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On the Search Efficiency of Parallel Lévy Walks on \mathbb{Z}^2

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Abstract

Lévy walk is a popular movement model where an agent repeatedly samples a direction uniformly at random, and then travels in that direction for a distance which follows a power law distribution with exponent $\alpha \in (1, +\infty)$. Lévy walks and some of its famous variants, such as Brownian motion, have been subject to extensive empirical investigations in the continuous framework (i.e. continuous time and support space) since, essentially, they play an important role in theoretical biology: for instance, their *search* efficiency has been investigated in terms of the *discovery rate* of food, where the latter corresponds to points distributed according to a given density over \mathbb{R}^2 . In that framework, it has been shown that the best efficiency is achieved by setting the parameter α to 2.

Motivated by their importance, we provide the first rigorous and comprehensive analysis of the efficiency of Lévy walks in the discrete setting, by estimating the search efficiency of k independent, parallel Lévy walks starting from the origin. In more detail, the search efficiency of this parallel process is here described in terms of the *hitting time* with respect to (only) one target node of the 2-dimensional infinite grid, and the consequent *total work*, i.e., the total number of steps performed by all the agents.

The study of distributed algorithms for k searching agents on an infinite grid that aim to minimize the hitting time of a target node has been considered in Distributed Computing under the name of ANTS Problem (Feinerman et al. PODC 2012). Our probabilistic analysis of Lévy walks implies the following main novel contributions:

- I. We show that Lévy walks provide a biologically well-motivated, time-invariant search protocol for the ANTS Problem which does not require any communication among the k agents, and which improves the state of the art in terms of efficiency, for a wide range of the parameter k .
- II. In contrast with the apparent general optimality of the setting $\alpha = 2$, suggested by the discovery rate mentioned above, we show that the best exponent with respect to a natural measure, such as the *total work* of the process, directly depends on the number of searching agents k .

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1 Introduction

Consider the following simple scenario: k agents, initially placed at the origin of an infinite grid (say, \mathbb{Z}^2), can move at each discrete time-step on a neighbouring node; their goal is, for any of them, to *hit* as soon as possible a unique node (from now on, *treasure*), whose location is *unknown*. Such fundamental *parallel search* problem has been investigated in the area of Distributed Computing under the name of Ants Nearby Treasure Search (ANTS) Problem, a name which is inspired by biological motivations. If we call *work* the natural measure defined as the total number of time-steps performed by all the agents (see Definition 4), it has been shown that for the ANTS Problem it is necessary and sufficient to perform a work which is $\Theta(\ell^2)$ with respect to the initial distance ℓ of the treasure from the origin (see Subsection 2.1). Interestingly enough, such performance can be achieved by *natural* algorithms which are as simple as alternating two different kinds of *random walk* processes.

On the other hand, the central question of modelling *animal movement* has led to a rich literature on random walk models, the most prominent of which is the Lévy walk. Consider the following general framework: an agent continuously moves towards a certain direction on the plane (say, \mathbb{R}^2), choosing a new direction uniformly at random at times $\{T_i\}_{i \in \mathbb{N}}$, where $\Delta_i = T_i - T_{i-1}$ are i.i.d. random variables.

When the underlying space is discrete and Δ_i is constant ($\Pr(\Delta_i = 1) = 1$), we retrieve the classical setting of a simple random walk on \mathbb{Z}^2 . When the underlying space is discrete but Δ_i has a power-law distribution with exponent α , we obtain instead a discrete variant of the Lévy walk process, which we call *Pareto walk* (Definition 3).

Considering again the continuous space \mathbb{R}^2 , familiarity with random walks and power law distribution may soon suggest the intuition that, choices of α for which Δ_i has finite variance, led to a Lévy walk process which behaves similarly to a *Brownian motion* (or *Wiener process*, which is the term we use in the following¹ [Dur10]). Ranges of α for which the second moment of Δ_i becomes infinite represents, mathematically, a more interesting regime. In fact, Lévy walks have been shown to achieve the best performance within such regime, for problems such as maximizing the expected discovery rate of random targets on \mathbb{R}^2 (distributed according to a spatial Poisson point process), or for minimizing expected hitting times on the real line. However, while the aforementioned results belongs to a rich literature in theoretical biology based on statistical mechanics techniques, the

¹Formally, the Brownian motion refers to a physical process, while the term Wiener process refers to the stochastic process whose study has been largely motivated by Brownian motion; while for historical reasons the term Brownian motion is often used in place of Wiener process, we prefer to avoid such abuse of terminology.

treatment in the discrete setting² has been limited to finite graphs. We discuss in detail all the above points in Section 4.

The purpose of this work is to provide, at the intersection of the two aforementioned contexts (the algorithmic study of the ANTS Problem on one side and the mathematical study of Lévy walks on the other), the *first rigorous, systematic treatment of the hitting time and work complexity of parallel Lévy walks in the discrete setting*. A major consequence of our analysis is an *optimal* solution to the biologically-motivated ANTS Problem, which is based on a classical model for animal movement in theoretical biology, and which is arguably simpler than previous solutions (Subsection 2.1). More generally, our contribution entails the analysis of novel random walk processes and aspects of sums of power law random variables which are of independent interest.

1.1 Overview of Our Results

In the current section we discuss in detail our contributions while providing, at the same time, an overview of the organization of the paper³.

It is well-known that the possible values of α which control the tail distribution of Lévy walks (in the discrete setting, Pareto walks, Definition 3) can be divided in three main regimes [VLR11]:

- the *diffusive* regime ($\alpha \geq 3$),
- the *super-diffusive* regime ($\alpha \in (2, 3)$) and
- the *ballistic* one ($\alpha \in (1, 2]$).

In the continuous time-space setting, the ballistic and diffusive regimes have been observed to be qualitatively equivalent to different processes such as the Wiener process and the *ballistic walk* in which an agent moves towards a fixed direction (Definition 6). In the discrete setting, the Wiener process corresponds⁴ to a simple random walk (Definition 5), while the ballistic walk has a natural discrete version (Definition 6). In Section 8 and Section 9, we prove such equivalence in a precise sense, by showing that for the ANTS Problem (Subsection 2.1), Pareto walks achieve the same performances as the corresponding discrete processes in terms of the hitting time and total work. In detail, we show that:

- For $\alpha \geq 3$, the Pareto walk behaves as a simple random walk (theorems 4 and 6);
- For $\alpha \in (1, 2]$, the Pareto walk behaves as a ballistic walk (theorems 5 and 7);
- Finally, for⁵ $\alpha \in (2, 3)$, the Pareto walk exhibits a peculiar behavior which clearly distinguishes it from the previous processes⁶.

The aforementioned equivalence results motivate the investigation of a Pareto walk over the full range of α . Given our focus on the total work of k agents on the infinite grid (that is, the ANTS Problem), with Lemma 2 we first observe that *no algorithm* can achieve an expected work which is less than ℓ^2 , where ℓ is the distance of the treasure node from the origin where all agents start the parallel search. We emphasize that such lower bound applies independently from the number

²Even in the original, continuous setting, we have not been able to identify a systematic, mathematically rigorous treatment.

³For the sake of readability, we often defer the reader to corresponding formal definitions and statements in the following sections, in our discrete setting.

⁴A classical convergence result is provided by Donsker's Theorem [Dud99].

⁵We remark here that, even though the threshold case $\alpha = 3$ turns out to be equivalent of the simple random walks, its mathematical analysis requires arguments similar to those we use for the case $(2, 3)$ (see Theorem 3).

⁶This is the regime of α which is sometimes implicitly assumed in the literature when using the term Lévy walk (Subsection 2.3)

of agents k , and even assuming that agents share all the information available to them at all time steps.

The central part of this paper is then the study of the work efficiency of Pareto walks with respect to the values of k and α . Remarkably, in theorems 2, 6 and 7 (and in the aforementioned, corresponding theorems 4 and 5), we show that, *for any value of α , there is a unique value of k , polynomial in ℓ , such that the work efficiency is optimal (up to polylogarithms)*: we summarize such statements in Table 1. Conversely, such results imply that, for such a natural aspect as the parallel hitting time, the α for which optimal work efficiency is achieved varies, as a function of the number of agents k , between 1 and 3. This should be contrasted with evidence provided by classical results on Lévy walks which suggested a key role for the value $\alpha = 2$ in biological applications (see Subsection 2.3).

We now discuss in more detail Theorem 2, which is probably the most intriguing technical scenario we address, namely the analysis of Pareto walks for the range $\alpha \in (2, 3)$. There, we prove that for the range $k = \tilde{\Theta}(\ell^{1+(\alpha-2)})$ the agents find the treasure within time $\tilde{\Theta}(\ell^{1-\alpha-2})$, making a total work of $\tilde{\Theta}(\ell^2)$, w.h.p. Furthermore, the result is almost-tight in a two-fold sense: if $k = \tilde{\Theta}(\ell^{1-(\alpha-2)-\epsilon})$ for any arbitrary constant $\epsilon \in (0, 1-(\alpha-2)]$, then the treasure is never hit by the agents, w.h.p., resulting in an infinite work. On the other hand, if $k = \tilde{\Theta}(\ell^{1-(\alpha-2)+\min(\epsilon+\frac{\alpha-2}{2}, \frac{3}{2}\epsilon)})$ for any arbitrary constant $\epsilon > 0$, then the treasure is found in time at least $\tilde{\Omega}(\ell^{1+(\alpha-2)-\epsilon})$, resulting in a work of order $\tilde{\Omega}(\ell^{2+\min(\frac{\alpha-2}{2}, \frac{\epsilon}{2})})$, w.h.p.

We conclude the section by remarking the independent interest of some technical contributions which are a byproduct of our analysis, such as Lemma 32 in which the following general *monotonicity property* is provided. Consider any random walk process which moves from one node u to any other node v , according to a probability distribution which is non-increasing with respect to the distance between u and v . Then, we prove that the corresponding agent's spatial distribution after t steps satisfies a monotonicity property which is essentially equivalent to that of the considered random walk process (see Appendix C).

α -Pareto walk	Equivalence	Optimal number of agents	First-hitting time	Total work
$\alpha \geq 3$ (theorems 3 and 6)	Simple random walk (Theorem 4)	$\log^{\Theta(1)}(\ell)$	$\tilde{\Theta}(\ell^2)$	$\tilde{\Theta}(\ell^2)$
$\alpha \in (2, 3)$ (Theorem 2)	Lévy walk (Theorem 2)	$\tilde{\Theta}(\ell^{1-(\alpha-2)})$	$\tilde{\Theta}(\ell^{1+(\alpha-2)})$	$\tilde{\Theta}(\ell^2)$
$\alpha \in (1, 2]$ (Theorem 7)	Ballistic walk (Theorem 5)	$\tilde{\Theta}(\ell)$	$\tilde{\Theta}(\ell)$	$\tilde{\Theta}(\ell^2)$

Table 1: Optimal settings. The values of the hitting time and of the work hold w.h.p.

2 Comparison with Related Work

Random searching strategies performed by simple agents is a topic that attracted strong attention of the researchers from several scientific fields. A huge amount of models, questions, analytical and experimental results are now available. We thus restrict our comparison to previous results that are more related to our setting.

2.1 The Ants Nearby Treasure Search (ANTS) Problem

In [FK17], Feinerman and Korman introduced⁷ and studied the Ants Nearby Treasure Search (ANTS) Problem, defined as follows. There are k probabilistic agents initially located at the origin (the nest) of the infinite grid \mathbb{Z}^2 and one treasure located in one arbitrary node of the grid. The goal is to find the treasure as fast as possible as function of k and the (unknown) distance ℓ of the treasure from the origin. The analysis in [FK17] provides the first known bounds on the trade-off between the agent memory size and the time complexity of the searching algorithm. They consider the case where agents are allowed, upon initialization (i.e. before leaving the origin), to make use of their local memory in order to coordinate their action. Such initial information is provided by a centralized oracle and, for instance, may consist of an approximation of the parameter k . They provide tight bounds on the memory/time trade-off. More in detail, it can be easily shown that $\Omega(\ell + \ell^2/k)$ time steps are necessary for any algorithm (see e.g. Lemma 2). In order to achieve such lower bound, $\log \log k + \Theta(1)$ for bits of local memory is proved to be necessary and sufficient, while larger time bounds are obtained for smaller memory size. As for their optimal algorithms, the authors remark that they are too complex to be considered realistic strategies for the biological agent systems the paper is inspired from.

2.1.1 Significance of our contribution

The Pareto walks studied in this paper are a natural discrete version of Lévy walks, a well-established model of animal movement [Rey18]. Indeed, they are significantly *simpler* than the algorithmic strategies considered in [FK17]. More precisely, in our parallel Pareto walks

- agents are not allowed to coordinate their search, not even upon initialization, and
- the rule they apply is *time-homogeneous*, in that it merely consists of moving directly towards a target which is always sampled according to the same distribution w.r.t. the position of the agent.

Despite their simplicity, we show that Pareto walks achieve optimal time-efficiency (see Table 1).

2.1.2 Harmonic search algorithm

[FK17] also consider a simpler strategy, the *Harmonic* algorithm and prove that it finds the treasure within a time which is not too large compared to the optimal one. In the Harmonic algorithm, agents are independent, perform the same strategy and do not use any initial information given by the oracle: it is thus somewhat comparable to the strategies studied in this paper. In detail, let $\alpha > 1$ be an arbitrarily small constant⁸, denote with $d(u)$ the ℓ_1 -distance of a node u from the origin, and let c_α be a normalization constant such that $p(u) = c_\alpha/d(u)^{1+\alpha}$ turns out to be a probability distribution over \mathbb{Z}^2 . Then, the Harmonic algorithm works in phases, each of them consisting of 3 consecutive actions. Each agent

- i) independently chooses a node u with probability $p(u)$ and reaches that node,
- ii) performs a local *spiral* search⁹ around u and
- iii) goes back to the nest.

In [FK17] the authors prove that the following holds.

⁷More precisely, the problem was first introduced in the conference version of the paper [FKLS12].

⁸We have adapted the notation [FK17] in so that the use of α is consistent with how we use it in our results.

⁹for details on this trajectory please see [FK17].

Theorem 1 (Theorem 5.8 in [FK17]). *Let δ and ϵ be arbitrarily-small positive constants and let $\alpha = 1 + \delta$. A positive real β exists such that, for any $k > \beta\ell^{\alpha-1}$, the Harmonic Search Algorithm finds the treasure within $\mathcal{O}(\ell + \ell^{1+\alpha}/k)$ time, with probability at least $1 - \epsilon$.*

A crucial difference between the Harmonic algorithm and our parallel Pareto walks lies in the fact that, in the former, agents deterministically go back to the origin at the end of every phase, so there is no chance for an agent to walk arbitrarily far from the nest for an infinite time. While such property ensure that, even for a single agent ($k = 1$), the treasure is eventually found with probability 1, such strategy requires that the agents precisely know, at all times, their relative position to the nest¹⁰. In the Pareto walks instead, as soon as an agent reaches its next way-point, it samples a new one and start moving towards it: hence, the algorithmic process does not require different phases in which different rules are applied. Such feature comes with a price: for $\alpha \in (1, 3)$, with positive probability an agent can walk arbitrarily far from the starting point (the nest) and does never return to it (this easily follows, for example, by taking $k = 1$ in our theorems 2 and 7).

Interestingly, the Harmonic Search Algorithm may be regarded as a combination of the two qualitatively opposite regimes of the Pareto walks algorithms: the first phase of the algorithm, which moves to a node at distance d in d steps, corresponds to the *ballistic regime* of the Pareto walks (cfr. theorems 5 and 7), while the second phase, where a spiral search takes place for essentially d^2 rounds, corresponds to the *diffusive regime* of the Pareto walks (cfr. theorems 4 and 6). Our analysis, summarized in Table 1, shows that Pareto walks achieve essentially the same trade-off¹¹ between the agent number and the hitting time as the above Theorem 5.8 in [FK17], showing that the same effect of combining the ballistic and diffusive regime can be obtained by considering a natural Pareto walk whose exponent lies between the twos.

2.2 Related work on finite graphs

The rigorous study of the *hitting time* of nodes on *finite* graphs has been introduced in [ER09], following the investigation of the *cover time* in [AAK⁺11]. In the finite-graph setting, standard techniques for bounding the hitting time involve the closely-related notion of *mixing time* of a single random walk [KMTS19, STM06]. In the infinite setting the mixing time is not defined, as a single random walk does not have a stationary distribution.

Recent studies have investigated the hitting time of Lévy walks (and Lévy flights, see Definition 7) on \mathbb{R}^2 (that is, in the continuous setting) [PBL⁺19]. We discuss the related work on the infinite setting in Subsection 2.3.

In [BGK⁺18], Boczkowski et al. analyse a single-agent searching strategy, called *m-intermittent* search, over the *ring topology* of n nodes. They provide some bounds on the cover time (and, thus, on the hitting time) as a function of the number m of different step lengths $\{L_i\}_{i \in [m]}$ an agent randomly selects for its jumps on every step, where L_i is chosen with probability p_i . They show that an expected hitting time which is linear in the size n of the ring can be achieved by choosing a distribution $\{p_i\}_{i \in [m]}$ on the step lengths according to a *Weierstrassian random walk* [HSM81]: The latter can be regarded as an efficient approximation of the ballistic regime of our Pareto walk ($\alpha = 1$). More in detail, a Weierstrassian random walk is obtained by fixing some integer $B > 1$ and choosing $m = \log n$, $L_i = B^i$ and $p_i \propto B^{-i}$. This setting results in an heavy tail w.r.t. n , formally $\Pr(\text{step length} > t) \geq \frac{1}{2t}$ for $t < \frac{n}{2}$. In [GK19], Guinard and Korman refined the results of [BGK⁺18] on the ring topology by providing bounds on the hitting time that can be achieved by considering small values of m . In particular, they show that by setting $B = n^{\frac{1}{m}}$ in the Weierstrassian random walk described above, the expected hitting time on the ring is $\tilde{\Theta}\left(n^{1+\frac{1}{m}}\right)$.

¹⁰Notice that, since the strategy is probabilistic, any bound m on the local memory available to the agents would imply a certain probability of failure in the event that an agents chooses a destination u such that $\lfloor \log u \rfloor \gg m$

¹¹In particular, we don't require the number of agents to be at least some polynomial ℓ^δ .

2.2.1 Mobility models

Agent’s movement models are fundamental tools in the study of mobile communication networks [LV06, Roy11] (in the corresponding research community, such models are called *mobility models*). Roughly speaking, analytical results in this area are available for the two most popular classes of mobility models, the random walk and the random way-point models, and some generalizations of them. In more detail, for the continuous space-time setting, stationary agents’ spatial distributions over finite support spaces have been derived in [LV06] for the general model known as *random trip model*. As for the discrete setting, similar results have been derived in [CMS11, CST14] for the general *Markovian trace model* under the assumption that the set of nodes of the (possibly infinite) support graph that admit feasible agent’s mobility paths has finite size.

2.2.2 Other related works in computer science

Our setting is reminiscent of the *Parallel Search without Coordination Problem* investigated in [FKR16], in which an infinite list of *boxes* is given, with a *treasure* hidden in one of them, where the boxes’ order reflects the importance of finding the treasure in a given box. At each time step, a search protocol executed by a searcher can *probe* one box, and see whether it contains the treasure. The author study the best running time achievable by *non-coordinating* algorithms, motivated by *robustness* requirements. Crucially, in their setting the searching agent are provided an approximate knowledge of the importance ordering according to which the treasure is present in a given box.

Finally, some readers may find some connection between our work and some research on the Small World phenomenon, which we discuss in Subsection 2.3.2.

2.3 Overview of literature on Lévy walks

It is reasonable to state that Lévy walks constitute nowadays the main mobility model in biology [Rey18], at least among models with comparable mathematical simplicity and elegance [VLR11]. In the following, we concisely summarize the history of the study of Lévy walks in order to provide the main context for the present work. For a rich monograph on the topic, we defer the reader to [VLR11].

Lévy flights (i.e. the turning points in which a Lévy walk changes trajectory) were originally investigated by Paul Lévy in his 1937 treatise¹² [Lé54], and named after him by his student Benoît Mandelbrot which investigated their fractal properties [Man82]. [SK86], which investigated the relation between Lévy flights and Lévy walks, was among the first studies to discuss the relevance of Lévy walks and Lévy flights as a biological mobility model. Many works, among which a 1996 Nature paper by Viswanathan et al., supported such hypothesis on the basis of analyses of empirical data [VAB⁺96]. Statistical flaws were later pointed out in several of these works [EPW⁺07], including [VAB⁺96]; however, subsequent works which avoided previous methodological issues continued to corroborate what has been known as *Lévy flight foraging hypothesis*:

Since Lévy flights and walks can optimize search efficiencies, therefore natural selection should have led to adaptations for Lévy flight foraging. - [VRdL08]

The surge of interest on Lévy walks motivated theoretical investigations. We have to point out here that, despite our efforts, *we failed to identify rigorous proofs for a large part of the main results* in the Lévy walks literature¹³.

The main mathematical models on which Lévy walks have been investigated in the literature involves a single agent continuously moving on the infinite real plane \mathbb{R}^2 , in continuous time. Targets

¹²The referenced citation refers to the second edition of the work.

¹³In fact, none of the works we mention in the following is organized in standard mathematical form involving clearly marked statements and proofs.

are randomly located in the plane according to a fixed density ρ . The agent chooses a way-point u.a.r. among those at distance d , where d is chosen according to a power-law distribution with tail $\Pr(d > t) = \Omega(\frac{1}{d^{\alpha-1}})$, and then moves towards it at a certain speed. If, at any time, any target happens to be within some distance radius r from the agent, the target is *found*¹⁴.

We remark that the above setting is quite different from the ANTS problem we consider. In particular, the optimality concept in the above framework is based on the *expected discovery rate*, that is the expected number of targets found per distance travelled, while our efficiency measure (Definition 4) involves the number of agents k and the *hitting time* of the target.

In the seminal paper [VBH⁺99], it is showed¹⁵ that a Pareto walk (in the language and notation of the present work) with exponent $\alpha \approx 2 - (\frac{1}{\log \frac{1}{2r^2\rho}})^2$ optimizes the expected number of targets found per distance travelled; they thus suggest that $\alpha = 2$ can be considered an optimal choice when the quantity $r^2\rho$ is small but not exactly known. Successively, [PCM14] showed that the choice¹⁶ $\alpha = 2$ does not have *universal* value in the sense of being rather sensible to the choice of the underlying model. In particular they show that, when in the one-dimensional case \mathbb{R} of the above framework an additional *bias* factor is considered (i.e. an *external drift* term in the jump distribution), the optimal choice for α varies in $(2, 3)$. We remark that our rigorous results corroborate such findings by showing how, for the ANTS Problem, the optimal value of α spans the entire interval $(1, 3]$, depending on the number of searching agents; in particular, we provide the first general setting for which the optimal α varies below 2.

2.3.1 Biological relevance of our analysis

The present work rigorously addresses the mathematical problem of investigating the efficiency of parallel, independent Pareto walks as a natural solution for the ANTS Problem. As for the general question of the biological relevance of Lévy walks as a realistic mobility model, we defer the reader to the discussion in [VLR11, Chapter 8]. Assuming that Lévy walks are a relevant mobility model, the main issue with respect to the biological relevance of our results is the absence of coordination among the agents. We argue here that, in some biological scenarios, it appears that the stochastic dependencies among the searching agents do not play a relevant role for the efficiency of the process. Experiments by Fourcassié et al. in [VBVT03] on a population of ants of the species *Messor sancta* consider groups of ants of different sizes and observe several aspects of their dispersion movements over a large arena. Their statistical results show that, while the geometry of an ant’s path appears to be *mechanically* affected by random *collisions* with other ants, the size of the area explored by each ant moving in a group and the average number of interactions among them is not significantly different from the explored area and number of encounters that would be observed if ants were moving on the area independently of each other. This fact leads the authors to conclude that there is no coordination (i.e. explicit information exchange) among the ants moving on the arena.

