Nevertheless, in this case we are conditioning on the event the the original two dimensional jump is bounded, and thus we cannot make use of Lemma 32.

is the probability that the original jump lies along the horizontal axis and has “length” exactly d (there are two such jumps if $d > 0$), and, for $k \geq 1 + d$, the terms

$$\frac{c}{2k(1+k)} \quad d=0 + \frac{c}{k(1+k)} \quad d \neq 0$$

are the probability that the original jump has “length” exactly k and its projection on the horizontal axis has “length” d (there are two such jumps if $d = 0$, and four such jumps if $d > 0$). Observe that (8) is of the order of

$$\Theta \left(\frac{1}{d(1+d)} + \sum_{k=1+d}^{(t \log t)^{\frac{1}{1-\alpha}}} \frac{1}{k(1+k)} \right).$$

By the integral test (Fact 1 in A), we know that this probability is

$$\mathbb{P}(S_j^X = \pm d \mid E_j) = \Theta \left(\frac{1}{(1+d)} \right).$$

Due to symmetry, it is easy to see that $\mathbb{E}[X_{t^\theta} \mid E] = 0$ for each time t^θ , while

$$\text{Var}(X_{t^\theta} \mid E) = \sum_{i=1}^{t^\theta} \text{Var}(S_i^X \mid E_i) = t^\theta \text{Var}(S_1^X \mid E_1)$$

since $S_1^X, \dots, S_{t^\theta}^X$ are i.i.d.

As for the case $\alpha \in (2, 3)$, the variance of S_1^X conditioned to the event $E_1 = \{S_1 < (t \log t)^{\frac{1}{1-\alpha}}\}$, can be bounded as follows

$$\begin{aligned} \text{Var}(S_1^X \mid E_1) &\leq \sum_{k=1}^{(t \log t)^{\frac{1}{1-\alpha}}} \mathcal{O} \left(\frac{k^2}{(1+k)} \right) \\ &\stackrel{(a)}{=} \mathcal{O} \left(\frac{1}{3-\alpha} \left[(t \log t)^{\frac{3}{1-\alpha}} - 2^3 \right] \right) + \mathcal{O} \left(\frac{1}{2^2} \right) \\ &= \mathcal{O} \left((t \log t)^{\frac{3}{1-\alpha}} \right), \end{aligned}$$

where, in (a), we used the integral test (Fact 1 in Appendix A). Observe that the event $E = \bigcap_{i=1}^t E_i$ has probability

$$\mathbb{P}(E) = 1 - \mathcal{O} \left(\frac{1}{\log t} \right).$$

Then, for each $t^\theta \leq t$, from the Chebyshev’s inequality and the fact that $\mathbb{E}[X_{t^\theta} \mid E] = 0$,

$$\mathbb{P} \left(|X_{t^\theta}| \geq (t \log t)^{\frac{1}{1-\alpha}} \mid E \right) \leq \frac{t^\theta \text{Var}(S_1^X \mid E_1)}{(t \log t)^{\frac{2}{1-\alpha}}} \leq \frac{t \text{Var}(S_1^X \mid E_1)}{(t \log t)^{\frac{2}{1-\alpha}}} = \mathcal{O} \left(\frac{1}{\log t} \right),$$

which implies that

$$\mathbb{P} \left(|X_{t^\theta}| \geq (t \log t)^{\frac{1}{1-\alpha}} \right) \leq \mathbb{P} \left(|X_{t^\theta}| \geq (t \log t)^{\frac{1}{1-\alpha}} \mid E \right) + \mathbb{P}(E^C) = \mathcal{O} \left(\frac{1}{\log t} \right).$$

Then, the probability that both X_{t^θ} and Y_{t^θ} are less than $(t \log t)^{\frac{1}{1-\alpha}}$ (call the corresponding events $A_{X:t^\theta}$ and $A_{Y:t^\theta}$, respectively) is

$$\mathbb{P}(A_{X:t^\theta} \cap A_{Y:t^\theta}) = \mathbb{P}(A_{X:t^\theta}) + \mathbb{P}(A_{Y:t^\theta}) - \mathbb{P}(A_{X:t^\theta} \cup A_{Y:t^\theta}) \geq 1 - \mathcal{O} \left(\frac{1}{\log t} \right),$$

for any $t^\theta \leq t$. Then, let $Z^\theta(t)$ be the random variable indicating the number of times the Pareto flight visits the set of nodes whose coordinates are both no less than $(t \log t)^{\frac{1}{\alpha}}$, until time t . Then,

$$\mathbb{E} [Z^\theta(t) | E] \leq \sum_{\substack{v=(x,y) \\ |x|+|y| \geq 2(t \log t)^{\frac{1}{\alpha}}}} \mathbb{E} [Z_v(t) | E],$$

and

$$\begin{aligned} \mathbb{E} [Z^\theta(t) | E] &= \sum_{i=0}^t \mathbb{E} [Z^\theta(i) | A_{x;i} \cap A_{y;i}, E] \mathbb{P}(A_{x;i} \cap A_{y;i} | E) \\ &\quad + \sum_{i=0}^t \mathbb{E} [Z^\theta(i) | (A_{x;i} \cap A_{y;i})^C, E] \mathbb{P}((A_{x;i} \cap A_{y;i})^C | E) \\ &= \sum_{i=0}^t \mathbb{E} [Z^\theta(i) | (A_{x;i} \cap A_{y;i})^C, E] \mathbb{P}((A_{x;i} \cap A_{y;i})^C | E) \\ &\leq t \cdot \mathcal{O} \left(\frac{1}{\log t} \right) = \mathcal{O} \left(\frac{t}{\log t} \right), \end{aligned}$$

which proves Eq. (6).

As for the case $\alpha = 3$, the variance of S_1^X conditional on E_1 is $\mathcal{O}(\log(t \log t))$. Then, we look at the probability that $|X_{t^\theta}|$ is at least $\sqrt{t} \cdot \log t$ conditional on E , which is, again, $\mathcal{O}(1/\log t)$. Finally, the proof proceeds in exactly the same way of the previous case, obtaining Eq. (7). \square

Let p be the node in which the treasure is located. For each node v in \mathcal{A}_2 we already know that $\mathbb{E} [Z_p(t) | E] \geq \mathbb{E} [Z_v(t) | E]$ thanks to Corollary 1. Then, we have the following result which formalizes Item 5 in Section 2.1.

Lemma 9. *Let $\alpha \in (2, 3)$, and let u be any node such that $d_u = \ell$. Then, for some $t = \Theta(\ell^{-1})$,*

$$ct + \mathbb{E} [Z_p(t) | E] \cdot 4(t \log t)^{\frac{2}{\alpha-1}} + \mathcal{O} \left(\frac{t}{\log t} \right) \geq t \quad \text{if } \alpha \in (2, 3); \quad (9)$$

$$ct + \mathbb{E} [Z_p(t) | E] \cdot 4t \log^2 t + \mathcal{O} \left(\frac{t}{\log t} \right) \geq t \quad \text{if } \alpha = 3. \quad (10)$$

Proof of Lemma 9. Suppose the agent has made t jumps for some $t = \Theta(\ell^{-1})$ (the same t of Lemma 6), thus visiting exactly t nodes. Then,

$$\mathbb{E} \left[\sum_{v \in \mathcal{Z}^2} Z_v(t) | E \right] = t.$$

As for Eq. (9), we observe that, from Lemma 6, the number of visits in $\mathcal{A}_1 = Q(\ell)$ until time t is at most ct , for some constant $c \in (0, 1)$. From Lemma 8, the number of visits in \mathcal{A}_3 is at most $\mathcal{O}(t/\log t)$. Thanks to Corollary 1, each of the remaining nodes, i.e., the nodes in \mathcal{A}_2 (whose size is at most $4(t \log t)^{\frac{2}{\alpha-1}}$), is visited by the agent at most $\mathbb{E} [Z_p(t) | E]$ times. It follows that

$$ct + \mathbb{E} [Z_p(t) | E] \cdot 4(t \log t)^{\frac{2}{\alpha-1}} + \mathcal{O} \left(\frac{t}{\log t} \right) \geq t.$$

As for Eq. (10), we proceed as for the first case above. We notice that, from Lemma 8, the number of visits in \mathcal{A}_2 is at most $\mathbb{E} [Z_p(t) | E] \cdot (4t \log^2 t)$. This gives Eq. (10). \square

The next two lemmas provide a clean relationship between the probability to hit a node u within time t to the average number of visits to the origin and to the average number of visits to u itself (this corresponds to Item 6 in Section 2.1). In particular, the first lemma estimate the average number of visits to the origin.

Lemma 10 (Visits in the origin). *For any $t \geq 0$ and $\alpha \in (2, 3]$, let $\mathbb{E}[Z_o(t) | E] = a_t(\alpha)$. Then,*

- if $\alpha \in (2, 3)$, then $a_t(\alpha) = \Theta(1)$ (i.e., it is constant w.r.t. t);
- if $\alpha = 3$, then $a_t(3) = \mathcal{O}(\log^2 t)$.

Proof of Lemma 10. As for the first claim, we proceed as follows. Since $\mathbb{E}[Z_o(t) | E] = \sum_{k=1}^t p_{o:k}$, it suffices to accurately bound the probability $p_{o:k}$ for each $k = 1, \dots, t$. Let us make a partition of the natural numbers in the following way

$$\mathbb{N} = \bigcup_{t^\ell=0}^7 \left[\mathbb{N} \cap [2t^\ell \log t^\ell, 2(t^\ell + 1) \log(t^\ell + 1)) \right].$$

For each $k \in \mathbb{N}$, there exists t^ℓ such that $k \in [2t^\ell \log t^\ell, 2(t^\ell + 1) \log(t^\ell + 1))$. Then, within $2t^\ell \log t^\ell$ steps, we claim that the walk has moved to distance $\lambda = \frac{(t^\ell)^{\frac{1}{\alpha-1}}}{2} - 1$ at least once, with probability $\Omega\left(\frac{1}{(t^\ell)^2}\right)$. Indeed, if there is one jump of length at least 2λ , then the walk has necessarily moved to a distance at least λ from the origin. We now bound the probability that one jump is at least 2λ . For the integral test, we get

$$\begin{aligned} \mathbb{P}\left(S_j \geq 2\lambda \mid S_j < (t \log t)^{\frac{1}{\alpha-1}}\right) &\geq \frac{1}{\mathbb{P}\left(S_j < (t \log t)^{\frac{1}{\alpha-1}}\right)} \left[\int_2^{(t \log t)^{\frac{1}{\alpha-1}} - 1} \frac{c}{(s+1)} ds \right] \\ &\geq \frac{c}{\alpha-1} \left(\frac{1}{t^\ell} - \frac{1}{t \log t} \right) \\ &\geq \frac{c}{\alpha-1} \left(\frac{1 - \frac{t^\ell}{t \log t}}{t^\ell} \right) \\ &\geq \frac{c}{\alpha-1} \left(\frac{1 - \frac{1}{2 \log(t^\ell) \log t}}{t^\ell} \right) \\ &= \Omega\left(\frac{1}{t^\ell}\right), \end{aligned}$$

where the last inequality holds since $2t^\ell \log t^\ell \leq t$. Thus, the probability that the first $2t^\ell \log t^\ell$ jumps are less than 2λ is

$$\begin{aligned} \mathbb{P}\left(\bigcap_{j=1}^{2t^\ell \log t^\ell} \{S_j < 2\lambda\} \mid E\right) &\stackrel{(a)}{=} \left[1 - \mathbb{P}\left(S_1 < 2\lambda \mid S_1 < (t \log t)^{\frac{1}{\alpha-1}} - 1\right) \right]^{2t^\ell \log t^\ell} \\ &\geq \left[1 - \Omega\left(\frac{1}{t^\ell}\right) \right]^{2t^\ell \log t^\ell} \\ &= \mathcal{O}\left(\frac{1}{(t^\ell)^2}\right), \end{aligned}$$

where in (a) we used the independence among the agent's jumps. Once the agent reaches such a distance, Lemma 7 implies that there are at least $\lambda^2 = \Omega\left(\frac{1}{(t^\ell)^{\frac{2}{\alpha-1}}}\right)$ different nodes that are at least as equally likely as o to be visited at any given future time. Thus, the probability to reach the origin at any future time is at most $\mathcal{O}\left(\frac{1}{(t^\ell)^{\frac{1}{\alpha-1}}}\right) = \mathcal{O}\left(\frac{1}{(t^\ell)^{\frac{1}{\alpha-1}}}\right)$ for some small constant $\epsilon > 0$:

in particular the bound holds for $p_{o:k}$. Observe that in an interval $[2t^\theta \log t^\theta, 2(t^\theta + 1) \log(t^\theta + 1))$ there are

$$2(t^\theta + 1) \log(t^\theta + 1) - 2t^\theta \log t^\theta = 2t^\theta \left[\log \left(1 + \frac{1}{t^\theta} \right) \right] + 2 \log(t^\theta + 1) = \mathcal{O}(\log t^\theta)$$

integers. Let \mathbf{P}_t^f be the two-dimensional random variable denoting the node visited at time t by an agent which started from the origin, and let H_{t^θ} be the event $\cup_{j=1}^{2t^\theta \log t^\theta} \{S_j \geq 2\lambda\}$. Observe that, by the law of total probability,

$$p_{o:k} = \mathbb{P} \left(\mathbf{P}_t^f = o \mid H_{t^\theta}, E \right) \mathbb{P} (H_{t^\theta} \mid E) + \mathbb{P} \left(\mathbf{P}_t^f = o \mid H_{t^\theta}^C, E \right) \mathbb{P} (H_{t^\theta}^C \mid E).$$

Thus, if $I_{t^\theta} = [2t^\theta \log t^\theta, 2(t^\theta + 1) \log(t^\theta + 1))$, we get

$$\begin{aligned} \sum_{k=1}^t p_{o:k} &\leq \sum_{t^\theta=0}^7 \sum_{k \in I_{t^\theta}} p_{o:k} \\ &\leq \sum_{t^\theta=0}^7 \left[\mathbb{P} \left(\mathbf{P}_t^f = o \mid H_{t^\theta}, E \right) \mathbb{P} (H_{t^\theta} \mid E) + \mathbb{P} \left(\mathbf{P}_t^f = o \mid H_{t^\theta}^C, E \right) \mathbb{P} (H_{t^\theta}^C \mid E) \right] \mathcal{O}(\log t^\theta) \\ &\leq \sum_{t^\theta=0}^7 \left[\mathcal{O} \left(\frac{1}{(t^\theta)^{1+}} \right) + \mathcal{O} \left(\frac{1}{(t^\theta)^2} \right) \right] \mathcal{O}(\log t^\theta) = \mathcal{O}(1). \end{aligned}$$

We have just proved that the quantity $\sum_{k=1}^t p_{o:k}$ is at most a constant. Conversely, the fact that this quantity is at least a constant is clear by observing that $p_{o:1}$ is constant.

As for the second claim, we can consider the same argument above for the first claim where we fix $\lambda = \sqrt{t^\theta}$. Then the proof proceeds as in the previous case by observing that the average number of visits until time t is, now, of magnitude $\mathcal{O}(\log^2 t)$. \square

Lemma 11. *Let $u \in \mathbb{Z}^2$ be any node. Then,*

- (i) $\mathbb{E} [Z_u(t) \mid E] \leq a_t(\alpha)$,
- (ii) $1 \leq \mathbb{E} [Z_u(t) \mid Z_u(t) > 0, E] \leq a_t(\alpha)$,
- (iii) $\mathbb{E} [Z_u(t) \mid E] / a_t(\alpha) \leq \mathbb{P} (Z_u(t) > 0 \mid E) \leq \mathbb{E} [Z_u(t) \mid E]$.

Proof of Lemma 11. Claim (i) is a direct consequence of Claim (ii), since $\mathbb{E} [Z_u(t) \mid Z_u(t) > 0, E] \geq \mathbb{E} [Z_u(t) \mid E]$. As for Claim (ii), let τ be the first time the agent visits u . Then, conditional on $Z_u(t) > 0$, τ is at most t , and

$$\mathbb{E} [Z_u(t) \mid Z_u(t) > 0, E] = \mathbb{E} [Z_o(t - \tau) \mid \tau \leq t, E] \leq \mathbb{E} [Z_o(t) \mid E] = a_t(\alpha).$$

Notice that this expectation is at least 1 since we have the conditional event. As for Claim (iii), let us explicitly write the term $\mathbb{E} [Z_u(t) \mid Z_u(t) > 0, E] \cdot \mathbb{P} (Z_u(t) > 0 \mid E)$:

$$\begin{aligned} &\sum_{i=1}^t i \mathbb{P} (Z_u(t) = i \mid Z_u(t) > 0, E) \cdot \mathbb{P} (Z_u(t) > 0 \mid E) \\ &= \sum_{i=1}^t i \frac{\mathbb{P} (Z_u(t) = i, Z_u(t) > 0, E)}{\mathbb{P} (Z_u(t) > 0, E)} \cdot \frac{\mathbb{P} (Z_u(t) > 0, E)}{\mathbb{P} (E)} \\ &= \sum_{i=1}^t i \frac{\mathbb{P} (Z_u(t) = i, Z_u(t) > 0, E)}{\mathbb{P} (E)} \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^t i \mathbb{P}(Z_u(t) = i \mid E) \\
&= \mathbb{E}[Z_u(t) \mid E].
\end{aligned}$$

Then,

$$\mathbb{E}[Z_u(t) \mid E] \geq \mathbb{P}(Z_u(t) > 0 \mid E) = \frac{\mathbb{E}[Z_u(t) \mid E]}{\mathbb{E}[Z_u(t) \mid Z_u(t) > 0, E]} \geq \frac{\mathbb{E}[Z_u(t) \mid E]}{a_t(\alpha)},$$

since, from Claim (ii), $\mathbb{E}[Z_u(t) \mid Z_u(t) > 0, E] \leq a_t(\alpha)$. \square

5.2.2 Wrap-up: proof of Proposition 1

We are now ready to prove Proposition 1. We first state the following

Corollary 2. *Let p be the node in which the treasure is located. For some $t = \Theta(\ell^{-1})$, the probability the agent visits the treasure at least once within time t , conditional to the event E , is*

$$\begin{aligned}
\mathbb{P}(Z_p(t) > 0 \mid E) &= \Omega\left(\frac{1}{t^{\frac{3}{\alpha-1}} [\log(t)]^{\frac{2}{\alpha-1}}}\right) \quad \text{if } \alpha \in (2, 3); \\
\mathbb{P}(Z_p(t) > 0 \mid E) &= \Omega\left(\frac{1}{\log^4 t}\right) \quad \text{if } \alpha = 3.
\end{aligned}$$

Proof of Corollary 2. The proof easily follows by combining Lemmas 9 to 11 (this corresponds to Item 8 in Section 2.1). \square

Then, to prove Proposition 1, it is sufficient to apply Corollary 2 by setting $t = \Theta(\ell^{-1}) = \Theta(\ell^{1+(\alpha-2)})$.

5.3 The Pareto run model with $\alpha \in (2, 3]$: proof of Proposition 4

Considering our task of searching the treasure, the Pareto run process is clearly at least as efficient as the Pareto walk one since the former can be seen as a Pareto walk which takes only one time unit to perform a jump, while visiting all the nodes in the path the agent chooses to reach the jump destination. To prove lower bounds on the hitting time of the Pareto walk, we can thus consider the Pareto run process. The main results of this section can be stated as follows.

Proposition 4 (Hitting time of Pareto run - case $\alpha \in (2, 3]$). *Let a single agent perform a Pareto run with $\alpha \in (2, 3]$. The followings hold:*

- (i) *If $\alpha \neq 3$, the agent never finds the treasure with probability $1 - \mathcal{O}(\log \ell / \ell^{1+(\alpha-2)})$;*
- (ii) *Let $c \geq 0$ be any arbitrary constant, and let t be any function in $\Theta(\ell^{1+(\alpha-2)} / (\log^c \ell))$. Then, the probability the agent finds the treasure within time t is $\mathcal{O}(1 / (\ell^{1+(\alpha-2)} \log^c \ell))$;*
- (iii) *For an arbitrary constant $\epsilon > 0$, the agent finds the treasure within time $\Theta(\ell^{1+(\alpha-2)-\epsilon})$ with probability:*
 - (a) $\mathcal{O}(1 / \ell^{1+(\alpha-2)+\min(\epsilon, 2)})$ if $\alpha \neq 3$;
 - (b) $\mathcal{O}(\log \ell / \ell^{\min(\epsilon+1, 2)})$ if $\alpha = 3$.

Our analysis proceeds as follows. We first use Lemma 5 in Section 4.1 to bound the probability that during *any* given jump the agent visits the treasure.

Lemma 12. *Let $i = 1, \dots, t$ denote the first $t \geq 1$ jumps performed by an agent, according to the Pareto run process with parameter $\alpha \in (2, 3]$, which starts at the origin. Then, for any $i = 1, \dots, t$, the probability that during the i -th jump the agent finds the treasure is $\mathcal{O}(1/\ell^2)$.*

Proof of Lemma 12. Consider the starting point v of the i -th jump. Our goal is to estimate the probability distribution of the distance of v from the origin. Consider the rhombus centered in \mathfrak{p} (i.e. the treasure node) of nodes that are within distance $\frac{\ell}{4}$ from \mathfrak{p} , namely

$$R_{\frac{\ell}{4}}(\mathfrak{p}) = \{w \in \mathbb{Z}^2 : d(w, \mathfrak{p}) \leq \ell/4\}.$$

For any $v \in R_{\frac{\ell}{4}}(\mathfrak{p})$, the probability that the i -th jump starts in v is at most $\mathcal{O}(1/\ell^2)$ due to Lemma 4 in Section 4.1, since the process restricted only to the jumps endpoints is a Pareto flight. Moreover, for any $1 \leq d \leq \ell/4$, there are at most $4d$ nodes in $R_{\frac{\ell}{4}}(\mathfrak{p})$ located at distance d from \mathfrak{p} . Then, from the chain rule and Lemma 5 in Section 4.1, the probability that the i -th jump starts from $R_{\frac{\ell}{4}}(\mathfrak{p})$ and the agent visits the treasure during the jump is bounded by

$$\mathcal{O}\left(\frac{1}{\ell^2}\right) \sum_{d=1}^{\frac{\ell}{4}} 4d \cdot \mathcal{O}\left(\frac{1}{d}\right) + \mathcal{O}\left(\frac{1}{\ell^2}\right) = \mathcal{O}\left(\frac{1}{\ell^2}\right),$$

where, in the first expression, the last term $\mathcal{O}(1/\ell^2)$ is the contribution of \mathfrak{p} itself. Then, according to the considered model with parameter α , for any fixed node $v \notin R_{\frac{\ell}{4}}(\mathfrak{p})$, the probability that a jump, starting from v , let the agent visit the treasure is at most $\mathcal{O}(1/\ell)$.

Define J_i as the event that the i -th jump, starting from v , let the agent visit the treasure, and V_i be the event that the starting point of the i -th jump is in $R_{\frac{\ell}{4}}(\mathfrak{p})$. Then, recalling that $2 < \alpha \leq 3$,

$$\mathbb{P}(J_i) \leq \mathbb{P}(J_i | V_i) \mathbb{P}(V_i) + \mathbb{P}(J_i | V_i^C) \leq \mathcal{O}\left(\frac{1}{\ell^2}\right) + \mathcal{O}\left(\frac{1}{\ell}\right) = \mathcal{O}\left(\frac{1}{\ell^2}\right),$$

which completes the proof. \square

We are ready to give a lower bound on the probability that an agent, performing a Pareto run, never finds the treasure.

Lemma 13. *Consider an agent performing a Pareto run (Definition 9) with parameter $\alpha \in (2, 3)$. Then, the probability that the agent never finds the treasure is $1 - \mathcal{O}(\log \ell / \ell^{1-\alpha})$.*

Proof of Lemma 13. Consider the first time t_i the agent is at distance at least $\lambda_i = 2^i \ell$ from the origin, for each $i \geq 1$. Define, for $i \geq 1$, the values $\tau_i = 2\lambda_i^{-1} \log \lambda_i$. The probability that $t_i \leq \tau_i$ is bounded from below by the probability that at least one between the first τ_i jumps has length at least λ_i . Since the jump lengths are independent, we get

$$\mathbb{P}(t_i \leq \tau_i) \geq 1 - \left[1 - \mathcal{O}\left(\frac{1}{\lambda_i^{1-\alpha}}\right)\right]^{2\lambda_i^{-1} \log \lambda_i} = 1 - \mathcal{O}\left(\frac{1}{\lambda_i^2}\right) = 1 - \mathcal{O}\left(\frac{1}{2^{2i} \ell^2}\right).$$

Then, thanks to Lemma 12, the expected number of visits to the treasure from time t_i until time t_{i+1} is $\mathcal{O}(\tau_{i+1}/\lambda_i^2) = \mathcal{O}(\tau_i/\lambda_i^2)$ since the agent starts at distance $\Theta(\lambda_i)$ from the target. Moreover, the same lemma implies that the average number of visits to the treasure until time t_1 is $\mathcal{O}(t_1/\ell^2) = \mathcal{O}(\tau_1/\ell^2)$, since the agent starts at distance ℓ from the treasure. From the facts above, we get that the expected total number of visits to the treasure is

$$\mathcal{O}(\tau_1/\ell^2) + \sum_{i=1}^{\infty} \mathcal{O}(\tau_i/\lambda_i^2) = \mathcal{O}(\log \ell / \ell^3) + \sum_{i=1}^{\infty} \mathcal{O}(\log(2^i \ell) / (2^i \ell)^3) = \mathcal{O}(\log \ell / \ell^3).$$

Finally, from Markov's inequality, the probability that the agent visits the treasure at least once is $\mathcal{O}(\log \ell / \ell^3)$, namely, $\mathcal{O}(\log \ell / \ell^{1-\alpha})$. \square

We next give a bound on the probability that an agent finds the treasure within time $\mathcal{O}(\ell^{1+(\alpha-2)}/(\log^c \ell))$, for any constant $c \geq 0$.

Lemma 14. *Consider a single agent performing a Pareto run with parameter $\alpha \in (2, 3]$. Let $c \geq 0$ be any arbitrary constant, and let t be any function in $\Theta(\ell^{1+(\alpha-2)}/(\log^c \ell))$. Then, the probability to find the treasure within time t is $\mathcal{O}(1/(\ell^{1+(\alpha-2)} \log^c \ell))$.*

Proof of Lemma 14. From Lemma 12 and the union bound, the expected number of visits to the treasure until time t is then $\mathcal{O}(t/\ell^2) = \mathcal{O}(1/(\ell^{1+(\alpha-2)} \log^c \ell))$, since the agent starts at distance $\Theta(\ell)$ from the treasure. Then, for the Markov inequality, the hitting probability is $\mathcal{O}(1/(\ell^{1+(\alpha-2)} \log^c \ell))$. \square

In the next lemma we show that the agent finds the treasure within time polynomially smaller than $\Theta(\ell^{1+(\alpha-2)})$ with a very small probability.

Lemma 15. *Consider a single agent performing a Pareto run with parameter $\alpha \in (2, 3]$. Let $\epsilon > 0$ be any arbitrary small constant, and let t be any function in $\Theta(\ell^{1+(\alpha-2)-\epsilon})$. Then, the probability to find the treasure within time t is:*

$$\mathcal{O}\left(\frac{1}{\ell^{1+(\alpha-2)+\min(\alpha-2, 2)}}\right) \quad \text{if } \alpha \neq 3; \quad (11)$$

$$\mathcal{O}\left(\frac{\log \ell}{\ell^{\min(\alpha+1, 2)}}\right) \quad \text{if } \alpha = 3. \quad (12)$$

Proof of Lemma 15. Let X_i be the x -coordinate of the agent at the end of the i -th jump. For any $i \leq t$, we bound the probability that $X_i > \ell/4$. The probability that there is a jump whose length is at least ℓ among the first i jumps is $\mathcal{O}(i/\ell^{-1})$. We first consider the case $\alpha \in (2, 3)$. Conditional on the event that the first i jump-lengths are all smaller than ℓ (event C_i), the expectation of X_i is zero and its variance is

$$i \cdot \sum_{d=0}^{\alpha-4} \Theta(d^2/(1+d)^\alpha) = \Theta(i\ell^3),$$

for the integral test (Fact 1 in Appendix A). Chebyshev's inequality implies that

$$\mathbb{P}(|X_i| \geq \ell/4 \mid C_i) \leq \frac{\Theta(i\ell^3)}{\Theta(\ell^2)} = \mathcal{O}(i/\ell^{-1}).$$

Since the conditional event has probability $1 - \mathcal{O}(i/\ell^{-1})$, then the ‘‘unconditional’’ probability of the event $|X_i| \leq \ell/4$ is

$$[1 - \mathcal{O}(i/\ell^{-1})]^2 = 1 - \mathcal{O}(1/\ell),$$

since $i \leq t = \Theta(\ell^{-1})$. The same result holds analogously for Y_i (the y -coordinate of the agent after the i -th jump), thus obtaining $|X_i| + |Y_i| \leq \ell/2$, with probability $1 - \mathcal{O}(1/\ell)$ by the union bound.

As for the first jump, thanks to Lemma 12, the probability it leads the agent to visit the treasure is $\mathcal{O}(1/\ell)$. Let $2 \leq i \leq t$ and consider the i -th jump. We want to estimate the probability the i -th jump leads the agent to visit the treasure, having the additional information that $t = \Theta(\ell^{1+(\alpha-2)-\epsilon})$. As in the proof of Lemma 12, we consider the node \mathfrak{p} where the treasure is located on, and the rhombus centered in \mathfrak{p} that contains the nodes within distance $\ell/4$ from \mathfrak{p} , namely

$$R_{\alpha-4}(\mathfrak{p}) = \{w \in \mathbb{Z}^2 : d(w, \mathfrak{p}) \leq \ell/4\}.$$

We define: J_i as the event that the i -th jump leads the agent to visit the treasure, K_{i-1} as the event that the $(i-1)$ -th jump ends in $R_{\alpha-4}(\mathfrak{p})$, and F_{i-1} as the event that the $(i-1)$ -th

jump ends at distance farther than $\ell/2$ from the origin. Let \mathbf{P}_i^r be the two-dimensional random variable denoting the coordinates of the node the agent is located on after the i -th jump. Then,

$$\begin{aligned} \mathbb{P}(J_i | K_{i-1}) \mathbb{P}(K_{i-1} | F_{i-1}) &= \sum_{v \in R_{\geq 4}(\mathfrak{p})} \mathbb{P}(J_i | \mathbf{P}_i^r = v) \mathbb{P}(\mathbf{P}_i^r = v | F_{i-1}) \\ &\leq \mathcal{O}\left(\frac{1}{\ell^2}\right) \sum_{v \in R_{\geq 4}(\mathfrak{p})} \mathbb{P}(J_i | \mathbf{P}_i^r = v), \end{aligned}$$

where in the above inequalities we used Lemma 4 in Section 4.1 (that holds since the process restricted to the jump endpoints is a Pareto flight, and since the lemma is independent of the event F_{i-1}), and the fact that, for each $v \in R_{\geq 4}(\mathfrak{p})$, there are at least $\Theta(\ell^2)$ nodes at distance at least $\ell/2$ from the origin which are more likely to be the destination of the i -th jump than v . Then, we proceed as in the proof of Lemma 12 and obtain

$$\mathbb{P}(J_i | K_{i-1}) \mathbb{P}(K_{i-1} | F_{i-1}) = \mathcal{O}\left(\frac{1}{\ell^2}\right). \quad (13)$$

By the law of total probabilities, we get

$$\begin{aligned} \mathbb{P}(J_i) &= \mathbb{P}(J_i | F_{i-1}) \mathbb{P}(F_{i-1}) + \mathbb{P}(J_i | F_{i-1}^C) \mathbb{P}(F_{i-1}^C) \\ &= [\mathbb{P}(J_i | F_{i-1}, K_{i-1}) \mathbb{P}(K_{i-1} | F_{i-1}) + \mathbb{P}(J_i | F_{i-1}, K_{i-1}^C) \mathbb{P}(K_{i-1}^C | F_{i-1})] \mathbb{P}(F_{i-1}) \\ &\quad + \mathbb{P}(J_i | F_{i-1}^C) \mathbb{P}(F_{i-1}^C) \\ &\stackrel{(a)}{\leq} [\mathbb{P}(J_i | K_{i-1}) \mathbb{P}(K_{i-1} | F_{i-1}) + \mathbb{P}(J_i | F_{i-1}, K_{i-1}^C)] \mathbb{P}(F_{i-1}) + \mathbb{P}(J_i | F_{i-1}^C) \mathbb{P}(F_{i-1}^C) \\ &\stackrel{(b)}{\leq} \left[\mathcal{O}\left(\frac{1}{\ell^2}\right) + \mathcal{O}\left(\frac{1}{\ell}\right) \right] \mathcal{O}\left(\frac{1}{\ell}\right) + \mathcal{O}\left(\frac{1}{\ell}\right) \left[1 - \mathcal{O}\left(\frac{1}{\ell}\right) \right] = \mathcal{O}\left(\frac{1}{\ell^{2+}} + \frac{1}{\ell}\right) \end{aligned} \quad (14)$$

where in (a) we used that $K_{i-1} \subset F_{i-1}$ and that $\mathbb{P}(K_{i-1}^C | F_{i-1}) \leq 1$, while in (b) we used Eq. (13), and that $\mathbb{P}(J_i | F_{i-1}, K_{i-1}^C) = \mathcal{O}(1/\ell)$, which is true because the jump starts in a node whose distance from the treasure is $\Omega(\ell)$, and that $\mathbb{P}(J_i | F_{i-1}^C) = \mathcal{O}(1/\ell)$, which is true for the same reason.

Thus, by the union bound and by Eq. (14), the probability that at least one between the t jumps leads the agent to find the treasure is

$$\begin{aligned} \frac{1}{\ell} + (t-1) \mathcal{O}\left(\frac{1}{\ell} + \frac{1}{\ell^{2+}}\right) &= \mathcal{O}(\ell^{-1}) \mathcal{O}\left(\frac{1}{\ell} + \frac{1}{\ell^{2+}}\right) \\ &= \mathcal{O}\left(\frac{1}{\ell^{1+}} + \frac{1}{\ell^{3+2}}\right) \\ &= \mathcal{O}\left(\frac{1}{\ell^{1 + (2) + \min(+(2), 2)}}\right), \end{aligned}$$

which is the first claim Eq. (11) of the lemma.

Consider now the case $\alpha = 3$. The proof proceeds exactly as in the first case, with the only key difference that the variance of X_i is $\Theta(i \log \ell)$. This means that the probability that $|X_i|$ is at least $\ell/4$ conditional to C_i is $\mathcal{O}(\log \ell / \ell^2)$, and the ‘‘unconditional’’ probability that $|X_i|$ is less than $\ell/4$ is $1 - \mathcal{O}(\log \ell / \ell)$. It thus follows that

$$\mathbb{P}(J_i) = \mathcal{O}\left(\frac{\log \ell}{\ell^{2+}} + \frac{\log \ell}{\ell}\right).$$

Then we get the second claimed bound of the lemma:

$$\mathcal{O}\left(\frac{\log \ell}{\ell^{\min(+(1), 2)}}\right).$$

□

5.3.1 Wrap-up: proof of Proposition 4

Lemma 13 immediately gives Claim i of the proposition. Lemma 14 is Claim ii, while Lemma 15 gives claims iiia and iiib.

5.4 On the coupling of Proposition 2

In this subsection, we show the proof of the coupling between the Pareto flight model and the Pareto walk model when $\alpha \in (2, 3]$. We first recall the result and then proceed proving it.

Proposition 2 (Coupling between Pareto flight and Pareto walk - case $\alpha \in (2, 3]$). *Suppose an agent performing the Pareto flight with any $\alpha \in (2, 3]$ finds the treasure within t steps with probability $p = p(t) > 0$, conditional on the event that all the performed jump lengths are less than $(t \log t)^{-\frac{1}{\alpha-1}}$. Then, another agent that performs the Pareto walk, with the same parameter α , finds the treasure within $\Theta(t)$ steps with probability at least $[1 - \mathcal{O}(1/\log t)] \cdot [p(t) - \exp(-t^{\Theta(1)})]$, without any conditional event.*

Proof of Proposition 2. Let S_j be the random variable denoting the j -th jump-length. First, we bound the random jump length in the following way:

$$\begin{aligned} \mathbb{P}(S_j \geq d) &= c \sum_k \frac{1}{d^{1+k}} \leq c \left[\frac{1}{(\alpha-1)(1+d)^{-1}} + \frac{1}{(1+d)} \right] \\ &= c \left[\frac{1+d+\alpha-1}{(\alpha-1)(1+d)} \right] \leq c \left[\frac{(\alpha-1)(1+d)}{(\alpha-1)(1+d)} \right] \\ &= c \frac{1}{(1+d)^{-1}}, \end{aligned}$$

where the first inequality holds for the integral test (Fact 1 in Appendix A). Thus, we get

$$\mathbb{P}\left(S_j \geq (t \log t)^{-\frac{1}{\alpha-1}}\right) \leq c \frac{1}{(1 + (t \log t)^{-\frac{1}{\alpha-1}})^{-1}} \leq c \frac{1}{t \log t}.$$

Let E_j be the event $\{S_j < (t \log t)^{-\frac{1}{\alpha-1}}\}$, and let E be the intersection of E_j for $j = 1, \dots, t$. Notice that, by the union bound, the probability of E is $1 - \mathcal{O}(1/\log t)$. We next apply the multiplicative form of the Chernoff bound to the sum of S_j , conditional on the event E . This is possible since the variable $S_j / \left((t \log t)^{-\frac{1}{\alpha-1}} - 1\right)$ takes values in $[0, 1]$. To this aim, we first bound the expectation of the sum of the random variables S_j , for $j = 1, \dots, t$ conditioned to E .

$$\begin{aligned} \mathbb{E}\left[\sum_{j=1}^t S_j \mid E\right] &= \sum_{j=1}^t \mathbb{E}[S_j \mid E] = t \frac{c}{\mathbb{P}(E)} \sum_{d=0}^{(t \log t)^{-\frac{1}{\alpha-1}} - 1} \frac{d}{(1+d)} \\ &\leq 2c t \sum_{d=1}^{(t \log t)^{-\frac{1}{\alpha-1}} - 1} \frac{1}{(1+d)^{-1}} \\ &\stackrel{(a)}{\leq} 2c t \left[\frac{1}{\alpha-2} \left(\frac{1}{2^{-2}} - \frac{1}{(t \log t)^{-\frac{2}{\alpha-1}}} \right) + \frac{1}{2^{-1}} \right] \\ &\leq 2c \frac{\alpha t}{(\alpha-2)2^{-1}} = \Theta(t), \end{aligned}$$

where (a) holds for the integral test (Fact 1 in Appendix A). Similarly, it holds that

$$\begin{aligned}\mathbb{E}\left[\sum_{j=1}^t S_j \mid E\right] &= \sum_{j=1}^t \mathbb{E}[S_j \mid E] = t \frac{c}{\mathbb{P}(E_j)} \sum_{d=0}^{(t \log t)^{\frac{1}{1-\alpha}} - 1} \frac{d}{(1+d)} \\ &\geq c t \sum_{d=1}^{(t \log t)^{\frac{1}{1-\alpha}} - 1} \frac{1}{2(1+d)} = \Theta(t),\end{aligned}$$

where we used the fact that the harmonic $(\alpha - 1)$ -series converges. Call $\mu = \mathbb{E}\left[\sum_{j=1}^t S_j \mid E\right] = \Theta(t)$. We next use the Chernoff bound (Lemma 30 in Appendix A) on the normalized sum of all jumps, to show that such a sum is linear in t with probability $1 - \exp(-t^{\Theta(1)})$, conditional on E . In formula,

$$\begin{aligned}\mathbb{P}\left(\sum_{j=1}^t S_j \geq 2\mu \mid E\right) &= \mathbb{P}\left(\frac{\sum_{j=1}^t S_j}{(t \log t)^{\frac{1}{1-\alpha}} - 1} \geq 2 \frac{\mu}{(t \log t)^{\frac{1}{1-\alpha}} - 1} \mid E\right) \leq \exp\left(-\frac{\Theta(t)}{3((t \log t)^{\frac{1}{1-\alpha}} - 1)}\right) \\ &\leq \exp\left(-\Theta\left(\frac{t^{-\frac{2}{1-\alpha}}}{(\log t)^{\frac{1}{1-\alpha}}}\right)\right) \leq \exp\left(-\Theta\left(t^{\frac{2}{2(1-\alpha)}}\right)\right).\end{aligned}$$

Then, define

$$\begin{aligned}F &= \left\{ \sum_{j=1}^t S_j = \mathcal{O}(t) \right\} \\ F_1 &= \{\text{the Pareto walk finds the treasure within } \Theta(t) \text{ steps}\} \text{ and} \\ F_2 &= \{\text{the Pareto flight finds the treasure within } t \text{ steps/jumps}\},\end{aligned}$$

where in F_1 the term $\Theta(t)$ counts also the jumps of length zero. Observe that the event $F \cap F_2$ implies the event

$$F_1 \cap \{\text{the process finds the treasure within } t \text{ jumps}\},$$

since if $F \cap F_2$ takes place, then the treasure is found at least in one among all the t jump destinations, and the overall amount of “travel” time is $\Theta(t)$. Thus

$$\begin{aligned}\mathbb{P}(F_1) &\geq \mathbb{P}(F_1, \text{the process finds the treasure within } t \text{ jumps}) \\ &\geq \mathbb{P}(F, F_2) \\ &\geq \mathbb{P}(F, F_2, E) \\ &\stackrel{(a)}{=} \mathbb{P}(E) [\mathbb{P}(F \mid E) + \mathbb{P}(F_2 \mid E) - \mathbb{P}(F \cup F_2 \mid E)] \\ &\stackrel{(b)}{\geq} \left(1 - \mathcal{O}\left(\frac{1}{\log t}\right)\right) [1 - \exp(-t^{\Theta(1)}) + p - 1] \\ &= \left(1 - \mathcal{O}\left(\frac{1}{\log t}\right)\right) (p - \exp(-t^{\Theta(1)})),\end{aligned}$$

where in (a) we used the definition of conditional probability and the inclusion-exclusion principle, and in (b) we used that $\mathbb{P}(E) = (1 - \mathcal{O}(1/\log t))$, $\mathbb{P}(F \mid E) \geq 1 - \exp(-t^{\Theta(1)})$, and $\mathbb{P}(F \cup F_2 \mid E) \leq 1$. \square

6 The Pareto Walk Model: Ballistic Regime

In this section, we analyze the search efficiency of the Pareto walk model in the ballistic regime, namely choosing the parameter $\alpha \in (1, 2]$. Afterwards, we analyze the ballistic walk model, showing that the latter is indeed “equivalent” to the Pareto walk model with $\alpha \in (1, 2]$ in terms of hitting time and work.

For the Pareto walk model, we prove the following theorem.

Theorem 4 (Ballistic regime). *Assume that the treasure is located in some node of the infinite grid at distance $\ell > 0$. Let k agents move performing mutually independent Pareto walks with $\alpha \in (1, 2]$. If $k = \tilde{\Theta}(\ell)$, then k agents find the treasure in time $\Theta(\ell)$, making a total work of $\tilde{\Theta}(\ell^2)$, w.h.p.¹³ Furthermore, the result is almost-tight in a two-fold sense:*

- (i) *If $k = \tilde{\Theta}(\ell^{1-\epsilon})$ for any arbitrary constant $\epsilon \in (0, 1]$, then the agents never find the treasure w.h.p., thus making an infinite work, w.h.p.;*
- (ii) *If $k = \tilde{\Theta}(\ell^{1+\epsilon})$ for any arbitrary constant $\epsilon > 0$, then the agents need time $\Theta(\ell)$ to find the treasure and total work $\tilde{\Theta}(\ell^{2+\epsilon})$, w.h.p.*

Interesting enough, the average jump-length in this case is infinite, thus, in average, we have that the Pareto walk has moved to distance $\Theta(t)$ from the origin in time $\Theta(t)$, exactly as the ballistic walk.

Next subsection is devoted to the proof of Theorem 4. Observe that the above theorem gives Claim (iii) of Theorem 1 in Section 1.1.

6.1 Analysis of the case $\alpha \in (1, 2]$: proof of Theorem 4

We need three lemmas. The first gives an upper bound on the hitting time of the treasure.