2.3.2 Power laws and stable distributions

In Lévy walks, the jump lengths are chosen according to a Pareto distribution, which naturally relates their analysis to the vast literature on *stable distribution* [Nol07]. [Nol19] provides an

¹⁴We emphasize that our high level description is omitting several details; e.g., in [VBH⁺99], when a target happens to be within radius r from the agent, the agent start moving towards such target until reaching its exact location.

¹⁵This is one of the cases in which, as mentioned above, we failed to reconstruct a rigours proof of the claimed result.

¹⁶We remark that in several works the exponent α correspond to the exponent of the survival function $\Pr(X > t)$ where X is the random distance chosen by the agent, rather than being the exponent of the density $\Pr(X = t)$; in this work, we adopt the latter convention.

extensive updated bibliography on such research area. In Subsection 2.3.3, we briefly argue that it does not appear to be possible to obtain our results using standard tools from such literature.

Another famous research endeavour evoked by the power-law jump distribution of Pareto walks is the research on the small-world phenomenon in social networks, in particular Kleinberg’s analysis of his generalized Watts-Strogatz model [Kle00b, Kle00a, EK10]. In its simplest form, Kleinberg’s model is a two-dimensional grid where, to each node u , a *long-range* edge is added by connecting it to another node v chosen according to a Pareto distribution w.r.t. the ℓ_1 distance $\text{dist}(u, v)$ between u and v . Kleinberg considers the simple routing algorithm according to which an agent greedily moves, at each step, across the edge which minimizes its distance from the target, and shows that, given a target node v , it achieves optimal performance ($\mathcal{O}(\log^2(\text{dist}(u, v)))$) when long-range connections of length ℓ are chosen with probability proportional to $\frac{1}{\ell}$. Notice that, besides the similarities given by the common presence of a Pareto distribution, our problem is of a different nature as our parallel Pareto walks do not know the location of the target.

2.3.3 Relation with the continuous setting and stable distributions

In this section we discuss how our analysis in the discrete setting relates to known results in the continuous setting. Essentially, we argue that, to the best of our knowledge, our results cannot be derived from known results for the classical, continuous version of Lévy walk.

The sum of i.i.d. random variables with Pareto distribution converges to a stable distribution¹⁷ for which there is no-known closed-form expression in terms of elementary function, except for few special cases (Gaussian, Cauchy and Lévy distributions) [Sam94, Nol07]. Hence, although it is a classical result that the *normalized* sum of random variables whose tail distribution decays as x^α converges to a distribution with the same tail [GK54, BB08], it does not seem possible to avoid *ad hoc* arguments in order to estimate the distribution of several steps of Pareto walks. More precisely, if we attempt to estimate the probability for the random walk process to be located on the target node by working directly with the distribution of the sum of the jumps, then we need to evaluate the latter distribution on constant values rather than estimating its tail.

3 Preliminaries

In this work we analyze and compare the search efficiency of some mobility models in the following setting. We have an infinite grid \mathbb{Z}^2 and a special node of the grid called the treasure, which is at distance $\ell > 0$ from the origin $o = (0, 0)$. Two nodes of the grid, (x_1, y_1) and (x_2, y_2) , are connected by an edge if their Manhattan distance is one, i.e. if $|x_1 - x_2| + |y_1 - y_2| = 1$. For any node $u = (u_x, u_y) \in \mathbb{Z}^2$, we write $|u|_1 = |u_x| + |u_y|$ for its Manhattan distance from the origin. At the same time, we write $|u|_2 = \sqrt{u_x^2 + u_y^2}$ to denote its Euclidean distance from the origin. With an abuse of notation, for a given set S , we denote as $|S|$ its cardinality.

Time is discrete and is marked by a global clock. Let $k \in \mathbb{N}$ be any positive integer. At time $t = 0$, k agents are positioned in the origin o and start moving, independently one from each other, over the edges of the grid, to search the treasure. We say an agent finds the treasure at time $t \geq 0$ if, at that time, the agent is located on the treasure (so we assume here that the agent’s detection radius is 0). We call step a move that takes one unit time. To introduce the considered mobility model over the infinite grid, we need a suitable notion of approximation of “walking direction”, where a direction is defined as the unique ray identified by a unit vector \vec{v} applied to some node u .

As usual, the length of a path is defined as the number of edges it consists of, and the distance between two nodes is the length of any of the shortest paths between them. Consider an agent in

¹⁷We remark how this fact is sometimes subject to misunderstandings in the literature [Lem08].

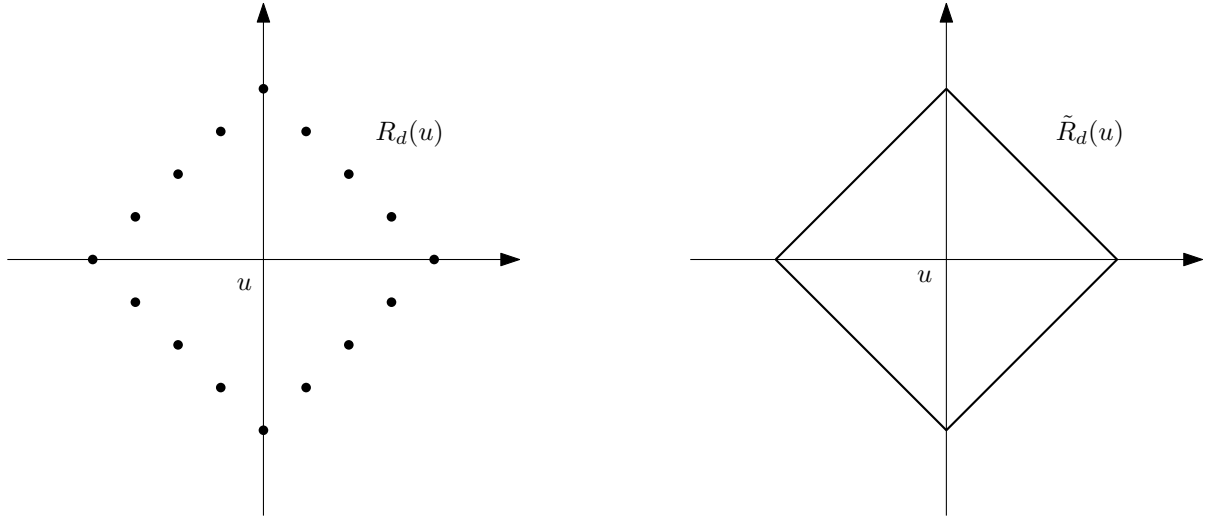


Figure 1: Examples of $R_d(u)$ and $\tilde{R}_d(u)$.

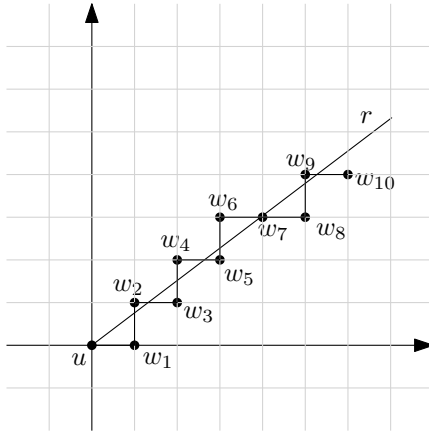


Figure 2: One direction-approximating path example.

some node u of the grid that wants to walk along a ray r . The agent chooses a path “approximating” r between the followings.

Definition 1 (*r-approximating path*). Let r be the unique ray identified by some unit vector \vec{v} applied to some node u . Consider, for $d \geq 0$, the sequence of rhombus centered at u

$$R_d(u) = \{v \in \mathbb{Z}^2 : |u - v|_1 = d\}.$$

For each $d \geq 0$, consider the “natural immersion” of the rhombus in the continuous plane, namely

$$\tilde{R}_d(u) = \{(x, y) \in \mathbb{R}^2 : (y + x + 1)(y + x - 1)(y - x + 1)(y - x - 1) = 0, |x| \leq d, |y| \leq d\},$$

as in Figure 1. Let v_d the intersection between r and $\tilde{R}_d(u)$. An *r-approximating path* is a simple path starting at u , whose d -th node is the node $w_d \in R_d(u)$ that minimize the distance $\min_{w \in R_d(u)} |w - v_d|_2$. Ties are broken uniformly at random.

A geometric description of such a path is given in Figure 2, while the well-posedness of Definition 1 is discussed in Appendix B. We now give a definition on what we mean when an agent chooses a direction and moves along it.

Definition 2 (Direction choice procedure). An agent at some node u chooses a direction r in the following way: it samples uniformly at random one node v of $\tilde{R}_1(u)$ and takes r as the unique ray starting in u and crossing v .

Observe that, in general, an approximating path is unique unless, for some $d \geq 0$, v_d is equidistant between two nodes of $R_d(u)$. Anyway, by choosing uniformly at random some direction according to the above procedure, the probability that this happens is equal to zero, since there are just a numerable quantity of rays that realize this kind of ambiguity. In Appendix B there is a formal argument proving this. Furthermore, we have the following lemma, whose proof is deferred to Appendix B.

Lemma 1. *Let u be any node of \mathbb{Z}^2 , $d \geq 1$, and $v \in R_d(u)$. Suppose an agent is on u and chooses a direction according to the procedure in Definition 2. Then, there is probability $1/(4d)$ that the corresponding direction-approximating path crosses v .*

We start defining the discrete version of Lévy walk, that we name Pareto walk.

Definition 3 (α -Pareto walk). Let $\alpha > 1$ be a real constant. At time $t = 0$, each agent chooses a distance $d \in \mathbb{N}$ with probability distribution $\frac{c_\alpha}{(1+d)^\alpha}$, where c_α is a normalization constant. Then, it chooses a direction according to the procedure in Definition 2, and walks along the corresponding direction-approximating path for d steps, reaching some node v . Once reached v , the agent repeats the procedure from the beginning, sampling a new distance and a new direction independently from the previous ones. If the chosen distance d is equal to zero, the agent keeps still for one time unit and then it repeats the procedure.

As discussed in the introduction, by varying α in the range $(1, +\infty)$, the Pareto walk “simulates” the behaviours of some popular movement models: the simple random walk (for $\alpha \geq 3$) and the ballistic walk (for $\alpha \in (1, 2]$), which we will define later. We are going to compare the search efficiency of each of these models as, essentially, a function of the number k of agents, the first hitting time, and the important notion of work we define below.

Definition 4 (Work). Let k be any positive integer. Suppose k agents move independently on the grid performing t_1, t_2, \dots, t_k steps, respectively. Then the (overall) work made by the agents is $\sum_{i=1}^k t_i$.

Notice that if k agents move on the grid for t steps, then the work is equal to $k \cdot t$. Informally, one crucial optimization aspect we consider here is to derive upper and lower bound on the best trade-off between the number k of agents and the hitting time of the corresponding parallel Pareto walk process. In this setting, we will show that the search efficiency of each of the above popular models is the same as that of the Pareto walk for a specific choice of $\alpha > 1$ (see Section 7 for more details). We first present a simple extension of the lower bound of [FK17] to the expected total work made by the agents performing *any* search algorithm.

Lemma 2 (Lower bound on the work). *Let ℓ be any real such that $\ell \geq 1$, and locate the treasure u.a.r. in one node of the infinite grid among those having distance at most ℓ from the origin. Then, for any $k \geq 1$, and for any search algorithm \mathcal{A} adopted by the k agents, the total average work required to find the treasure is $\Omega(\ell^2)$.*

Proof of Lemma 2. Suppose we have k agents looking for the treasure, and let H be the random variable denoting the first hitting time of the treasure. Let $t = \ell^2/(4k)$. Within time $2t$, the agents can cover at most $2kt = \ell^2/2$ nodes. Since the treasure is located u.a.r. on one out of ℓ^2 nodes, there is probability at least $1/2$ that it is not found within $2t$ time steps. Then,

$$\mathbb{E}[kH] = k \sum_{i \geq 1} i \mathbb{P}(H = i) \geq k \sum_{i > 2t} i \mathbb{P}(H = i) > 2kt \mathbb{P}(H > 2t) \geq \ell^2/4.$$

□

Our analysis shows that this lower bound can be achieved up to polylogarithmic factor, w.h.p., by the Pareto walk in some optimal setting, and in the different mobility models we are now defining.

The simple random walk is a well known process, but we are going to present a definition as well.

Definition 5 (*simple random walk*). At each round, the agent, located on node u , chooses one (grid) neighbor v of u u.a.r., and takes a step toward it.

The simple random walk is a “reliable” process to search for any specific node (i.e. the treasure) of the grid since it guarantees that with probability one the treasure will be found. On the other hand, this property is achieved at a very-high cost in terms of hitting time since it has high redundancy: the expected number of times the agent visits the same nodes is very large. As we will see in the next sections, the Pareto walk obviates this problem in the regime $\alpha \in (1, 3)$, having a smaller redundancy. Last, we also consider the ballistic walk which is the basic mobility model having redundancy zero.

Definition 6 (*Ballistic walk*). At the first round, the agent chooses a direction according to the procedure in Definition 2, and walks along the corresponding direction-approximating path forever, moving over one edge at each step.

We remark that the Ballistic walk process is just intended to challenge the search efficiency of the other processes, because an agent moving according to it finds the treasure in time exactly ℓ with probability $\Theta(1/\ell)$. If it does not find the treasure within that time, then it will never find it. As discussed in the introduction, the above three processes can be seen as particular cases of the Pareto walk process, for some specific choices of $\alpha > 1$.

Informally, if we set $\alpha \geq 3$, the Pareto walk behaves like a simple random walk; if instead $\alpha \in (1, 2]$, the ballistic walk is a good approximation of the Pareto walk. To provide concrete argument for the above claims we need some preliminaries. We name S_j the random variable denoting the j -th jump-length of a Pareto walk. Then, for the expected value of a jump-length, the following bound holds:

$$\mathbb{E}[S_j] = \sum_{d=0}^{\infty} \frac{c_\alpha d}{(1+d)^\alpha} = \Theta\left(\sum_{d=0}^{\infty} \frac{1}{(1+d)^{\alpha-1}}\right). \quad (1)$$

Notice that the latter bound is finite if $\alpha > 2$, and is infinite if $1 < \alpha \leq 2$. We can also calculate the variance of a jump length (which is well-defined only if the mean is finite) as follows:

$$\text{Var}(S_j) = \sum_{d=0}^{\infty} \frac{c_\alpha d^2}{(1+d)^\alpha} - \left[\sum_{d=0}^{\infty} \frac{c_\alpha d}{(1+d)^\alpha}\right]^2 = \Theta\left(\sum_{d=0}^{\infty} \frac{1}{(1+d)^{\alpha-2}}\right), \quad (2)$$

which takes finite values if $\alpha > 3$, and is infinite if $2 < \alpha \leq 3$. Looking at this threshold behaviour of the expectation and the variance is the key to show the equivalence between the various Pareto walks and the mobility models we focus on. In detail, if $1 < \alpha \leq 2$, within $\Theta(t)$ steps, the Pareto walk has moved to a distance $\Theta(t)$ from the origin, in average, since the expected step-length is infinite, like the ballistic walk. If $\alpha > 3$, the variance of the jump-length is finite, so the random jump-length is concentrated around its expectation: this fact makes this case of Pareto walks similar to a simple random walk with a longer step-size. Another intuition on the “equivalence” in this case is that, since the variance of a jump-length is finite, the distribution of the Pareto walk at the

i -th jump is a two-dimensional Gaussian distribution, due to the Central Limit Theorem, similarly to the case of simple random walk. The case $\alpha = 3$ is a threshold case in which we observe the transition from a super-diffusive regime to the diffusive one, yielding roughly the same results as the simple random walk. Such equivalences will be formally discussed in Section 4 (case $\alpha \in (2, 3]$) and Section 7 (cases $\alpha \in (1, 2]$ and $\alpha \geq 3$), showing that the work done by the Pareto walk and that of the corresponding mobility model (according to the choice of α) are essentially the same.

A useful fact we use several times throughout our analysis is the following. A single agent, performing a Pareto walk, chooses a jump of length at least $d \geq 0$ with probability

$$\mathbb{P}(S_1 \geq d) = \sum_{h=d}^{\infty} \frac{c_\alpha}{(1+h)^\alpha} = \Theta\left(\frac{1}{(1+d)^{\alpha-1}}\right), \quad (3)$$

where the last equation follows by applying the standard integral test (Fact 1 in the Appendix A).

We remark that most of the asymptotic bounds we will obtain in this paper hold *with high probability* (in short, *w.h.p.*) with respect to the parameter ℓ : as usual, considering increasing values of ℓ , we say that an event E holds w.h.p. if $\mathbb{P}(E) \geq 1 - 1/\ell^{\Theta(1)}$. Furthermore, each event whose probability is dominated by a factor $1/\ell^{\Theta(1)}$ is said to hold with *negligible* probability, while any other event is said to hold with *non-negligible* probability.

Finally, one remark on the meaning of the statements of the theorems we are going to present.

Remark 1. We emphasize that with the statement “if $k = \tilde{\Theta}(f(\ell))$, then k agents find the treasure in time $\tilde{\Theta}(g(\ell))$, w.h.p.”, we mean that there exists at least one k in the family $\tilde{\Theta}(f(\ell))$ such that the agents find the treasure in time $\tilde{\mathcal{O}}(g(\ell))$, w.h.p., and for all k in the above family, the treasure is found in time $\tilde{\Omega}(g(\ell))$ with probability at least $1 - o(1)$. For theorems 4 to 7, the lower bound actually holds w.h.p. Furthermore, from the respective analyses, it will be clear that

- (i) if we increase the optimal k by multiplying by polylogarithmic factors, the upper bound on the hitting time still holds w.h.p.;
- (ii) if we decrease the optimal k by dividing by polylogarithmic factors, the upper bound on the hitting time holds with non-negligible probability.

4 Pareto Walk Model: Case $\alpha \in (2, 3]$

The aim of this section is to analyze the Pareto walk model for the parameter range $\alpha \in (2, 3]$ on the infinite grid and prove the following results.

Theorem 2 (Performances of Pareto walks - Case $\alpha \in (2, 3)$). *Let $\alpha \in (2, 3)$ be a real constant and assume that the treasure is located in some node of the infinite grid at distance $\ell > 0$. Let k agents perform mutually independent Pareto walks with parameter α . If $k = \tilde{\Theta}(\ell^{1-(\alpha-2)})$, then the agents find the treasure in time $\tilde{\Theta}(\ell^{1+(\alpha-2)})$, making total work $\tilde{\Theta}(\ell^2)$, w.h.p.¹⁸ Furthermore, the result is almost-tight in a two-fold sense.*

- (i) *Let $k = \tilde{\Theta}(\ell^{1-(\alpha-2)-\epsilon})$ for an arbitrary constant $\epsilon \in (0, 1 - (\alpha - 2)]$. Then the treasure is never hit by the agents, w.h.p., letting the work to be infinite, w.h.p.*
- (ii) *Let $k = \tilde{\Theta}\left(\ell^{1-(\alpha-2)+\min(\epsilon+\frac{\alpha-2}{2}, \frac{3}{2}\epsilon)}\right)$ for an arbitrary constant $\epsilon > 0$. Then the treasure is found in time at least $\tilde{\Omega}(\ell^{1+(\alpha-2)-\epsilon})$, letting the work to be $\tilde{\Omega}\left(\ell^{2+\min(\frac{\alpha-2}{2}, \frac{\epsilon}{2})}\right)$, w.h.p.*

¹⁸For a formal comprehension of this statement, we refer to Remark 1 in the Preliminaries (Section 3).

The Pareto walk model for $\alpha = 3$ represents a “threshold” case since, as we will see in the next theorem, it shows the transition from the super-diffusive regime (for $\alpha \in (2, 3)$ the hitting time is sub-quadratic) to the diffusive one. Indeed, its hitting time has roughly the same distribution of that of the simple random walks. However, its analysis cannot rely on the same arguments we will use for the simple random walks since the variance of the jump length is infinite. Instead, we will exploit the same tools used for the case of $\alpha \in (2, 3)$ leading to slightly different results from that of the simple random walks. This technical closeness leads us to present the analysis of the two cases above in the same section.

Theorem 3 (Performances of Pareto walks - Case $\alpha = 3$). *Assume that the treasure is located in some node of the infinite grid at distance $\ell > 0$. Let k agents perform mutually independent Pareto walks with parameter $\alpha = 3$. If $k = \log^{\mathcal{O}(1)} \ell$, then k agents find the treasure in time $t = \tilde{\Theta}(\ell^2)$, making a total work of $\tilde{\Theta}(\ell^2)$, w.h.p.¹⁹ Furthermore, the result is almost-tight in the following sense: if $k = \tilde{\Theta}\left(\ell^{\min(\epsilon + \frac{\alpha-2}{2}, \frac{3}{2}\epsilon)}\right)$ for any arbitrary constant $\epsilon > 0$, then k agents find the treasure in time at least $\tilde{\Omega}(\ell^{2-\epsilon})$, letting the work to be $\tilde{\Omega}\left(\ell^{2+\min(\frac{1}{2}, \frac{\epsilon}{2})}\right)$, w.h.p.*

4.1 Proofs of Theorem 2 and Theorem 3: main tools and general scheme

The random process $\{\mathbf{P}_t\}_{t \in \mathbb{N}}$, where \mathbf{P}_t represents coordinates of a Pareto walk at round t , is not a Markov chain over the state space \mathbb{Z}^2 : \mathbf{P}_t in fact depends not only on \mathbf{P}_{t-1} , but, also, on the state of the agent, and its previous jump decision. Furthermore, it is hard to understand which nodes are actually visited during one iteration of the procedure in Definition 3.

To address the issues above we consider two simpler movement models, the Pareto flight (the discrete version of the well-known Lévy flight, which also has a per-se interest) and the Pareto run, and provide bounds on the performances of such models. Then, using coupling arguments, we show how such bounds can be exploited to get similar bounds for the original Pareto walk model.

Definition 7 (α -Pareto flight). Let $\alpha > 1$ be a real constant. At each round, the agent chooses a distance d with distribution $\frac{c_\alpha}{(1+d)^\alpha}$, where c_α is a normalization constant, and chooses u.a.r. one node u among the $4d$ nodes of the grid at distance d from its current position. Then, in one step/unit time, the agent reaches u . Once reached u , the agent repeats the procedure above, and so on. If the chosen distance d is equal to zero, the agent keeps still for one time unit and then it repeats the procedure.

By defining the two-dimensional random variable \mathbf{P}_t^f as the coordinates of the node the Pareto flight visits at time t , we can easily observe that the process $\{\mathbf{P}_t^f\}_{t \in \mathbb{N}}$ is a Markov chain over the state space \mathbb{Z}^2 . This important property will be exploited in Subsection 4.2 to prove the following result.

Proposition 1 (Hitting time of Pareto flight - Case $\alpha \in (2, 3]$). *Consider a single agent that performs t steps of the Pareto flight with parameter $\alpha \in (2, 3]$. For some $t = \Theta(\ell^{\alpha-1})$, conditional on the event that the lengths of all performed jumps are less than $(t \log t)^{\frac{1}{\alpha-1}}$, the agent finds the treasure within the t -th step, with probability*

$$(i) \quad \Omega\left(\left(\ell^{1-(\alpha-2)}(\log \ell)^{\frac{2}{\alpha-1}}\right)^{-1}\right) \text{ if } \alpha \neq 3;$$

$$(ii) \quad \Omega\left((\log^4 \ell)^{-1}\right) \text{ if } \alpha = 3.$$

The next proposition shows a useful coupling between the Pareto flight process the Pareto walk one: its proof will be given in Subsection 4.4.