Lemma 16. *Let $k \in \mathbb{N}$ be any integer such that $k = \Theta(\ell \log^2 \ell)$. Then, k agents performing independent Pareto walks with parameter $\alpha \in (1, 2]$ find the treasure in time $t = \Theta(\ell)$, w.h.p., letting the work to be $\Theta(\ell^2 \log^2 \ell)$, w.h.p.*

Proof of Lemma 16. Consider a single agent moving according the Pareto walk with parameter $\alpha \in (1, 2]$. By Equation (4) in the preliminaries (Section 4), the probability the agent chooses a jump of length at least d is of the order of $\Theta(1/(1+d)^{\alpha-1})$. Thus, for some constant $c > 0$, we look at the probability of choosing a jump of length no less than $c\ell$, which is $\Theta(1/(1+c\ell)^{\alpha-1}) = \Theta(1/\ell^{\alpha-1})$. This means that an agent chooses such a distance at least once in $\ell^{\alpha-1}/\log(c\ell)$ jumps with probability

$$1 - \left(1 - \Theta\left(\frac{1}{\ell^{\alpha-1}}\right)\right)^{\frac{1}{\log(c\ell)}} = o(1).$$

Let E_i be the event that the all the jumps until the i -th one (included) have length less than $c\ell$. By what we have said before, it is true that

$$\mathbb{P}(E_i) = 1 - o(1) \quad \text{for all } i \leq \ell^{\alpha-1}/\log(c\ell).$$

Then, we show that, conditional on the event E_i , the sum of the first i jumps is at most $\ell/2$ with constant probability. Indeed, if $j < i$, the expected value of S_j is

$$\mathbb{E}[S_j | E_i] = \sum_{d=0}^{c-1} \frac{c-d}{(1+d)} = \mathcal{O}(c^2 \log(c\ell))$$

¹³See Remark 1 in the preliminaries (Section 4) for some formal details.

for the integral test (Fact 1 in Appendix A), where the $\log(c\ell)$ factor takes care of the case $\alpha = 2$. Thus,

$$\mathbb{E} \left[\sum_{j=1}^i S_j \mid E_i \right] \leq \sum_{j=1}^{\frac{1}{\log(c)}} \mathbb{E}[S_j \mid E_i] = \mathcal{O}(c\ell).$$

We choose c small enough so that this expression is less than $\ell/2$. Conditional on E_i , the $\{S_j\}_{j=1}^i$ random variables are non negative and bounded from above since all jump indexes are less than $\ell^{-1}/\log(c\ell)$. Then, we can use the Chernoff bound on their sum (normalized dividing by $c\ell$) to concentrate the probability around its normalized expectation (Lemma 30 in Appendix A). We have that

$$\mathbb{P} \left(\sum_{j=1}^i \frac{S_j}{c\ell} \geq \frac{\ell}{2} \cdot \frac{1+1/2}{c\ell} \mid E_i \right) \leq \exp \left(-\frac{1}{3 \cdot 4 \cdot c} \right),$$

which is a constant. Then it is guaranteed that there is at least constant probability the agent has displacement at most $3\ell/4$ from the origin in time $\mathcal{O}(\ell)$ (since the sum of all jumps is at most linear in ℓ), without any conditional event. Indeed,

$$\begin{aligned} \mathbb{P} \left(\sum_{j=1}^i S_j \leq 3\ell/4 \right) &\geq \mathbb{P} \left(\sum_{j=1}^i S_j \leq 3\ell/4 \mid E_i \right) \mathbb{P}(E_i) \\ &\geq \Theta(1) (1 - o(1)) = \Theta(1), \end{aligned}$$

for each $i \leq \ell^{-1}/\log(c\ell)$.

Let $F_i = \{\sum_{j=1}^i S_j \leq 3\ell/4\}$. We now want to compute the probability that, given $i \leq \ell^{-1}/\log(c\ell)$, the i -th jump leads the agent finding the treasure. Let J_i be such an event. Then, since

$$\mathbb{P}(J_i, F_{i-1}) = \mathbb{P}(J_i \mid F_{i-1}) \mathbb{P}(F_{i-1})$$

we estimate $\mathbb{P}(J_i \mid F_{i-1})$. Let \mathbf{P}_t be the two-dimensional random variable representing the coordinates of the nodes the Pareto walk visits at time t . If t_i is the time the agent starts the i -th jump, we have

$$\mathbb{P}(J_i \mid F_{i-1}) \geq \sum_{v \in \mathcal{Q}(\ell/4)} \mathbb{P}(J_i \mid \mathbf{P}_{t_i} = v, F_{i-1}) \mathbb{P}(\mathbf{P}_{t_i} = v \mid F_{i-1}).$$

By Lemma 5 in Section 4.1, the term $\mathbb{P}(J_i \mid \mathbf{P}_{t_i} = v, F_{i-1})$ is $\Theta(1/\ell)$. At the same time, by Lemma 4 in Section 4.1, we have that $\mathbb{P}(\mathbf{P}_{t_i} = v \mid F_{i-1}) = \Omega(1/\ell^2)$ since $v \in \mathcal{Q}(\ell/4)$. Then

$$\sum_{v \in \mathcal{Q}(\ell/4)} \mathbb{P}(J_i \mid \mathbf{P}_{t_i} = v, F_{i-1}) \mathbb{P}(\mathbf{P}_{t_i} = v \mid F_{i-1}) \geq \Theta \left(\frac{1}{\ell} \right) \cdot \sum_{v \in \mathcal{Q}(\ell/4)} \Omega \left(\frac{1}{\ell^2} \right) = \Omega \left(\frac{1}{\ell} \right),$$

implying $\mathbb{P}(J_i) = \Omega \left(\frac{1}{\ell} \right)$ for all $i \leq \ell^{-1}/\log(c\ell)$. Then, for the chain rule, the probability that none of the events $F_i \cap F_{i-1}$ holds for each $i \leq \ell^{-1}/\log(c\ell)$ is

$$\begin{aligned} \mathbb{P} \left(\bigcup_{i=1}^{\frac{1}{\log(c)}} (J_i \cap F_{i-1}) \right) &= 1 - \mathbb{P} \left(\bigcap_{i=1}^{\frac{1}{\log(c)}} (J_i^C \cup F_{i-1}^C) \right) \\ &= 1 - \prod_{i=1}^{\frac{1}{\log(c)}} \mathbb{P} \left(J_i^C \cup F_{i-1}^C \mid \bigcap_{j=1}^{i-1} (J_j^C \cup F_{j-1}^C) \right) \end{aligned}$$

$$\begin{aligned}
&= 1 - \prod_{i=1}^{\lfloor \frac{1}{\log(c)} \rfloor} \left(1 - \mathbb{P} \left(J_i \cap F_{i-1} \mid \bigcap_{j=i-1} (J_j^C \cup F_j^C) \right) \right) \\
&= 1 - \prod_{i=1}^{\lfloor \frac{1}{\log(c)} \rfloor} \left(1 - \mathbb{P} \left(J_i \cap F_{i-1} \mid \bigcap_{j=i-1} (J_j^C \cup F_j^C) \right) \right) \\
&\stackrel{(a)}{\geq} 1 - \prod_{i=1}^{\lfloor \frac{1}{\log(c)} \rfloor} \left(1 - \mathbb{P} \left(J_i \cap F_{i-1}, \bigcap_{j=i-1} (J_j^C \cup F_j^C) \right) \right) \\
&\stackrel{(b)}{=} 1 - \prod_{i=1}^{\lfloor \frac{1}{\log(c)} \rfloor} (1 - \mathbb{P}(J_i \cap F_{i-1})) \\
&= 1 - \left(1 - \Omega \left(\frac{1}{\ell} \right) \right)^{\lfloor \frac{1}{\log(c)} \rfloor} \\
&= e^{-\Omega \left(\frac{1}{\log \ell} \right)} = 1 - \mathcal{O} \left(\frac{1}{\ell \log \ell} \right),
\end{aligned}$$

where, (a) holds since $\mathbb{P}(A \mid B) \geq \mathbb{P}(A, B)$, (b) holds since $F_{i-1} \subseteq (F_{i-1}^C \cup J_{i-1}^C)$ for $j \leq i-1$, and last equality holds for the Taylor decomposition of $f(x) = e^x$. Then, there is probability at least $\Omega(1/(\ell \log \ell))$ to find the treasure within time $\mathcal{O}(\ell)$. On the other hand, it is trivial that the process needs at least t steps to reach the treasure. Thus, if $k = \Theta(\ell \log^2 \ell)$, then k agents find the treasure within time $\Theta(\ell)$, w.h.p., letting the work to be $\Theta(\ell^2 \log^2 \ell)$. \square

The next two lemmas aim at giving a lower bound on the hitting time of the treasure.

Lemma 17. *Let $i = 1, \dots, t$, denote the t jumps an agent performing a Pareto walk with $\alpha \in (1, 2]$ which starts at the origin takes. The probability that during the i -th jump the agent finds the treasure is $\mathcal{O}(\log \ell / \ell)$.*

Proof of Lemma 17. Consider the starting point v of the i -th jump. We want to give probabilities to the distance at which v is from the origin. Call u the node in which the treasure is located, and consider the rhombus centered in u of nodes that are distant at most $\frac{\ell}{4}$ from u , namely

$$R_{\leq \frac{\ell}{4}}(u) = \{w \in \mathbb{Z}^2 : d(w, u) \leq \ell/4\}.$$

For any $v \in R_{\leq \frac{\ell}{4}}(u)$, the probability that the i -th jump starts in v is at most $\mathcal{O}(1/\ell^2)$ due to Lemma 4 in Section 4.1. At the same time, for any distance $1 \leq d \leq \ell/4$, there are at most $4d$ nodes in $R_{\leq \frac{\ell}{4}}(u)$ at distance d from u . Then, for the expression of conditional probability and Lemma 5 in Section 4.1, the probability that the i -th jump starts from $R_{\leq \frac{\ell}{4}}(u)$ and the agent visits the treasure during the jump is

$$\mathcal{O} \left(\frac{1}{\ell^2} \right) \sum_{d=1}^{\lfloor \frac{\ell}{4} \rfloor} 4d \cdot \mathcal{O} \left(\frac{1}{d} \right) + \mathcal{O} \left(\frac{1}{\ell^2} \right) = \mathcal{O} \left(\frac{\log \ell}{\ell} \right),$$

where, in the first expression, the last term $\mathcal{O}(1/\ell^2)$ is the contribution of u itself. If v is outside $R_{\leq \frac{\ell}{4}}(u)$, then the probability that a jump that starts from v leads the agent to visit the treasure is at most $\mathcal{O}(1/\ell)$.

Let J_i be the event that the i -th jump (which starts in v) leads the agent to visit the treasure, and V_i be the event that the starting point of the i -th jump is in $R_{\leq \frac{\ell}{4}}(u)$. Then

$$\mathbb{P}(J_i) \leq \mathbb{P}(J_i \mid V_i) \mathbb{P}(V_i) + \mathbb{P}(J_i \mid V_i^C) \leq \mathcal{O} \left(\frac{\log \ell}{\ell} \right) + \mathcal{O} \left(\frac{1}{\ell} \right) = \mathcal{O} \left(\frac{\log \ell}{\ell} \right),$$

which is the thesis. \square

Next lemma give us the probability an agent never finds the treasure.

Lemma 18. *Consider a single agent performing a Pareto walk with $\alpha \in (1, 2]$. The probability that the agent never finds the treasure is $1 - \mathcal{O}(\log^2(\ell)/\ell)$.*

Proof of Lemma 18. Consider the first time t_i the agent is at distance at least $\lambda_i = 2^i \ell$ from the origin, for each $i \geq 1$. Define, for $i \geq 1$, $\tau_i = 2\lambda_i^{-1} \log \lambda_i$. Then,

$$\mathbb{P}(t_i \leq \tau_i) \geq 1 - \left[1 - \mathcal{O}\left(\frac{1}{\lambda_i^{-1}}\right)\right]^{2^i \log \lambda_i} = 1 - \mathcal{O}\left(\frac{1}{\lambda_i^2}\right) = 1 - \mathcal{O}\left(\frac{1}{2^{2i} \ell^2}\right).$$

Then, the expected number of visits to the treasure from time t_i until time t_{i+1} is then $\mathcal{O}(\tau_{i+1} \log(\ell)/\lambda_i) = \mathcal{O}(\tau_i \log(\ell)/\lambda_i)$ by Lemma 17, since the agent starts at distance $\Theta(\lambda_i)$ from the target. At the same time, the average number of visits to the treasure until time t_1 is $\mathcal{O}(t_1 \log(\ell)/\ell) = \mathcal{O}(\tau_1 \log(\ell)/\ell)$. Combining the above, we have that the expected total number of visits to the treasure is

$$\mathcal{O}\left(\frac{\tau_1 \log \ell}{\ell}\right) + \sum_{i=1}^{\infty} \mathcal{O}\left(\frac{\tau_i \log \ell}{\lambda_i}\right) = \mathcal{O}\left(\frac{\log^2 \ell}{\ell}\right) + \sum_{i=1}^{\infty} \mathcal{O}\left(\frac{\log(2^i) \log^2 \ell}{2^i \ell}\right) = \mathcal{O}\left(\frac{\log^2 \ell}{\ell}\right).$$

Thus, for the Markov property, the probability that the agent visits the treasure at least once is $\mathcal{O}(\log^2(\ell)/\ell)$. \square

We are ready to prove our main result.

6.1.1 Wrap-up: proof of Theorem 4

Proof of Theorem 4. As for the main result, Lemma 16 tells us that $k = \Theta(\ell \log^2 \ell)$ agents find the treasure in time $\Theta(\ell)$, making a total work of $\Theta(\ell^2 \log^2 \ell)$, w.h.p. Furthermore, if we increase the number of agents by multiplying by any polylogarithmic factor, the same upper bound on the hitting time holds, w.h.p., while if we decrease it by dividing by any polylogarithmic factor, the upper bound on the hitting time holds with non-negligible probability.

At the same time, for all k in the family $\tilde{\Theta}(\ell)$, we have that it is needed at least time $\Omega(\ell)$ to find the treasure, making a total work of $\tilde{\Omega}(\ell^2)$, almost surely.

Claim (i) is a direct consequence of Lemma 18. Indeed, $\tilde{\Theta}(\ell^1)$ agents eventually find the treasure with probability $\tilde{\mathcal{O}}(\log^2(\ell)/\ell)$ for the union bound. Thus, w.h.p., the work is infinite.

Claim (ii) comes from the fact that the minimum time needed to find the treasure is ℓ almost surely. \square

6.2 The Ballistic Walk Model

In this section we analyze the search efficiency of the ballistic walks and prove the following result.

Theorem 5 (Hitting time - ballistic walks). *Assume that the treasure is located in some node of the infinite grid at distance $\ell > 0$. Let k agents move performing mutually independent ballistic walks. If $k = \tilde{\Theta}(\ell)$, then k agents find the treasure in time $\Theta(\ell)$, making a total work of $\tilde{\Theta}(\ell^2)$, w.h.p.¹⁴ Furthermore, the result is almost-tight in a two-fold sense:*

- (i) *If $k = \tilde{\Theta}(\ell^1)$ for any arbitrary constant $\epsilon \in (0, 1]$, then the agents never find the treasure, thus making an infinite work, w.h.p.;*

¹⁴See Remark 1 in the preliminaries (Section 4) for some formal details.

(ii) If $k = \tilde{\Theta}(\ell^{1+})$ for any arbitrary constant $\epsilon > 0$, then the agents need time $\Theta(\ell)$ to find the treasure, making a total work of $k = \tilde{\Theta}(\ell^{2+})$, w.h.p.

As discussed in Section 6, we see that the performance of the Pareto walks for $\alpha \in (1, 2]$ is the same as that of the ballistic walks. The reader may compare the above theorem with Theorem 4 in Section 6, and see that optimal work is reached for the same range of k both in the Pareto walks with $\alpha \in (1, 2]$ and in the ballistic walks. Moreover, any polynomial variation in the value of k make the work worsen by at least a polynomial factor in both models, w.h.p.

We first observe the following simple fact.

Lemma 19. *Consider an agent that performs the ballistic walk starting at the origin. Then, the probability it hits the treasure is $\Theta(1/\ell)$.*

Proof of Lemma 19. At time $t = 0$ the agent chooses any direction according to the procedure in Definition 3. For Lemma 1 in Section 4, the probability the agent hits any node at distance ℓ is $\Theta(1/\ell)$. \square

Now, the proof of the theorem follows.

Proof of Theorem 5. From Lemma 19, an agent that starts at the origin reaches the treasure in time $\Theta(\ell)$ with probability $\Theta(1/\ell)$. Thus, $\Theta(\ell \log \ell)$ agents reach the treasure in time $\Theta(\ell)$, letting the work to be $\Theta(\ell^2 \log \ell)$, w.h.p. Furthermore, if we increase the number of agents by any polylogarithmic factor, the same upper bound on the hitting time holds, w.h.p., while if we decrease it by dividing by any polylogarithmic factor, the upper bound on the hitting time holds with non-negligible probability.

At the same time, each agent needs time $\Theta(\ell)$ to reach distance $\Theta(\ell)$, almost surely. Thus, $\tilde{\Theta}(\ell)$ agents find the treasure in time $\Omega(\ell)$, making a total work of $\tilde{\Omega}(\ell^2)$, w.h.p.

As for the almost-tight results, if k is any function in $\tilde{\Theta}(\ell^1)$ for any $\epsilon \in (0, 1]$, then k agents eventually find the treasure with probability $\tilde{O}(1/\ell)$, letting the work to be infinite w.h.p. While, if k is any function in $\tilde{\Theta}(\ell^{1+})$ for any $\epsilon \in (0, 1]$, the agents will find the treasure in time $\Theta(\ell)$, letting the work to be $\tilde{\Theta}(\ell^{2+})$, w.h.p. \square

7 The Pareto Walk Model: Diffusive Regime

In this section, we analyze the search efficiency of the Pareto walk model in the diffusive regime, namely choosing the parameter $\alpha \in (3, +\infty)$. The analysis for the case $\alpha = 3$ has already been provided in Section 5. Afterwards, we analyze the simple random walk model, showing that the latter is indeed “equivalent” to the Pareto walk model with $\alpha \in [3, +\infty)$ in terms of hitting time and work.

For the Pareto walk model, we prove the following theorem.

Theorem 6 (Diffusive regime). *Assume that the treasure is located in some node of the infinite grid at distance $\ell > 0$. Let k agents move performing mutually independent Pareto walks with $\alpha > 3$. If $k = \log^{O(1)}(\ell)$, then k agents find the treasure in time $\tilde{\Theta}(\ell^2)$, making a total work of $\tilde{\Theta}(\ell^2)$, w.h.p.¹⁵ Furthermore, the result is almost-tight in the following senses:*

- (i) *If $k = \tilde{\Theta}(\ell)$ for any fixed constant $\epsilon \in [0, 3 - \alpha)$, then the agents need time $\tilde{\Omega}(\ell^2)$ to find the treasure and total work $\tilde{\Omega}(\ell^{2+})$, w.h.p.;*
- (ii) *If $k = \tilde{\Theta}\left(\ell^{-3 + \min(\frac{3}{2}, \frac{1}{2} + \epsilon)}\right)$ for any $\epsilon \geq 0$, then k agents need time at least $\Omega(\ell^2)$ to find the treasure, w.h.p., making a total work of $\tilde{\Omega}\left(\ell^{2 + (-3 + \min(\frac{1}{2}, \frac{1}{2}))}\right)$.*

¹⁵See Remark 1 in the preliminaries (Section 4) for some formal details.

Interestingly, since the expected jump-length is finite when $\alpha > 3$, by the central limit theorem (Theorem 8 in Appendix A), we get that the point-wise distribution of the position of a Pareto walk after long time is a normal distribution. The same (up to constant factors) holds for the simple random walk, suggesting the equivalence between the two models. We remark that the above theorem gives Claim (ii) (for $\alpha > 3$) of Theorem 1 in Section 1.1.

7.1 Main tools and general scheme

Some tools that are valid for this regime of α have already been discussed in Section 4.1. In order to prove Theorem 6, we need three more results which we are going to discuss in Sections 7.2 and 7.3. Let p be the node in which the treasure is located, with $d_p = \ell$. The first result is an upper bound on the hitting time of the treasure.

Proposition 5. *For some $t = \Theta(\ell^2 \log^2 \ell)$, the probability an agent performing the Pareto walk with $\alpha > 3$ visits the treasure within time t is $\Omega(1/(\log^4 \ell))$.*

Section 7.2 is devoted to the proof of this result. Furthermore we need the two following lower bounds, which will be proved in Section 7.3.

Proposition 6. *Let k be any integer such that $k = \mathcal{O}(\ell)$ for some constant $\epsilon \in [0, \alpha - 3]$. Then, k agents performing the Pareto walk with parameter $\alpha > 3$ need time at least $\Omega(\ell^2/(\log \ell))$ to find the treasure, w.h.p.*

Proposition 7. *Consider a single agent performing a Pareto walk for $\alpha > 3$. Let $\epsilon \geq 0$ be any arbitrary small constant, and let $t = \Theta(\ell^{2+\epsilon})$. Then, the probability to find the treasure within time t is $\mathcal{O}(1/\ell^{3+\min(2, 1+\epsilon)})$.*

7.1.1 Wrap-up: proof of Theorem 6

We are now ready to prove our main result.

Proof of Theorem 6. First, notice that if t is some $\Theta(\ell^2 \log^2 \ell)$, then, by Proposition 5, the probability an agent performing a Pareto walk with $\alpha > 3$ visits the treasure at least once within time t is $\Omega(1/(\log^4 \ell))$. Then, it is clear that $\Theta(\log^5 \ell)$ agents performing Pareto walks find the treasure within time t , w.h.p., making a total work $\mathcal{O}(\ell^2 \log^7 \ell)$. Furthermore, if we increase the number of agents by multiplying by any polylogarithmic factor, the same upper bound on the hitting time holds, w.h.p., while if we decrease it by dividing by any polylogarithmic factor, the upper bound on the hitting time holds with non-negligible probability.

Instead, let $k = \tilde{\Theta}(\ell)$ for any $\epsilon \in [0, \alpha - 3]$. Proposition 6 tells us that the time k agents need to find the treasure is at least $\Omega(\ell^2/\log \ell)$, w.h.p., thus making a total work equal to $\tilde{\Omega}(\ell^{2+\epsilon})$ (this part covers also the case $k = \log^{\mathcal{O}(1)}(\ell)$), giving us both the lower bound of the main claim and Claim i.

On the other hand, let $k = \tilde{\Theta}(\ell^{3+\min(\frac{3}{2}, \frac{1}{2}+\epsilon)})$ for any $\epsilon \geq 0$. For Proposition 7, we have that the probability k agents find the treasure within time $t = \Theta(\ell^{2+\epsilon})$ is, by the union bound, $\tilde{\mathcal{O}}(1/\ell^{\min(\frac{1}{2}, \frac{1}{2}+\epsilon)})$. Thus, w.h.p., the agents find the treasure in time at least $\Omega(\ell^{2+\epsilon})$, and the total work is $\tilde{\Omega}(\ell^{2+(\alpha-3)+\min(\frac{1}{2}, \frac{1}{2}+\epsilon)})$, giving us Claim ii. \square

7.2 Analysis of the case $\alpha \in (3; +\infty)$: proof of Proposition 5

This section aims at proving the following result.

Proposition 5. *For some $t = \Theta(\ell^2 \log^2 \ell)$, the probability an agent performing the Pareto walk with $\alpha > 3$ visits the treasure within time t is $\Omega(1/(\log^4 \ell))$.*

The analysis is very similar to that made in Section 5.2. In order to show the result, we have to analyze the Pareto flight for $\alpha > 3$, and then link the results for the Pareto flight to the Pareto walk through a coupling.

We look at a single agent moving on the grid \mathbb{Z}^2 performing a Pareto flight with $\alpha > 3$ which starts at the origin $o = (0, 0)$. We introduce some definitions and notations we use throughout the analysis. For any node $u = (u_x, u_y)$ of the grid, define the random variable

$$Z_u(t) = \text{number of agent's visits at node } u \text{ within } t \text{ steps.}$$

In order to bound the probability that the node u has been visited at least once at time t , namely $\mathbb{P}(Z_u(t) > 0)$, we define

$$p_{u,i} = \mathbb{P}(\text{the agent is in node } u \text{ at step } i).$$

By the definitions above, we easily get that

$$\mathbb{E}[Z_u(t)] = \sum_{i=0}^t p_{u,i}.$$

7.2.1 Road-map of the analysis

The scheme of the proof follows the same structure and main ideas of that in Section 2.1. We omit such an informal description and go directly with the main steps of the proof. Let p be the node in which the treasure is located. We divide \mathbb{Z}^2 in three different regions. The first one contains all nodes having distance from the origin roughly smaller than ℓ , i.e.

$$\mathcal{A}_1 = Q(\ell) = \{(x, y) \in \mathbb{Z}^2 : \max(|x|, |y|) \leq \ell\}.$$

The second region consists, instead, of all nodes whose distance from the origin ranges from ℓ and at most a logarithmic factor further. Its formal definition depends on the current time step t the process is running on. In detail, we wait until the process performs $t = \Omega(\ell^2)$, and we define, for any fixed $\delta \geq 0$,

$$\mathcal{A}_2 = \{v \in \mathbb{Z}^2 : |v|_1 \leq 4\sqrt{2(1+\delta)t \log t}\} \setminus \mathcal{A}_1.$$

Finally, the third region, which consists of all other further nodes, is defined as follows: for any $t = \Omega(\ell^2)$ and any $\delta \geq 0$,

$$\mathcal{A}_3 = \{v \in \mathbb{Z}^2 : |v|_1 > 4\sqrt{2(1+\delta)t \log t}\}.$$

Our analysis proceeds along the following technical steps.

1. **Upper bound for the number of visits in \mathcal{A}_1 .** We first show that $\mathbb{E}[Z_o(t)] = b_t = \mathcal{O}(\log^2 t)$ (see Lemma 20 for details). Then we show $\mathbb{E}[Z_u(t)] \leq b_t$ for all u (see Lemma 21 for details), which implies that the average number of visits in \mathcal{A}_1 until time t is at most $b_t \cdot m_p$, where $m_p = |Q(d_p)| = |Q(\ell)|$.
2. **Upper bound for the number of visits in \mathcal{A}_2 .** From Lemma 4 in Section 4.1, $\mathbb{E}[Z_u(t)] \geq \mathbb{E}[Z_v(t)]$ for all but $m_u = |Q(d_u)|$ many v (see Corollary 3), where $Q(d_u) = \{(x, y) : \max(|x|, |y|) \leq d_u\}$. This means that the average number of visits in \mathcal{A}_2 until time t is at most $\mathbb{E}[Z_u(t)] \cdot (32(1+\delta)t \log^2 t)$.
3. **Upper bound for the number of visits in \mathcal{A}_3 .** Using Chernoff-Hoeffding bounds, for any fixed $\delta > 0$, we can show

$$\sum_{v \in \mathbb{Z}^2 : |v|_1 > 4\sqrt{2(1+\delta)t \log t}} \mathbb{E}[Z_v(t)] = \mathcal{O}\left(t^{1-\frac{3}{2}}\right).$$

(see Lemma 22 for details), thus bounding the average number of visits to \mathcal{A}_3 until time t .

4. **Lower bound on the number of visits in p .** If p is the node where the treasure is located, from Items 1 to 3 we get that

$$m_p \cdot b_t + \mathbb{E}[Z_p(t)] \cdot (32(1 + \delta)t \log^2 t) + \mathcal{O}\left(t^{1 - \frac{3}{2}}\right) \geq t,$$

for any $t = \Omega(\ell^2)$ (see Lemma 23 for details), which implies

$$\mathbb{E}[Z_p(t)] \geq \frac{t - m_p b_t - \mathcal{O}\left(t^{1 - \frac{3}{2}}\right)}{(32(1 + \delta)t \log^2 t) - m_p},$$

for any $t = \Omega(\ell^2)$.

5. **Hitting probability via expected number of visits.** Using the result in Item 1, we show that (see Lemma 21 for details)

$$\mathbb{E}[Z_u(t)] \geq \mathbb{P}(Z_u(t) > 0) = \frac{\mathbb{E}[Z_u(t)]}{\mathbb{E}[Z_u(t) \mid Z_u(t) > 0]} \geq \frac{\mathbb{E}[Z_u(t)]}{b_t}.$$

6. **Wrap-up.** From Items 4 and 5,

$$\mathbb{P}(Z_p(t) > 0) \geq \frac{t - m_p b_t - \mathcal{O}\left(t^{1 - \frac{3}{2}}\right)}{(32(1 + \delta)t \log^2 t - m_p) b_t},$$

for any $t = \Omega(\ell^2)$ (see Corollary 4 for details). We show that the same holds for the Pareto walk with the same α through a coupling result (Lemma 24). Substituting $t = \Theta(\ell^2 \log^2 \ell)$, we get Proposition 5.

7.2.2 Full analysis

We start estimating the average number of visits to the origin until time t (Item 1 in Section 7.2.1).

Lemma 20. *For any $t \geq 0$, $\mathbb{E}[Z_o(t)] = b_t = \mathcal{O}(\log^2 t)$.*

Proof of Lemma 20. First, we show the following. Let $\mathbf{P}_{t^\theta}^f$ be the two dimensional random variable representing the coordinates of the agent performing the Pareto flight at time t^θ . Consider the projection of the Pareto flight on the x -axis, namely the random variable X_{t^θ} such that $\mathbf{P}_{t^\theta}^f = (X_{t^\theta}, Y_{t^\theta})$. The random variable X_{t^θ} can be expressed as the sum of t^θ random variables S_j^X , $j = 1, \dots, t^\theta$, representing the projection of the jumps (with sign) of the agent on the x -axis at times $j = 1, \dots, t^\theta$. The partial distribution of the jumps along the x -axis is given by Lemma 32 in Appendix D, and states that, for any given $d \geq 0$, we have

$$\mathbb{P}(S_j^X = \pm d) = \Theta\left(\frac{1}{(1 + d)}\right).$$

Since $\mathbb{E}[Z_o(t)] = \sum_{k=1}^t p_{o,k}$, it suffices to accurately bound the probability $p_{o,k}$ for each $k = 1, \dots, t$. Let us partition the natural numbers in the following way

$$\mathbb{N} = \bigcup_{t^\theta=0}^t \left[\mathbb{N} \cap [2t^\theta \log t^\theta, 2(t^\theta + 1) \log(t^\theta + 1)) \right].$$

For each $k \in \mathbb{N}$, there exists t^θ such that $k \in [2t^\theta \log t^\theta, 2(t^\theta + 1) \log(t^\theta + 1))$. Then, within $2t^\theta \log t^\theta$ steps the walk has moved to distance $\Theta\left(\sqrt{t^\theta}\right)$ at least once, with probability $\Omega\left(\frac{1}{(t^\theta)^2}\right)$. Indeed,

the sequence $\{S_j^x\}_{j=1}^{t^\theta}$ consists of i.i.d. r.v.s with zero mean and constant variance (which comes from the fact that $\alpha > 3$). Thus, the central limit theorem (Theorem 8 in Appendix A) says to us that, for t^θ large enough, the variable

$$\frac{S_1^x + \dots + S_{t^\theta}^x}{\sigma\sqrt{t^\theta}}$$

converges in distribution to a standard normal random variable Z . Let $\epsilon > 0$ be a small enough constant, then there exists a \bar{t}^θ large enough, such that for all $t^\theta \geq \bar{t}^\theta$ it holds that

$$\mathbb{P}\left(\frac{S_1^x + \dots + S_{t^\theta}^x}{\sigma\sqrt{t^\theta}} \geq \sigma\sqrt{t}\right) \geq \mathbb{P}(Z \geq 1) - \epsilon,$$

which is a constant since $\mathbb{P}(Z \geq 1)$ is a constant. The symmetrical results in which the normalized sum is less than $-\sigma\sqrt{t}$ holds analogously. Thus, for all $t^\theta \geq \bar{t}^\theta$, we have that $\sum_{j=1}^{t^\theta} |S_j^x| \geq \sigma\sqrt{t^\theta}$ with constant probability $c > 0$. In $2t^\theta \log t^\theta$ jumps, we have $2 \log t^\theta$ sets of t^θ consequent i.i.d. such jumps. For independence, the probability that at least in one round before round $2t^\theta \log t^\theta$ the Pareto flight has displacement $\Theta(\sqrt{t})$ from the origin is at least

$$1 - (1 - c)^{2 \log t^\theta} = 1 - \mathcal{O}\left(\frac{1}{(t^\theta)^2}\right).$$

Once reached such a distance, there are at least $\lambda^2 = \Theta(t^\theta)$ different nodes that are at least as equally likely as o to be visited at any given future time (from Lemma 4 in Section 4.1). Thus, the probability to reach the origin at any future time is at most $\mathcal{O}(1/t^\theta)$, in particular the bounds holds for $p_{o,k}$. Observe that in an interval $[2t^\theta \log t^\theta, 2(t^\theta + 1) \log(t^\theta + 1))$ there are

$$2(t^\theta + 1) \log(t^\theta + 1) - 2t^\theta \log t^\theta = 2t^\theta \left[\log\left(1 + \frac{1}{t^\theta}\right) \right] + 2 \log(t^\theta + 1) = \mathcal{O}(\log t^\theta)$$

integers. Let H_{t^θ} be the event that in any time before $2t^\theta \log t^\theta$ the Pareto flight has displacement at least $\Theta(\sqrt{t})$. Observe that

$$p_{o,k} = \mathbb{P}\left(\mathbf{P}_t^f = o \mid H_{t^\theta}\right) \mathbb{P}(H_{t^\theta}) + \mathbb{P}\left(\mathbf{P}_t^f = o \mid H_{t^\theta}^c\right) \mathbb{P}(H_{t^\theta}^c),$$

by the law of total probability. Thus, if $I_{t^\theta} = [2t^\theta \log t^\theta, 2(t^\theta + 1) \log(t^\theta + 1))$, we have

$$\begin{aligned} \sum_{k=1}^t p_{o,k} &\leq \sum_{t^\theta=0}^t \sum_{k \in I_{t^\theta}} p_{o,k} \\ &\leq \sum_{t^\theta=0}^t \left[\mathbb{P}\left(\mathbf{P}_t^f = o \mid H_{t^\theta}\right) \mathbb{P}(H_{t^\theta}) + \mathbb{P}\left(\mathbf{P}_t^f = o \mid H_{t^\theta}^c\right) \mathbb{P}(H_{t^\theta}^c) \right] \mathcal{O}(\log t^\theta) \\ &\leq \sum_{t^\theta=0}^t \left[\mathcal{O}\left(\frac{1}{t^\theta}\right) + \mathcal{O}\left(\frac{1}{(t^\theta)^2}\right) \right] \mathcal{O}(\log t^\theta) = \mathcal{O}(\log^2 t). \end{aligned}$$

□

We have also the the following (Items 1 and 5 in Section 7.2.1).

Lemma 21. *For any node $u \in \mathbb{Z}^2$, it holds that*

(i) $\mathbb{E}[Z_u(t)] \leq bt$;

(ii) $1 \leq \mathbb{E}[Z_u(t) \mid Z_u(t) > 0] \leq bt$;

(iii) $\mathbb{E}[Z_u(t)]/b_t \leq \mathbb{P}(Z_u(t) > 0) \leq \mathbb{E}[Z_u(t)]$.

Proof of Lemma 21. Claim (i) directly comes from Claim (ii), since $\mathbb{E}[Z_u(t)] \leq \mathbb{E}[Z_u(t) | Z_u(t) > 0]$. Consider Claim (ii), and let τ be the random variable denoting the first time the simple random walk visits u . Observe that, conditional on $Z_u(t) > 0$, $\tau \leq t$ with probability 1. Then we have

$$\mathbb{E}[Z_u(t) | Z_u(t) > 0] = \mathbb{E}[Z_o(t - \tau) | \tau \leq t] \leq \mathbb{E}[Z_o(t) | \tau \leq t] = \mathbb{E}[Z_o(t) | \tau \leq t],$$

where last inequality holds for independence. At the same time, $\mathbb{E}[Z_u(t) | Z_u(t) > 0] \geq 1$ due to the conditional event. As for Claim (iii), let us explicitly express the term $\mathbb{E}[Z_u(t) | Z_u(t) > 0] \cdot \mathbb{P}(Z_u(t) > 0)$. This is equal to

$$\begin{aligned} & \sum_{i=1}^t i \mathbb{P}(Z_u(t) = i | Z_u(t) > 0) \cdot \mathbb{P}(Z_u(t) > 0) \\ &= \sum_{i=1}^t i \frac{\mathbb{P}(Z_u(t) = i, Z_u(t) > 0)}{\mathbb{P}(Z_u(t) > 0)} \mathbb{P}(Z_u(t) > 0) \\ &= \sum_{i=1}^t i \mathbb{P}(Z_u(t) = i, Z_u(t) > 0) \\ &= \sum_{i=1}^t i \mathbb{P}(Z_u(t) = i) \\ &= \mathbb{E}[Z_u(t)]. \end{aligned}$$

Then,

$$\mathbb{E}[Z_u(t)] \geq \mathbb{P}(Z_u(t) > 0) = \frac{\mathbb{E}[Z_u(t)]}{\mathbb{E}[Z_u(t) | Z_u(t) > 0]} \geq \frac{\mathbb{E}[Z_u(t) | E]}{b_t},$$

since $\mathbb{E}[Z_u(t) | Z_u(t) > 0] \leq b_t$ for the first claim. \square

We thus have that the total number of visits in the \mathcal{A}_1 is upper bounded by $m_p b_t$, where $m_p = |Q(\ell)|$. The last lemma we also be used later.

Furthermore, from Lemma 4 in Section 4.1, the following holds (Item 2 in Section 7.2.1).

Corollary 3. *For any u in \mathbb{Z}^2 , we have $\mathbb{E}[Z_u(t)] \geq \mathbb{E}[Z_v(t)]$ for all $v \notin Q(d_u)$ (see Fig. 3 for geometrical details).*

Namely, almost all the nodes that are “further” than u from the origin are less likely to be visited at any given future time. This easily gives an upper bound on the total number of visits in \mathcal{A}_2 until time t , namely, by taking $u = \mathfrak{p}$ and by observing that each $v \in \mathcal{A}_2$ lies outside $Q(\ell)$, we get that the average number of visits in \mathcal{A}_2 is at most the expected number of visits on the treasure \mathfrak{p} (i.e. $\mathbb{E}[Z_{\mathfrak{p}}(t)]$) times (any upper bound of) the size of \mathcal{A}_2 : in formula, it is upper bounded by $\mathbb{E}[Z_{\mathfrak{p}}(t)] \cdot 32(1 + \delta)t \log^2 t$.

We also give a bound to the average number of visits to nodes that are further roughly $\sqrt{t} \cdot \log t$ from the origin (Item 3 in Section 7.2.1).

Lemma 22. *There exists a constant $\delta > 0$ such that*

$$\sum_{v \in \mathbb{Z}^2 : 4\sqrt{2(1+\delta)}t \log t} \mathbb{E}[Z_v(t)] = \mathcal{O}\left(t^{1-\frac{3}{2}}\right).$$

Proof of Item 3. Consider a single agent moving according the Pareto walk with parameter $\alpha \in (3, +\infty)$. Then, by Equations (2) and (3) in the preliminaries (Section 4), the expectation and the variance of a single jump-length is finite and the variance is finite. By Equation (4) in

the preliminaries (Section 4), the probability a jump length is at least \sqrt{t} is $\Theta\left(1/t^{-2}\right)$. Let us call A_j the event that the j -th jump-length is less than \sqrt{t} . Let us also define \mathbf{P}_j^f the random variable denoting the coordinates of the nodes the corresponding Pareto flight visits at the j -th jump. We can see this random variable as a couple (X_j, Y_j) , where X_j is x -coordinate of the ballistic Pareto walk after the j -th jump, and Y_j is the y -coordinate. Then, X_j can be seen as the sum $\sum_{i=1}^j S_i^\theta$ of j random variables representing the projections of the jumps along the x -axis. For symmetry, $\mathbb{E}[X_j] = 0$ for each j , while $\text{Var}(X_j) = j\text{Var}(S_1^\theta) = \Theta(j)$ since S_1^θ has finite variance. This comes by observing that $S_1^\theta \leq S_1$. Then, conditional on $A = \cap_{j=1}^t A_j$, we can apply the Chernoff bound (Lemma 31 in Appendix A) on the sum of the first j jumps, for $j \leq t$. We have

$$\begin{aligned} \mathbb{P}\left(|X_t| \geq 2\sqrt{2(1+\delta)t} \log t \mid A\right) &\leq 2 \exp\left(-\frac{8(1+\delta)t \cdot \log^2 t}{\Theta(t) + \Theta\left(\sqrt{(1+\delta)t} \cdot \log t\right) \sqrt{t}}\right) \\ &= 2 \exp\left(-\Theta\left(\sqrt{1+\delta} \cdot \log t\right)\right) \\ &\leq \frac{2}{t^{\Theta\left(\frac{1}{1+\delta}\right)}}, \end{aligned}$$

which is less than $1/t^{-2}$ if we choose δ big enough. The same result holds for the random variable X_j for each $j < t$, since the variance of X_j is smaller than the variance of X_t . Notice that

$$\begin{aligned} \mathbb{P}\left(\cap_{j=1}^t \{|X_j| < 2\sqrt{2(1+\delta)t} \log t\} \mid A\right) &= 1 - \mathbb{P}\left(\cup_{j=1}^t \{|X_t| \geq 2\sqrt{2(1+\delta)t} \log t\} \mid A\right) \\ &\geq 1 - \frac{t}{t^{\frac{1}{2}}} = 1 - \frac{1}{t^{\frac{3}{2}}}, \end{aligned}$$

and that

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \mathbb{P}\left(\cup_{j=1}^t A_j^c\right) \geq 1 - \mathcal{O}\left(\frac{t}{t^{\frac{1}{2}}}\right) = 1 - \mathcal{O}\left(\frac{1}{t^{\frac{3}{2}}}\right).$$

An analogous argument holds for the random variable Y_t conditioned to the event A . Then,

$$\begin{aligned} &\mathbb{P}\left(\cap_{j=1}^t \{|X_j|_1 < 2\sqrt{2(1+\delta)t} \cdot \log t\}, \cap_{j=1}^t \{|Y_j|_1 < 2\sqrt{2(1+\delta)t} \cdot \log t\}\right) \\ &\geq \mathbb{P}\left(\cap_{j=1}^t \{|X_j|_1 < 2\sqrt{2(1+\delta)t} \cdot \log t\}, \cap_{j=1}^t \{|Y_j|_1 < 2\sqrt{2(1+\delta)t} \cdot \log t\} \mid A\right) \mathbb{P}(A) \\ &\geq \left(2\mathbb{P}\left(\cap_{j=1}^t \{|X_j|_1 < 2\sqrt{2(1+\delta)t} \cdot \log t\} \mid A\right) - 1\right) \mathbb{P}(A) \\ &\stackrel{(a)}{\geq} \left[2\left(1 - \frac{1}{t^{\frac{3}{2}}}\right) - 1\right] \left(1 - \mathcal{O}\left(\frac{1}{t^{\frac{3}{2}}}\right)\right) \\ &\geq \left(1 - \frac{1}{t^{\frac{3}{2}}}\right) \left(1 - \mathcal{O}\left(\frac{1}{t^{\frac{3}{2}}}\right)\right) \\ &\geq 1 - \mathcal{O}\left(\frac{1}{t^{\frac{3}{2}}}\right), \end{aligned}$$

where (a) holds for symmetry (the distribution of Y_t is the same as the one of X_t) and for the union bound. Thus, in t jumps (which take at least time t), the walk has never reached distance $4\sqrt{2(1+\delta)t} \cdot \log t$, w.h.p. We denote this event as E . The average number of visits until time t to nodes at distance at least $4\sqrt{2(1+\delta)t} \cdot \log t$ is then less than $t \cdot \mathcal{O}\left(1/t^{-2}\right) = \mathcal{O}\left(t^{-\frac{3}{2}}\right)$. \square

The following puts together the previous estimations in order to get a lower bound on the average number of visits the treasure ρ (Item 4 in Section 7.2.1).