¹⁹For a formal comprehension of this statement, we refer to Remark 1 in the Preliminaries (Section 3).

Proposition 2 (Coupling between Pareto flight and Pareto walk - Case $\alpha \in (2, 3]$). *Suppose an agent performing the Pareto flight with any $\alpha \in (2, 3]$ finds the treasure within t steps with probability $p = p(t) > 0$, conditional on the event that all the performed jump lengths are less than $(t \log t)^{\frac{1}{\alpha-1}}$. Then, another agent that performs the Pareto walk, with the same parameter α , finds the treasure within $\Theta(t)$ steps with probability at least $[1 - \mathcal{O}(1/\log t)] \cdot [p(t) - \exp(-t^{\Theta(1)})]$, without any conditioning event.*

Informally, thanks to the above coupling, we can transform the upper bound in Proposition 1 on the Pareto flight hitting time into an (unconditional) upper bounds on the Pareto walk hitting time. We now need an equivalent framework to derive a lower bound for the Pareto walk. To this aim, we define another similar process, the Pareto run, which we show to be at least as efficient as the Pareto walk.

Definition 8 (α -Pareto run). Let $\alpha > 1$ be a real constant. At each round, the agent chooses a distance d with distribution $\frac{c_\alpha}{(1+d)^\alpha}$, where c_α is a normalization constant, and chooses a direction according to the procedure in Definition 2. Then, it walks along the corresponding direction-approximating path (visiting all the path nodes) *in one step/unit time* until it reaches the end-point v of the path at distance d . Once v is reached, the agent repeats all the procedure, and so on. If the chosen distance d is equal to zero, the agent keeps still for one time unit and then it repeats the procedure.

Similarly to Pareto flight, let \mathbf{P}_t^r be the two-dimensional random variable denoting the coordinates of the node a Pareto run visits at time t . It is easy to see that it is a Markov chain on the space \mathbb{Z}^2 . Furthermore, we know that at each iteration of the procedure in Definition 8, the agent takes just one time unit to visit all the nodes of the chosen path: this allows us to avoid dealing with the time needed to cover the path. For an agent performing a Pareto run, the following holds.

Proposition 3 (Hitting time of Pareto run - Case $\alpha \in (2, 3]$). *Let a single agent perform a Pareto run with $\alpha \in (2, 3]$.*

- (i) *If $\alpha \neq 3$, the agent never finds the treasure with probability $1 - \mathcal{O}(\log \ell / \ell^{1-(\alpha-2)})$;*
- (ii) *let $c \geq 0$ be any arbitrary constant, and let t be any function in $\Theta(\ell^{1+(\alpha-2)}/(\log^c \ell))$. Then, the probability the agent finds the treasure within time t is $\mathcal{O}(1/(\ell^{1-(\alpha-2)} \log^c \ell))$;*
- (iii) *for an arbitrary constant $\epsilon > 0$, the agent finds the treasure within time $\Theta(\ell^{\alpha-1-\epsilon})$ with probability:*
 - (a) $\mathcal{O}(1/\ell^{1-(\alpha-2)+\min(\epsilon+(\alpha-2), 2\epsilon)})$ if $\alpha \neq 3$;
 - (b) $\mathcal{O}(\log \ell / \ell^{\min(\epsilon+1, 2\epsilon)})$ if $\alpha = 3$.

Subsection 4.3 is devoted to the proof of the above proposition, while in Subsection 4.4 we give the proof of the next proposition that links the Pareto run to the Pareto walk.

Proposition 4 (Coupling: Pareto run into Pareto walk). *Suppose an agent a_1 that moves according to the Pareto run with parameter $\alpha > 1$ finds the treasure within t steps with probability $p > 0$. Then, another agent a_2 that moves according to the Pareto walk with the same parameter α finds the treasure within $\Omega(t)$ steps with probability p . Furthermore, if a_1 never finds the treasure with probability q , then a_2 never finds the treasure with probability at least q .*

Thanks to the above results, we can now prove the main result on the Pareto walks for $\alpha \in (2, 3)$. The proof for the case $\alpha = 3$ follows in the successive subsection.

4.1.1 Wrap-up I: proof of Theorem 2

As for the claimed upper bound on the hitting time, the proof proceeds as follows. From Proposition 1 and Proposition 2, we get that a single agent, that moves according to the Pareto walk and perform $t = \Theta(\ell^{1+(\alpha-2)})$ steps, finds the treasure within the t -th step with probability at least $\Omega\left(\left(\ell^{1-(\alpha-2)}(\log \ell)^{\frac{2}{\alpha-1}}\right)^{-1}\right)$. Then, if we take k mutually-independent Pareto walk agents, with

$$k = \Theta\left(\ell^{1-(\alpha-2)}(\log \ell)^{\frac{2}{\alpha-1}} \log \ell\right) = \Theta\left(\ell^{1-(\alpha-2)}(\log \ell)^{\frac{\alpha+1}{\alpha-1}}\right),$$

the treasure will be found by at least one of such agents within time $t = \Theta(\ell^{1+(\alpha-2)})$, w.h.p., making a total work of $\tilde{\mathcal{O}}(\ell^2)$. Furthermore, if we increase the number of agents by multiplying by any polylogarithmic factor, the same upper bound on the hitting time holds, w.h.p., while if we decrease it by dividing by any polylogarithmic factor, the upper bound on the hitting time holds with non-negligible probability.

As for the lower bound, if we set the number of agents to be $\Theta(\ell^{1-(\alpha-2)} \log^c \ell)$, then, for Proposition 3, we have that the agent will not find the treasure within time $t = \Theta(\ell^{1+(\alpha-2)}/\log^{2c} \ell)$ with probability

$$\left[1 - \mathcal{O}\left(\frac{1}{\ell^{1-(\alpha-2)} \log^{2c} \ell}\right)\right]^{\Theta(\ell^{1-(\alpha-2)} \log^c \ell)} = \exp\left(-\mathcal{O}\left(\frac{1}{\log^c \ell}\right)\right) = 1 - \mathcal{O}\left(\frac{1}{\log^c \ell}\right),$$

where the last equality comes from the Taylor's expansion. Then, the agents need time at least $\tilde{\Omega}(\ell^{1+(\alpha-2)})$ to find the treasure with probability $1 - o(1)$, making a total work of $\tilde{\Omega}(\ell^2)$. On the other hand, if we decrease the number of agents to be $k = \Theta(\ell^{1-(\alpha-2)}/\log^c \ell)$ for some $c > 0$, then, trivially, the hitting time cannot improve, with at least the same probability.

As for Claim (i), propositions 3 and 4 tell us that an agent performing a Pareto walk has probability $\mathcal{O}(\log \ell / \ell^{1-(\alpha-2)})$ to eventually find the treasure. Let $\epsilon \in (0, 1 - (\alpha - 2)]$ be a constant, and $k = \tilde{\Theta}(\ell^{1-(\alpha-2)-\epsilon})$. Then, k agents ever find the treasure with probability $\tilde{\mathcal{O}}(1/\ell^\epsilon)$ (thus the work is infinite, w.h.p.).

As for Claim (ii), let $\epsilon > 0$ be any positive constant, and $k = \tilde{\Theta}\left(\ell^{1-(\alpha-2)+\min(\epsilon+\frac{\alpha-2}{2}, \frac{3}{2}\epsilon)}\right)$. An agent performing the Pareto walk has probability $\mathcal{O}(1/\ell^{1-(\alpha-2)+\min(\epsilon+(\alpha-2), 2\epsilon)})$ to find the treasure within time $t = \Theta(\ell^{1+(\alpha-2)-\epsilon})$, for propositions 3 and 4. Thus, $k = \tilde{\Theta}\left(\ell^{1-(\alpha-2)+\min(\epsilon+\frac{\alpha-2}{2}, \frac{3}{2}\epsilon)}\right)$ agents performing mutually independent Pareto walks find the treasure in time $\tilde{\Omega}(\ell^{1+(\alpha-2)-\epsilon})$, w.h.p. for the union bound, letting the work to be $\Omega\left(\ell^{2+\min(\frac{\alpha-2}{2}, \frac{\epsilon}{2})}\right)$.

4.1.2 Wrap-up II: proof of Theorem 3

From Proposition 1 and Proposition 2, we get that a single agent, that moves according to the Pareto walk and perform $t = \Theta(\ell^2)$ steps, finds the treasure within the t -th step with probability at least $\Omega\left((\log^4 \ell)^{-1}\right)$. Then, if we take k mutually-independent Pareto walk agents, with

$$k = \Theta(\log^5 \ell),$$

the treasure will be found by at least one of such agents within time $t = \Theta(\ell^2)$, w.h.p., making a total work of $\tilde{\mathcal{O}}(\ell^2)$. Furthermore, if we increase the number of agents by multiplying by any polylogarithmic factor, the same upper bound on the hitting time holds, w.h.p., while if we decrease it by dividing by any polylogarithmic factor, the upper bound on the hitting time holds with non-negligible probability.

As for the lower bound, if we set the number of agents to be $\Theta(\ell^{1-(\alpha-2)} \log^c \ell)$, then, for Proposition 3, we have that the agent will not find the treasure within time $t = \Theta(\ell^{1+(\alpha-2)}/\log^{2c} \ell)$ with probability

$$\left[1 - \mathcal{O}\left(\frac{1}{\ell^{1-(\alpha-2)} \log^{2c} \ell}\right)\right]^{\Theta(\ell^{1-(\alpha-2)} \log^c \ell)} = \exp\left(-\mathcal{O}\left(\frac{1}{\log^c \ell}\right)\right) = 1 - \mathcal{O}\left(\frac{1}{\log^c \ell}\right),$$

where the last equality comes from the Taylor's expansion. Then, the agents need time at least $\tilde{\Omega}(\ell^{1+(\alpha-2)})$ to find the treasure with probability $1 - o(1)$, making a total work of $\tilde{\Omega}(\ell^2)$. On the other hand, if we decrease the number of agents to be $k = \Theta(\ell^{1-(\alpha-2)}/\log^c \ell)$ for some $c > 0$, then, trivially, the hitting time cannot improve with at least the same probability.

As for the almost-tightness result, let $\epsilon > 0$ be any positive constant, and $k = \tilde{\Theta}(\ell^{\min(\epsilon+\frac{1}{2}, \frac{3}{2}\epsilon)})$. An agent performing the Pareto walk has probability $\mathcal{O}(\log \ell / \ell^{\min(\epsilon+1, 2\epsilon)})$ to find the treasure within time $t = \Theta(\ell^{2-\epsilon})$ for propositions 3 and 4. Thus, $k = \tilde{\Theta}(\ell^{\min(\epsilon+\frac{1}{2}, \frac{3}{2}\epsilon)})$ agents performing mutually independent Pareto walks find the treasure in time $\tilde{\Omega}(\ell^{2-\epsilon})$, w.h.p. for the union bound, letting the work to be $\Omega(\ell^{2+\min(\frac{1}{2}, \frac{\epsilon}{2})})$.

4.2 Pareto flight model - case $\alpha \in (2, 3]$: proof of Proposition 1

In this subsection, we analyze the Pareto flight for any fixed $\alpha \in (2, 3]$ and prove Proposition 1 that, as discussed in the previous subsection, represents the key-ingredient to derive the upper bounds on the performances of the Pareto walk model. We proceed as follows.

We consider the following sequence of random variables

$$S_i = \text{length of the } i\text{-th jump of the agent, } i = 1, \dots, t.$$

For each $i = 1, \dots, t$, we define the events

$$E_i = \{S_i < (t \log t)^{\frac{1}{\alpha-1}}\} \text{ and } E(t) = \bigcap_{i=1}^t E_i.$$

For brevity's sake, we write $E(t) = E$ when the dependency from t is clear from context. Then, for any node $u = (u_x, u_y)$ of the grid, we also define the random variable

$$Z_u(t) = \text{number of agent's visits at node } u \text{ within } t \text{ steps.}$$

In order to bound the probability that the node u has been visited at least once at time t , namely $\mathbb{P}(Z_u(t) > 0 \mid E)$, we define the following agent's spatial distribution

$$p_{u,i} = \mathbb{P}(\text{the agent is in node } u \text{ at step } i \mid E),$$

and we note that

$$\mathbb{E}[Z_u(t) \mid E] = \sum_{i=0}^t p_{u,i}. \quad (4)$$

4.2.1 Road-map of the analysis

The key idea of our approach is to estimate the expected number of visits on the treasure the agent does within the first t steps of the Pareto flight process, where we recall that the treasure is

an arbitrary node of the infinite grid located at distance $\ell > 0$ from the origin. We first prove a “monotonicity” property of the agent’s spatial distribution of Pareto flight: this property essentially implies that, on average, the agent visits nodes “closer” to the origin more frequently than those who are “farther”. Second, we make a partition of the grid in three concentric regions: the first region, named \mathcal{A}_1 , consists of all the nodes having distance from the origin smaller than ℓ ; the second one, named \mathcal{A}_2 , consists of all nodes whose distance from the origin is roughly ℓ and at most a logarithmic factor farther; finally, the third region, named \mathcal{A}_3 , consists of all other (farther) nodes. Then, we exploit the monotonicity property of the agent’s spatial distribution to get some upper bounds on the average number of agent’s visits in each of the three regions, until time t . In particular, thanks to the monotonicity property, we show that the above upper bound for the region \mathcal{A}_2 can be written as the average number of visits to the treasure times the number of nodes in \mathcal{A}_2 . Being the total number of agent’s visits (within the first t steps) clearly equal to t , the obtained upper bounds for the three regions allow us to derive a lower bound on the average number of visits to the treasure itself. This expectation is then used to provide a lower bound on the probability that the treasure has been visited at least once within the first t rounds. Indeed, we prove that the latter probability is related to the average number of visits to the treasure and the average number of visits to the origin within time t .

Recall that the treasure is located in some node T at distance ℓ from the origin. We name the three regions in the following way. The first one is

$$\mathcal{A}_1 = Q(\ell) = \{(x, y) : \max(|x|, |y|) \leq \ell\};$$

the second region is defined for any $t = \Theta(\ell^{\alpha-1})$ and is (remember that α is fixed)

$$\mathcal{A}_2 = \{v \in \mathbb{Z}^2 : |v|_1 \leq 2(t \log t)^{\frac{1}{\alpha-1}} \text{ if } \alpha \in (2, 3), \quad |v|_1 \leq 2\sqrt{t} \log t \text{ if } \alpha = 3\} \setminus \mathcal{A}_1;$$

and the third one, defined for any $t = \Theta(\ell^{\alpha-1})$ as well, is

$$\mathcal{A}_3 = \{v \in \mathbb{Z}^2 : |v|_1 > 2(t \log t)^{\frac{1}{\alpha-1}} \text{ if } \alpha \in (2, 3), \quad |v|_1 > 2\sqrt{t} \log t \text{ if } \alpha = 3\}.$$

What follows is a road-map of the analysis to keep track of main idea behind the lemmas and the proofs that follow. The analysis proceeds along the following technical steps.

1. The spatial distribution $p_{u,t}$ has a useful monotonicity property that can be stated as follows. For any $u = (u_x, u_y)$, define $d_u = |u_x| + |u_y|$. Then, we show that $p_{u,t} \geq p_{v,t}$ for all nodes v lying outside the square $Q(d_u) = \{(x', y') : \max(|x'|, |y'|) \leq d_u\}$ (see Lemma 3 and Figure 3 for details).
2. Thanks to eq. (4) and Step 1 above, we get that, for a node u , $\mathbb{E}[Z_u(t) | E] \geq \mathbb{E}[Z_v(t) | E]$ for all nodes v outside $Q(d_u)$ (see Corollary 1). Thus, the average number of visits in \mathcal{A}_2 is at most the cardinality of \mathcal{A}_2 times $\mathbb{E}[Z_T(t) | E]$, thus upper bounded by $\mathbb{E}[Z_T(t) | E] \cdot 4(t \log t)^{\frac{2}{\alpha-1}}$ if $\alpha \in (2, 3)$, and by $\mathbb{E}[Z_T(t) | E] \cdot 4t \log^2 t$ if $\alpha = 3$.
3. For some $t = \Theta(\ell^{\alpha-1})$ we get the following upper bound on the average number of visits to region \mathcal{A}_1 : $\mathbb{E}\left[\sum_{v \in Q(\ell)} Z_v(t) | E\right] \leq ct$ for some constant $c \in (0, 1)$ (see Lemma 4).
4. Using Chebyshev, we get the following upper bound to the average number of visits to region \mathcal{A}_3 : we show that (see Lemma 5)

$$\sum_{\substack{v=(x,y): \\ |x|+|y| \geq 2(t \log t)^{\frac{1}{\alpha-1}}}} \mathbb{E}[Z_v(t) | E] = \mathcal{O}\left(\frac{t}{\log t}\right) \text{ if } \alpha \in (2, 3);$$

$$\sum_{\substack{v=(x,y): \\ |x|+|y|\geq 2\sqrt{t}\log t}} \mathbb{E}[Z_v(t) | E] = \mathcal{O}\left(\frac{t}{\log t}\right) \text{ if } \alpha = 3;$$

5. From steps 2 to 4, for some $t = \Theta(\ell^{\alpha-1})$ we have (see Lemma 6)

$$ct + \mathbb{E}[Z_T(t) | E] \cdot 4(t \log t)^{\frac{2}{\alpha-1}} + \mathcal{O}\left(\frac{t}{\log t}\right) \geq t \text{ if } \alpha \in (2, 3);$$

$$ct + \mathbb{E}[Z_T(t) | E] \cdot 4t \log^2 t + \mathcal{O}\left(\frac{t}{\log t}\right) \geq t \text{ if } \alpha = 3.$$

These expressions imply the followings:

$$\mathbb{E}[Z_T(t) | E] = \Omega\left(\frac{1}{t^{\frac{3-\alpha}{\alpha-1}} [\log(t)]^{\frac{2}{\alpha-1}}}\right) \text{ if } \alpha \in (2, 3);$$

$$\mathbb{E}[Z_T(t) | E] = \Omega\left(\frac{1}{\log^2 t}\right) \text{ if } \alpha = 3.$$

6. Using Step 2, we prove that the expected number of visits in the origin $\mathbb{E}[Z_o(t) | E]$ is a positive constant w.r.t. t , i.e., $\mathbb{E}[Z_o(t) | E] = a_t(\alpha) = \Theta(1)$, for any choice of $\alpha \in (2, 3)$, while this expectation grows at most as a poly-logarithmic function for the case $\alpha = 3$, i.e. $a_t(3) = \mathcal{O}(\log^2 t)$ (see Lemma 7 for more details).

Furthermore, by simple calculations (Lemma 8) we link the probability to visit a node u within time t with the average number of visits to that node within time t and $a_t(\alpha)$, showing that

$$\mathbb{E}[Z_u(t) | E] \geq \mathbb{P}(Z_u(t) > 0 | E) \geq \mathbb{E}[Z_u(t) | E] / a_t(\alpha).$$

7. From steps 5 and 6, for some $t = \Theta(\ell^{\alpha-1})$, we get that

$$\mathbb{P}(Z_T(t) > 0 | E) = \Omega\left(\frac{1}{t^{\frac{3-\alpha}{\alpha-1}} [\log(t)]^{\frac{2}{\alpha-1}}}\right) \text{ if } \alpha \in (2, 3), \text{ and}$$

$$\mathbb{P}(Z_T(t) > 0 | E) = \Omega\left(\frac{1}{\log^4 t}\right) \text{ if } \alpha = 3,$$

which yield the claim of Proposition 1 substituting $t = \Theta(\ell^{\alpha-1})$ (see Corollary 2).

4.2.2 Full analysis

The spatial distribution yielded by the Pareto flight process has a useful geometric shape (described in Step 1) that can be “roughly” characterized as follows. For any node $u = (u_x, u_y)$, we let $d_u = |u_x| + |u_y|$ and consider the square

$$Q(d_u) = \{(x', y') \in \mathbb{Z}^2 : \max(|x'|, |y'|) \leq d_u\}$$

(See Figure 3 in the proof of Lemma 3). Then, the following geometric property holds.

Lemma 3 (Monotonicity). *Let $u \in \mathbb{Z}^2$ be an arbitrary node. Then, for each node $v \notin Q(d_u)$ and each step t , it holds that $p_{u,t} \geq p_{v,t}$.*

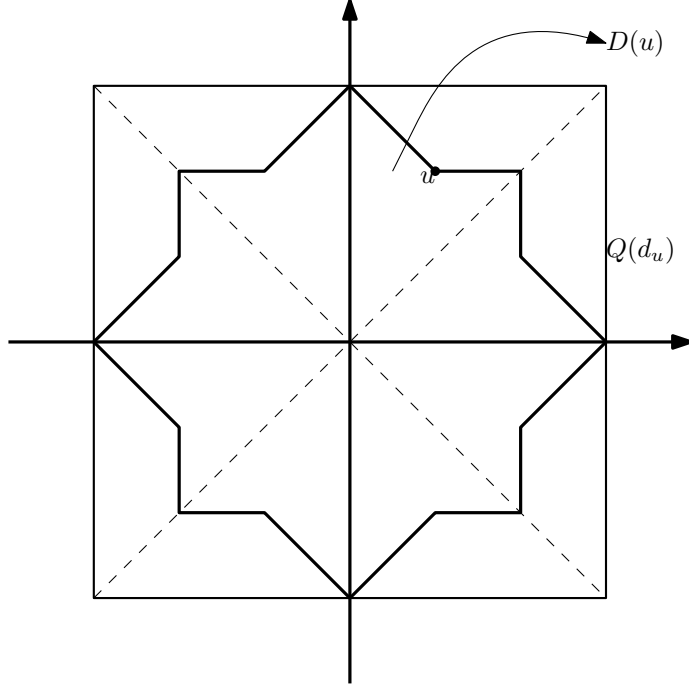


Figure 3: The set $D(u)$, consisting in all inner nodes of the “star”, and the square $Q(d_u)$.

Proof of Lemma 3. The proof follows from Lemma 32 in Appendix C, observing that the Pareto flight model satisfies the hypothesis of that lemma, and that the conditioning event E does not interfere with the proof. \square

Notice that, from $\mathbb{E}[Z_v(t) | E] = \sum_{i=0}^t p_{v,i}$, we have the following corollary (described in Step 2 in the scheme), whose proof is trivial.

Corollary 1. $\mathbb{E}[Z_u(t) | E] \geq \mathbb{E}[Z_v(t) | E]$ for all $v \notin Q(d_u)$.

Namely, the more the node is “distant” (according to the sequence of squares $\{Q(d)\}_{d \in \mathbb{N}}$) from the origin, the less it is visited in average. Thus, each node is visited at most as many times as the origin, in average.

We now divide the grid in three subset, namely

$$\begin{aligned} \mathcal{A}_1 &= Q(\ell) = \{(x, y) : \max(|x|, |y|) \leq \ell\}, \\ \mathcal{A}_2 &= \{v \in \mathbb{Z}^2 : |v|_1 \leq 2(t \log t)^{\frac{1}{\alpha-1}} \text{ if } \alpha \in (2, 3), \quad |v|_1 \leq 2\sqrt{t} \log t \text{ if } \alpha = 3\} \setminus \mathcal{A}_1, \\ \mathcal{A}_3 &= \{v \in \mathbb{Z}^2 : |v|_1 > 2(t \log t)^{\frac{1}{\alpha-1}} \text{ if } \alpha \in (2, 3), \quad |v|_1 > 2\sqrt{t} \log t \text{ if } \alpha = 3\}, \end{aligned}$$

where \mathcal{A}_2 and \mathcal{A}_3 are defined for any $t = \Theta(\ell^{\alpha-1})$ (remember that α is fixed). Then, we estimate some upper bound on the expected number of visits the agent makes in the first and the third of these sets until time t , considering that the sum of these upper bounds must be at least t . As for the nodes inside $Q(\ell)$, the following holds (Step 3 in the scheme).