Lemma 23. *Let \mathfrak{p} be the node in which the treasure is located. For any $t = \Omega(\ell^2)$, the following holds:*

$$m_{\mathfrak{p}}b_t + \mathbb{E}[Z_{\mathfrak{p}}(t)] \cdot 32(1 + \delta)(t \log^2 t) + \mathcal{O}\left(t^{1 - \frac{3}{2}}\right) \geq t.$$

Proof of Lemma 23. Suppose the agent has made t jumps, thus visiting t nodes. Then,

$$\mathbb{E}\left[\sum_{v \in \mathbb{Z}^2} Z_v(t)\right] = t.$$

We divide the plane in different zones, and we bound the number of visits over each zone in expectation. From Lemma 21, the number of visits inside $\mathcal{A}_1 = Q(\ell)$ until time t is at most $m_{\mathfrak{p}}b_t$, where $m_{\mathfrak{p}} = |Q(\ell)| = 4\ell^2$. From Lemma 22, the number of visits \mathcal{A}_3 is at most $\mathcal{O}\left(t^{1 - \frac{3}{2}}\right)$. Each of the remaining nodes, i.e. the nodes in \mathcal{A}_2 , which are at most $32(1 + \delta)(t \log^2 t)$ in total, is visited by the agent at most $\mathbb{E}[Z_u(t)]$ times, for Corollary 3. Then, we have that

$$m_{\mathfrak{p}}b_t + \mathbb{E}[Z_{\mathfrak{p}}(t)] \cdot 32(1 + \delta)(t \log^2 t) + \mathcal{O}\left(t^{1 - \frac{3}{2}}\right) \geq t.$$

□

Finally, we prove (Item 6 in Section 7.2.1) the following.

Corollary 4. *For any $t = \Omega(\ell^2)$, the probability to have visited \mathfrak{p} within time t is*

$$\mathbb{P}(Z_{\mathfrak{p}}(t) > 0) \geq \frac{t - m_{\mathfrak{p}}b_t - \mathcal{O}\left(t^{1 - \frac{3}{2}}\right)}{(t \log t - m_{\mathfrak{p}})b_t}.$$

Proof of Corollary 4. The proof follows from the combination of Lemma 23 and Lemma 21. □

7.2.3 Coupling

We next prove Item 6 in Section 7.2.1, i.e., a coupling between the Pareto flight and the Pareto walk: essentially, this coupling shows that the Pareto walk is at least as efficient as the Pareto flight. Indeed, an agent that performs t jumps of the Pareto walk (thus, it has moved t steps according to the Pareto flight) with $\alpha \in (3, +\infty)$, has in fact walked for a time $\Theta(t)$, w.h.p.

Lemma 24. *Consider an agent that performs t jumps according to the Pareto flight with $\alpha \in (3, +\infty)$. Then, the sum of all the jump lengths is $\Theta(t)$, w.h.p.*

Proof of Lemma 24. If S_i is the random variable yielding the i -th jump length, then it has finite expectation and variance. This means that the sum $\bar{S}_t = \sum_{i=1}^t S_i$ has expectation $\Theta(t)$ and variance $\Theta(t)$. Then, from Chebyshev's inequality,

$$\mathbb{P}(\bar{S}_t \geq \Theta(t) + t) \leq \frac{\text{Var}(\bar{S}_t)}{t^2} = \mathcal{O}\left(\frac{1}{t}\right).$$

Thus, w.h.p., the sum of all the jump lengths is $\Theta(t)$. □

7.2.4 Wrap-up: proof of Proposition 5

Let \mathfrak{p} be the node in which the treasure is located ($d_{\mathfrak{p}} = \ell$). Let $t = t(\ell)$ be some function in $\Theta(\ell^2 \log^2 \ell)$. Then, from Corollary 4, we easily get that the probability the Pareto flight visits the treasure within time t is $\Omega(1/\log^3 \ell)$. For Lemma 24, and considering also the jumps in which the agent keeps still (which are at most $\Theta(t)$), the same holds for an agent performing the Pareto walk.

7.3 Analysis of the case $\alpha \in (3; +\infty)$: proof of Propositions 6 and 7

In this subsection, we aim at proving the two results giving lower bounds on the hitting time of the treasure. First, we prove a result saying that $\Theta(\ell)$ agents need time at least $\tilde{\Omega}(\ell^2)$ to find the treasure, w.h.p., for $\epsilon \in [0, 3\alpha)$, letting the work to be $\tilde{\Omega}(\ell^{2+\epsilon})$.

Proposition 6. *Let k be any integer such that $k = \mathcal{O}(\ell)$ for some constant $\epsilon \in [0, \alpha - 3)$. Then, k agents performing the Pareto walk with parameter $\alpha > 3$ need time at least $\Omega(\ell^2/(\log \ell))$ to find the treasure, w.h.p.*

Proof of Proposition 6. Consider a single agent moving according the Pareto walk with parameter $\alpha \in (3, +\infty)$. Then, by Eqs. (2) and (3) in Section 4, the expectation and the variance of a single jump-length is finite and the variance is finite. By Eq. (4) in Section 4, the probability a jump length is less than $\ell/(\log \ell)$ is $1 - \Theta(\ell^{-1}/\ell^{-1})$. Let us call A_j the event that the j -th jump length is less than $\ell/(\log \ell)$. Let us also define \mathbf{P}_t^f the random variable denoting the coordinates of the nodes the corresponding Pareto flight visits at the t -th jump. We can see this random variable as a couple (X_t, Y_t) , where X_t is x -coordinate of the ballistic Pareto walk after the t -th jump, and Y_t is the y -coordinate. Then, X_t can be seen as the sum $\sum_{j=1}^t S_j^\theta$ of t random variables representing the projections of the jumps along the x -axis. For symmetry, $\mathbb{E}[X_t] = 0$ for each t , while $\text{Var}(X_t) = t\text{Var}(S_1^\theta) = \Theta(t)$ since S_1^θ has finite variance. This comes by observing that $S_1^\theta \leq S_1$. Then, conditional on $A = \bigcap_{j=1}^{\lfloor \delta \ell^2 / (\log \ell) \rfloor} A_j$, we can apply the Chernoff bound (Lemma 31 in Appendix A) on the sum of the first $\delta \ell^2 / (\log \ell)$ jumps, for some δ accurately chosen. We have

$$\mathbb{P}(|X_{\lfloor \delta \ell^2 / (\log \ell) \rfloor}| \geq \ell \mid A) \leq 2 \exp\left(-\frac{\ell^2}{\Theta(\delta \ell^2 / (\log \ell)) + (\ell / (\log \ell)) \ell}\right) = 2 \exp\left(-\Theta\left(\frac{\log \ell}{\delta}\right)\right) \leq \frac{2}{\ell^{\Theta(1-\epsilon)}},$$

the latter bound can be made less than $1/\ell^{-1}$ by fixing a sufficiently small δ . The same result holds for the random variable X_t for each $t < \delta \ell^2 / (\log \ell)$, since the variance of X_t is smaller than the variance of $X_{\lfloor \delta \ell^2 / (\log \ell) \rfloor}$. Notice that

$$\mathbb{P}\left(\bigcap_{t=1}^{\lfloor \delta \ell^2 / (\log \ell) \rfloor} \{|X_t| < \ell\} \mid A\right) = 1 - \mathbb{P}\left(\bigcup_{t=1}^{\lfloor \delta \ell^2 / (\log \ell) \rfloor} \{|X_t| \geq \ell\}\right) \geq 1 - \frac{\delta \ell^2 / (\log \ell)}{\ell^{-1}} = 1 - \frac{\delta}{\ell^{-3} \log \ell},$$

and that

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^C) = 1 - \mathbb{P}\left(\bigcup_{j=1}^{\lfloor \delta \ell^2 / (\log \ell) \rfloor} A_j^C\right) \geq 1 - \frac{\delta \ell^2 (\log \ell)}{\ell^{-1} \log \ell} = 1 - \frac{\delta (\log \ell)}{\ell^{-3}}.$$

An analogous argument holds for the random variable Y_t conditional on the event A . Then,

$$\begin{aligned} \mathbb{P}\left(\bigcap_{t=1}^{\lfloor \delta \ell^2 / (\log \ell) \rfloor} \{X_t < \ell\}, \bigcap_{t=1}^{\lfloor \delta \ell^2 / (\log \ell) \rfloor} \{Y_t < \ell\}\right) &\geq \mathbb{P}\left(\bigcap_{t=1}^{\lfloor \delta \ell^2 / (\log \ell) \rfloor} \{X_t < \ell\}, \bigcap_{t=1}^{\lfloor \delta \ell^2 / (\log \ell) \rfloor} \{Y_t < \ell\} \mid A\right) \mathbb{P}(A) \\ &\geq \left(2\mathbb{P}\left(\bigcap_{t=1}^{\lfloor \delta \ell^2 / (\log \ell) \rfloor} \{X_t < \ell\} \mid A\right) - 1\right) \mathbb{P}(A) \\ &\stackrel{(a)}{\geq} \left[2\left(1 - \frac{\delta}{\ell^{-3} \log \ell}\right) - 1\right] \left(1 - \frac{\delta (\log \ell)}{\ell^{-3}}\right)^2 \\ &\geq \left(1 - \frac{\delta}{\ell^{-3} \log \ell}\right) \left(1 - \frac{\delta (\log \ell)}{\ell^{-3}}\right)^2 \\ &\geq 1 - \frac{\delta (1 + (\log \ell)^{-1})}{\ell^{-3} \log \ell}, \end{aligned}$$

where (a) holds for symmetry (the distribution of Y_t is the same as the one of X_t) and for the union bound. Thus, in $\delta \ell^2 / (\log \ell)$ jumps (which take at least time $\delta \ell^2 / (\log \ell)$), the walk has never reached distance ℓ , w.h.p. If the number of agents is $\mathcal{O}(\ell)$ for some $\epsilon \in [0, 3 - \alpha)$, there is probability $1 - \tilde{\mathcal{O}}(1/\ell^{-3})$ that the treasure is not found within time $\delta \ell^2 / (\log \ell)$. \square

The next result estimates the probability to find the treasure within time $t = \Theta(\ell^2)$, for any $\epsilon > 0$.

Proposition 7. *Consider a single agent performing a Pareto walk for $\alpha > 3$. Let $\epsilon \geq 0$ be any arbitrary small constant, and let $t = \Theta(\ell^2)$. Then, the probability to find the treasure within time t is $\mathcal{O}(1/\ell^{-3+\min(2, 1+\epsilon)})$.*

Proof of Proposition 7. First consider an agent performing the Pareto run with the same parameter α . Let X_i be the x -coordinate of the agent at the end of the i -th jump. For any $i \leq t$, we bound the probability that $X_i > \ell/4$. The probability that there is a jump whose length is at least ℓ among the first i jumps is $\mathcal{O}(i/\ell^{-1})$. Conditional on the event that the first i jump lengths are all smaller than ℓ (event C_i), the expectation of X_i is zero and its variance is

$$i \cdot \sum_{d=0}^{\ell-4} \Theta(d^2/(1+d)^{-\alpha}) = \Theta(i\ell^3)^{-\alpha},$$

for the integral test (Fact 1 in Appendix A). Chebyshev's inequality implies that

$$\mathbb{P}(|X_i| \geq \ell/4 \mid C_i) \leq \frac{\Theta(i\ell^3)^{-\alpha}}{\Theta(\ell^2)} = \Theta(i/\ell^{-1})^{-\alpha}.$$

Since the conditional event has probability $1 - \mathcal{O}(i/\ell^{-1})$, then the ‘‘unconditional’’ probability that of the event $|X_i| \leq \ell/4$ is

$$[1 - \mathcal{O}(i/\ell^{-1})]^{-\alpha} = 1 - \mathcal{O}(1/\ell^{-3+\epsilon}),$$

since $i \leq t = \Theta(\ell^2)$. The same result holds analogously for Y_i (the y -coordinate of the agent after the i -th jump), obtaining that $|X_i| + |Y_i| \leq \ell/2$ with probability $1 - \mathcal{O}(1/\ell^{-3+\epsilon})$ by the union bound.

Consider the first jump. The probability it leads the agent to visit the treasure is $\mathcal{O}(1/\ell)$ for Lemma 5 in Section 4.1. Now, let $2 \leq i \leq t$ and consider the i -th jump. We want to estimate the probability the jump leads the agent to visit the treasure. We call u the node in which the treasure is located, and we consider the rhombus centered in u that contains the nodes at distance at most $\frac{\ell}{4}$ from u , namely

$$R_{\frac{\ell}{4}}(u) = \{w \in \mathbb{Z}^2 : d(w, u) \leq \ell/4\}.$$

We call J_i the event that the i -th jump leads the agent to visit the treasure, R_{i-1} the event that the $(i-1)$ -th jump ends in $R_{\frac{\ell}{4}}(u)$, and F_{i-1} the event that the $(i-1)$ -th jump ends at distance farther than $\ell/2$ from the origin. Then, by the law of total probabilities, we have

$$\begin{aligned} \mathbb{P}(J_i) &= \mathbb{P}(J_i \mid F_{i-1}) \mathbb{P}(F_{i-1}) + \mathbb{P}(J_i \mid F_{i-1}^C) \mathbb{P}(F_{i-1}^C) \\ &= [\mathbb{P}(J_i \mid F_{i-1}, R_{i-1}) \mathbb{P}(R_{i-1} \mid F_{i-1}) + \mathbb{P}(J_i \mid F_{i-1}, R_{i-1}^C) \mathbb{P}(R_{i-1}^C \mid F_{i-1})] \mathbb{P}(F_{i-1}) \\ &\quad + \mathbb{P}(J_i \mid F_{i-1}^C) \mathbb{P}(F_{i-1}^C) \\ &\stackrel{(a)}{\leq} [\mathbb{P}(J_i \mid R_{i-1}) \mathbb{P}(R_{i-1} \mid F_{i-1}) + \mathbb{P}(J_i \mid F_{i-1}, R_{i-1}^C)] \mathbb{P}(F_{i-1}) + \mathbb{P}(J_i \mid F_{i-1}^C) \mathbb{P}(F_{i-1}^C) \\ &\stackrel{(b)}{\leq} \left[\mathcal{O}\left(\frac{1}{\ell^2}\right) + \mathcal{O}\left(\frac{1}{\ell}\right) \right] \mathcal{O}\left(\frac{1}{\ell^{-3+\epsilon}}\right) + \mathcal{O}\left(\frac{1}{\ell}\right) \left[1 - \mathcal{O}\left(\frac{1}{\ell^{-3+\epsilon}}\right) \right] = \mathcal{O}\left(\frac{1}{\ell^{-1+\epsilon}} + \frac{1}{\ell}\right) \end{aligned} \tag{15}$$

where in (a) we used that $R_{i-1} \subset F_{i-1}$ and that $\mathbb{P}(R_{i-1}^C \mid F_{i-1}) \leq 1$, while in (b) we used that

$$\mathbb{P}(J_i \mid R_{i-1}) \mathbb{P}(R_{i-1} \mid F_{i-1}) = \mathcal{O}\left(\frac{1}{\ell^2}\right), \text{ (the proof is below)}$$

that $\mathbb{P}(J_i | F_{i-1}, R_{i-1}^C) = \mathcal{O}(1/\ell)$ because the jump starts in a node whose distance from the treasure is $\Omega(\ell)$, and that $\mathbb{P}(J_i | F_{i-1}^C) = \mathcal{O}(1/\ell)$ for the same reason. As for the term $\mathbb{P}(J_i | R_{i-1})\mathbb{P}(R_{i-1} | F_{i-1})$ we observe the following. Let \mathbf{P}_i^r be the two-dimensional random variable denoting the coordinates of nodes the agent is located on after the i -th jump. Then

$$\begin{aligned} \mathbb{P}(J_i | R_{i-1})\mathbb{P}(R_{i-1} | F_{i-1}) &= \sum_{v \in R_{\geq 4}(u)} \mathbb{P}(J_i | \mathbf{P}_i^r = v)\mathbb{P}(\mathbf{P}_i^r = v | F_{i-1}) \\ &\leq \mathcal{O}\left(\frac{1}{\ell^2}\right) \sum_{v \in R_{\geq 4}(u)} \mathbb{P}(J_i | \mathbf{P}_i^r = v), \end{aligned}$$

since Lemma 4 (Section 4.1) holds in a consequent way conditional on F_{i-1} , and since, for each $v \in R_{\geq 4}(u)$, there are at least $\Theta(\ell^2)$ nodes at distance at least $\ell/2$ from the origin which are more probable to be visited than v . Then, we proceed like in the proof of Lemma 12 (Section 5.3) showing that $\sum_{v \in R_{\geq 4}(u)} \mathbb{P}(J_i | \mathbf{P}_i^r = v) = \mathcal{O}(1)$ and we obtain $\mathbb{P}(J_i | R_{i-1})\mathbb{P}(R_{i-1} | F_{i-1}) = \mathcal{O}(1/\ell^2)$.

Thus, by the union bound and by the inequality (15), the probability that at least one between the t jumps leads the agent to find the treasure is

$$\begin{aligned} \frac{1}{\ell} + (t-1)\mathcal{O}\left(\frac{1}{\ell^{1+}} + \frac{1}{\ell}\right) &= \mathcal{O}(\ell^{-2})\mathcal{O}\left(\frac{1}{\ell^{1+}} + \frac{1}{\ell}\right) \\ &= \mathcal{O}\left(\frac{1}{\ell^{3+2}} + \frac{1}{\ell^{2+}}\right) \\ &= \mathcal{O}\left(\frac{1}{\ell^{3+\min(2, 1+)}}\right). \end{aligned}$$

We conclude observing that the Pareto run is at least as efficient as the Pareto walk, since the first takes just one time unit to perform a jump, while the latter takes a time equal to the jump length. \square

7.4 The Simple Random Walk Model

In this section we aim at proving the following theorem on the search efficiency of simple random walks. The reader may compare the following theorem with Theorem 3 in Section 5 and Theorem 6 in Section 7 to observe that the performance of the Pareto walks for $\alpha \in [3, +\infty)$ is essentially the same as that of the simple random walks.

Theorem 7 (Hitting time - simple random walks). *Assume that the treasure is located in some node of the infinite grid at distance $\ell > 0$. Let k agents move performing mutually independent simple random walks. If $k = \log^{\mathcal{O}(1)}(\ell)$, then the agents find the treasure within time $\tilde{\Theta}(\ell^2)$, making a total work of $\tilde{\Theta}(\ell^2)$, w.h.p.¹⁶ Furthermore, the result is tight in the following sense: for all $k = \tilde{\Theta}(\ell)$ for any fixed constant $\epsilon > 0$, then the agents need time at least $\tilde{\Omega}(\ell^2)$ to find the treasure, thus making a total work of $\tilde{\Omega}(\ell^{2+})$, w.h.p.*

In order to prove the above result, in the next two subsections we present the analysis of two technical results, which we state here. The first one is an ‘‘upper bound’’ on the hitting time of the treasure. Let \mathfrak{p} be the node in which the treasure is located, with $d_{\mathfrak{p}} = \ell$.

Proposition 8. *For some $t = \Theta(\ell^2 \log \ell)$, the probability one agent performing a simple simple random walk visits the treasure within time t is $\Omega(1/(\log^3 \ell))$.*

Section 7.4.2 is devoted to the proof of such proposition. On the other hand, Section 7.4.6 aims at proving the following, which is a ‘‘lower bound’’ on the hitting time of the treasure.

¹⁶See Remark 1 in the preliminaries (Section 4) for some formal details.

Proposition 9. *Let $k = \Theta(\ell^\epsilon)$ for any fixed constant $\epsilon \geq 0$. Then, k agents need at least time $\Omega(\ell^2/(\log^2 \ell))$ to find the treasure, w.h.p.*

With these two propositions, we are ready to prove the main result of this section.

7.4.1 Wrap-up: proof of Theorem 7

Proof of Theorem 7. Proposition 8 implies that $k = \Theta(\log^4 \ell)$ agents find the treasure in time $\mathcal{O}(\ell^2 \log \ell)$ with probability

$$1 - \left[1 - \mathcal{O}\left(\frac{1}{\log^3 \ell}\right) \right]^{\Theta(\log^4 \ell)} = 1 - \mathcal{O}\left(\frac{1}{\ell}\right).$$

Furthermore, if we increase the number of agents by multiplying by any polylogarithmic factor, the same upper bound on the hitting time holds, w.h.p., while if we decrease it by dividing by any polylogarithmic factor, the upper bound on the hitting time holds with non-negligible probability.

As for the almost-tightness result, let k be any function in $\tilde{\Theta}(\ell \log^{\mathcal{O}(1)} \ell)$ for any fixed $\epsilon \geq 0$. Then, for Proposition 9, k agents need at least time $\tilde{\Omega}(\ell^2)$ to find the treasure, w.h.p. Indeed, since the result holds for $\Theta(\ell^{\epsilon+1})$ agents, it holds for k too by observing $k = \tilde{\Theta}(\ell \log^{\mathcal{O}(1)} \ell) \leq \Theta(\ell^{\epsilon+1})$. \square

7.4.2 Analysis of the simple random walk model: proof of Proposition 8

Let p be the node in which the treasure is located, with $d_p = \ell$. This subsection aims at proving the following result.

Proposition 8. *For some $t = \Theta(\ell^2 \log \ell)$, the probability one agent performing a simple simple random walk visits the treasure within time t is $\Omega(1/(\log^3 \ell))$.*

We look at a single agent moving on the grid \mathbb{Z}^2 performing a simple (symmetric) random walk which starts at the origin $o = (0, 0)$. We are going to introduce some definitions and notations we use throughout the analysis. For any node $u = (u_x, u_y)$ of the grid, define the random variable

$$Z_u(t) = \text{number of agent's visits at node } u \text{ within } t \text{ steps.}$$

In order to bound the probability that the node u has been visited at least once at time t , namely $\mathbb{P}(Z_u(t) > 0)$, we define

$$p_{u,i} = \mathbb{P}(\text{the agent is in node } u \text{ at step } i).$$

By the definitions above, we easily get that

$$\mathbb{E}[Z_u(t)] = \sum_{i=0}^t p_{u,i}.$$

Notice that at a generic round t , the simple random walk can visit only nodes whose distance from the origin has the same parity of t . Thus, given $u \in \mathbb{Z}^2$, we only compare the probability to be on u in a given round t with the probability to be in any v in the set

$$P_u = \{v \in \mathbb{Z}^2 : |d_v - d_u| \text{ is even}\}.$$

As for the main result of the theorem, we present a road-map of the analysis to keep track of main idea behind the lemmas and the proofs that follow, while the almost-tightness results will be later discussed and shown.