Lemma 4 (Bound on visits in \mathcal{A}_1). *For some $t = \Theta(\ell^{\alpha-1})$, it holds that*

$$\sum_{v \in Q(\ell)} \mathbb{E}[Z_v(t) | E] \leq ct$$

for some constant $c \in (0, 1)$.

Proof of Lemma 4. Let u , t , and $Q(\ell)$ be as in the hypotheses. We estimate the probability the walk has moved to distance $\frac{5}{2}\ell$ at least once within time $t = \Theta(\ell^{\alpha-1})$ bounding it by the probability that at least one of the jumps made is no less than 5ℓ (event H). Indeed, if there is a jump of length at least 5ℓ , the walk moves necessarily to distance no less than $\frac{5}{2}\ell$.

$$\begin{aligned} \mathbb{P}\left(S_j \geq 5\ell \mid S_j < (t \log t)^{\frac{1}{\alpha-1}}\right) &= \sum_{k=5\ell}^{(t \log t)^{\frac{1}{\alpha-1}} - 1} \frac{c_\alpha}{(1+k)^\alpha} \\ &\stackrel{(a)}{\geq} \frac{c_\alpha}{\alpha-1} \left(\frac{1}{(5\ell+1)^{\alpha-1}} - \frac{1}{t \log t} \right) \\ &\stackrel{(b)}{\geq} \frac{c_\alpha}{2(\alpha-1)(5\ell)^{\alpha-1}}, \end{aligned}$$

where (a) holds for the integral test (Fact 1 in Appendix A), and (b) holds for ℓ large enough, since $t = \Theta(\ell^{\alpha-1})$. Due to independence, there exists some constant $c' > 0$ such that there is probability

$$1 - \left[1 - \frac{c_\alpha}{2(\alpha-1)(5\ell)^{\alpha-1}} \right]^{c' \frac{2(\alpha-1)(5\ell)^{\alpha-1}}{c_\alpha}} \geq \frac{3}{4}$$

that the desired jump takes place within time $c' \cdot 2(\alpha-1)(5\ell)^{\alpha-1}/c_\alpha$. If we take t to be $t \geq 4c' \cdot 2(\alpha-1)(5\ell)^{\alpha-1}/c_\alpha$, the desired jump takes place probability $\frac{3}{4}$ within time $\frac{t}{4}$. Once reached such a distance (conditioning on such an event), there are at least other 3 disjoint zones at least as equally likely as $Q(\ell)$ to be visited at any future time, as Figure 4 shows. Thus, the probability to

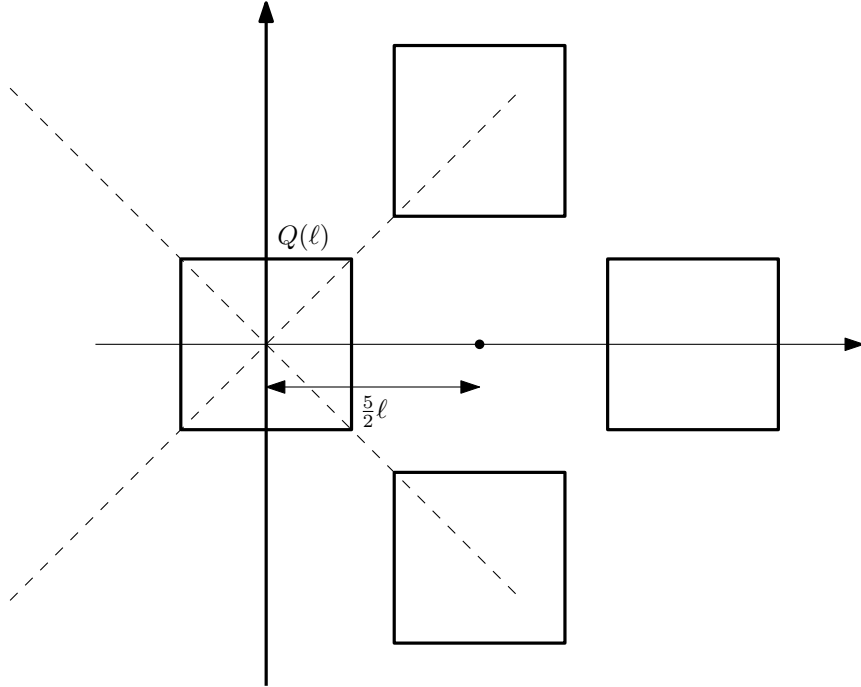


Figure 4: The disjoint zones at least as equally likely as $Q(\ell)$ to be visited.

visit $Q(\ell)$ at any future time is at most $\frac{1}{4}$. To get the thesis, observe that

$$\mathbb{E} \left[\sum_{v \in Q(\ell)} Z_v(t) \mid E \right] = \mathbb{E} \left[\sum_{v \in Q(\ell)} Z_v(t) \mid H, E \right] \mathbb{P}(H \mid E) + \mathbb{E} \left[\sum_{v \in Q(\ell)} Z_v(t) \mid H^C, E \right] \mathbb{P}(H^C \mid E)$$

$$\begin{aligned}
&\leq \left(\frac{1}{4}t + \frac{1}{4} \cdot \frac{3}{4}t \right) \frac{3}{4} + t \cdot \frac{1}{4} \\
&= \frac{t}{4} \left(1 + \frac{3}{4} + \frac{9}{16} \right) \\
&= \frac{37}{64}t,
\end{aligned}$$

which, setting $c = \frac{37}{64}$, is the thesis. \square

Next lemma focuses on the third set, \mathcal{A}_3 (Step 4 in the scheme).

Lemma 5 (Bound on visits in \mathcal{A}_3). *For $\alpha \in (2, 3]$, it holds that*

$$\sum_{\substack{v=(x,y): \\ |x|+|y| \geq 2(t \log t)^{\frac{1}{\alpha-1}}}} \mathbb{E}[Z_v(t) \mid E] = \mathcal{O}\left(\frac{t}{\log t}\right) \quad \text{if } \alpha \in (2, 3); \quad (5)$$

$$\sum_{\substack{v=(x,y): \\ |x|+|y| \geq 2\sqrt{t} \log t}} \mathbb{E}[Z_v(t) \mid E] = \mathcal{O}\left(\frac{t}{\log t}\right) \quad \text{if } \alpha = 3;. \quad (6)$$

Proof of Lemma 5. Let $\mathbf{P}_{t'}^f$ be the two dimensional random variable representing the coordinates of the agent performing the Pareto flight at time t' . Consider the projection of the Lévy flight on the x -axis, namely the random variable $X_{t'}$ such that $\mathbf{P}_{t'}^f = (X_{t'}, Y_{t'})$. The random variable $X_{t'}$ can be expressed as the sum of t' random variables S_j^x , $j = 1, \dots, t'$, representing the jumps (with sign) the projection of the walk takes at each of the t' rounds. The partial distribution of the jumps along the x -axis, conditioning on the event E , is given by the following. For any $0 \leq d \leq (t \log t)^{\frac{1}{\alpha-1}} - 1$

$$\mathbb{P}\left(S_j^x = \pm d \mid S_j < (t \log t)^{\frac{1}{\alpha-1}}\right) = \frac{c_\alpha}{4d(1+d)^\alpha} \mathbb{1}_{d \neq 0} + \frac{c_\alpha}{2^\alpha} \mathbb{1}_{d=0} + \sum_{k=1+d}^{(t \log t)^{\frac{1}{\alpha-1}} - 1} \frac{c_\alpha}{2k(1+k)^\alpha}, \quad (7)$$

where $\mathbb{1}_{d \in A}$ yields 1 if $d \in A$ and 0 otherwise, the term $\frac{c_\alpha}{4d(1+d)^\alpha} \mathbb{1}_{d \neq 0} + \frac{c_\alpha}{2^\alpha} \mathbb{1}_{d=0}$ is the probability that the original jump lies along the horizontal axis and has “length” exactly d (or $-d$), and the terms $\frac{c_\alpha}{2k(1+k)^\alpha}$ are the probability that the original jump has “length” exactly k and its projection on the horizontal axis has “length” d (or $-d$). Quantity (7) is at least

$$\frac{c_\alpha}{4} \left(\frac{1}{d(1+d)^\alpha} + \sum_{k=1+d}^{(t \log t)^{\frac{1}{\alpha-1}} - 1} \frac{2}{k(1+k)^\alpha} \right),$$

and at most

$$c_\alpha \left(\frac{1}{d(1+d)^\alpha} + \sum_{k=1+d}^{(t \log t)^{\frac{1}{\alpha-1}} - 1} \frac{2}{k(1+k)^\alpha} \right).$$

By the integral test (Fact 1 in A) we know that this probability is

$$\mathbb{P}(S_j^x = \pm d \mid E_j) = \Theta\left(\frac{1}{(1+d)^\alpha}\right).$$

Due to symmetry, it is trivial to see that $\mathbb{E}[X_{t'} | E] = 0$ for each time t' , while

$$\text{Var}(X_{t'} | E) = \sum_{i=1}^{t'} \text{Var}(S_i^x | E_i) = t' \text{Var}(S_1^x | E_1)$$

because $S_1^x, \dots, S_{t'}^x$ are i.i.d. Suppose now $\alpha \in (2, 3)$. The variance of S_1^x conditioned to the event $E_1 = \left\{ S_1 < (t \log t)^{\frac{1}{\alpha-1}} \right\}$ is

$$\begin{aligned} \text{Var}(S_1^x | E_1) &\leq \sum_{k=1}^{(t \log t)^{\frac{1}{\alpha-1}} - 1} \mathcal{O}\left(\frac{k^2}{(1+k)^\alpha}\right) \\ &\stackrel{(a)}{=} \mathcal{O}\left(\frac{1}{3-\alpha} \left[(t \log t)^{\frac{3-\alpha}{\alpha-1}} - 2^{3-\alpha} \right]\right) + \mathcal{O}\left(\frac{1}{2^{\alpha-2}}\right) \\ &= \mathcal{O}\left(t \log t\right)^{\frac{3-\alpha}{\alpha-1}}, \end{aligned}$$

where, in (a), we used the integral test (Fact 1 in Appendix A). Observe that the event $E = \bigcap_{i=1}^t E_i$ has probability

$$\mathbb{P}(E) = 1 - \mathcal{O}\left(\frac{1}{\log t}\right).$$

Then, for each $t' \leq t$, from the Chebyshev's inequality and the fact that $\mathbb{E}[X_{t'} | E] = 0$,

$$\mathbb{P}\left(|X_{t'}| \geq (t \log t)^{\frac{1}{\alpha-1}} | E\right) \leq \frac{t' \text{Var}(S_1^x | E_1)}{(t \log t)^{\frac{2}{\alpha-1}}} \leq \frac{t \text{Var}(S_1^x | E_1)}{(t \log t)^{\frac{2}{\alpha-1}}} = \mathcal{O}\left(\frac{1}{\log t}\right),$$

which implies that

$$\mathbb{P}\left(|X_{t'}| \geq (t \log t)^{\frac{1}{\alpha-1}}\right) \leq \mathbb{P}\left(|X_{t'}| \geq (t \log t)^{\frac{1}{\alpha-1}} | E\right) + \mathbb{P}(E^C) = \mathcal{O}\left(\frac{1}{\log t}\right).$$

Then, the probability that both $X_{t'}$ and $Y_{t'}$ are less than $(t \log t)^{\frac{1}{\alpha-1}}$ (call the events $A_{x,t'}$ and $A_{y,t'}$, respectively) is

$$\mathbb{P}(A_{x,t'} \cap A_{y,t'}) = \mathbb{P}(A_{x,t'}) + \mathbb{P}(A_{y,t'}) - \mathbb{P}(A_{x,t'} \cup A_{y,t'}) \geq 1 - \mathcal{O}\left(\frac{1}{\log t}\right),$$

for $t' \leq t$. Then, let $Z'(t)$ indicate the number of times the Pareto flight visits the set of nodes whose coordinates are both no less than $(t \log t)^{\frac{1}{\alpha-1}}$ until time t . Then,

$$\mathbb{E}[Z'(t) | E] \leq \sum_{\substack{v=(x,y) \\ |x|+|y| \geq 2(t \log t)^{\frac{1}{\alpha-1}}}} \mathbb{E}[Z_v(t) | E],$$

and

$$\begin{aligned} \mathbb{E}[Z'(t) | E] &= \sum_{i=0}^t \mathbb{E}[Z'(i) | A_{x,i} \cap A_{y,i}, E] \mathbb{P}(A_{x,i} \cap A_{y,i} | E) \\ &\quad + \sum_{i=0}^t \mathbb{E}[Z'(i) | (A_{x,i} \cap A_{y,i})^C, E] \mathbb{P}((A_{x,i} \cap A_{y,i})^C | E) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^t \mathbb{E} [Z'(i) \mid (A_{x,i} \cap A_{y,i})^C, E] \mathbb{P}((A_{x,i} \cap A_{y,i})^C \mid E) \\
&\leq t \cdot \mathcal{O}\left(\frac{1}{\log t}\right) = \mathcal{O}\left(\frac{t}{\log t}\right),
\end{aligned}$$

which leads to eq. (5).

If $\alpha = 3$, the variance of S_1^x conditioned to E_1 becomes $\mathcal{O}(\log(t \log t))$. Then we look at the probability that $|X_{t'}|$ is at least $\sqrt{t} \cdot \log t$ conditional to E , which is, again, $\mathcal{O}(1/\log t)$. Then the proof proceeds in exactly the same way, obtaining eq. (6). \square

Let T be the node in which the treasure is located. For each node v in \mathcal{A}_2 we already know that $\mathbb{E}[Z_T(t) \mid E] \geq \mathbb{E}[Z_v(t) \mid E]$ thanks to Corollary 1. Then, we have the following (Step 5 in the scheme).

Lemma 6. *Let $\alpha \in (2, 3)$, and let u be any node such that $d_u = \ell$. For some $t = \Theta(\ell^{\alpha-1})$, the following holds:*

$$ct + \mathbb{E}[Z_T(t) \mid E] \cdot 4(t \log t)^{\frac{2}{\alpha-1}} + \mathcal{O}\left(\frac{t}{\log t}\right) \geq t \quad \text{if } \alpha \in (2, 3); \quad (8)$$

$$ct + \mathbb{E}[Z_T(t) \mid E] \cdot 4t \log^2 t + \mathcal{O}\left(\frac{t}{\log t}\right) \geq t \quad \text{if } \alpha = 3. \quad (9)$$

Proof of Lemma 6. Suppose the agent has made $t = \Theta(\ell^{\alpha-1})$ jumps (the same t of Lemma 4), thus visiting t nodes. Then,

$$\mathbb{E} \left[\sum_{v \in \mathcal{Z}^2} Z_v(t) \mid E \right] = t.$$

First, we focus on eq. (8). From Lemma 4, the number of visits inside $\mathcal{A}_1 = Q(\ell)$ until time t is at most ct , for some constant $c \in (0, 1)$. From Lemma 5, the number of visits in \mathcal{A}_3 is at most $\mathcal{O}(t/\log t)$. Each of the remaining nodes, i.e. the nodes in \mathcal{A}_2 , which are at most $4(t \log t)^{\frac{2}{\alpha-1}}$ in total, is visited by the agent at most $\mathbb{E}[Z_T(t) \mid E]$ times, for Corollary 1. Then, we have that

$$ct + \mathbb{E}[Z_T(t) \mid E] \cdot 4(t \log t)^{\frac{2}{\alpha-1}} + \mathcal{O}\left(\frac{t}{\log t}\right) \geq t.$$

Now we prove eq. (9). We proceed in the exactly same way, noticing that, from Lemma 5, the number of visits in \mathcal{A}_2 is at most $\mathbb{E}[Z_T(t) \mid E] \cdot (4t \log^2 t)$. This gives eq. (9). \square

The following two results aim at relating the probability to hit a node u within time t to the average number of visits to the origin and to the average number of visits to u itself (Step 6 in the scheme). The first lemma we show estimate the average number of visits to the origin.

Lemma 7 (Visits in the origin). *For any $t \geq 0$ and $\alpha \in (2, 3]$, let $\mathbb{E}[Z_o(t) \mid E] = a_t(\alpha)$. We have that*

1. if $\alpha \in (2, 3)$, then $a_t(\alpha) = \Theta(1)$;
2. if $\alpha = 3$, then $a_t(3) = \mathcal{O}(\log^2 t)$.

Proof of Lemma 7. We first go for Step 1. We want to prove that $\mathbb{E}[Z_o(t) | E]$ is constant. This involves an intricate argument. Since $\mathbb{E}[Z_o(t) | E] = \sum_{k=1}^t p_{o,k}$, it suffices to accurately bound the probability $p_{o,k}$ for each $k = 1, \dots, t$. Let partition the natural numbers in the following way

$$\mathbb{N} = \bigcup_{t'=0}^{\infty} \left[\mathbb{N} \cap [2t' \log t', 2(t'+1) \log(t'+1)) \right].$$

For each $k \in \mathbb{N}$, there exists t' such that $k \in [2t' \log t', 2(t'+1) \log(t'+1))$. Then, within $2t' \log t'$ steps the walk has moved to distance $\lambda = \frac{(t')^{\frac{1}{\alpha-1}-1}}{2}$ at least once, with probability $\Omega\left(\frac{1}{(t')^2}\right)$. Indeed, if there is one jump of length at least 2λ , we are sure the walk has moved to distance at least λ from the origin. We compute the probability that one jump is at least 2λ . For the integral test, we have

$$\begin{aligned} \mathbb{P}\left(S_j \geq 2\lambda \mid S_j < (t \log t)^{\frac{1}{\alpha-1}}\right) &\geq \frac{1}{\mathbb{P}\left(S_j < (t \log t)^{\frac{1}{\alpha-1}}\right)} \left[\int_{2\lambda}^{(t \log t)^{\frac{1}{\alpha-1}-1}} \frac{c_\alpha}{(s+1)^\alpha} ds \right] \\ &\geq \frac{c_\alpha}{\alpha-1} \left(\frac{1}{t'} - \frac{1}{t \log t} \right) \\ &\geq \frac{c_\alpha}{\alpha-1} \left(\frac{1 - \frac{t'}{t \log t}}{t'} \right) \\ &\geq \frac{c_\alpha}{\alpha-1} \left(\frac{1 - \frac{1}{2 \log(t') \log t}}{t'} \right) \\ &= \Omega\left(\frac{1}{t'}\right) \end{aligned}$$

where the last inequality holds because $2t' \log t' \leq t$. Thus, the probability that the first $2t' \log t'$ jumps are less than 2λ is

$$\begin{aligned} \mathbb{P}\left(\bigcap_{j=1}^{2t' \log t'} \{S_j < 2\lambda\} \mid E\right) &\stackrel{(a)}{=} \left[1 - \mathbb{P}\left(S_1 < 2\lambda \mid S_1 < (t \log t)^{\frac{1}{\alpha-1}} - 1\right) \right]^{2t' \log t'} \\ &\geq \left[1 - \Omega\left(\frac{1}{t'}\right) \right]^{2t' \log t'} \\ &= \mathcal{O}\left(\frac{1}{(t')^2}\right), \end{aligned}$$

where (a) holds for independence. Once reached such a distance, there are at least $\lambda^2 = \Omega\left(\frac{2}{(t')^{\frac{2}{\alpha-1}}}\right)$ different nodes that are at least as equally likely as o to be visited at any given future time (from Lemma 3). Thus, the probability to reach the origin at any future time is at most $\mathcal{O}\left(\frac{1}{(t')^{\frac{2}{\alpha-1}}}\right) = \mathcal{O}\left(\frac{1}{(t')^{1+\epsilon}}\right)$ for some small constant $\epsilon > 0$, in particular the bounds holds for $p_{o,k}$. Observe that in an interval $[2t' \log t', 2(t'+1) \log(t'+1))$ there are

$$2(t'+1) \log(t'+1) - 2t' \log t' = 2t' \left[\log\left(1 + \frac{1}{t'}\right) \right] + 2 \log(t'+1) = \mathcal{O}(\log t')$$

integers. Let \mathbf{P}_t^f be the two-dimensional random variable denoting the node a Pareto flight which has started in the origin visits at time t , and let $H_{t'}$ be the event $\bigcup_{j=1}^{2t' \log t'} \{S_j \geq 2\lambda\}$. Observe that

$$p_{o,k} = \mathbb{P}\left(\mathbf{P}_t^f = o \mid H_{t'}, E\right) \mathbb{P}(H_{t'} \mid E) + \mathbb{P}\left(\mathbf{P}_t^f = o \mid H_{t'}^C, E\right) \mathbb{P}(H_{t'}^C \mid E),$$

by the law of total probability. Thus, if $I_{t'} = [2t' \log t', 2(t' + 1) \log(t' + 1))$, we have

$$\begin{aligned} \sum_{k=1}^t p_{o,k} &\leq \sum_{t'=0}^{\infty} \sum_{k \in I_{t'}} p_{o,k} \\ &\leq \sum_{t'=0}^{\infty} \left[\mathbb{P}(\mathbf{P}_t^f = o \mid H_{t'}, E) \mathbb{P}(H_{t'} \mid E) + \mathbb{P}(\mathbf{P}_t^f = o \mid H_{t'}^C, E) \mathbb{P}(H_{t'}^C \mid E) \right] \mathcal{O}(\log t') \\ &\leq \sum_{t'=0}^{\infty} \left[\mathcal{O}\left(\frac{1}{(t')^{1+\epsilon}}\right) + \mathcal{O}\left(\frac{1}{(t')^2}\right) \right] \mathcal{O}(\log t') = \mathcal{O}(1). \end{aligned}$$

Finally, the fact that this quantity is at least a constant comes from observing that $p_{o,1}$ is constant.

As for Step 2, notice that in the previous proof we have to put $\lambda = \sqrt{t'}$ and the proof follows almost analogously, giving that the average number of visits until time t is of the order of $\mathcal{O}(\log^2 t)$. \square

The following results relate the probability to hit some node u within some time t with the expected number of visits to that node and to the average number of visits to the origin (Step 6 in the scheme).

Lemma 8. *Let $u \in \mathbb{Z}^2$ be any node. Then, we have the followings.*

- (i) $\mathbb{E}[Z_u(t) \mid E] \leq a_t(\alpha)$,
- (ii) $1 \leq \mathbb{E}[Z_u(t) \mid Z_u(t) > 0, E] \leq a_t(\alpha)$,
- (iii) $\mathbb{E}[Z_u(t) \mid E] / a_t(\alpha) \leq \mathbb{P}(Z_u(t) > 0 \mid E) \leq \mathbb{E}[Z_u(t) \mid E]$.

Proof of Lemma 8. Item (i) comes directly from Item (ii), since $\mathbb{E}[Z_u(t) \mid Z_u(t) > 0, E] \geq \mathbb{E}[Z_u(t) \mid E]$. Consider Item (ii), and let τ be the first time the agent visits u . Then, conditioning on $Z_u(t) > 0$, τ is at most t . We have

$$\mathbb{E}[Z_u(t) \mid Z_u(t) > 0, E] = \mathbb{E}[Z_o(t - \tau) \mid \tau \leq t, E] \leq \mathbb{E}[Z_o(t) \mid E] = a_t(\alpha).$$

The fact that this expectation is at least 1 comes from the conditional event. As for Item (iii), let us explicitly express the term $\mathbb{E}[Z_u(t) \mid Z_u(t) > 0, E] \cdot \mathbb{P}(Z_u(t) > 0 \mid E)$. This is equal to

$$\begin{aligned} &\sum_{i=1}^t i \mathbb{P}(Z_u(t) = i \mid Z_u(t) > 0, E) \cdot \mathbb{P}(Z_u(t) > 0 \mid E) \\ &= \sum_{i=1}^t i \frac{\mathbb{P}(Z_u(t) = i, Z_u(t) > 0, E)}{\mathbb{P}(Z_u(t) > 0, E)} \cdot \frac{\mathbb{P}(Z_u(t) > 0, E)}{\mathbb{P}(E)} \\ &= \sum_{i=1}^t i \frac{\mathbb{P}(Z_u(t) = i, Z_u(t) > 0, E)}{\mathbb{P}(E)} \\ &= \sum_{i=1}^t i \mathbb{P}(Z_u(t) = i \mid E) \\ &= \mathbb{E}[Z_u(t) \mid E]. \end{aligned}$$

Then,

$$\mathbb{E}[Z_u(t) \mid E] \geq \mathbb{P}(Z_u(t) > 0 \mid E) = \frac{\mathbb{E}[Z_u(t) \mid E]}{\mathbb{E}[Z_u(t) \mid Z_u(t) > 0, E]} \geq \frac{\mathbb{E}[Z_u(t) \mid E]}{a_t(\alpha)},$$

since $\mathbb{E}[Z_u(t) \mid Z_u(t) > 0, E] \leq a_t(\alpha)$ for the Item (ii). \square

4.2.3 Wrap-up: proof of Proposition 1

From Lemma 6, we can easily compute the expectation $\mathbb{E}[Z_u(t) | E]$ and, thus, the sought probability (Step 7 in the scheme).

Corollary 2. *Let T be the node in which the treasure is located. For some $t = \Theta(\ell^{\alpha-1})$, the probability the agent visits the treasure at least once within time t , conditioning to the event E , is*

$$\begin{aligned} \mathbb{P}(Z_T(t) > 0 | E) &= \Omega\left(\frac{1}{t^{\frac{3-\alpha}{\alpha-1}} [\log(t)]^{\frac{2}{\alpha-1}}}\right) \quad \text{if } \alpha \in (2, 3); \\ \mathbb{P}(Z_T(t) > 0 | E) &= \Omega\left(\frac{1}{\log^4 t}\right) \quad \text{if } \alpha = 3. \end{aligned}$$

Proof of Corollary 2. The proof easily follows by combining lemmas 6 to 8. \square

We are finally ready to prove Proposition 1. Indeed, it is sufficient to apply Corollary 2 substituting $t = \Theta(\ell^{\alpha-1}) = \Theta(\ell^{1+(\alpha-2)})$.

4.3 Pareto run model - case $\alpha \in (2, 3]$: proof of Proposition 3

The Pareto run is at least as efficient as the Pareto walk in finding the treasure, indeed it can be seen as a Pareto walk which takes just a time unit to perform a jump, visiting all the nodes in the path chosen to reach the jump-destination. In this section we find lower bounds on the hitting time of the treasure for the Pareto run, and we exploit its characteristics to gain the same lower bound for the Pareto walk. The result we are going to prove is the following.

Proposition 3 (*Hitting time of Pareto run - Case $\alpha \in (2, 3]$*). *Let a single agent perform a Pareto run with $\alpha \in (2, 3]$.*

- (i) *If $\alpha \neq 3$, the agent never finds the treasure with probability $1 - \mathcal{O}(\log \ell / \ell^{1-(\alpha-2)})$;*
- (ii) *let $c \geq 0$ be any arbitrary constant, and let t be any function in $\Theta(\ell^{1+(\alpha-2)} / (\log^c \ell))$. Then, the probability the agent finds the treasure within time t is $\mathcal{O}(1 / (\ell^{1-(\alpha-2)} \log^c \ell))$;*
- (iii) *for an arbitrary constant $\epsilon > 0$, the agent finds the treasure within time $\Theta(\ell^{\alpha-1-\epsilon})$ with probability:*
 - (a) $\mathcal{O}(1 / \ell^{1-(\alpha-2)+\min(\epsilon+(\alpha-2), 2\epsilon)})$ if $\alpha \neq 3$;
 - (b) $\mathcal{O}(\log \ell / \ell^{\min(\epsilon+1, 2\epsilon)})$ if $\alpha = 3$.

Now, a little analysis to prove the above result. First, we state again Lemma 1 which we will use.

Lemma 1. *Let u be any node of \mathbb{Z}^2 , $d \geq 1$, and $v \in R_d(u)$. Suppose an agent is on u and chooses a direction according to the procedure in Definition 2. Then, there is probability $1/(4d)$ that the corresponding direction-approximating path crosses v .*

Then, we estimate the probability that, during a given jump which starts at some distance from some node u , the Pareto run actually visits u .

Lemma 9. *Consider an agent performing a Pareto run which is located at distance $d \geq 0$ from any node u . Then, the probability that it visits u during the next jump is $\Theta(1/d^\alpha)$.*

Proof of Lemma 9. For Lemma 1, the probability to choose a direction leading to u is $\Theta(1/d)$. Independently, the probability to choose to walk for a distance at least d across the chosen direction is $\Theta(1/d^{\alpha-1})$ by Equation (3). Thus, the probability to eventually visit u is $\Theta(1/d^\alpha)$. \square

Through the above lemma, we find the probability that during *any* given jump the agent actually visits the treasure.

Lemma 10. *Let $i = 1, \dots, t$, denote the t jumps an agent performing a Pareto run with parameter $\alpha \in (2, 3]$ which starts at the origin takes. The probability that during the i -th jump the agent finds the treasure is $\mathcal{O}(1/\ell^2)$.*

Proof of Lemma 10. Consider the starting point v of the i -th jump. We want to give probabilities to the distance at which v is from the origin. Call T the node in which the treasure is located, and consider the rhombus centered in T of nodes that are distant at most $\frac{\ell}{4}$ from T , namely

$$R_{\ell/4}^*(T) = \{w \in \mathbb{Z}^2 : d(w, T) \leq \ell/4\}.$$

For any $v \in R_{\ell/4}^*(T)$, the probability that the i -th jump starts in v is at most $\mathcal{O}(1/\ell^2)$ due to Lemma 3. At the same time, for any distance $1 \leq d \leq \ell/4$, there are at most $4d$ nodes in $R_{\ell/4}^*(T)$ at distance d from T . Then, for the expression of conditional probability and Lemma 9, the probability that the i -th jump starts from $R_{\ell/4}^*(T)$ and the agent visits the treasure during the jump is

$$\mathcal{O}\left(\frac{1}{\ell^2}\right) \sum_{d=1}^{\ell/4} 4d \cdot \mathcal{O}\left(\frac{1}{d^\alpha}\right) + \mathcal{O}\left(\frac{1}{\ell^2}\right) = \mathcal{O}\left(\frac{1}{\ell^2}\right),$$

where, in the first expression, the last term $\mathcal{O}(1/\ell^2)$ is the contribution of T itself. If v is outside $R_{\ell/4}^*(T)$, then the probability that a jump that starts from v leads the agent to visit the treasure is at most $\mathcal{O}(1/\ell^\alpha)$.

Let J_i be the event that the i -th jump (which starts in v) leads the agent to visit the treasure, and V_i be the event that the starting point of the i -th jump is in $R_{\ell/4}^*(T)$. Then, recalling that $2 < \alpha \leq 3$,

$$\mathbb{P}(J_i) \leq \mathbb{P}(J_i | V_i) \mathbb{P}(V_i) + \mathbb{P}(J_i | V_i^C) \leq \mathcal{O}\left(\frac{1}{\ell^2}\right) + \mathcal{O}\left(\frac{1}{\ell^\alpha}\right) = \mathcal{O}\left(\frac{1}{\ell^2}\right),$$

which is the thesis. \square

We are ready to give a lower bound on the probability that an agent performing a Pareto run never finds the treasure.

Lemma 11. *Let $\alpha \in (2, 3)$, and consider a single agent performing a Pareto run (Definition 8). The probability that the agent never finds the treasure is $1 - \mathcal{O}(\log \ell / \ell^{1-(\alpha-2)})$.*

Proof of Lemma 11. Consider the first time t_i the agent is at distance at least $\lambda_i = 2^i \ell$ from the origin, for each $i \geq 1$. Define, for $i \geq 1$, $\tau_i = 2\lambda_i^{\alpha-1} \log \lambda_i$. Then,

$$\mathbb{P}(t_i \leq \tau_i) \geq 1 - \left[1 - \mathcal{O}\left(\frac{1}{\lambda_i^{\alpha-1}}\right)\right]^{2\lambda_i^{\alpha-1} \log \lambda_i} = 1 - \mathcal{O}\left(\frac{1}{\lambda_i^2}\right) = 1 - \mathcal{O}\left(\frac{1}{2^{2i} \ell^2}\right).$$

Then, the expected number of visits to the treasure from time t_i until time t_{i+1} is then $\mathcal{O}(\tau_{i+1}/\lambda_i^2) = \mathcal{O}(\tau_i/\lambda_i^2)$ by Lemma 10, since the agent starts at distance $\Theta(\lambda_i)$ from the target. At the same time,

the average number of visits to the treasure until time t_1 is $\mathcal{O}(t_1/\ell^2) = \mathcal{O}(\tau_1/\ell^2)$. Combining the above, we have that the expected total number of visits to the treasure is

$$\mathcal{O}(\tau_1/\ell^2) + \sum_{i \geq 1} \mathcal{O}(\tau_i/\lambda_i^2) = \mathcal{O}(\log \ell / \ell^{3-\alpha}) + \sum_{i \geq 1} \mathcal{O}(\log(2^i \ell) / (2^i \ell)^{3-\alpha}) = \mathcal{O}(\log \ell / \ell^{3-\alpha}).$$

Thus, for the Markov property, the probability that the agent visits the treasure at least once is $\mathcal{O}(\log \ell / \ell^{3-\alpha})$, which is equal to $\mathcal{O}(\log \ell / \ell^{1-(\alpha-2)})$. \square

We also give a bound on the probability that an agent finds the treasure in time $\mathcal{O}(\ell^{1+(\alpha-2)} / (\log^c \ell))$, for any constant $c \geq 0$.

Lemma 12. *Consider a single agent performing a Pareto run with parameter $\alpha \in (2, 3]$ (Definition 8). Let $c \geq 0$ be any arbitrary constant, and let t be any function in $\Theta(\ell^{1+(\alpha-2)} / (\log^c \ell))$. Then, the probability to find the treasure within time t is $\mathcal{O}(1 / (\ell^{1-(\alpha-2)} \log^c \ell))$.*

Proof of Lemma 12. By Lemma 10 and the union bound, the expected number of visits to the treasure until time t is then $\mathcal{O}(t/\ell^2) = \mathcal{O}(1 / (\ell^{1-(\alpha-2)} \log^c \ell))$, since the agent starts at distance $\Theta(\ell)$ from the treasure. Then, for the Markov property, the sought probability is $\mathcal{O}(1 / (\ell^{1-(\alpha-2)} \log^c \ell))$. \square

At the same time, in the next lemma we show that the agent finds the treasure within time polynomially smaller than $\Theta(\ell^{1+(\alpha-2)})$ with really low probability.

Lemma 13. *Let $\alpha \in (2, 3)$, and consider a single agent performing a Pareto run (Definition 8). Let $\epsilon > 0$ be any arbitrary small constant, and let t be any function in $\Theta(\ell^{1+(\alpha-2)-\epsilon})$. Then, the probability to find the treasure within time t is*

$$\mathcal{O}\left(\frac{1}{\ell^{1-(\alpha-2)+\min(\epsilon+(\alpha-2), 2\epsilon)}}\right) \quad \text{if } \alpha \neq 3; \quad (10)$$

$$\mathcal{O}\left(\frac{\log \ell}{\ell^{\min(\epsilon+1, 2\epsilon)}}\right) \quad \text{if } \alpha = 3. \quad (11)$$

Proof of Lemma 13. Let X_i be the x -coordinate of the agent at the end of the i -th jump. For any $i \leq t$, we bound the probability that $X_i > \ell/4$. The probability that there is a jump whose length is at least ℓ among the first i jumps is $\mathcal{O}(i/\ell^{\alpha-1})$. First we consider the case in which $\alpha \in (2, 3)$. Conditioning on the event that the first i jump-length are all smaller than ℓ (event C_i), the expectation of X_i is zero and its variance is

$$i \cdot \sum_{d=0}^{\ell/4} \Theta(d^2 / (1+d)^\alpha) = \Theta(i\ell^{3-\alpha}),$$

for the integral test (Fact 1 in Appendix A). Chebyshev's inequality implies that

$$\mathbb{P}(|X_i| \geq \ell/4 \mid C_i) \leq \frac{\Theta(i\ell^{3-\alpha})}{\Theta(\ell^2)} = \Theta(i/\ell^{\alpha-1}).$$

Since the conditioning event has probability $1 - \mathcal{O}(i/\ell^{\alpha-1})$, then the ‘‘unconditional’’ probability that of the event $|X_i| \leq \ell/4$ is

$$[1 - \mathcal{O}(i/\ell^{\alpha-1})]^2 = 1 - \mathcal{O}(1/\ell^\epsilon),$$

since $i \leq t = \Theta(\ell^{\alpha-1-\epsilon})$. The same result holds analogously for Y_i (the y -coordinate of the agent after the i -th jump), obtaining that $|X_i| + |Y_i| \leq \ell/2$ with probability $1 - \mathcal{O}(1/\ell^\epsilon)$ by the union bound.

Consider the first jump. The probability it leads the agent to visit the treasure is $\mathcal{O}(1/\ell^\alpha)$ for Lemma 9. Now, let $2 \leq i \leq t$ and consider the i -th jump. We want to estimate the probability the jump leads the agent to visit the treasure. As in the proof of Lemma 10, we call T the node in which the treasure is located, and we consider the rhombus centered in T that contains the nodes at distance at most $\frac{\ell}{4}$ from T , namely

$$R_{\ell/4}^*(T) = \{w \in \mathbb{Z}^2 : d(w, T) \leq \ell/4\}.$$

We call J_i the event that the i -th jump leads the agent to visit the treasure, K_{i-1} the event that the $(i-1)$ -th jump ends in $R_{\ell/4}^*(T)$, and F_{i-1} the event that the $(i-1)$ -th jump ends at distance farther than $\ell/2$ from the origin. First, let \mathbf{P}_i^r be the two-dimensional random variable denoting the coordinates of the node the agent is located on after the i -th jump. Then

$$\begin{aligned} \mathbb{P}(J_i | K_{i-1}) \mathbb{P}(K_{i-1} | F_{i-1}) &= \sum_{v \in R_{\ell/4}^*(T)} \mathbb{P}(J_i | \mathbf{P}_i^r = v) \mathbb{P}(\mathbf{P}_i^r = v | F_{i-1}) \\ &\leq \mathcal{O}\left(\frac{1}{\ell^2}\right) \sum_{v \in R_{\ell/4}^*(T)} \mathbb{P}(J_i | \mathbf{P}_i^r = v), \end{aligned}$$

since Lemma 3 holds in a consequent way conditioning on F_{i-1} , and since, for each $v \in R_{\ell/4}^*(T)$, there are at least $\Theta(\ell^2)$ nodes at distance at least $\ell/2$ from the origin which are more probable to be the destination of the i -th jump than v . Then, we proceed like in the proof of Lemma 10 and we obtain

$$\mathbb{P}(J_i | K_{i-1}) \mathbb{P}(K_{i-1} | F_{i-1}) = \mathcal{O}\left(\frac{1}{\ell^2}\right). \quad (12)$$

By the law of total probabilities, we have

$$\begin{aligned} \mathbb{P}(J_i) &= \mathbb{P}(J_i | F_{i-1}) \mathbb{P}(F_{i-1}) + \mathbb{P}(J_i | F_{i-1}^C) \mathbb{P}(F_{i-1}^C) \\ &= [\mathbb{P}(J_i | F_{i-1}, K_{i-1}) \mathbb{P}(K_{i-1} | F_{i-1}) + \mathbb{P}(J_i | F_{i-1}, K_{i-1}^C) \mathbb{P}(K_{i-1}^C | F_{i-1})] \mathbb{P}(F_{i-1}) \\ &\quad + \mathbb{P}(J_i | F_{i-1}^C) \mathbb{P}(F_{i-1}^C) \\ &\stackrel{(a)}{\leq} [\mathbb{P}(J_i | K_{i-1}) \mathbb{P}(K_{i-1} | F_{i-1}) + \mathbb{P}(J_i | F_{i-1}, K_{i-1}^C)] \mathbb{P}(F_{i-1}) + \mathbb{P}(J_i | F_{i-1}^C) \mathbb{P}(F_{i-1}^C) \\ &\stackrel{(b)}{\leq} \left[\mathcal{O}\left(\frac{1}{\ell^2}\right) + \mathcal{O}\left(\frac{1}{\ell^\alpha}\right) \right] \mathcal{O}\left(\frac{1}{\ell^\epsilon}\right) + \mathcal{O}\left(\frac{1}{\ell^\alpha}\right) \left[1 - \mathcal{O}\left(\frac{1}{\ell^\epsilon}\right)\right] = \mathcal{O}\left(\frac{1}{\ell^{2+\epsilon}} + \frac{1}{\ell^\alpha}\right) \end{aligned} \quad (13)$$

where in (a) we used that $K_{i-1} \subset F_{i-1}$ and that $\mathbb{P}(K_{i-1}^C | F_{i-1}) \leq 1$, while in (b) we used eq. (12), and that $\mathbb{P}(J_i | F_{i-1}, K_{i-1}^C) = \mathcal{O}(1/\ell^\alpha)$, which is true because the jump starts in a node whose distance from the treasure is $\Omega(\ell)$, and that $\mathbb{P}(J_i | F_{i-1}^C) = \mathcal{O}(1/\ell^\alpha)$, which is true for the same reason.

Thus, by the union bound and by inequality (13), the probability that at least one between the t jumps leads the agent to find the treasure is

$$\begin{aligned} \frac{1}{\ell^\alpha} + (t-1) \mathcal{O}\left(\frac{1}{\ell^\alpha} + \frac{1}{\ell^{2+\epsilon}}\right) &= \mathcal{O}(\ell^{\alpha-1-\epsilon}) \mathcal{O}\left(\frac{1}{\ell^\alpha} + \frac{1}{\ell^{2+\epsilon}}\right) \\ &= \mathcal{O}\left(\frac{1}{\ell^{1+\epsilon}} + \frac{1}{\ell^{3-\alpha+2\epsilon}}\right) \end{aligned}$$

$$= \mathcal{O}\left(\frac{1}{\ell^{1-(\alpha-2)+\min(\epsilon+(\alpha-2), 2\epsilon)}}\right),$$

which is eq. (10).

Consider now the case in which $\alpha = 3$. The proof proceeds exactly as before, but the variance of X_i is $\Theta(i \log \ell)$. This means that the probability that $|X_i|$ is at least $\ell/4$ conditional to C_i is $\mathcal{O}(\log \ell/\ell^2)$, and the “unconditional” probability that $|X_i|$ is less than $\ell/4$ is $1 - \mathcal{O}(\log \ell/\ell^\epsilon)$. It follows that

$$\mathbb{P}(J_i) = \mathcal{O}\left(\frac{\log \ell}{\ell^{2+\epsilon}} + \frac{\log \ell}{\ell^\alpha}\right).$$

Thus, the sought probability results in

$$\mathcal{O}\left(\frac{\log \ell}{\ell^{\min(\epsilon+1, 2\epsilon)}}\right),$$

which is eq. (11). □

Finally, we have that lemmas 11 and 13 imply Proposition 3, which is necessary for the main result on the Pareto walk.

4.3.1 Wrap-up: proof of Proposition 3

Lemma 11 immediately gives Step i of the proposition. Lemma 12 is Step ii, while Lemma 13 gives Step iiia and Step iiib.

4.4 Coupling results

In this subsection, we show the proof of the coupling propositions, namely propositions 2 and 4. We start with Proposition 2, which we recall.

Proposition 2 (Coupling between Pareto flight and Pareto walk - Case $\alpha \in (2, 3]$). *Suppose an agent performing the Pareto flight with any $\alpha \in (2, 3]$ finds the treasure within t steps with probability $p = p(t) > 0$, conditional on the event that all the performed jump lengths are less than $(t \log t)^{\frac{1}{\alpha-1}}$. Then, another agent that performs the Pareto walk, with the same parameter α , finds the treasure within $\Theta(t)$ steps with probability at least $[1 - \mathcal{O}(1/\log t)] \cdot [p(t) - \exp(-t^{\Theta(1)})]$, without any conditioning event.*

Proof of Proposition 2. Let S_j be the random variable denoting the j -th jump-length. First, we bound a jump-length in the following way:

$$\begin{aligned} \mathbb{P}(S_j \geq d) &= c_\alpha \sum_{k \geq d} \frac{1}{(1+k)^\alpha} \leq c_\alpha \left[\frac{1}{(\alpha-1)(1+d)^{\alpha-1}} + \frac{1}{(1+d)^\alpha} \right] \\ &= c_\alpha \left[\frac{1+d+\alpha-1}{(\alpha-1)(1+d)^\alpha} \right] \leq c_\alpha \left[\frac{(\alpha-1)(1+d)}{(\alpha-1)(1+d)^\alpha} \right] \\ &= c_\alpha \frac{1}{(1+d)^{\alpha-1}}, \end{aligned}$$

where the first inequality holds for the integral test (Fact 1 in Appendix A). Thus, we have

$$\mathbb{P}\left(S_j \geq (t \log t)^{\frac{1}{\alpha-1}}\right) \leq c_\alpha \frac{1}{(1 + (t \log t)^{\frac{1}{\alpha-1}})^{\alpha-1}} \leq c_\alpha \frac{1}{t \log t}.$$

Denote by E_j the event $\left\{S_j < (t \log t)^{\frac{1}{\alpha-1}}\right\}$, and by E the intersection of E_j for $j = 1, \dots, t$. Note that the probability of E is $1 - \mathcal{O}(1/\log t)$ by the union bound. We want to apply the multiplicative form of Chernoff bound to the sum of S_j , conditioning on the event E . This is possible since the variable $S_j / \left((t \log t)^{\frac{1}{\alpha-1}} - 1\right)$ takes values in $[0, 1]$.