7.4.3 Road-map of the analysis

The scheme of the proof follows, again, the same structure and main ideas of that in Section 2.1. We omit such an informal description and go directly with the main steps of the proof. Let \mathfrak{p} be the node in which the treasure is located and $P_{\mathfrak{p}}$ the set of nodes that have even distance from \mathfrak{p} . We divide $P_{\mathfrak{p}}$ in three different regions. The first one contains all nodes having distance from the origin roughly smaller than ℓ , i.e.

$$\mathcal{A}_1 = Q(\ell) \cap P_{\mathfrak{p}} = \{(x, y) \in P_{\mathfrak{p}} : \max(|x|, |y|) \leq \ell\}.$$

The second region consists, instead, of all nodes whose distance from the origin ranges from ℓ and at most a logarithmic factor further. Its formal definition depends on the current time step t the process is running on. In detail, we wait until the process performs $t = \Omega(\ell^2)$, and we define, for any fixed $\delta \geq 0$,

$$\mathcal{A}_2 = \{v \in P_{\mathfrak{p}} : |v|_1 \leq 4\sqrt{2(1+\delta)t \log t}\} \setminus \mathcal{A}_1.$$

Finally, the third region, which consists of all other further nodes, is defined as follows: for any $t = \Omega(\ell^2)$ and any $\delta \geq 0$,

$$\mathcal{A}_3 = \{v \in P_{\mathfrak{p}} : |v|_1 > 4\sqrt{2(1+\delta)t \log t}\}.$$

Our analysis proceeds along the following technical steps.

1. **Decomposition of the simple random walk.** We show that the two dimensional simple random walk can be decomposed into two independent one dimensional simple random walks.
2. **Upper bound for the number of visits in \mathcal{A}_1 .** Using Item 1, we show that $\mathbb{E}[Z_o(t)] = c_t = \Theta(\log t)$ (see Lemma 26 for details). By conditional on the first arrival time at any point u , we get that $\mathbb{E}[Z_u(t)] \leq c_t$ for all u (see Lemma 27 for details). Then, the upper bound on the average number of visits in \mathcal{A}_1 until time t is $m_{\mathfrak{p}}c_t$, where $m_{\mathfrak{p}} = |Q(d_{\mathfrak{p}})| = |Q(\ell)|$.
3. **Monotonicity of $p_{u,t}$.** Using Item 1, for any $u = (x, y)$, if $|t - d_u|$ is even, we argue that $p_{u,t} \geq p_{v,t}$ for all $v \in \mathbb{Z}^2$ outside the square $Q(d_u) = \{(x, y) \in \mathbb{Z}^2 : \max(|x|, |y|) \leq d_u\}$ (see Lemma 25 and Fig. 7 for details).
4. **Upper bound for the number of visits in \mathcal{A}_2 .** From Item 3, $\mathbb{E}[Z_u(t)] \geq \mathbb{E}[Z_v(t)]$ for all $v \in P_u$ but at most $m_u = |Q(d_u)|$ (see Corollary 5 for details). Thus, we get that the average number of visits in \mathcal{A}_2 until time t is at most its cardinality times $\mathbb{E}[Z_{\mathfrak{p}}(t)]$, thus bounded by $\mathbb{E}[Z_{\mathfrak{p}}(t)] \cdot 32(1+\delta)t \log t$.
5. **Upper bound for the number of visits in \mathcal{A}_3 .** Using Chernoff-Hoeffding bounds and Item 1, for any fixed $\delta \geq 0$, we can show (see Lemma 28 for details) the following upper bound on the average number of visits in \mathcal{A}_3 until time t :

$$\sum_{v \in \mathbb{Z}^2: d_v > 4\sqrt{2(1+\delta)t \log t}} \mathbb{E}[Z_v(t)] < 1.$$

6. **Lower bound on the number of visits in \mathfrak{p} .** Combining Items 2, 4 and 5, it holds that, for any $t = \Omega(\ell^2)$,

$$m_{\mathfrak{p}} \cdot b_{\mathfrak{p}} + \mathbb{E}[Z_{\mathfrak{p}}(t)] \cdot (32(1+\delta)t \log t) + 1 \geq t,$$

(see Lemma 29 for details). Then, for any $t = \Omega(\ell^2)$, we get

$$\mathbb{E}[Z_p(t)] \geq \frac{t - m_p c_t - 1}{(32(1 + \delta)t \log t) - m_u}.$$

7. **Hitting probability via expected number of visits.** From Item 2 and simple calculations, we show (see Lemma 27 for details) that

$$\mathbb{E}[Z_u(t)] \geq \mathbb{P}(Z_u(t) > 0) = \frac{\mathbb{E}[Z_u(t)]}{\mathbb{E}[Z_u(t) | Z_u(t) > 0]} \geq \frac{\mathbb{E}[Z_u(t)]}{c_t}.$$

8. **Wrap-up.** Combining Items 6 and 7, we have (see Corollary 6 for details) that, for any $t = \Omega(\ell^2)$,

$$\mathbb{P}(Z_p(t) > 0) \geq \frac{\lfloor \frac{t}{2} \rfloor - 4\ell^2 \cdot c_t - 1}{(32(1 + \delta)t \log t) \cdot c_t}.$$

By letting t to be some function in $\Theta(\ell^2 \log \ell)$, we have Proposition 8.

7.4.4 Full analysis

As a preliminary, we show (Item 1 in Section 7.4.3) a natural decomposition of the two-dimensional simple random walk into two mutually independent one-dimensional simple random walks, which can be found in [Nor97]). Let $R^w(t)$ be the random variable denoting the coordinates of the node the two-dimensional simple random walk visits at time t . The two new “axes” are the two bisectors of the quadrants of the grid \mathbb{Z}^2 , namely r and s , as in Fig. 6. As

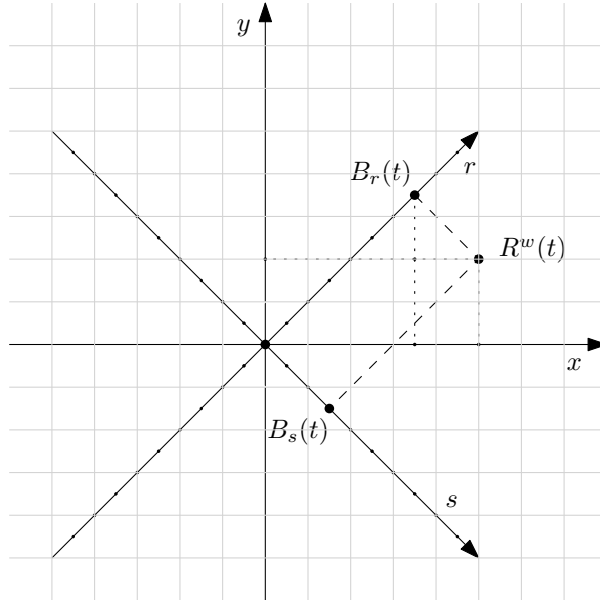


Figure 6: Decomposition of a two-dimensional simple random walk into two i.i.d. one-dimensional simple random walks over the bisectors of the quadrants.

a convention, we fix the part of r in the first quadrant and that of s in the fourth quadrant to be the positive side of r and s . Over these two strict lines, consider a sequence of nodes in all directions such that two subsequent nodes have Euclidean distance $\sqrt{2}/2$ between them, starting from the origin, in the positive and in the negative directions. Let $B_r(t)$ and $B_s(t)$ be two independent simple random walks on r and s , respectively, whose steps are over the nodes

we just inserted. More precisely, let $\{S_j^r\}_{j \in \mathbb{Z}^+}$ be a sequence of i.i.d. random variables with values in $\{-\sqrt{2}/2, +\sqrt{2}/2\}$ denoting the step increment of $B_r(t)$ at round j , and $\{S_j^s\}_{j \in \mathbb{Z}^+}$ the same for $B_s(t)$, with S_j^r and S_j^s mutually independent for each $i, j \geq 1$. Then $B_r(t) = \sum_{j=1}^t S_j^r$ and $B_s(t) = \sum_{j=1}^t S_j^s$. We have that

$$R^w(t) = \left(\cos \frac{\pi}{4} (B_r(t) + B_s(t)), \sin \frac{\pi}{4} (B_r(t) - B_s(t)) \right) = \left(\frac{\sqrt{2}}{2} \sum_{j=1}^t (S_j^r + S_j^s), \frac{\sqrt{2}}{2} \sum_{j=1}^t (S_j^r - S_j^s) \right).$$

Note that in a generic round $t - 1$, we have that the probability the increment is $(1, 0)$ is

$$\begin{aligned} \mathbb{P}(R^w(t) - R^w(t-1) = (1, 0)) &= \mathbb{P}(S_t^r + S_t^s = \sqrt{2}, S_t^r - S_t^s = 0) \\ &= \mathbb{P}(S_t^r = \sqrt{2} - S_t^s, S_t^r = S_t^s) \\ &= \mathbb{P}\left(S_t^r = \frac{\sqrt{2}}{2}, S_t^s = \frac{\sqrt{2}}{2}\right) = \frac{1}{4} \end{aligned}$$

for independence between S_t^r and S_t^s . The same holds for $(-1, 0)$, $(0, 1)$, and $(0, -1)$. Thus, the two-dimensional simple random walk can be seen as a combination of two independent and identically distributed one-dimensional simple random walks.

We are going to present a result (Item 3 in Section 7.4.3) which is very similar to Lemma 3 in Appendix C, and describes a ‘‘monotonicity’’ of the point-wise distribution at a generic time $t \geq 0$. Indeed, according to the previous notation, for any node $u = (u_x, u_y)$, we let $d_u = |u_x| + |u_y|$ and consider the square

$$Q(d_u) = \{(x, y) \in \mathbb{Z}^2 : \max(|x|, |y|) \leq d_u\}$$

(see Fig. 7 in the proof of Lemma 25). Then, the following geometric property holds.

Lemma 25. *Let $u \in \mathbb{Z}^2$ be an arbitrary node. Then, for each node $v \notin Q(d_u)$ and each step t such that $t - d_u$ is even, it holds that $p_{u:t} \geq p_{v:t}$.*

The difference here is that the simple random walk has probability equal to zero to stay still at one round, i.e. it always moves one step towards one of its neighbors, while the request we make for Lemma 3 in Appendix C is that the mobility model has a non-increasing step-length distribution; thus, we have to look for a different proof.

Proof of Lemma 25. For any given distance $d \geq 0$, consider the rhombus $R_d(o) = \{(x, y) \in \mathbb{Z}^2 : |x| + |y| \leq d\}$. For any point (x, y) in \mathbb{Z}^2 , it is defined a square $T(x, y) = \{(x^\theta, y^\theta) \in \mathbb{Z}^2 : 0 \leq \max(|x^\theta|, |y^\theta|) \leq \max(|x|, |y|)\}$. Then, for $u \in \mathbb{Z}^2$, define

$$D(u) = R_{d_u}(o) \cup T(u)$$

(see Fig. 7 for details). We will show that $p_{u:t} \geq p_{v:t}$ for each $v \notin D(u)$ through an induction argument on d_u , which will imply the thesis of the Lemma.

Let v be any other node on the grid, and let $d_v = |v|_1$. Then, if $|d_v - d_u|$ is odd, the thesis is trivial. For $|d_v - d_u|$ even, we show a more complicated argument: without loss of generality, suppose u is in the first quadrant and not below the main bisector, namely in the set $\{(x, y) \in \mathbb{Z}^2 : y \geq 0, x \geq y\}$ (Fig. 8).

According to the decomposition we have showed in Fig. 6, we have that

$$p_{u:t} = \mathbb{P}\left(\frac{\sqrt{2}}{2}(B_r(t) + B_s(t)) = u_x, \frac{\sqrt{2}}{2}(B_r(t) - B_s(t)) = u_y\right)$$

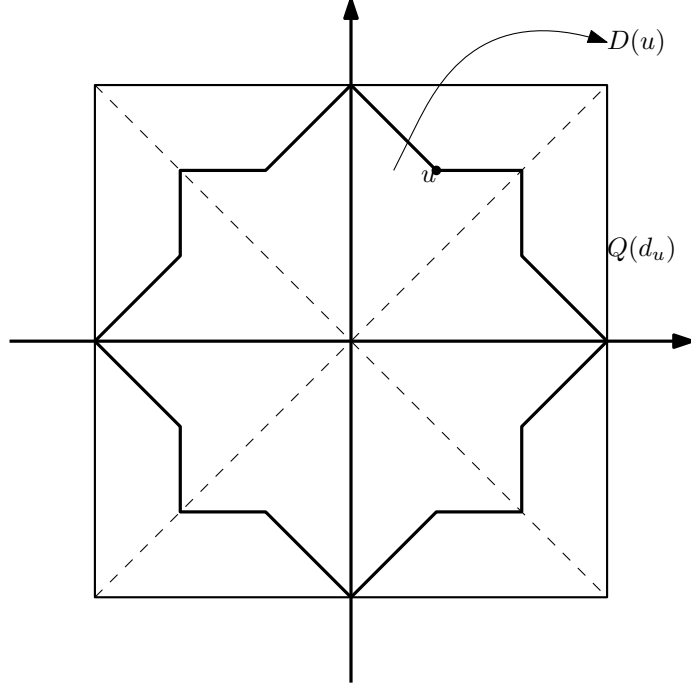


Figure 7: The set $Q(d_u)$ and the set $D(u)$.

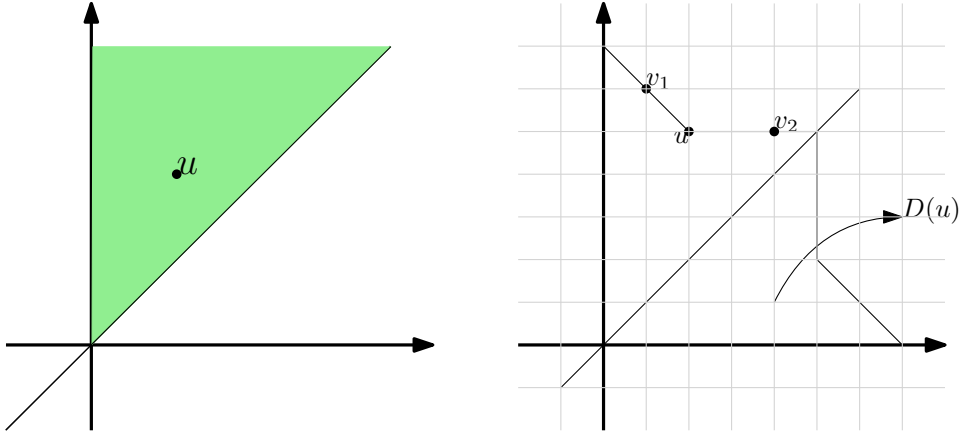


Figure 8: The “area” in which we take u , and the choices v_1, v_2 .

$$= \mathbb{P} \left(B_r(t) = \frac{u_x + u_y}{\sqrt{2}} \right) \cdot \mathbb{P} \left(B_s(t) = \frac{u_x - u_y}{\sqrt{2}} \right).$$

We first show the thesis taking v in the set $\{v_1 = (u_x - 1, u_y + 1), v_2 = (u_x + 2, u_y)\}$ (as long as they still lie in the highlighted zone in Fig. 8). If $t = d_u$, the thesis is trivial for v_1 and v_2 . Indeed, v_2 cannot be reached in t steps since $d_{v_2} > t$, while there are more possible paths to get to u in d_u steps than to v_1 in d_u steps. So, we assume $t \geq d_u + 2$. Keeping in mind that $u_y \geq u_x$ for the choice of u , we have that

$$\begin{aligned} p_{v_1; t} &= \mathbb{P} \left(B_r(t) = \frac{u_x + u_y}{\sqrt{2}} \right) \cdot \mathbb{P} \left(B_s(t) = \frac{u_x - u_y - 2}{\sqrt{2}} \right) \\ &\stackrel{(a)}{=} \binom{t}{\frac{t+u_x+u_y}{2}} \frac{1}{2^t} \cdot \binom{t}{\frac{t+u_y}{2} \quad u_x + 1} \frac{1}{2^t} \\ &= \binom{t}{\frac{t+u_x+u_y}{2}} \binom{t}{\frac{t+u_y}{2} \quad u_x + 1} \frac{1}{2^{2t}} \end{aligned}$$

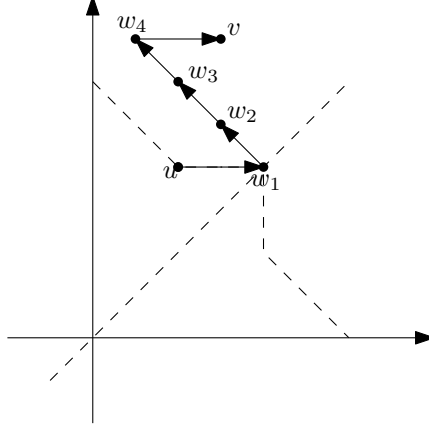


Figure 9: One “path” example.

$$\begin{aligned}
&\stackrel{(b)}{\leq} \binom{t}{\frac{t+u_x+u_y}{2}} \binom{t}{\frac{t+u_y}{2} \ u_x} \frac{1}{2^{2t}} \\
&= \mathbb{P} \left(B_r(t) = \frac{u_x + u_y}{\sqrt{2}} \right) \cdot \mathbb{P} \left(B_s(t) = \frac{u_x - u_y}{\sqrt{2}} \right) \\
&= p_{u;t},
\end{aligned}$$

where (a) is true because $|u_x - u_y - 2| = u_y - u_x + 2$, and (b) is true for Fact 2 in Appendix A. Then, if $v = v_1$ we have that $p_{u;t} \geq p_{v;t}$. As for v_2 , we have two cases. If $u_y \geq u_x + 2$, it holds that

$$\begin{aligned}
p_{v_2;t} &= \mathbb{P} \left(B_r(t) = \frac{u_x + u_y + 2}{\sqrt{2}} \right) \cdot \mathbb{P} \left(B_s(t) = \frac{u_x - u_y + 2}{\sqrt{2}} \right) \\
&\stackrel{(a)}{=} \binom{t}{\frac{t+u_x+u_y}{2} + 1} \frac{1}{2^t} \cdot \binom{t}{\frac{t+u_y}{2} \ u_x - 1} \frac{1}{2^t} \\
&= \binom{t}{\frac{t+u_x+u_y}{2} + 1} \binom{t}{\frac{t+u_y}{2} \ u_x - 1} \frac{1}{2^{2t}} \\
&\stackrel{(b)}{\leq} \binom{t}{\frac{t+u_x+u_y}{2}} \binom{t}{\frac{t+u_y}{2} \ u_x} \frac{1}{2^{2t}} \\
&= \mathbb{P} \left(B_r(t) = \frac{u_x + u_y}{\sqrt{2}} \right) \cdot \mathbb{P} \left(B_s(t) = \frac{u_x - u_y}{\sqrt{2}} \right) \\
&= p_{u;t},
\end{aligned}$$

where (a) is true because $|u_x - u_y + 2| = u_y - u_x - 2$, and (b) is true for Fact 2 in Appendix A. Else, if $u_y < u_x + 2$ (which basically means $u_y \leq u_x$ since we have to keep the parity), we have

$$\begin{aligned}
p_{v_2;t} &= \mathbb{P} \left(B_r(t) = \frac{u_x + u_y + 2}{\sqrt{2}} \right) \cdot \mathbb{P} \left(B_s(t) = \frac{u_x - u_y + 2}{\sqrt{2}} \right) \\
&\stackrel{(a)}{=} \binom{t}{\frac{t+u_x+u_y}{2} + 1} \frac{1}{2^t} \cdot \binom{t}{\frac{t+u_x}{2} \ u_y + 1} \frac{1}{2^t} \\
&= \binom{t}{\frac{t+u_x+u_y}{2} + 1} \binom{t}{\frac{t+u_x}{2} \ u_y + 1} \frac{1}{2^{2t}} \\
&\stackrel{(b)}{\leq} \binom{t}{\frac{t+u_x+u_y}{2}} \binom{t}{\frac{t+u_y}{2} \ u_x} \frac{1}{2^{2t}} \\
&= \mathbb{P} \left(B_r(t) = \frac{u_x + u_y}{\sqrt{2}} \right) \cdot \mathbb{P} \left(B_s(t) = \frac{u_x - u_y}{\sqrt{2}} \right)
\end{aligned}$$

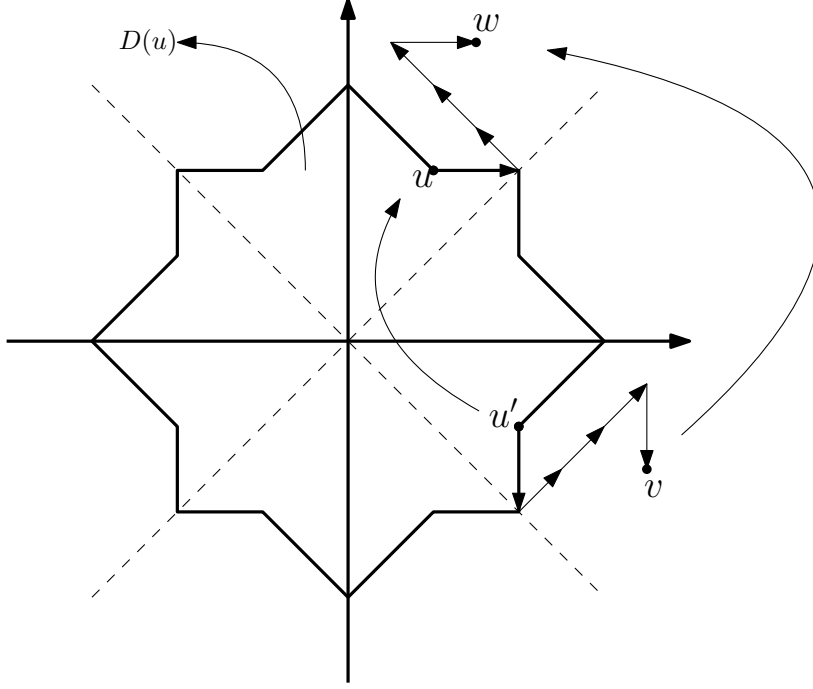


Figure 10: Symmetrical argument.

$$= p_{u;t},$$

where (a) is true because $|u_x - u_y + 2| = u_x - u_y + 2$, and (b) is true for Fact 2 in Appendix A. We thus have that $p_{u;t} \geq p_{v;t}$. For each other v in the highlighted area in Fig. 8, there exists a sequence of nodes $u = w_0, w_1, \dots, w_k = v$, all lying in the same area above, such that w_i belongs to the set $\{((w_{i-1})_x - 1, (w_{i-1})_y + 1), ((w_{i-1})_x, (w_{i-1})_y + 2)\}$ (see Fig. 9).

Then,

$$p_{u;t} = p_{w_0;t} \geq p_{w_1;t} \geq \dots \geq p_{w_k;t} = p_{v;t}.$$

For any other $v \notin D(u)$, we have a symmetrical argument shown in Fig. 10 that implies the thesis. \square

Recalling that $P_u = \{v \in \mathbb{Z}^2 : |d_v - d_u| \text{ is even}\}$, the following corollary holds (which is Item 4 in Section 7.4.3).