Let us now compute the expectation of the sum of the random variables S_j , for $j = 1, \dots, t$ conditioned to E .

$$\begin{aligned} \mathbb{E} \left[\sum_{j=1}^t S_j \mid E \right] &= \sum_{j=1}^t \mathbb{E}[S_j \mid E] = t \frac{c_\alpha}{\mathbb{P}(E)} \sum_{d=0}^{(t \log t)^{\frac{1}{\alpha-1}} - 1} \frac{d}{(1+d)^\alpha} \\ &\leq 2c_\alpha t \sum_{d=1}^{(t \log t)^{\frac{1}{\alpha-1}} - 1} \frac{1}{(1+d)^{\alpha-1}} \\ &\stackrel{(a)}{\leq} 2c_\alpha t \left[\frac{1}{\alpha-2} \left(\frac{1}{2^{\alpha-2}} - \frac{1}{(t \log t)^{\frac{\alpha-2}{\alpha-1}}} \right) + \frac{1}{2^{\alpha-1}} \right] \\ &\leq 2c_\alpha \frac{\alpha t}{(\alpha-2)2^{\alpha-1}} = \Theta(t), \end{aligned}$$

where (a) holds for the integral test (Fact 1 in Appendix A). Similarly, it holds that

$$\begin{aligned} \mathbb{E} \left[\sum_{j=1}^t S_j \mid E \right] &= \sum_{j=1}^t \mathbb{E}[S_j \mid E] = t \frac{c_\alpha}{\mathbb{P}(E_j)} \sum_{d=0}^{(t \log t)^{\frac{1}{\alpha-1}} - 1} \frac{d}{(1+d)^\alpha} \\ &\geq c_\alpha t \sum_{d=1}^{(t \log t)^{\frac{1}{\alpha-1}} - 1} \frac{1}{2(1+d)^{\alpha-1}} = \Theta(t), \end{aligned}$$

because the harmonic $(\alpha-1)$ -series converges. Call $\mu = \mathbb{E} \left[\sum_{j=1}^t S_j \mid E \right] = \Theta(t)$. Here we use the Chernoff bound (Lemma 30 in Appendix A) on the normalized sum of all jumps, to show that it (the sum) is linear in t with probability $1 - \exp(-t^{\Theta(1)})$, conditional on E .

$$\begin{aligned} \mathbb{P} \left(\sum_{j=1}^t S_j \geq 2\mu \mid E \right) &= \mathbb{P} \left(\frac{\sum_{j=1}^t S_j}{(t \log t)^{\frac{1}{\alpha-1}} - 1} \geq 2 \frac{\mu}{(t \log t)^{\frac{1}{\alpha-1}} - 1} \mid E \right) \leq \exp \left(-\frac{\Theta(t)}{3 \left((t \log t)^{\frac{1}{\alpha-1}} - 1 \right)} \right) \\ &\leq \exp \left(-\Theta \left(\frac{t^{\frac{\alpha-2}{\alpha-1}}}{(\log t)^{\frac{1}{\alpha-1}}} \right) \right) \leq \exp \left(-\Theta \left(t^{\frac{\alpha-2}{2(\alpha-1)}} \right) \right). \end{aligned}$$

Then, define

$$F = \left\{ \sum_{j=1}^t S_j = \mathcal{O}(t) \right\}$$

$F_1 = \{\text{the Pareto walk finds the treasure within } \Theta(t) \text{ steps}\}$ and

$F_2 = \{\text{the Pareto flight finds the treasure within } t \text{ steps/jumps}\}$,

where in F_1 the term $\Theta(t)$ counts also the jumps of length zero. Observe that the event $F \cap F_2$ implies the event

$$F_1 \cap \{\text{the process finds the treasure within } t \text{ jumps}\},$$

because if $F \cap F_2$ takes place, then the treasure is found at least in one between all the t jump destinations, at the overall amount of time spent to “travel” is $\Theta(t)$. Thus

$$\begin{aligned}
\mathbb{P}(F_1) &\geq \mathbb{P}(F_1, \text{the process finds the treasure within } t \text{ jumps}) \\
&\geq \mathbb{P}(F, F_2) \\
&\geq \mathbb{P}(F, F_2, E) \\
&\stackrel{(a)}{=} \mathbb{P}(E) [\mathbb{P}(F | E) + \mathbb{P}(F_2 | E) - \mathbb{P}(F \cup F_2 | E)] \\
&\stackrel{(b)}{\geq} \left(1 - \mathcal{O}\left(\frac{1}{\log t}\right)\right) [1 - \exp(-t^{\Theta(1)}) + p - 1] \\
&= \left(1 - \mathcal{O}\left(\frac{1}{\log t}\right)\right) (p - \exp(-t^{\Theta(1)})),
\end{aligned}$$

where in (a) we used the definition of conditional probability and the inclusion-exclusion principle, and in (b) we used that $\mathbb{P}(E) = (1 - \mathcal{O}(1/\log t))$, $\mathbb{P}(F | E) \geq 1 - \exp(-t^{\Theta(1)})$, and $\mathbb{P}(F \cup F_2 | E) \leq 1$. \square

We then have the couple proposition that links the Pareto run to the Pareto walk. We recall it.

Proposition 4 (Coupling: Pareto run into Pareto walk). *Suppose an agent a_1 that moves according to the Pareto run with parameter $\alpha > 1$ finds the treasure within t steps with probability $p > 0$. Then, another agent a_2 that moves according to the Pareto walk with the same parameter α finds the treasure within $\Omega(t)$ steps with probability p . Furthermore, if a_1 never finds the treasure with probability q , then a_2 never finds the treasure with probability at least q .*

Proof of Proposition 4. The proof is trivial, since the Pareto walk behaves exactly like a Pareto run taking more time to perform a jump. \square

5 Simple Random Walk Model

In this section we aim at proving the following theorem on the search efficiency of simple random walks. In sections 4 and 7, we will see that the performance of the Pareto walks for $\alpha \in [3, +\infty)$ is the same as that of the simple random walks. The reader may compare the following theorem with Theorem 3 in Section 4 and Theorem 6 in Section 7.

Theorem 4 (Hitting time - simple simple random walks). *Assume that the treasure is located in some node of the infinite grid at distance $\ell > 0$. Let k agents move performing mutually independent simple simple random walks. If $k = \log^{\mathcal{O}(1)}(\ell)$, then the agents find the treasure within time $\tilde{\Theta}(\ell^2)$, making a total work of $\tilde{\Theta}(\ell^2)$, w.h.p.²⁰ Furthermore, the result is tight in the following sense: for all $k = \tilde{\Theta}(\ell^\epsilon)$ for any fixed constant $\epsilon > 0$, then the agents need time at least $\tilde{\Omega}(\ell^2)$ to find the treasure, thus making a total work of $\tilde{\Omega}(\ell^{2+\epsilon})$, w.h.p.*

5.1 Proof of Theorem 4: main tools and general scheme

In order to do this, in the next two subsections we present the analysis of two technical results, which we state here. The first one is an “upper bound” on the hitting time of the treasure. Let T be the node in which the treasure is located, with $d_T = \ell$.

Proposition 5. *For some $t = \Theta(\ell^2 \log \ell)$, the probability one agent performing a simple simple random walk visits the treasure within time t is $\Omega(1/(\log^3 \ell))$.*

²⁰For a formal comprehension of this statement, we refer to Remark 1 in the Preliminaries (Section 3).

Subsection 5.2 is devoted to the proof of such proposition. On the other hand, Subsection 5.2.4 aims at proving the following, which is a “lower bound” on the hitting time of the treasure.

Proposition 6. *Let $k = \Theta(\ell^\epsilon)$ for any fixed constant $\epsilon \geq 0$. Then, k agents need at least time $\Omega(\ell^2/(\log^2 \ell))$ to find the treasure, w.h.p.*

With these two propositions, we are ready to prove the main result of this section.

5.1.1 Wrap-up: proof of Theorem 4

Proof of Theorem 4. Proposition 5 implies that $k = \Theta(\log^4 \ell)$ agents find the treasure in time $\mathcal{O}(\ell^2 \log \ell)$ with probability

$$1 - \left[1 - \mathcal{O}\left(\frac{1}{\log^3 \ell}\right) \right]^{\Theta(\log^4 \ell)} = 1 - \mathcal{O}\left(\frac{1}{\ell}\right).$$

Furthermore, if we increase the number of agents by multiplying by any polylogarithmic factor, the same upper bound on the hitting time holds, w.h.p., while if we decrease it by dividing by any polylogarithmic factor, the upper bound on the hitting time holds with non-negligible probability.

As for the almost-tightness result, let k be any function in $\tilde{\Theta}(\ell^\epsilon \log^{\mathcal{O}(1)} \ell)$ for any fixed $\epsilon \geq 0$. Then, for Proposition 6, k agents need at least time $\tilde{\Omega}(\ell^2)$ to find the treasure, w.h.p. Indeed, since the result holds for $\Theta(\ell^{\epsilon+1})$ agents, it holds for k too by observing $k = \tilde{\Theta}(\ell^\epsilon \log^{\mathcal{O}(1)} \ell) \leq \Theta(\ell^{\epsilon+1})$. \square

5.2 Analysis of the simple random walk model: proof of Proposition 5

Let T be the node in which the treasure is located, with $d_T = \ell$. This subsection aims at proving the following result.

Proposition 5. *For some $t = \Theta(\ell^2 \log \ell)$, the probability one agent performing a simple simple random walk visits the treasure within time t is $\Omega(1/(\log^3 \ell))$.*

We look at a single agent moving on the grid \mathbb{Z}^2 performing a simple (symmetric) random walk which starts at the origin $o = (0, 0)$. We are going to introduce some definitions and notations we use throughout the analysis. For any node $u = (u_x, u_y)$ of the grid, define the random variable

$$Z_u(t) = \text{number of agent's visits at node } u \text{ within } t \text{ steps.}$$

In order to bound the probability that the node u has been visited at least once at time t , namely $\mathbb{P}(Z_u(t) > 0)$, we define

$$p_{u,i} = \mathbb{P}(\text{the agent is in node } u \text{ at step } i).$$

By the definitions above, we easily get that

$$\mathbb{E}[Z_u(t)] = \sum_{i=0}^t p_{u,i}.$$

Notice that at a generic round t , the simple random walk can visit only nodes whose distance from the origin has the same parity of t . Thus, given $u \in \mathbb{Z}^2$, we only compare the probability to be on u in a given round t with the probability to be in any v in the set

$$P_u = \{v \in \mathbb{Z}^2 : |d_v - d_u| \text{ is even}\}.$$

As for the main result of the theorem, we present a road-map of the analysis to keep track of main idea behind the lemmas and the proofs that follow, while the almost-tightness results will be later discussed and shown.

5.2.1 Road-map of the analysis

The scheme of the proof follows the same structure and main ideas of that in Subsection 4.2. We omit such an informal description and go directly with the main steps of the proof. Let T be the node in which the treasure is located and P_T the set of nodes that have even distance from T . We divide P_T in three different regions, namely, for any $\delta \geq 0$ and any $t = \Omega(\ell^2)$,

$$\begin{aligned}\mathcal{A}_1 &= Q(\ell) \cap P_T = \{(x, y) \in P_T : \max(|x|, |y|) \leq \ell\} \\ \mathcal{A}_2 &= \{v \in P_T : |v|_1 \leq 4\sqrt{2(1+\delta)t \log t}\} \setminus \mathcal{A}_1; \\ \mathcal{A}_3 &= \{v \in P_T : |v|_1 > 4\sqrt{2(1+\delta)t \log t}\}.\end{aligned}$$

Intuitively, \mathcal{A}_1 contains all nodes of P_T which are “close” to the origin than the treasure; \mathcal{A}_3 contains all nodes of P_T which are distant from the origin at least a polylogarithmic factor more than the treasure; \mathcal{A}_2 contains all nodes of P_T in the between.

The key steps of the analysis are the following.

1. We show that the two dimensional simple random walk can be decomposed into two independent one dimensional simple random walks.
2. Using Step 1, for any $u = (x, y)$, if $|t - d_u|$ is even, we argue that $p_{u,t} \geq p_{v,t}$ for all $v \in \mathbb{Z}^2$ outside the square $Q(d_u) = \{(x, y) \in \mathbb{Z}^2 : \max(|x|, |y|) \leq d_u\}$ (see Lemma 14 and Figure 6 for details).
3. From Step 2, $\mathbb{E}[Z_u(t)] \geq \mathbb{E}[Z_v(t)]$ for all $v \in P_u$ but at most $m_u = |Q(d_u)|$ (see Corollary 3 for details). Thus, we get that the average number of visits in \mathcal{A}_2 is at most its cardinality times $\mathbb{E}[Z_T(t)]$, thus bounded by $\mathbb{E}[Z_T(t)] \cdot 32(1+\delta)t \log t$.
4. Using Step 1, we show that $\mathbb{E}[Z_o(t)] = b_t = \Theta(\log t)$ (see Lemma 15 for details). By conditioning on the first arrival time at any point u , we get that $\mathbb{E}[Z_u(t)] \leq b_t$ for all u (see Lemma 16 for details). Then, from Step 3, the upper bound on the average number of visits in \mathcal{A}_1 is $m_u b_t$.
5. Using Chernoff-Hoeffding bounds and Step 1, for any fixed $\delta \geq 0$ and any $t \geq 4$, we can show (see Lemma 17 for details) the following upper bound on the average number of visits in \mathcal{A}_3 :

$$\sum_{v \in \mathbb{Z}^2: d_v \geq 4\sqrt{2(1+\delta)t \log t}} \mathbb{E}[Z_v(t)] < 1.$$

6. Combining steps 3 to 5, it holds that, for any $t = \Omega(\ell^2)$,

$$m_T \cdot b_t + \mathbb{E}[Z_T(t)] \cdot (32(1+\delta)t \log t) + 1 \geq t,$$

(see Lemma 18 for details). Then, for any $t = \Omega(\ell^2)$, we get

$$\mathbb{E}[Z_T(t)] \geq \frac{t - m_T b_t - 1}{(32(1+\delta)t \log t) - m_u}.$$

7. From Step 4 and simple calculations, we show (see Lemma 16 for details) that

$$\mathbb{E}[Z_u(t)] \geq \mathbb{P}(Z_u(t) > 0) = \frac{\mathbb{E}[Z_u(t)]}{\mathbb{E}[Z_u(t) \mid Z_u(t) > 0]} \geq \frac{\mathbb{E}[Z_u(t)]}{b_t}.$$

8. Combining steps 6 and 7, we have (see Corollary 4 for details) that, for any $t = \Omega(\ell^2)$,

$$\mathbb{P}(Z_T(t) > 0) \geq \frac{\lfloor \frac{t}{2} \rfloor - 4\ell^2 \cdot b_t - 1}{(32(1 + \delta)t \log t) \cdot b_t}.$$

By letting t to be some function in $\Theta(\ell^2 \log \ell)$, we have Proposition 5.

5.2.2 Full analysis

As a preliminary, we show (Step 1 of the scheme) a natural decomposition of the two-dimensional simple random walk into two mutually independent one-dimensional simple random walks, which can be found in [Nor97]). Let $R^w(t)$ be the random variable denoting the coordinates of the node the two-dimensional simple random walk visits at time t . The two new “axes” are the two bisectors of the quadrants of the grid \mathbb{Z}^2 , namely r and s , as in Figure 5. As a convention, we fix the part

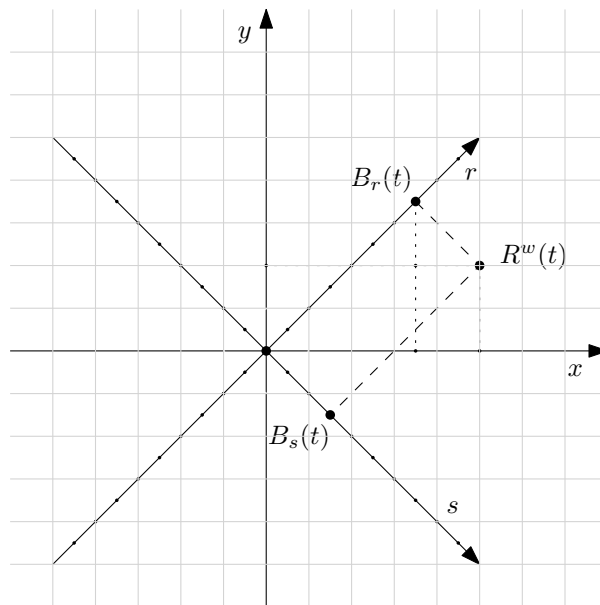


Figure 5: Decomposition of a two-dimensional simple random walk into two i.i.d. one-dimensional simple random walks over the bisectors of the quadrants.

of r in the first quadrant and that of s in the fourth quadrant to be the positive side of r and s . Over these two strict lines, consider a sequence of nodes in all directions such that two subsequent nodes have Euclidean distance $\sqrt{2}/2$ between them, starting from the origin, in the positive and in the negative directions. Let $B_r(t)$ and $B_s(t)$ be two independent simple random walks on r and s , respectively, whose steps are over the nodes we just inserted. More precisely, let $\{S_j^r\}_{j \in \mathbb{N}}$ be a sequence of i.i.d. random variables with values in $\{-\sqrt{2}/2, +\sqrt{2}/2\}$ denoting the step increment of $B_r(t)$ at round j , and $\{S_j^s\}_{j \in \mathbb{N}}$ the same for $B_s(t)$, with S_j^r and S_j^s mutually independent for each $i, j \geq 1$. Then $B_r(t) = \sum_{j=1}^t S_j^r$ and $B_s(t) = \sum_{j=1}^t S_j^s$. We have that

$$R^w(t) = \left(\cos \frac{\pi}{4} (B_r(t) + B_s(t)), \sin \frac{\pi}{4} (B_r(t) - B_s(t)) \right) = \left(\frac{\sqrt{2}}{2} \sum_{j=1}^t (S_j^r + S_j^s), \frac{\sqrt{2}}{2} \sum_{j=1}^t (S_j^r - S_j^s) \right).$$

Note that in a generic round $t - 1$, we have that the probability the increment is $(1, 0)$ is

$$\mathbb{P}(R^w(t) - R^w(t-1) = (1, 0)) = \mathbb{P}(S_t^r + S_t^s = \sqrt{2}, S_t^r - S_t^s = 0)$$

$$\begin{aligned}
&= \mathbb{P} \left(S_t^r = \sqrt{2} - S_t^s, S_t^r = S_t^s \right) \\
&= \mathbb{P} \left(S_t^r = \frac{\sqrt{2}}{2}, S_t^s = \frac{\sqrt{2}}{2} \right) = \frac{1}{4}
\end{aligned}$$

for independence between S_t^r and S_t^s . The same holds for $(-1, 0)$, $(0, 1)$, and $(0, -1)$. Thus, the two-dimensional simple random walk can be seen as a combination of two independent and identically distributed one-dimensional simple random walks.

We are going to present a result (Step 2 in the scheme) which is very similar to Lemma 32 in Appendix C, and describes a “monotonicity” of the point-wise distribution at a generic time $t \geq 0$. Indeed, according to the previous notation, for any node $u = (u_x, u_y)$, we let $d_u = |u_x| + |u_y|$ and consider the square

$$Q(d_u) = \{(x, y) \in \mathbb{Z}^2 : \max(|x|, |y|) \leq d_u\}$$

(see Figure 6 in the proof of Lemma 14). Then, the following geometric property holds.

Lemma 14. *Let $u \in \mathbb{Z}^2$ be an arbitrary node. Then, for each node $v \notin Q(d_u)$ and each step t such that $t - d_u$ is even, it holds that $p_{u,t} \geq p_{v,t}$.*

The difference here is that the simple random walk has probability equal to zero to stay still at one round, i.e. it always moves one step towards one of its neighbors, while the request we make for Lemma 32 in Appendix C is that the mobility model has a non-increasing step-length distribution; thus, we have to look for a different proof.

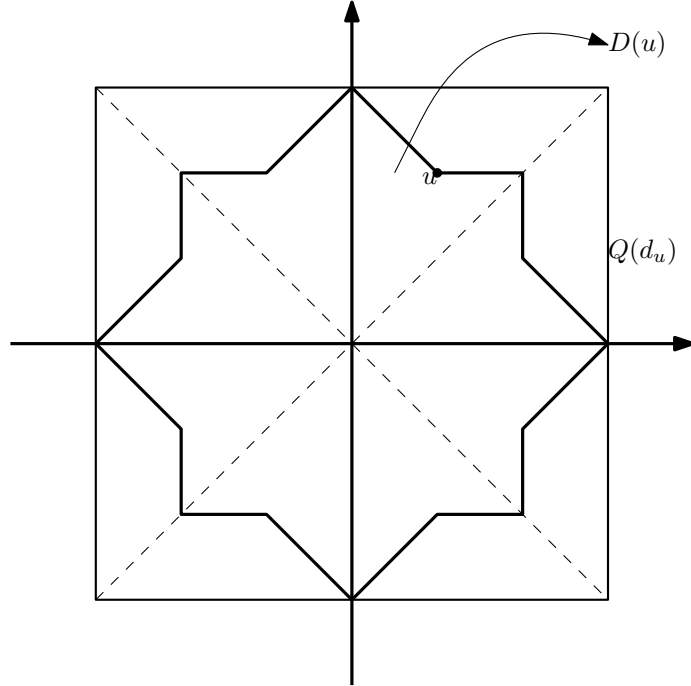


Figure 6: The set $Q(d_u)$ and the set $D(u)$.

Proof of Lemma 14. For any given distance $d \geq 0$, consider the rhombus $R_d^*(o) = \{(x, y) \in \mathbb{Z}^2 : |x| + |y| \leq d\}$. For any point (x, y) in \mathbb{Z}^2 , it is defined a square $T(x, y) = \{(x', y') \in \mathbb{Z}^2 : 0 \leq \max(|x'|, |y'|) \leq \max(|x|)\}$. Then, for $u \in \mathbb{Z}^2$, define

$$D(u) = R_{d_u}^*(o) \cup T(u)$$

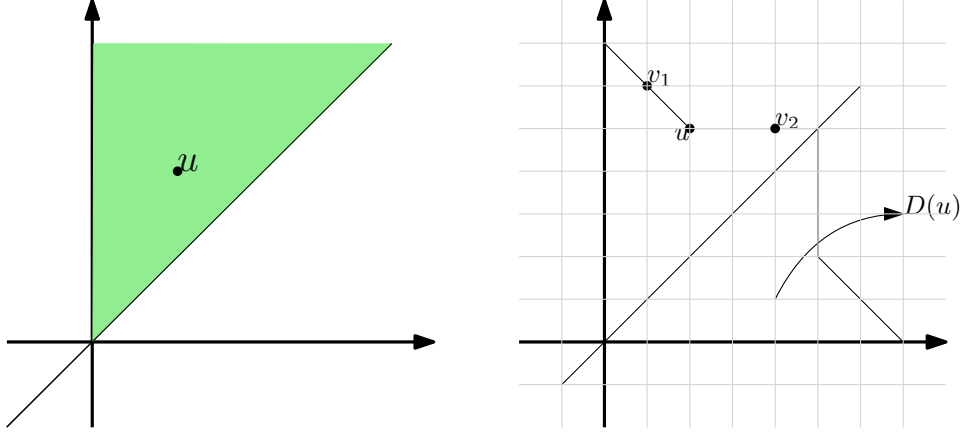


Figure 7: The “area” in which we take u , and the choices v_1, v_2 .

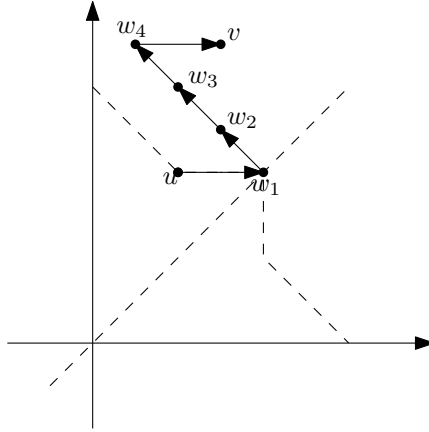


Figure 8: One “path” example.

(see Figure 6 for details). We will show that $p_{u,t} \geq p_{v,t}$ for each $v \notin D(u)$ through an induction argument on d_u , which will imply the thesis of the Lemma.

Let v be any other node on the grid, and let $d_v = |v|_1$. Then, if $|d_v - d_u|$ is odd, the thesis is trivial. For $|d_v - d_u|$ even, we show a more complicated argument: without loss of generality, suppose u is in the first quadrant and not below the main bisector, namely in the set $\{(x, y) \in \mathbb{Z}^2 : y \geq 0, x \geq y\}$ (Figure 7).

According to the decomposition we have showed in Figure 5, we have that

$$\begin{aligned} p_{u,t} &= \mathbb{P} \left(\frac{\sqrt{2}}{2}(B_r(t) + B_s(t)) = u_x, \frac{\sqrt{2}}{2}(B_r(t) - B_s(t)) = u_y \right) \\ &= \mathbb{P} \left(B_r(t) = \frac{u_x + u_y}{\sqrt{2}} \right) \cdot \mathbb{P} \left(B_s(t) = \frac{u_x - u_y}{\sqrt{2}} \right). \end{aligned}$$

We first show the thesis taking v in the set $\{v_1 = (u_x - 1, u_y + 1), v_2 = (u_x + 2, u_y)\}$ (as long as they still lie in the highlighted zone in Figure 7). If $t = d_u$, the thesis is trivial for v_1 and v_2 . Indeed, v_2 cannot be reached in t steps since $d_{v_2} > t$, while there are more possible paths to get to u in d_u steps than to v_1 in d_u steps. So, we assume $t \geq d_u + 2$. Keeping in mind that $u_y \geq u_x$ for the choice of u , we have that

$$p_{v_1,t} = \mathbb{P} \left(B_r(t) = \frac{u_x + u_y}{\sqrt{2}} \right) \cdot \mathbb{P} \left(B_s(t) = \frac{u_x - u_y - 2}{\sqrt{2}} \right)$$

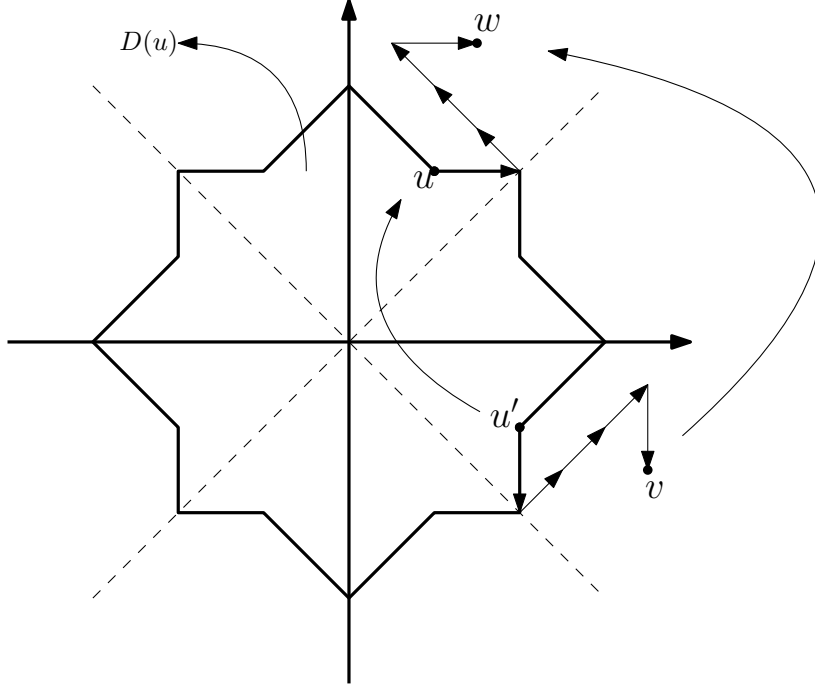


Figure 9: Symmetrical argument.

$$\begin{aligned}
&\stackrel{(a)}{=} \binom{t}{\frac{t+u_x+u_y}{2}} \frac{1}{2^t} \cdot \binom{t}{\frac{t+u_y-u_x}{2} + 1} \frac{1}{2^t} \\
&= \binom{t}{\frac{t+u_x+u_y}{2}} \binom{t}{\frac{t+u_y-u_x}{2} + 1} \frac{1}{2^{2t}} \\
&\stackrel{(b)}{\leq} \binom{t}{\frac{t+u_x+u_y}{2}} \binom{t}{\frac{t+u_y-u_x}{2}} \frac{1}{2^{2t}} \\
&= \mathbb{P} \left(B_r(t) = \frac{u_x + u_y}{\sqrt{2}} \right) \cdot \mathbb{P} \left(B_s(t) = \frac{u_x - u_y}{\sqrt{2}} \right) \\
&= p_{u,t},
\end{aligned}$$

where (a) is true because $|u_x - u_y - 2| = u_y - u_x + 2$, and (b) is true for Fact 2 in Appendix A. Then, if $v = v_1$ we have that $p_{u,t} \geq p_{v,t}$. As for v_2 , we have two cases. If $u_y \geq u_x + 2$, it holds that

$$\begin{aligned}
p_{v_2,t} &= \mathbb{P} \left(B_r(t) = \frac{u_x + u_y + 2}{\sqrt{2}} \right) \cdot \mathbb{P} \left(B_s(t) = \frac{u_x - u_y + 2}{\sqrt{2}} \right) \\
&\stackrel{(a)}{=} \binom{t}{\frac{t+u_x+u_y}{2} + 1} \frac{1}{2^t} \cdot \binom{t}{\frac{t+u_y-u_x}{2} - 1} \frac{1}{2^t} \\
&= \binom{t}{\frac{t+u_x+u_y}{2} + 1} \binom{t}{\frac{t+u_y-u_x}{2} - 1} \frac{1}{2^{2t}} \\
&\stackrel{(b)}{\leq} \binom{t}{\frac{t+u_x+u_y}{2}} \binom{t}{\frac{t+u_y-u_x}{2}} \frac{1}{2^{2t}} \\
&= \mathbb{P} \left(B_r(t) = \frac{u_x + u_y}{\sqrt{2}} \right) \cdot \mathbb{P} \left(B_s(t) = \frac{u_x - u_y}{\sqrt{2}} \right) \\
&= p_{u,t},
\end{aligned}$$

where (a) is true because $|u_x - u_y + 2| = u_y - u_x - 2$, and (b) is true for Fact 2 in Appendix A.

Else, if $u_y < u_x + 2$ (which basically means $u_y \leq u_x$ since we have to keep the parity), we have

$$\begin{aligned}
p_{v_2,t} &= \mathbb{P}\left(B_r(t) = \frac{u_x + u_y + 2}{\sqrt{2}}\right) \cdot \mathbb{P}\left(B_s(t) = \frac{u_x - u_y + 2}{\sqrt{2}}\right) \\
&\stackrel{(a)}{=} \binom{t}{\frac{t+u_x+u_y}{2} + 1} \frac{1}{2^t} \cdot \binom{t}{\frac{t+u_x-u_y}{2} + 1} \frac{1}{2^t} \\
&= \binom{t}{\frac{t+u_x+u_y}{2} + 1} \binom{t}{\frac{t+u_x-u_y}{2} + 1} \frac{1}{2^{2t}} \\
&\stackrel{(b)}{\leq} \binom{t}{\frac{t+u_x+u_y}{2}} \binom{t}{\frac{t+u_y-u_x}{2}} \frac{1}{2^{2t}} \\
&= \mathbb{P}\left(B_r(t) = \frac{u_x + u_y}{\sqrt{2}}\right) \cdot \mathbb{P}\left(B_s(t) = \frac{u_x - u_y}{\sqrt{2}}\right) \\
&= p_{u,t},
\end{aligned}$$

where (a) is true because $|u_x - u_y + 2| = u_x - u_y + 2$, and (b) is true for Fact 2 in Appendix A. We thus have that $p_{u,t} \geq p_{v_2,t}$. For each other v in the highlighted area in Figure 7, there exists a sequence of nodes $u = w_0, w_1, \dots, w_k = v$, all lying in the same area above, such that w_i belongs to the set $\{((w_{i-1})_x - 1, (w_{i-1})_y + 1), ((w_{i-1})_x, (w_{i-1})_y + 2)\}$ (see Figure 8).

Then,

$$p_{u,t} = p_{w_0,t} \geq p_{w_1,t} \geq \dots \geq p_{w_k,t} = p_{v,t}.$$

For any other $v \notin D(u)$, we have a symmetrical argument shown in Figure 9 that implies the thesis. \square

Recalling that $P_u = \{v \in \mathbb{Z}^2 : |d_v - d_u| \text{ is even}\}$, the following corollary holds (which is Step 3 in the scheme).

Corollary 3. *For all $v \in P_u$ but at most $4d_u^2$, i.e. those who lie in $Q(d_u)$, it holds that*

$$\mathbb{E}[Z_u(t)] \geq \mathbb{E}[Z_v(t)]$$

for any $t \geq 0$.

Proof of Corollary 3. The proof is a simple application of Lemma 14. \square

Next lemma gives the average number of agent's returns to the origin (Step 4 in the scheme).

Lemma 15. *The average number of visits to the origin until time $t > 0$ is*

$$\mathbb{E}[Z_o(t)] = b_t = \Theta(\log t).$$

Proof of Lemma 15. Consider the decomposition we showed with Figure 5. Then, the simple random walk is at the origin at time i if and only if both $B_r(i) = 0$ and $B_s(i) = 0$. In other words,

$$p_{o,i} = \mathbb{P}(B_r(i) = 0) \cdot \mathbb{P}(B_s(i) = 0)$$

for independence. First, notice that if i is odd, then $B_r(i) \neq 0$. We then consider only even timings, i.e. $i = 2k$. By Stirling's formula

$$\mathbb{P}(B_r(2k) = 0) = \binom{2k}{k} \frac{1}{2^{2k}} = \frac{(2k)!}{(k!)^2} \frac{1}{2^{2k}} \sim \frac{\sqrt{4\pi k} (2k/e)^{2k}}{2\pi k (k/e)^{2k}} \frac{1}{2^{2k}} = \frac{1}{\sqrt{\pi k}}.$$

Thus, there exists a large enough $\bar{k} > 0$, such that

$$\frac{1}{2} \leq \mathbb{P}(B_r(2k) = 0) \cdot \sqrt{\pi k} \leq 2$$

for any $k > \bar{k}$, and the same holds for $B_s(2k)$ for symmetry. Then we have that

$$\sum_{k=0}^{\lfloor \frac{t}{2} \rfloor} p_{o,2k} = \sum_{k=0}^{\lfloor \frac{t}{2} \rfloor} [\mathbb{P}(B_r(2k) = 0)]^2 \leq \bar{k} + \sum_{k=\bar{k}+1}^{\lfloor \frac{t}{2} \rfloor} \frac{4}{\pi k} = \mathcal{O}(\log t),$$

and, at the same time

$$\sum_{k=0}^{\lfloor \frac{t}{2} \rfloor} p_{o,2k} = \sum_{k=0}^{\lfloor \frac{t}{2} \rfloor} [\mathbb{P}(B_r(2k) = 0)]^2 \geq \sum_{k=\bar{k}+1}^{\lfloor \frac{t}{2} \rfloor} \frac{1}{4\pi k} = \Omega(\log t),$$

where the latter inequalities holds for the integral test, and because \bar{k} is a constant. We thus have that $\mathbb{E}[Z_o(t)] = \Theta(\log t)$. \square

We use this result to give a bound on the average number of visits the simple random walk makes to nodes that are in \mathcal{A}_1 . Indeed, we will exploit the first item of next result. Call $b_t = \mathbb{E}[Z_o(t)]$. Then, we have the following (steps 4 and 7 in the scheme).

Lemma 16. *Let $u \in \mathbb{Z}^2$ be any node. It holds that*

- (i) $\mathbb{E}[Z_u(t)] \leq b_t$
- (ii) $1 \leq \mathbb{E}[Z_u(t) \mid Z_u(t) > 0] \leq b_t$;
- (iii) $\mathbb{E}[Z_u(t)] / b_t \leq \mathbb{P}(Z_u(t) > 0) \leq \mathbb{E}[Z_u(t)]$.

Proof of Lemma 16. Item (i) directly comes from Item (ii), since $\mathbb{E}[Z_u(t)] \leq \mathbb{E}[Z_u(t) \mid Z_u(t) > 0]$. Consider Item (ii), and let τ be the random variable denoting the first time the simple random walk visits u . Observe that, conditional on $Z_u(t) > 0$, $\tau \leq t$ with probability 1. Then we have

$$\mathbb{E}[Z_u(t) \mid Z_u(t) > 0] = \mathbb{E}[Z_o(t - \tau) \mid \tau \leq t] \leq \mathbb{E}[Z_o(t) \mid \tau \leq t] = \mathbb{E}[Z_o(t) \mid \tau \leq t],$$

where last inequality holds for independence. At the same time, $\mathbb{E}[Z_u(t) \mid Z_u(t) > 0] \geq 1$ due to the conditional event. As for the Item (iii), let us explicitly express the term $\mathbb{E}[Z_u(t) \mid Z_u(t) > 0] \cdot \mathbb{P}(Z_u(t) > 0)$. This is equal to

$$\begin{aligned} & \sum_{i=1}^t i \mathbb{P}(Z_u(t) = i \mid Z_u(t) > 0) \cdot \mathbb{P}(Z_u(t) > 0) \\ &= \sum_{i=1}^t i \frac{\mathbb{P}(Z_u(t) = i, Z_u(t) > 0)}{\mathbb{P}(Z_u(t) > 0)} \mathbb{P}(Z_u(t) > 0) \\ &= \sum_{i=1}^t i \mathbb{P}(Z_u(t) = i, Z_u(t) > 0) \\ &= \sum_{i=1}^t i \mathbb{P}(Z_u(t) = i) \\ &= \mathbb{E}[Z_u(t)]. \end{aligned}$$

Then,

$$\mathbb{E}[Z_u(t)] \geq \mathbb{P}(Z_u(t) > 0) = \frac{\mathbb{E}[Z_u(t)]}{\mathbb{E}[Z_u(t) | Z_u(t) > 0]} \geq \frac{\mathbb{E}[Z_u(t) | E]}{b_t},$$

since $\mathbb{E}[Z_u(t) | Z_u(t) > 0] \leq b_t$ for the first item. \square

Now we are going to give a bound on the average number of visits to nodes in \mathcal{A}_3 the simple random walk does (Step 5 in the scheme).

Lemma 17. *If $t \geq 4$, for any constant $\delta \geq 0$ it holds that*

$$\sum_{v \in \mathbb{Z}^2: d_v \geq 4\sqrt{2(1+\delta)t \log t}} \mathbb{E}[Z_v(t)] < 1.$$

Proof of Lemma 17. Once again, we exploit the decomposition shown in Figure 5. The one-dimensional simple random walk $B_r(i)$ is such that $\mathbb{E}[B_r(i)] = 0$ and $\text{Var}(B_r(i)) = i/\sqrt{2}$. Furthermore, each step S_j^r , for $j \leq i$, has mean equal to zero and is less than $1/\sqrt{2}$. Let $i \leq t$ and $\delta \geq 0$ an arbitrary constant (it will be useful for a later lemma): we use a particular form of Chernoff bound (Lemma 31 in Appendix A) to get that

$$\mathbb{P}\left(B_r(i) \geq 2\sqrt{(1+\delta)t \log t}\right) \leq \exp\left(\frac{4(1+\delta)t \log t}{i + \frac{2\sqrt{t \log t}}{3\sqrt{2}}}\right) \leq \frac{1}{t^{3+\delta}}.$$

For symmetry, we get also that $\mathbb{P}\left(B_r(i) \leq -2\sqrt{(1+\delta)t \log t}\right) \leq \frac{1}{t^{3+\delta}}$, thus

$$\mathbb{P}\left(B_r(i) \geq 2\sqrt{(1+\delta)t \log t}\right) \leq \frac{2}{t^{3+\delta}}.$$

The same result holds analogously for $B_s(i)$, for each $i \leq t$. Since $|R^w(i)|_1$ from the origin is at most $\sqrt{2} \cdot (B_r(i) + B_s(i))$ we have that with probability at least $1 - 4/t^{3+\delta}$ this distance is bounded by $4\sqrt{2(1+\delta)t \log t}$. Then, the probability that in any of the first t rounds, the walk has ever gone further than distance $4\sqrt{2(1+\delta)t \log t}$ from the origin is $4/t^{2+\delta}$. Denote by F_t the event $\cap_{i=0}^t \{|R^w(i)|_1 \leq 4\sqrt{2(1+\delta)t \log t}\}$. It follows that

$$\begin{aligned} \sum_{v \in \mathbb{Z}^2: d_v \geq 4\sqrt{2(1+\delta)t \log t}} \mathbb{E}[Z_v(t)] &= \mathbb{E}\left[\sum_{v \in \mathbb{Z}^2: d_v \geq 4\sqrt{2(1+\delta)t \log t}} Z_v(t)\right] \\ &= \mathbb{E}\left[\sum_{v \in \mathbb{Z}^2: d_v \geq 4\sqrt{2(1+\delta)t \log t}} Z_v(t) \mid F_t\right] \mathbb{P}(F_t) + \mathbb{E}\left[\sum_{v \in \mathbb{Z}^2: d_v \geq 4\sqrt{2(1+\delta)t \log t}} Z_v(t) \mid F_t^C\right] \mathbb{P}(F_t^C) \\ &\leq 0 + t \cdot \left(\frac{4}{t^{2+\delta}}\right) = \left(\frac{4}{t^{1+\delta}}\right), \end{aligned}$$

and we have the thesis for $t \geq 4$. \square

We are finally ready to give a lower bound on the expected number of visits to the treasure (Step 6 in the scheme).

Lemma 18. *Let T be the node in which the treasure is located. Then, for any $t = \Omega(\ell^2)$ and any $\delta \geq 0$, it holds that*

$$4\ell^2 \cdot b_t + \mathbb{E}[Z_T(t)] \cdot (32(1+\delta)t \log t) + 1 \geq \left\lfloor \frac{t}{2} \right\rfloor.$$

Proof of Lemma 18. At a generic round t , the simple random walk can only visit nodes whose distances from the origin have the same parity of t . Then, we have

$$\sum_{v \in P_u} \mathbb{E}[Z_v(t)] \geq \left\lfloor \frac{t}{2} \right\rfloor.$$

Lemma 15 and Lemma 16 give us that

$$\sum_{v \in \mathcal{A}_1} \mathbb{E}[Z_v(t)] \leq 4d_u^2 \cdot b_t.$$

Lemma 14 and Corollary 3 imply that

$$\sum_{v \in \mathcal{A}_2} \mathbb{E}[Z_v(t)] \leq \mathbb{E}[Z_T(t)] \cdot (32(1 + \delta)t \log t),$$

since $|\mathcal{A}_2| \leq 32(1 + \delta)t \log t$, while Lemma 17 tells us that

$$\sum_{v \in \mathcal{A}_3} \mathbb{E}[Z_v(t)] < 1$$

for $t \geq 4$. Then, we get the thesis for t large enough ($\Omega(\ell^2)$ is sufficient). \square

5.2.3 Wrap-up: proof of Proposition 5

We are ready to compute a lower bound on the probability to hit the treasure (Step 8 in the scheme).

Corollary 4. *Let T be the node in which the treasure is located. Then, for $t = \Theta(\ell^2 \log \ell)$, it holds that*

$$\mathbb{P}(Z_T(t) > 0) \geq \frac{\left\lfloor \frac{t}{2} \right\rfloor - 4\ell^2 \cdot b_t - 1}{(32(1 + \delta)t \log t) \cdot b_t}.$$

Proof of Corollary 4. Lemma 15 tells us that

$$\mathbb{P}(Z_T(t) > 0) \geq \frac{\mathbb{E}[Z_T(t)]}{b_t}.$$

Then, Lemma 18 gives us that

$$\mathbb{P}(Z_T(t) > 0) \geq \frac{\left\lfloor \frac{t}{2} \right\rfloor - 4\ell^2 \cdot b_t - 1}{(32(1 + \delta)t \log t) \cdot b_t}.$$

\square

Then, the proof of Proposition 5 comes from Corollary 4, keeping in mind that $b_t = \Theta(\log t)$ due to Lemma 15, setting t to be some function in $\Theta(\ell^2 \log \ell)$, and $\delta = 1$, we get the thesis.

5.2.4 Analysis of the simple random walk model: proof of Proposition 6

Here we prove the following result.

Proposition 6. *Let $k = \Theta(\ell^\epsilon)$ for any fixed constant $\epsilon \geq 0$. Then, k agents need at least time $\Omega(\ell^2/(\log^2 \ell))$ to find the treasure, w.h.p.*

Proof of Proposition 6. let $\delta = \epsilon$. Lemma 17 in Subsection 5.2, tells us that a single agent performing a simple random walk within time $t = \Theta(\ell^2/\log^2 \ell)$ never gets at distance $4\sqrt{2(1 + \epsilon)t \log t} = \mathcal{O}(\ell^2/\log \ell)$ with probability

$$1 - \mathcal{O}(1/t^{1+\epsilon}) = 1 - \mathcal{O}(1/\ell^{1+\epsilon}).$$

Then, $\Theta(\ell^\epsilon)$ agents have probability at most $\mathcal{O}(1/\ell)$ to go further than distance $\mathcal{O}(\ell^2/\log \ell)$ within time t . \square

6 Ballistic Walk Model

In this section we show the search-efficiency of the ballistic walks. In Section 7, we will see that the performance of the Pareto walks for $\alpha \in (1, 2]$ is the same as that of the ballistic walks. The reader may compare the following theorem with Theorem 7 in Section 7.

Theorem 5 (Hitting time - ballistic walks). *Assume that the treasure is located in some node of the infinite grid at distance $\ell > 0$. Let k agents move performing mutually independent ballistic walks. If $k = \tilde{\Theta}(\ell)$, then k agents find the treasure in time $\Theta(\ell)$, making a total work of $\tilde{\Theta}(\ell^2)$, w.h.p.²¹ Furthermore, the result is almost-tight in a two-fold sense:*

- (i) *if $k = \tilde{\Theta}(\ell^{1-\epsilon})$ for any arbitrary constant $\epsilon \in (0, 1]$, then the agents never find the treasure, thus making an infinite work, w.h.p.;*
- (ii) *if $k = \tilde{\Theta}(\ell^{1+\epsilon})$ for any arbitrary constant $\epsilon > 0$, then the agents need time $\Theta(\ell)$ to find the treasure, making a total work of $k = \tilde{\Theta}(\ell^{2+\epsilon})$, w.h.p.*

This theorem is maybe the most intuitive and easiest to prove. We start with one technical lemma.

Lemma 19. *Suppose we have one agent performing the ballistic walk which starts at the origin. The probability it hits the treasure is $\Theta(1/\ell)$.*

Proof of Lemma 19. At time $t = 0$ the agent chooses any direction according to the procedure in Definition 2. For Lemma 1 in the Preliminaries (Section 3), the probability the agent hits any node at distance ℓ is $\Theta(1/\ell)$. \square

The proof of the theorem follows.

Proof of Theorem 5. One agent who starts at the origin reaches the treasure in time $\Theta(\ell)$ with probability $\Theta(1/\ell)$, due to Lemma 19. Thus, we need $\Theta(\ell \log \ell)$ agents to reach the treasure in time $\Theta(\ell)$, letting the work to be $\Theta(\ell^2 \log \ell)$, w.h.p. Furthermore, if we increase the number of agents by multiplying by any polylogarithmic factor, the same upper bound on the hitting time holds, w.h.p., while if we decrease it by dividing by any polylogarithmic factor, the upper bound on the hitting time holds with non-negligible probability.

At the same time, each agent needs time $\Theta(\ell)$ to reach distance $\Theta(\ell)$, almost surely. Thus, $\tilde{\Theta}(\ell)$ agents find the treasure in time $\Omega(\ell)$, making a total work of $\tilde{\Omega}(\ell^2)$, w.h.p.

As for the almost-tightness results, if k is any function in $\tilde{\Theta}(\ell^{1-\epsilon})$ for any $\epsilon \in (0, 1]$, then k agents eventually find the treasure with probability $\tilde{O}(1/\ell^\epsilon)$, letting the work to be infinite w.h.p.

In, instead, if k is any function in $\tilde{\Theta}(\ell^{1+\epsilon})$ for any $\epsilon \in (0, 1]$, the agents will find the treasure in time $\Theta(\ell)$, letting the work to be $\tilde{\Theta}(\ell^{2+\epsilon})$, w.h.p. \square

7 Pareto Walk Model: Equivalences

In this section we show how the search-efficiency and the performance of the Pareto walk model (Definition 3 in Section 3), for cases in which $\alpha \in (3, +\infty)$ and $\alpha \in (1, 2]$, is essentially equivalent to those of, respectively, the simple random walk model and the ballistic walk model.