Corollary 5. *For all $v \in P_u$ but at most $4d_u^2$, i.e. those who lie in $Q(d_u)$, it holds that*

$$\mathbb{E}[Z_u(t)] \geq \mathbb{E}[Z_v(t)]$$

for any $t \geq 0$.

Proof of Corollary 5. The proof is a simple application of Lemma 25. \square

This easily gives an upper bound on the total number of visits in \mathcal{A}_2 until time t , namely, by taking $u = \mathfrak{p}$ and by observing that each $v \in \mathcal{A}_2$ lies outside $Q(\ell)$, we get that the average number of visits in \mathcal{A}_2 is at most the expected number of visits on the treasure \mathfrak{p} (i.e. $\mathbb{E}[Z_{\mathfrak{p}}(t)]$) times (any upper bound of) the size of \mathcal{A}_2 : in formula, it is upper bounded by $\mathbb{E}[Z_{\mathfrak{p}}(t)] \cdot 32(1+\delta)t \log t$.

Next lemma gives the average number of agent's returns to the origin (Item 2 in Section 7.4.3).

Lemma 26. *The average number of visits to the origin until time $t > 0$ is*

$$\mathbb{E}[Z_o(t)] = c_t = \Theta(\log t).$$

Proof of Lemma 26. Consider the decomposition we showed with Fig. 6. Then, the simple random walk is at the origin at time i if and only if both $B_r(i) = 0$ and $B_s(i) = 0$. In other words,

$$p_{o;i} = \mathbb{P}(B_r(i) = 0) \cdot \mathbb{P}(B_s(i) = 0)$$

for independence. First, notice that if i is odd, then $B_r(i) \neq 0$. We then consider only even timings, i.e. $i = 2k$. By Stirling's formula

$$\mathbb{P}(B_r(2k) = 0) = \binom{2k}{k} \frac{1}{2^{2k}} = \frac{(2k)!}{(k!)^2} \frac{1}{2^{2k}} \sim \frac{\sqrt{4\pi k} (2k/e)^{2k}}{2\pi k (k/e)^{2k}} \frac{1}{2^{2k}} = \frac{1}{\sqrt{\pi k}}.$$

Thus, there exists a large enough $\bar{k} > 0$, such that

$$\frac{1}{2} \leq \mathbb{P}(B_r(2k) = 0) \cdot \sqrt{\pi k} \leq 2$$

for any $k > \bar{k}$, and the same holds for $B_s(2k)$ for symmetry. Then we have that

$$\sum_{k=0}^{\lfloor \frac{t}{2} \rfloor} p_{o;2k} = \sum_{k=0}^{\lfloor \frac{t}{2} \rfloor} [\mathbb{P}(B_r(2k) = 0)]^2 \leq \bar{k} + \sum_{k=\bar{k}+1}^{\lfloor \frac{t}{2} \rfloor} \frac{4}{\pi k} = \mathcal{O}(\log t),$$

and, at the same time

$$\sum_{k=0}^{\lfloor \frac{t}{2} \rfloor} p_{o;2k} = \sum_{k=0}^{\lfloor \frac{t}{2} \rfloor} [\mathbb{P}(B_r(2k) = 0)]^2 \geq \sum_{k=\bar{k}+1}^{\lfloor \frac{t}{2} \rfloor} \frac{1}{4\pi k} = \Omega(\log t),$$

where the latter inequalities holds for the integral test, and because \bar{k} is a constant. We thus have that $\mathbb{E}[Z_o(t)] = \Theta(\log t)$. \square

We use this result to give a bound on the average number of visits the simple random walk makes to nodes that are in \mathcal{A}_1 . Indeed, we will exploit the first claim of the next result. Call $c_t = \mathbb{E}[Z_o(t)]$. Then, we have the following (Items 2 and 7 in Section 7.4.3).

Lemma 27. *Let $u \in \mathbb{Z}^2$ be any node. It holds that*

- (i) $\mathbb{E}[Z_u(t)] \leq c_t$
- (ii) $1 \leq \mathbb{E}[Z_u(t) \mid Z_u(t) > 0] \leq c_t$;
- (iii) $\mathbb{E}[Z_u(t)] / c_t \leq \mathbb{P}(Z_u(t) > 0) \leq \mathbb{E}[Z_u(t)]$.

Proof of Lemma 27. The proof is exactly the same as that of Lemma 21 in Section 7.2. \square

We thus have that the total number of visits in the \mathcal{A}_1 is upper bounded by $m_p c_t$, where $m_p = |Q(\ell)|$. The last lemma we also be used later.

Now we are going to give a bound on the average number of visits to nodes in \mathcal{A}_3 the simple random walk does (Item 5 in Section 7.4.3).

Lemma 28. *For any $t \geq 4$ and for any constant $\delta \geq 0$,*

$$\sum_{v \in \mathbb{Z}^2: d_v \geq 4\sqrt{2(1+\delta)t \log t}} \mathbb{E}[Z_v(t)] < 1.$$

Proof of Lemma 28. Once again, we exploit the decomposition shown in Fig. 6. The one-dimensional simple random walk $B_r(i)$ is such that $\mathbb{E}[B_r(i)] = 0$ and $\text{Var}(B_r(i)) = i/\sqrt{2}$. Furthermore, each step S_j^r , for $j \leq i$, has mean equal to zero and is less than $1/\sqrt{2}$. Let $i \leq t$ and $\delta \geq 0$ an arbitrary constant (it will be useful for a later lemma): we use a particular form of Chernoff bound (Lemma 31 in Appendix A) to get that

$$\mathbb{P}\left(B_r(i) \geq 2\sqrt{(1+\delta)t \log t}\right) \leq \exp\left(\frac{4(1+\delta)t \log t}{i + \frac{2^{\frac{t \log t}{3^{\frac{1}{2}}}}}{2}}\right) \leq \frac{1}{t^{3+}}.$$

By symmetry, we get also that $\mathbb{P}\left(B_r(i) \leq -2\sqrt{(1+\delta)t \log t}\right) \leq \frac{1}{t^{3+}}$, thus

$$\mathbb{P}\left(B_r(i) \geq 2\sqrt{(1+\delta)t \log t}\right) \leq \frac{2}{t^{3+}}.$$

The same result holds analogously for $B_s(i)$, for each $i \leq t$. Since $|R^w(i)|_1$ from the origin is at most $\sqrt{2} \cdot (B_r(i) + B_s(i))$ we have that with probability at least $1 - 4/t^{3+}$ this distance is bounded by $4\sqrt{2(1+\delta)t \log t}$. Then, the probability that in any of the first t rounds, the walk has ever gone further than distance $4\sqrt{2(1+\delta)t \log t}$ from the origin is $4/t^{2+}$. Denote by F_t the event $\cap_{i=0}^t \{|R^w(i)|_1 \leq 4\sqrt{2(1+\delta)t \log t}\}$. It follows that

$$\begin{aligned} & \sum_{v \in 2\mathbb{Z}^2: d_v} \frac{\mathbb{E}[Z_v(t)]}{4\sqrt{2(1+\delta)t \log t}} = \mathbb{E}\left[\sum_{v \in 2\mathbb{Z}^2: d_v} \frac{Z_v(t)}{4\sqrt{2(1+\delta)t \log t}}\right] \\ &= \mathbb{E}\left[\sum_{v \in 2\mathbb{Z}^2: d_v} \frac{Z_v(t)}{4\sqrt{2(1+\delta)t \log t}} \mid F_t\right] \mathbb{P}(F_t) + \mathbb{E}\left[\sum_{v \in 2\mathbb{Z}^2: d_v} \frac{Z_v(t)}{4\sqrt{2(1+\delta)t \log t}} \mid F_t^C\right] \mathbb{P}(F_t^C) \\ &\leq 0 + t \cdot \left(\frac{4}{t^{2+}}\right) = \left(\frac{4}{t^{1+}}\right), \end{aligned}$$

and we have the thesis for $t \geq 4$. \square

We are finally ready to give a lower bound on the expected number of visits to the treasure (Item 6 in Section 7.4.3).

Lemma 29. *Let p be the treasure node. Then, for any $t = \Omega(\ell^2)$ and any $\delta \geq 0$,*

$$4\ell^2 \cdot c_t + \mathbb{E}[Z_p(t)] \cdot (32(1+\delta)t \log t) + 1 \geq \left\lfloor \frac{t}{2} \right\rfloor.$$

Proof of Lemma 29. At any round t , the simple random walk can only visit nodes whose distances from the origin have the same parity of t . Then,

$$\sum_{v \in 2P_p} \mathbb{E}[Z_v(t)] \geq \left\lfloor \frac{t}{2} \right\rfloor.$$

Lemma 26 and Lemma 27 give us that

$$\sum_{v \in 2A_1} \mathbb{E}[Z_v(t)] \leq 4d_U^2 \cdot c_t.$$

Lemma 25 and Corollary 5 imply that

$$\sum_{v \in 2A_2} \mathbb{E}[Z_v(t)] \leq \mathbb{E}[Z_p(t)] \cdot (32(1+\delta)t \log t),$$

since $|\mathcal{A}_2| \leq 32(1 + \delta)t \log t$. Moreover, Lemma 28 implies that

$$\sum_{v \in \mathcal{A}_3} \mathbb{E}[Z_v(t)] < 1,$$

for any $t \geq 4$. Then, we get the thesis for any large enough t ($\Omega(\ell^2)$ is sufficient). \square

7.4.5 Wrap-up: proof of Proposition 8

We are ready to derive a lower bound on the probability to hit the treasure (Item 8 in Section 7.4.3).

Corollary 6. *Let p be the treasure node. Then, for $t = \Theta(\ell^2 \log \ell)$,*

$$\mathbb{P}(Z_p(t) > 0) \geq \frac{\lfloor \frac{t}{2} \rfloor - 4\ell^2 \cdot c_t - 1}{(32(1 + \delta)t \log t) \cdot c_t}.$$

Proof of Corollary 6. Lemma 26 implies that

$$\mathbb{P}(Z_p(t) > 0) \geq \frac{\mathbb{E}[Z_p(t)]}{c_t}.$$

Then, from Lemma 29,

$$\mathbb{P}(Z_p(t) > 0) \geq \frac{\lfloor \frac{t}{2} \rfloor - 4\ell^2 \cdot c_t - 1}{(32(1 + \delta)t \log t) \cdot c_t}.$$

\square

Finally, the proof of Proposition 8 is a consequence of Corollary 6 by observing that, thanks to Lemma 26, $c_t = \Theta(\log t)$. Then, setting t to be some function in $\Theta(\ell^2 \log \ell)$ and $\delta = 1$, we get the thesis.

7.4.6 Analysis of the simple random walk model: proof of Proposition 9

Here we prove the following result.

Proposition 9. *Let $k = \Theta(\ell)$ for any fixed constant $\epsilon \geq 0$. Then, k agents need at least time $\Omega(\ell^2 / (\log^2 \ell))$ to find the treasure, w.h.p.*

Proof of Proposition 9. Let $\delta = \epsilon$. Lemma 28 in Section 7.4.2, tells us that a single agent performing a simple random walk within time $t = \Theta(\ell^2 / \log^2 \ell)$ never gets at distance $4\sqrt{2(1 + \epsilon)t \log t} = \mathcal{O}(\ell^2 / \log \ell)$ with probability

$$1 - \mathcal{O}(1/t^{1+\epsilon}) = 1 - \mathcal{O}(1/\ell^{1+\epsilon}).$$

Then, $\Theta(\ell)$ agents have probability at most $\mathcal{O}(1/\ell)$ to go further than distance $\mathcal{O}(\ell^2 / \log \ell)$ within time t . \square

A Tools

Fact 1. *Let $\alpha > 0$ be a constant, $d > 0$ a positive integer, and let $d_{\max} > d$ be another integer. The followings hold*

$$\frac{1}{(\alpha - 1)(d - 1)} \leq \sum_k \frac{1}{d^k} \leq \frac{1}{(\alpha - 1)(d - 1)} + \frac{1}{d} \quad \text{and} \quad (16)$$

$$\frac{1}{(\alpha-1)} \left(\frac{1}{d^{\alpha-1}} - \frac{1}{d_{\max}^{\alpha-1}} \right) \leq \sum_{k=d}^{d_{\max}} \frac{1}{k} \leq \frac{1}{(\alpha-1)} \left(\frac{1}{d^{\alpha-1}} - \frac{1}{d_{\max}^{\alpha-1}} \right) + \frac{1}{d} \quad \text{for } \alpha > 1, \quad (17)$$

and, at the same time,

$$\log \left(\frac{d_{\max}}{d} \right) \leq \sum_{k=d}^{d_{\max}} \frac{1}{k} \leq \log \left(\frac{d_{\max}}{d} \right) + \frac{1}{d} \quad (\text{case } \alpha = 1), \quad \text{and} \quad (18)$$

$$\frac{(d_{\max})^1 - d^1}{1 - \alpha} \leq \sum_{k=d}^{d_{\max}} \frac{1}{k} \leq \frac{(d_{\max})^1 - d^1}{1 - \alpha} + \frac{1}{d} \quad \text{for } \alpha < 1. \quad (19)$$

Proof. By the integral test, it holds that

$$\int_d^{d_{\max}} \frac{1}{k} dk \leq \sum_{k=d}^{d_{\max}} \frac{1}{k} \leq \int_d^{d_{\max}} \frac{1}{k} dk + \frac{1}{d}.$$

Straightforward calculations give the result for Eqs. (17) to (19). As for Eq. (16), it comes from the integral test letting $d_{\max} \rightarrow \infty$. \square

Fact 2. Let $n \geq 0$ be any integer, and let $a, b \in [0, n-2]$ such that $n - a - b$ is even and non-negative, and $a \geq b$. Then, it holds that

$$\binom{n}{\frac{n+a+b}{2}} \geq \binom{n}{\frac{n+a+b}{2} + 1}, \quad \text{and} \quad (20)$$

$$\binom{n}{\frac{n+a+b}{2}} \binom{n}{\frac{n+a-b}{2}} \geq \binom{n}{\frac{n+a+b}{2} + 1} \binom{n}{\frac{n+a-b}{2} - 1}. \quad (21)$$

Proof of Fact 2. We first prove Eq. (20). Expressing the binomial coefficients, we have

$$\begin{aligned} \binom{n}{\frac{n+a+b}{2}} &\geq \binom{n}{\frac{n+a+b}{2} + 1} \quad \text{iff} \\ \frac{n!}{\left(\frac{n+a+b}{2}\right)! \left(\frac{n-a-b}{2}\right)!} &\geq \frac{n!}{\left(\frac{n+a+b}{2} + 1\right)! \left(\frac{n-a-b}{2} - 1\right)!} \quad \text{iff} \\ \frac{1}{\left(\frac{n-a-b}{2}\right)} &\geq \frac{1}{\left(\frac{n+a+b}{2} + 1\right)} \quad \text{iff} \\ \frac{n+a+b}{2} + 1 &\geq \frac{n-a-b}{2}, \end{aligned}$$

which is true. As for Eq. (21), we have

$$\begin{aligned} \binom{n}{\frac{n+a+b}{2}} \binom{n}{\frac{n+a-b}{2}} &\geq \binom{n}{\frac{n+a+b}{2} + 1} \binom{n}{\frac{n+a-b}{2} - 1} \quad \text{iff} \\ \frac{n!}{\left(\frac{n+a+b}{2}\right)! \left(\frac{n-a-b}{2}\right)!} \cdot \frac{n!}{\left(\frac{n+a-b}{2}\right)! \left(\frac{n+a+b}{2}\right)!} &\geq \frac{n!}{\left(\frac{n+a+b}{2} + 1\right)! \left(\frac{n-a-b}{2} - 1\right)!} \cdot \frac{n!}{\left(\frac{n+a-b}{2} - 1\right)! \left(\frac{n+a+b}{2} + 1\right)!} \quad \text{iff} \\ \frac{1}{\left(\frac{n-a-b}{2}\right)} \cdot \frac{1}{\left(\frac{n+a-b}{2}\right)} &\geq \frac{1}{\left(\frac{n+a+b}{2} + 1\right)} \cdot \frac{1}{\left(\frac{n-a-b}{2} - 1\right)} \quad \text{iff} \\ \left(\frac{n+a+b}{2} + 1\right) \left(\frac{n-a-b}{2} - 1\right) &\geq \left(\frac{n-a-b}{2}\right) \left(\frac{n+a-b}{2}\right), \end{aligned}$$

which is true (compare the second factor on the left with the first factor on the right, and the first factor on the left with the second factor on the right). \square

As for the probabilistic tools, first we state the well-known central limit theorem, which can be found in [Fel68] (Chapter X).

Theorem 8 (Central limit theorem). *Let $\{X_k\}_{k=1}^n$ be a sequence of i.i.d. random variables. Let $\mu = \mathbb{E}[X_1]$, $\sigma^2 = \text{Var}(X_1)$, and $S_n = \sum_{k=1}^n X_k$ for any $n \geq 1$. Let $\Phi : \mathbb{R} \rightarrow [0, 1]$ be the cumulative distribution function of a standard normal distribution. Then, for any $\beta \in \mathbb{R}$, it holds that*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{S_n - n\mu}{\sigma\sqrt{n}} < \beta \right) = \Phi(\beta).$$

Furthermore, we give some forms of the Chernoff bounds. The first form can be found in the appendix of [DP09], and can be stated as follow.

Lemma 30 (Multiplicative forms of Chernoff bounds). *Let X_1, X_2, \dots, X_n be independent random variables taking values in $[0, 1]$. Let $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[X]$. Then:*

(i) *For any $\delta > 0$ and $\mu \leq \mu_+ \leq n$, it holds that*

$$P(X \geq (1 + \delta)\mu_+) \leq e^{-\frac{\delta^2}{3}\mu_+}, \quad (22)$$

(ii) *For any $\delta \in (0, 1)$ and $0 \leq \mu_- \leq \mu$, it holds that*

$$P(X \leq (1 - \delta)\mu_-) \leq e^{-\frac{\delta^2}{2}\mu_-}. \quad (23)$$

We also use the following form of Chernoff bound, which can be found in [CL06] (Theorem 3.4).

Lemma 31 (Additive form of Chernoff bound using variance). *Let X_1, \dots, X_n be independent random variables satisfying $X_i \leq \mathbb{E}[X_i] + M$ for some $M \geq 0$, for all $i = 1, \dots, n$. Let $X = \sum_{i=1}^n X_i$, $\mu = \mathbb{E}[X]$, and $\sigma^2 = \text{Var}(X)$. Then, for any $\lambda > 0$, it holds that*

$$\mathbb{P}(X \geq \mu + \lambda) \leq \exp \left(-\frac{\lambda^2}{\sigma^2 + \frac{M}{3}} \right). \quad (24)$$

B Proofs: Preliminaries

In this part of the appendix, we first show that the direction choice procedure leads to an unique approximating path with probability 1. We then prove Lemma 1. We recall what an approximating path is and what we mean by an agent that chooses a direction.

Definition 2 (r -approximating path). *Let r be the unique ray identified by some unit vector \vec{v} applied to some node u . Consider, for $d \geq 0$, the rhombus centered at u*

$$R_d(u) = \{v \in \mathbb{Z}^2 : |u - v|_1 = d\}.$$

For each $d \geq 0$, consider the “natural immersion” of the rhombus in the continuous plane, namely

$$\tilde{R}_d(u) = \{(x, y) \in \mathbb{R}^2 : (y + x + 1)(y + x - 1)(y - x + 1)(y - x - 1) = 0, |x| \leq d, |y| \leq d\},$$

as in Figure 1. Let v_d the intersection between r and $\tilde{R}_d(u)$. An r -approximating path is a simple path which starts at u and whose d -th node, for any d , is the node $w_d \in R_d(u)$ that minimize the distance $\min_{w \in R_d(u)} |w - v_d|_2$. Ties are broken uniformly at random.

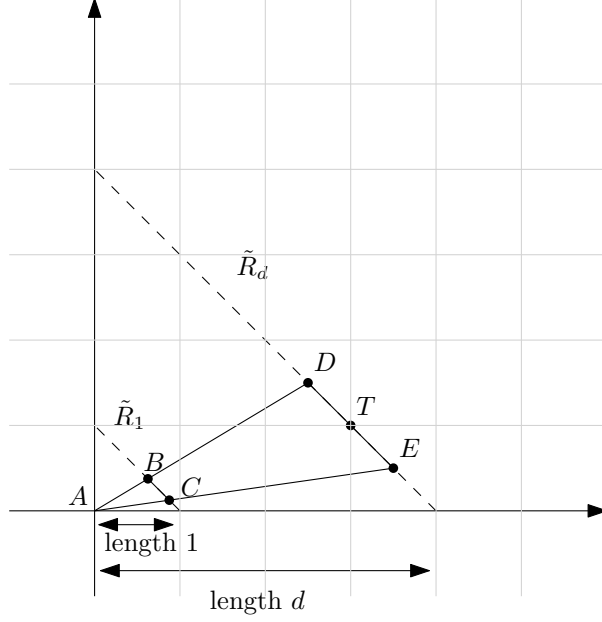


Figure 11: Mapping from $\tilde{R}_d(u)$ to $\tilde{R}_1(u)$

Definition 3 (Direction choice procedure). *An agent at some node u chooses a direction \vec{v} in the following way: it samples uniformly at random one node w of $\tilde{R}_1(u)$ and takes $\vec{v} = \vec{w} - \vec{u}$ as the unit vector from u to w .*

Consider an agent starting at some node u and a direction r which itself starts at u chosen according to Definition 3. We first argue that there is probability equal to zero that Definition 2 leads to an ambiguity. Indeed, for any $d \geq 0$, consider the mapping $f_d : \tilde{R}_d(u) \rightarrow \tilde{R}_1(u)$ such that $f(w) = w/d$. This clearly is a homothetic transformation, which is a similarity and a bijection. For each $d \geq 0$, define

$$A_d = \{v \in \tilde{R}_d(u) : \exists w_1 \neq w_2 \in R_d(u) \text{ such that } |v - w_1|_2 = |v - w_2|_2\}.$$

The probability that there is an ambiguity in the determination of the r -approximating path is equal to the probability that the ray r crosses one point of A_d for some $d \geq 0$. But the set $\cup_{d \geq 0} A_d$ is countable (each A_d has cardinality equal to $4d$) and so is the set $\cup_{d \geq 0} f_d(A_d)$, thus there is probability equal to zero that a point chosen u.a.r. in $\tilde{R}_1(u)$ lies in $\cup_{d \geq 0} f_d(A_d)$ (i.e., it leads to ambiguity). Since we look at the mobility models for a countable amount of time and for a countable number of agents, during the whole process there is probability zero that the procedure to choose a direction leads to ambiguity.

Thus, Definition 2 is well-posed with probability one, and we are ready to prove Lemma 1.

Lemma 1. *Let u be any node of \mathbb{Z}^2 , $d \geq 1$, and $v \in R_d(u)$. Suppose an agent is on u and chooses a direction according to the procedure in Definition 3. Then, there is probability $1/(4d)$ that the corresponding approximating path crosses v .*

Proof of Lemma 1. Consider an agent starting at some node u and a direction r which itself starts at u chosen according to the procedure in Definition 3. We argue that there is probability exactly $1/(4d)$ that a given node $v \in R_d(u)$ belongs to a corresponding r -approximating path. Indeed, for $d \geq 0$, consider the same mapping $f_d : \tilde{R}_d(u) \rightarrow \tilde{R}_1(u)$ such that $f(w) = w/d$. As we said, this is a homothetic transformation, which is a similarity, and the probability that the r -approximating path crosses v is the probability that a point chosen u.a.r. in $\tilde{R}_1(u)$ lies in the segment BC in Fig. 11 (the boundaries have probability zero to be chosen). Since the rhombuses $\tilde{R}_1(u)$ and $\tilde{R}_d(u)$ are similar with a dilatation factor of d , and so are the triangles ABC and

ADE , then the probability that a point chosen u.a.r. in $\tilde{R}_1(u)$ lies in BC is the same as the probability that a point chosen u.a.r. in $\tilde{R}_d(u)$ lies in DE , which is $\overline{DE}/(4d\sqrt{2}) = 1/(4d)$. \square

C Monotonicity Property

Consider any mobility model fulfilling the following definition.

Definition 8 (Radially monotone mobility model). *We say that a mobility model is radially monotone if it exists a non-increasing distribution function $\rho : \mathbb{N} \rightarrow \mathbb{R}$, such that, given any two nodes u and v , the probability to jump in just one step from u to v is $p(u, v) = \rho(|u - v|_1)$.*

For any node $u = (u_x, u_y)$, let $d_u = |u_x| + |u_y|$, and define the square

$$Q(d_u) = \{(x^\theta, y^\theta) \in \mathbb{Z}^2 : \max(|x^\theta|, |y^\theta|) \leq d_u\}$$

(See Fig. 12 below). Furthermore, for any node $u \in \mathbb{Z}^2$, let $p_{u,t}$ be the probability that the agent is located in u at time t . Then, the following geometric property holds.

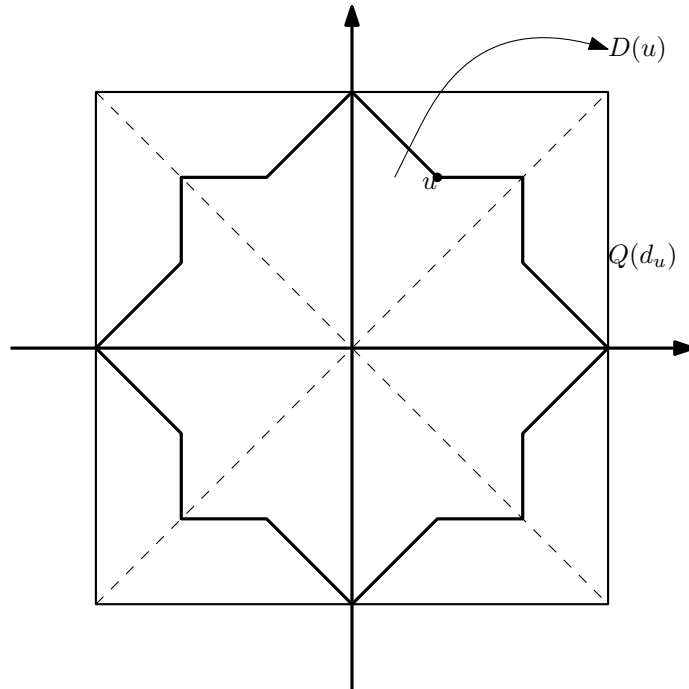


Figure 12: The set $D(u)$, consisting in all inner nodes of the “star”, and the square $Q(d_u)$.

Lemma 3 (Monotonicity property). *Let $u \in \mathbb{Z}^2$ be an arbitrary node, and consider an agent which moves according to a radially monotone mobility model, starting at the origin. Then, for each node $v \notin Q(d_u)$ and each time t , it holds that $p_{u,t} \geq p_{v,t}$.*

Proof of Lemma 3. For a given distance $d \geq 0$, consider the rhombus $R_d(o) = \{(x, y) \in \mathbb{Z}^2 : |x| + |y| \leq d\}$. For any point (x, y) we also define a square $T(x, y) = \{(x^\theta, y^\theta) \in \mathbb{Z}^2 : \max(|x^\theta|, |y^\theta|) \leq \max(|x|, |y|)\}$.

Let $u = (u_x, u_y)$ be any point in \mathbb{Z}^2 , $d_u = |u_x| + |u_y|$ its distance from the origin, and $t \geq 1$ any time step. Let \mathbf{X}_t be a the random variable representing the coordinate of the node the agent is located on at time t . We are now going to prove a stronger result which will imply the thesis. In particular, we show that, for any $v \in \mathbb{Z}^2$ which is outside the set $D(u) = R_{d_u}(o) \cup T(u)$,

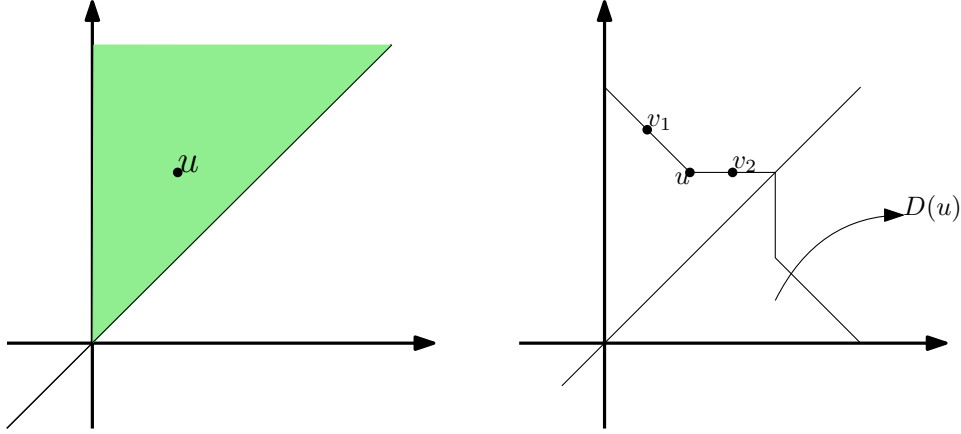


Figure 13: The “area” in which we take u , and the possible choices of v .

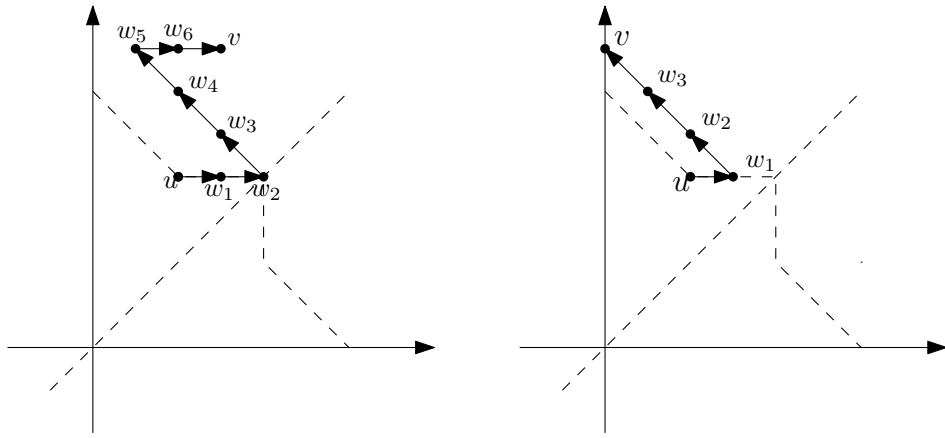


Figure 14: Two “path” examples.

or, at most, on its “border”,¹⁷ it holds that

$$\mathbb{P}(\mathbf{X}_t = u) \geq \mathbb{P}(\mathbf{X}_t = v), \quad (25)$$

for any $t \geq 1$. Let $Q(d_U) = \{(x^\theta, y^\theta) : \max(|x^\theta|, |y^\theta|) \leq d_U\}$, and note that $D(u) \subseteq Q(d_U)$. In Fig. 12 such sets are plotted.

Without loss of generality, suppose u is in the first quadrant and not below the main bisector, namely in the set $\{(x, y) \in \mathbb{Z}^2 : y \geq 0, x \geq y\}$ (Fig. 13). First, we argue that it is sufficient to show the statement for

$$v \in \{v_1 = (u_x - 1, u_y + 1), v_2 = (u_x + 1, u_y)\},$$

as in Fig. 13. Indeed, for any $v \notin D(u)$ that “lives” in the highlighted area in Fig. 13, there exists a sequence of nodes $u = w_0, w_1, \dots, w_k = v$ from u to v such that w_{i+1} belongs to the set

$$\{(x_{w_i} - 1, y_{w_i} + 1), (x_{w_i} + 1, y_{w_i})\},$$

where $w_i = (x_{w_i}, y_{w_i})$, as Fig. 14 shows. Thus, if the thesis is true for $v \in \{v_1, v_2\}$, then it is true also for all $v \notin D(u)$ in the highlighted area in Fig. 13. At the same time, for any other $v \notin D(u)$, outside the highlighted area in Fig. 13, there exists a symmetrical argument explained in Fig. 15. Thus, if the thesis is true for all $v \notin D(u)$ in the highlighted area in Fig. 13, then it is also true for any $v \notin D(u)$. We now consider some geometric constructions which will be used in the proof, one for each choice of v . The following description is showed in Fig. 16 .

¹⁷By “border” of $D(u)$ we mean the set $R_{d_u}(u) \cap T^0(u)$, where $R_{d_u}(u) = \{(x, y) \in \mathbb{Z}^2 : |x| + |y| = d_u\}$ and $T^0(u) = \{(x, y) \in \mathbb{Z}^2 : \max(|x|, |y|) = \max(|u_x|, |u_y|)\}$.

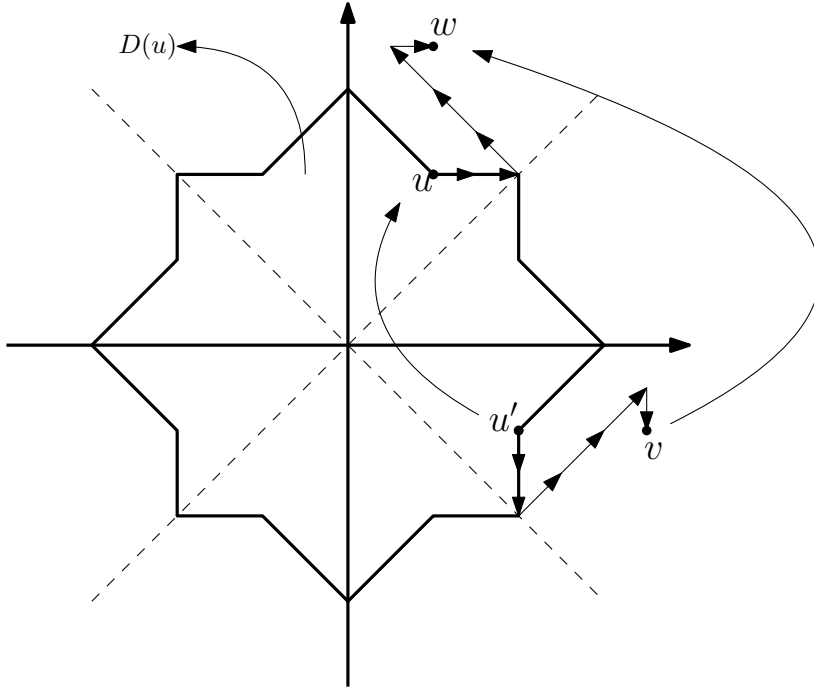


Figure 15: Symmetrical argument.

- (i) $v = (u_x - 1, u_y + 1)$: consider the strict line defined by $r : y = x + (u_y - u_x) + 1$ (i.e. the line in \mathbb{R}^2 which is the set of points that are equidistant from u and v according to the Euclidean distance). Call $V \subset \mathbb{Z}^2$ the set of nodes that are “above” this line, namely the ones that are closer to v than u . Define $U = \mathbb{Z}^2 \setminus (V \cup r)$ the complementary set without line r . Consider the injective function $f : V \rightarrow U$ such that $f(x, y) = (y - (u_y - u_x) - 1, x + (u_y - u_x) + 1)$, which is the symmetry with respect to r . It trivially holds that for any $w \in V$, $|w - v|_1 = |f(w) - u|_1$ and $|w - u|_1 = |f(w) - v|_1$. Furthermore, it holds that for each $w \in V$, either $w \notin D(f(w))$, or w lies on the “border” of $D(f(w))$. All these properties are well-shown in Fig. 17.
- (ii) $v = (u_x + 1, u_y)$: the same construction can be done in this case. Indeed, the strict line will be $x = u_x + \frac{1}{2}$, and the injective function $f(x, y) = (2u_x + 1 - x, y)$. The same properties we have seen in the previous case hold here too.

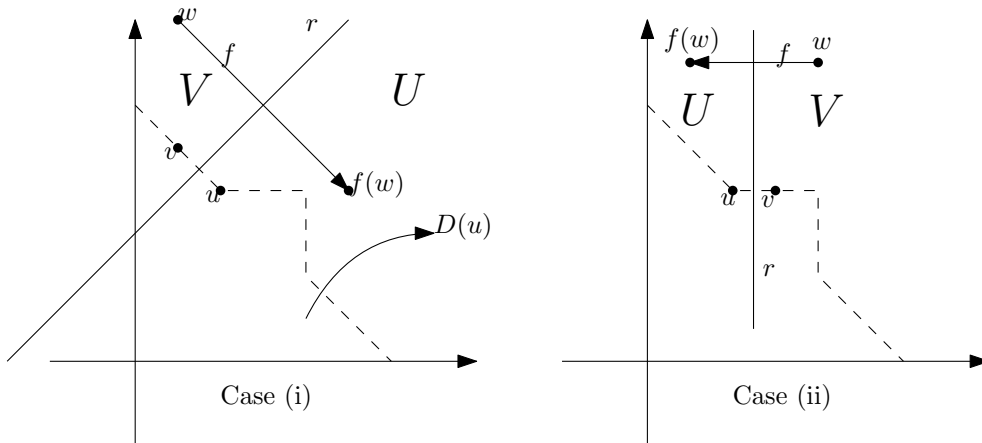


Figure 16: Geometric constructions in the three cases.

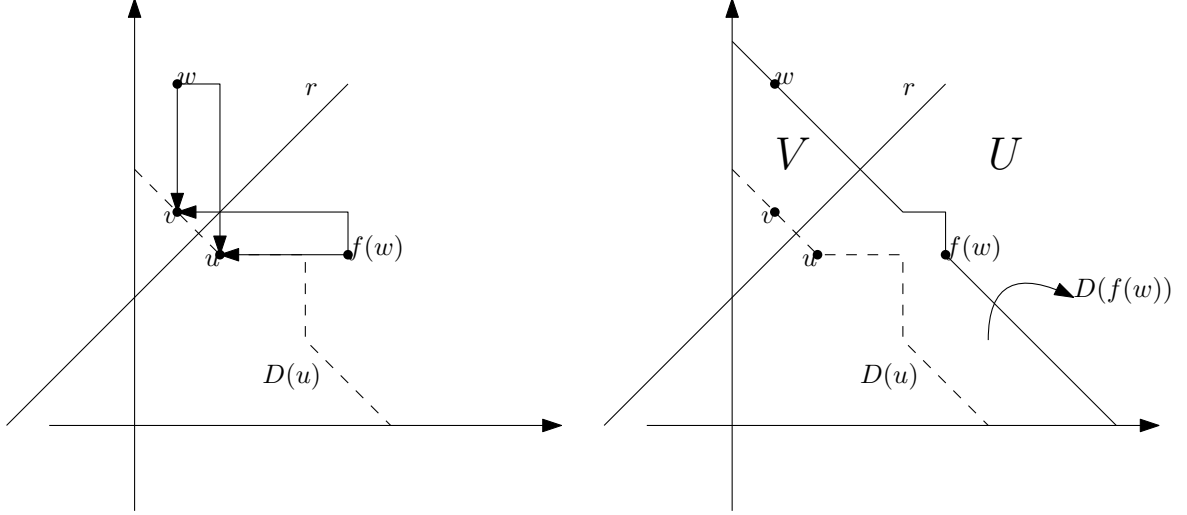


Figure 17: First, equivalence between distances. Second, $w \notin D(f(w))$.

Now we go for the proof. For any time i , and any two nodes $u^\ell, v^\ell \in \mathbb{Z}^2$, define

$$p^i(u^\ell, v^\ell) = \mathbb{P}(\mathbf{X}_i = v^\ell \mid \mathbf{X}_0 = u^\ell).$$

Let o be the origin, u be any node, and $v \in \{v_1, v_2\}$. We show that $p^t(o, u) \geq p^t(o, v)$ by induction on t . The base case is $t = 1$. From the hypothesis on the mobility model, we know that

$$p^1(o, u) - p^1(o, v) \geq 0$$

for any u and v in \mathbb{Z}^2 such that $|u|_1 \leq |v|_1$. We now suppose $t \geq 2$ and the thesis true for $t - 1$. Fix u and v as in Fig. 13; then, for the geometric construction we made above, it holds that

$$\begin{aligned} p^t(o, u) - p^t(o, v) &= \sum_{w \in \mathbb{Z}^2} p^{t-1}(o, w) (p^1(w, u) - p^1(w, v)) \\ &\geq \sum_{w \in U} p^{t-1}(o, w) (p^1(w, u) - p^1(w, v)) + \sum_{w \in V} p^{t-1}(o, w) (p^1(w, u) - p^1(w, v)) \end{aligned}$$

where last inequality is immediate for case (ii), indeed the line r does not contain elements of \mathbb{Z}^2 , while in case (i) the sum over nodes in line r is zero. Then, the previous value is equal to

$$\sum_{w \in V} p^{t-1}(o, f(w)) (p^1(f(w), u) - p^1(f(w), v)) + \sum_{w \in V} p^{t-1}(o, w) (p^1(w, u) - p^1(w, v))$$

because of the definition of $f : V \rightarrow U$, and, changing the sign of the second sum, we obtain

$$\sum_{w \in V} p^{t-1}(o, f(w)) (p^1(f(w), u) - p^1(f(w), v)) - \sum_{w \in V} p^{t-1}(o, w) (p^1(w, v) - p^1(w, u)).$$

Now, observe that the definition of f implies that for each $w \in V$, $|w - v|_1 = |f(w) - u|_1$ and $|f(w) - v|_1 = |w - u|_1$ (Fig. 17). Thus we can group out the term $p^1(f(w), u) - p^1(f(w), v) = p^1(w, v) - p^1(w, u)$, and we have

$$\sum_{w \in V} (p^{t-1}(o, f(w)) - p^{t-1}(o, w)) (p^1(f(w), u) - p^1(f(w), v)). \quad (26)$$

We observe that $p^{t-1}(o, f(w)) - p^{t-1}(o, w) \geq 0$ by the inductive hypothesis, since either $w \notin D(f(w))$ or w lies on the “border” of $D(f(w))$ (Fig. 17), and $p^1(f(w), u) - p^1(f(w), v) \geq 0$ by definition of f , since the distance between $f(w)$ and u is no more than the distance between $f(w)$ and v . It follows that (26) is non-negative, and, thus, the thesis. \square

D Projection of a Pareto Flight Jump

Let \mathbf{P}_t^f be the two dimensional random variable representing the coordinates of an agent performing an α -Pareto flight at time t , for any $\alpha > 1$. Consider the projection of the Pareto flight on the x -axis, namely the random variable X_t such that $\mathbf{P}_t^f = (X_t, Y_t)$. The random variable X_t can be expressed as the sum of t random variables S_j^X , $j = 1, \dots, t$, representing the projection of the jumps (with sign) of the agent on the x -axis at times $j = 1, \dots, t$. With the next lemma, we prove that the jump projection length has the same tail distribution as the original jump length.

Lemma 32. *The probability that a jump S_j^X has length equal to d is $\Theta(1/(1+d))$.*

Proof. The partial distribution of the jumps along the x -axis is given by the following. For any $d \geq 0$,

$$\mathbb{P}(S_j^X = \pm d) = \left[c + \sum_{k=1}^{\mathbb{1}_{d=0}} \frac{c}{2k(1+k)} \right] \mathbb{1}_{d=0} + \left[\frac{c}{2d(1+d)} + \sum_{k=1+d}^{\mathbb{1}_{d \neq 0}} \frac{c}{k(1+k)} \right] \mathbb{1}_{d \neq 0}, \quad (27)$$

where $\mathbb{1}_A$ returns 1 if $d \in A$ and 0 otherwise, the term

$$c \mathbb{1}_{d=0} + \frac{c}{2d(1+d)} \mathbb{1}_{d \neq 0}$$

is the probability that the original jump lies along the horizontal axis and has “length” exactly d (there are two such jumps if $d > 0$), and, for $k \geq 1+d$, the terms

$$\frac{c}{2k(1+k)} \mathbb{1}_{d=0} + \frac{c}{k(1+k)} \mathbb{1}_{d \neq 0}$$

are the probability that the original jump has “length” exactly k and its projection on the horizontal axis has “length” d (there are two such jumps if $d = 0$, and four such jumps if $d > 0$). Quantity (27) is at least

$$\frac{c}{2} \left(\frac{1}{(1+d)^{\alpha+1}} + \sum_{k=1+d}^{\mathbb{1}_{d \neq 0}} \frac{1}{k(1+k)} \right),$$

and at most

$$2c \left(\frac{1}{(1+d)^{\alpha+1}} + \sum_{k=1+d}^{\mathbb{1}_{d \neq 0}} \frac{1}{k(1+k)} \right).$$

By the integral test (Fact 1 in Appendix A) we know that quantity (27) is

$$\mathbb{P}(S_j^X = \pm d) = \Theta\left(\frac{1}{(1+d)}\right).$$

□

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