The first result we present is on the “diffusive regime”, namely the case in which $\alpha \in (3, +\infty)$ (the case $\alpha = 3$ has already been discussed in Section 4). The reader may compare the following theorem with that for the simple random walk model (Theorem 4 in Section 5).

²¹For a formal comprehension of this statement, we refer to Remark 1 in the Preliminaries (Section 3).

Theorem 6 (Diffusive regime). *Assume that the treasure is located in some node of the infinite grid at distance $\ell > 0$. Let k agents move performing mutually independent Pareto walks with $\alpha > 3$. If $k = \log^{\mathcal{O}(1)}(\ell)$, then k agents find the treasure in time $\tilde{\Theta}(\ell^2)$, making a total work of $\tilde{\Theta}(\ell^2)$, w.h.p.²² Furthermore, the result is almost-tight in the following sense:*

- (i) *if $k = \tilde{\Theta}(\ell^\epsilon)$ for any fixed constant $\epsilon \in [0, 3 - \alpha)$, then the agents need time $\tilde{\Omega}(\ell^2)$ to find the treasure and total work $\tilde{\Omega}(\ell^{2+\epsilon})$, w.h.p.;*
- (ii) *if $k = \tilde{\Theta}(\ell^{\alpha-3+\min(\frac{3}{2}\epsilon, \frac{1}{2}+\epsilon)})$ for any $\epsilon \geq 0$, then k agents need time at least $\Omega(\ell^{2-\epsilon})$ to find the treasure, w.h.p., making a total work of $\tilde{\Omega}(\ell^{2+(\alpha-3)+\min(\frac{1}{2}\epsilon, \frac{1}{2})})$.*

This first result is proved in Section 8. As for the “ballistic regime”, the reader may compare the following theorem with that for the ballistic walk model (Theorem 5 in Section 6).

Theorem 7 (Ballistic regime). *Assume that the treasure is located in some node of the infinite grid at distance $\ell > 0$. Let k agents move performing mutually independent Pareto walks with $\alpha \in (1, 2]$. If $k = \tilde{\Theta}(\ell)$, then k agents find the treasure in time $\Theta(\ell)$, making a total work of $\tilde{\Theta}(\ell^2)$, w.h.p.²³ Furthermore, the result is almost-tight in a two-fold sense:*

- (i) *for all $k = \tilde{\Theta}(\ell^{1-\epsilon})$ for any arbitrary constant $\epsilon \in (0, 1]$, then the agents never find the treasure w.h.p., thus making an infinite work, w.h.p.;*
- (ii) *for all $k = \tilde{\Theta}(\ell^{1+\epsilon})$ for any arbitrary constant $\epsilon > 0$, then the agents need time $\Theta(\ell)$ to find the treasure and total work $k = \tilde{\Theta}(\ell^{2+\epsilon})$, w.h.p.*

7.1 Main tools

Let \mathbf{P}_t be the two-dimensional random variable representing the coordinates of the nodes a Pareto walk visits at time t . For the sake of the analysis, we will consider the Pareto flight and Pareto run processes we defined in Section 4 (definitions 7 and 8, respectively), whose definitions we recall.

Definition 7 (α -Pareto flight). Let $\alpha > 1$ be a real constant. At each round, the agent chooses a distance d with distribution $\frac{c_\alpha}{(1+d)^\alpha}$, where c_α is a normalization constant, and chooses u.a.r. one node u among the $4d$ nodes of the grid at distance d from its current position. Then, in one step/unit time, the agent reaches u . Once reached u , the agent repeats the procedure above, and so on. If the chosen distance d is equal to zero, the agent keeps still for one time unit and then it repeats the procedure.

Definition 8 (α -Pareto run). Let $\alpha > 1$ be a real constant. At each round, the agent chooses a distance d with distribution $\frac{c_\alpha}{(1+d)^\alpha}$, where c_α is a normalization constant, and chooses a direction according to the procedure in Definition 2. Then, it walks along the corresponding direction-approximating path (visiting all the path nodes) in one step/unit time until it reaches the end-point v of the path at distance d . Once v is reached, the agent repeats all the procedure, and so on. If the chosen distance d is equal to zero, the agent keeps still for one time unit and then it repeats the procedure.

Let \mathbf{P}_t^f be the two-dimensional random variable representing the coordinates of the nodes this process visits after the t -th jump. Let also \mathbf{P}_t^r be the two-dimensional random variable representing the coordinates of this process after the t -th jump.

²²For a formal comprehension of this statement, we refer to Remark 1 in the Preliminaries (Section 3).

²³For a formal comprehension of this statement, we refer to Remark 1 in the Preliminaries (Section 3).

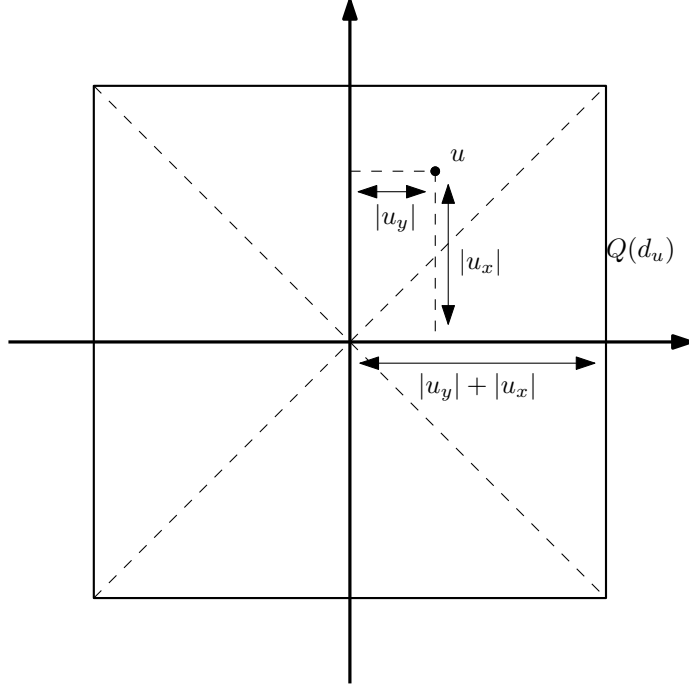


Figure 10: The set $Q(d_u)$.

In this subsection, we show some preliminaries necessary to the analysis of the Pareto walk model in the cases $\alpha \in (1, 2]$ and $\alpha \in (3, +\infty)$. We will make use of Lemma 1 in the Preliminaries (Section 3), which we state again.

Lemma 1. *Let u be any node of \mathbb{Z}^2 , $d \geq 1$, and $v \in R_d(u)$. Suppose an agent is on u and chooses a direction according to the procedure in Definition 2. Then, there is probability $1/(4d)$ that the corresponding direction-approximating path crosses v .*

Starting from this, we can prove the following.

Lemma 20. *Consider an agent performing a Pareto walk for any parameter $\alpha > 1$ which is located at distance $d \geq 0$ from the some node v . Then, the probability that it visits v during the next jump is $\Theta(1/d^\alpha)$.*

Proof of Lemma 20. Then, for Lemma 1, the probability to choose a direction leading to T is $\Theta(1/d)$. Independently, the probability to choose to walk for a distance at least d across the chosen direction is $\Theta(1/d^{\alpha-1})$ by Equation (3). Thus, the probability to eventually reach v is $\Theta(1/d^\alpha)$. \square

Furthermore, the following “monotonicity” property holds.

Lemma 21. *Consider an agent performing a Pareto flight for any parameter $\alpha > 1$ starting at the origin. For any $w \in \mathbb{Z}^2$, let $p_{w,t}$ be the probability that, at the end of the t -th jump, the agent is in w . Let $u \in \mathbb{Z}^2$ be an arbitrary node. Then, for each node $v \notin Q(d_u)$ and each step t , it holds that $p_{u,t} \geq p_{v,t}$, where*

$$Q(d_u) = \{(x, y) \in \mathbb{Z}^2 : \max(|x|, |y|) \leq d_u\}.$$

Proof of Lemma 21. The Pareto flight mobility model fulfill the hypothesis of Lemma 32 in Appendix C. The latter gives the desired result. \square

Figure 10 gives an idea of the geometrical shape of this “monotonicity” property.

8 Pareto Walk Model: Case $\alpha \in (3, +\infty)$

This section is devoted to the proof of Theorem 6, which we state again.

Theorem 6 (Diffusive regime). *Assume that the treasure is located in some node of the infinite grid at distance $\ell > 0$. Let k agents move performing mutually independent Pareto walks with $\alpha > 3$. If $k = \log^{\mathcal{O}(1)}(\ell)$, then k agents find the treasure in time $\tilde{\Theta}(\ell^2)$, making a total work of $\tilde{\Theta}(\ell^2)$, w.h.p.²⁴ Furthermore, the result is almost-tight in the following sense:*

- (i) *if $k = \tilde{\Theta}(\ell^\epsilon)$ for any fixed constant $\epsilon \in [0, 3 - \alpha)$, then the agents need time $\tilde{\Omega}(\ell^2)$ to find the treasure and total work $\tilde{\Omega}(\ell^{2+\epsilon})$, w.h.p.;*
- (ii) *if $k = \tilde{\Theta}\left(\ell^{\alpha-3+\min(\frac{3}{2}\epsilon, \frac{1}{2}+\epsilon)}\right)$ for any $\epsilon \geq 0$, then k agents need time at least $\Omega(\ell^{2-\epsilon})$ to find the treasure, w.h.p., making a total work of $\tilde{\Omega}\left(\ell^{2+(\alpha-3)+\min(\frac{1}{2}\epsilon, \frac{1}{2})}\right)$.*

We invite the reader to compare the above theorem with Theorem 4, in order to see that the performances of the Pareto walks with $\alpha \in (3, +\infty)$ are the same as that of the simple random walks. Furthermore, from the central limit theorem (Theorem 8 in Appendix A), it is clear that the point-wise distribution after long time is roughly the same between the two models.

8.1 Main tools and general scheme

In order to prove the above theorem, we need three results which we are going to prove in the following subsections. Let T be the node in which the treasure is located, with $d_T = \ell$. The first result is an upper bound on the hitting time of the treasure.

Proposition 7. *For some $t = \Theta(\ell^2 \log^2 \ell)$, the probability an agent performing the Pareto walk with $\alpha > 3$ visits the treasure within time t is $\Omega(1/(\log^4 \ell))$.*

Subsection 8.2 is devoted to the proof of this result. Furthermore we need the two following lower bounds, which will be proved in Subsection 8.3.

Proposition 8. *Let k be any integer such that $k = \mathcal{O}(\ell^\epsilon)$ for some constant $\epsilon \in [0, \alpha - 3)$. Then, k agents performing the Pareto walk with parameter $\alpha > 3$ find the treasure in time at least $\Omega(\ell^2/(\log \ell))$, w.h.p.*

Proposition 9. *Consider a single agent performing a Pareto walk for $\alpha > 3$. Let $\epsilon \geq 0$ be any arbitrary small constant, and let $t = \Theta(\ell^{2-\epsilon})$. Then, the probability to find the treasure within time t is $\mathcal{O}(1/\ell^{\alpha-3+\min(2\epsilon, 1+\epsilon)})$.*

8.1.1 Wrap-up: proof of Theorem 6

We are now ready to prove our main result.

Proof of Theorem 6. First, notice that if t is some $\Theta(\ell^2 \log^2 \ell)$, then, by Proposition 7, the probability an agent performing a Pareto walk with $\alpha > 3$ visits the treasure at least once within time t is $\Omega(1/(\log^4 \ell))$. Then, it is clear that $\Theta(\log^5 \ell)$ agents performing Pareto walks find the treasure within time t , w.h.p., making a total work $\mathcal{O}(\ell^2 \log^6 \ell)$. Furthermore, if we increase the number of agents by multiplying by any polylogarithmic factor, the same upper bound on the hitting time holds, w.h.p., while if we decrease it by dividing by any polylogarithmic factor, the upper bound on the hitting time holds with non-negligible probability.

²⁴For a formal comprehension of this statement, we refer to Remark 1 in the Preliminaries (Section 3).

At the same time, let $k = \tilde{\Theta}(\ell^\epsilon)$ for any $\epsilon \in [0, \alpha - 3)$. Proposition 8 tells us that the time k agents need to find the treasure is at least $\Omega(\ell^2/\log \ell)$, w.h.p., thus making a total work equal to $\tilde{\Omega}(\ell^{2+\epsilon})$ (this includes also each $k = \log^{\mathcal{O}(1)}(\ell)$), giving us both the lower bound of the main claim and Step i.

On the other hand, let $k = \tilde{\Theta}(\ell^{\alpha-3+\min(\frac{3}{2}\epsilon, \frac{1}{2}+\epsilon)})$ for any $\epsilon \geq 0$. For Proposition 9, we have that the probability k agents find the treasure within time $t = \Theta(\ell^{2-\epsilon})$ is, by the union bound, $\tilde{\mathcal{O}}(1/\ell^{\min(\frac{1}{2}\epsilon, \frac{1}{2})})$. Thus, w.h.p., the agents find the treasure in time at least $\Omega(\ell^{2-\epsilon})$, and the total work is $\tilde{\Omega}(\ell^{2+(\alpha-3)+\min(\frac{1}{2}\epsilon, \frac{1}{2})})$, giving us Step ii. \square

8.2 Analysis of the case $\alpha \in (3, +\infty)$: proof of Proposition 7

This section aims at proving the following result.

Proposition 7. *For some $t = \Theta(\ell^2 \log^2 \ell)$, the probability an agent performing the Pareto walk with $\alpha > 3$ visits the treasure within time t is $\Omega(1/(\log^4 \ell))$.*

The analysis we're going through is very similar to that made in Subsection 4.2. In order to show the result, we have to analyze the Pareto flight for $\alpha > 3$, and then link the results for the Pareto flight to the Pareto walk through a coupling result.

We look at a single agent moving on the grid \mathbb{Z}^2 performing a Pareto flight with $\alpha > 3$ which starts at the origin $o = (0, 0)$. We are going to introduce some definitions and notations we use throughout the analysis. For any node $u = (u_x, u_y)$ of the grid, define the random variable

$$Z_u(t) = \text{number of agent's visits at node } u \text{ within } t \text{ steps.}$$

In order to bound the probability that the node u has been visited at least once at time t , namely $\mathbb{P}(Z_u(t) > 0)$, we define

$$p_{u,i} = \mathbb{P}(\text{the agent is in node } u \text{ at step } i).$$

By the definitions above, we easily get that

$$\mathbb{E}[Z_u(t)] = \sum_{i=0}^t p_{u,i}.$$

8.2.1 Road-map of the analysis

The scheme of the proof follows the same structure and main ideas of that in Subsection 4.2. We omit such an informal description and go directly with the main steps of the proof. Let T be the node in which the treasure is located. We divide \mathbb{Z}^2 in three different regions, namely, for any $\delta \geq 0$ and any $t = \Omega(\ell^2)$,

$$\begin{aligned} \mathcal{A}_1 &= Q(\ell) = \{(x, y) : \max(|x|, |y|) \leq \ell\}; \\ \mathcal{A}_2 &= \{v \in \mathbb{Z}^2 : |v|_1 \leq 4\sqrt{2(1+\delta)t} \cdot \log t\} \setminus \mathcal{A}_1; \\ \mathcal{A}_3 &= \{v \in \mathbb{Z}^2 : |v|_1 > 4\sqrt{2(1+\delta)t} \cdot \log t\}. \end{aligned}$$

Intuitively, \mathcal{A}_1 contains all the nodes which are ‘‘close’’ to the origin than the treasure; \mathcal{A}_3 contains all the nodes which are distant from the origin at least a polylogarithmic factor more than the treasure; \mathcal{A}_2 contains all the nodes in the between.

The key steps of the analysis are the following.

1. We first show that $\mathbb{E}[Z_o(t)] = c_t = \mathcal{O}(\log^2 t)$ (see Lemma 22 for details). Then we show $\mathbb{E}[Z_u(t)] \leq c_t$ for all u (see Lemma 23 for details), which implies that the average number of visits in \mathcal{A}_1 until time t is at most $c_t \cdot |Q(\ell)|$.
2. From Lemma 21, $\mathbb{E}[Z_u(t)] \geq \mathbb{E}[Z_v(t)]$ for all but $m_u = |Q(d_u)|$ many v (see Corollary 5), where $Q(d_u) = \{(x, y) : \max(|x|, |y|) \leq d_u\}$. This means that the average number of visits in \mathcal{A}_2 is at most $\mathbb{E}[Z_u(t)] \cdot (32(1 + \delta)t \log^2 t)$.
3. Using Chernoff-Hoeffding bounds, for any fixed $\delta > 0$, we can show

$$\sum_{v \in \mathbb{Z}^2 : 4\sqrt{2(1+\delta)t \log t}} \mathbb{E}[Z_v(t)] = \mathcal{O}\left(t^{1-\frac{\alpha-3}{2}}\right).$$

for any $t > 0$ (see Lemma 24 for details), thus bounding the average number of visits to \mathcal{A}_3 .

4. If T is the node where the treasure is located, from steps 1 to 3 we get that

$$m_T \cdot c_t + \mathbb{E}[Z_T(t)] \cdot (32(1 + \delta)t \log^2 t) + \mathcal{O}\left(t^{1-\frac{\alpha-3}{2}}\right) \geq t,$$

for any $t = \Omega(\ell^2)$ (see Lemma 25 for details), which implies

$$\mathbb{E}[Z_T(t)] \geq \frac{t - m_T c_t - \mathcal{O}\left(t^{1-\frac{\alpha-3}{2}}\right)}{(32(1 + \delta)t \log^2 t) - m_T},$$

for any $t = \Omega(\ell^2)$.

5. Using the result in Step 1, we show that (see Step 5 for details)

$$\mathbb{E}[Z_u(t)] \geq \mathbb{P}(Z_u(t) > 0) = \frac{\mathbb{E}[Z_u(t)]}{\mathbb{E}[Z_u(t) | Z_u(t) > 0]} \geq \frac{\mathbb{E}[Z_u(t)]}{c_t}.$$

6. From steps 4 and 5,

$$\mathbb{P}(Z_T(t) > 0) \geq \frac{t - m_T c_t - \mathcal{O}\left(t^{1-\frac{\alpha-3}{2}}\right)}{(32(1 + \delta)t \log^2 t - m_T) c_t},$$

for any $t = \Omega(\ell^2)$ (see Corollary 6 for details). We show that the same holds for the Pareto walk with the same α through a coupling result (Lemma 26). Substituting $t = \Theta(\ell^2 \log^2 \ell)$, we get Proposition 7.

8.2.2 Full analysis

We start estimating the average number of visits to the origin until time t (Step 1 in the scheme).

Lemma 22. *For any $t \geq 0$, $\mathbb{E}[Z_o(t)] = c_t = \mathcal{O}(\log^2 t)$.*

Proof of Lemma 22. First, we show the following. Let $\mathbf{P}_{t'}^f$ be the two dimensional random variable representing the coordinates of the agent performing the Pareto flight at time t' . Consider the projection of the Pareto flight on the x -axis, namely the random variable $X_{t'}$ such that $\mathbf{P}_{t'}^f = (X_{t'}, Y_{t'})$. The random variable $X_{t'}$ can be expressed as the sum of t' random variables S_j^x , $j =$

$1, \dots, t'$, representing the jumps (with sign) the projection of the walk takes at each of the t' rounds. The partial distribution of the jumps along the x -axis is given by the following. For any $d \geq 0$,

$$\mathbb{P}(S_j^x = \pm d) = \frac{c_\alpha}{4d(1+d)^\alpha} \mathbb{1}_{d \neq 0} + \frac{c_\alpha}{2^\alpha} \mathbb{1}_{d=0} + \sum_{k=1+d}^{\infty} \frac{c_\alpha}{2k(1+k)^\alpha}, \quad (14)$$

where $\mathbb{1}_{d \in A}$ yields 1 if $d \in A$ and 0 otherwise, the term

$$\frac{c_\alpha}{4d(1+d)^\alpha} \mathbb{1}_{d \neq 0} + \frac{c_\alpha}{2^\alpha} \mathbb{1}_{d=0}$$

is the probability that the original jump lies along the horizontal axis and has “length” exactly d (or $-d$), and the terms $\frac{c_\alpha}{2k(1+k)^\alpha}$ are the probability that the original jump has “length” exactly k and its projection on the horizontal axis has “length” d (or $-d$). Quantity (14) is at least

$$\frac{c_\alpha}{4} \left(\frac{1}{(1+d)^\alpha} + \sum_{k=1+d}^{\infty} \frac{1}{k(1+k)^\alpha} \right),$$

and at most

$$c_\alpha \left(\frac{1}{(1+d)^\alpha} + \sum_{k=1+d}^{\infty} \frac{1}{k(1+k)^\alpha} \right).$$

By the integral test (Fact 1 in Appendix A) we know that this probability is

$$\mathbb{P}(S_j^x = \pm d) = \Theta \left(\frac{1}{(1+d)^\alpha} \right).$$

Since $\mathbb{E}[Z_o(t)] = \sum_{k=1}^t p_{o,k}$, it suffices to accurately bound the probability $p_{o,k}$ for each $k = 1, \dots, t$. Let partition the natural numbers in the following way

$$\mathbb{N} = \bigcup_{t'=0}^{\infty} \left[\mathbb{N} \cap [2t' \log t', 2(t'+1) \log(t'+1)) \right].$$

For each $k \in \mathbb{N}$, there exists t' such that $k \in [2t' \log t', 2(t'+1) \log(t'+1))$. Then, within $2t' \log t'$ steps the walk has moved to distance $\Theta(\sqrt{t'})$ at least once, with probability $\Omega\left(\frac{1}{(t')^2}\right)$. Indeed, the sequence $\{S_j^x\}_{1 \leq j \leq t'}$ consists of i.i.d. r.v.s with zero mean and constant variance (which comes from the fact that $\alpha > 3$). Thus, the central limit theorem (Theorem 8 in Appendix A) says to us that, for t' large enough, the variable

$$\frac{S_1^x + \dots + S_{t'}^x}{\sigma \sqrt{t'}}$$

converges in distribution to a standard normal random variable Z . Let $\epsilon > 0$ be a small enough constant, then there exists a \bar{t}' large enough, such that for all $t' \geq \bar{t}'$ it holds that

$$\mathbb{P} \left(\frac{S_1^x + \dots + S_{t'}^x}{\sigma \sqrt{t'}} \geq \sigma \sqrt{t} \right) \geq \mathbb{P}(Z \geq 1) - \epsilon,$$

which is a constant since $\mathbb{P}(Z \geq 1)$ is a constant. The symmetrical results in which the normalized sum is less than $-\sigma \sqrt{t}$ holds analogously. Thus, for all $t' \geq \bar{t}'$, we have that $\sum_{j=1}^{t'} |S_j^x| \geq \sigma \sqrt{t'}$ with constant probability $c > 0$. In $2t' \log t'$ jumps, we have $2 \log t'$ sets of t' consequent i.i.d. such jumps.

For independence, the probability that at least in one round before round $2t' \log t'$ the Pareto flight has displacement $\Theta(\sqrt{t})$ from the origin is at least

$$1 - (1 - c)^{2 \log t'} = 1 - \mathcal{O}\left(\frac{1}{(t')^2}\right).$$

Once reached such a distance, there are at least $\lambda^2 = \Theta(t')$ different nodes that are at least as equally likely as o to be visited at any given future time (from Lemma 21 in Subsection 7.1). Thus, the probability to reach the origin at any future time is at most $\mathcal{O}(1/t')$, in particular the bounds holds for $p_{o,k}$. Observe that in an interval $[2t' \log t', 2(t'+1) \log(t'+1))$ there are

$$2(t'+1) \log(t'+1) - 2t' \log t' = 2t' \left[\log\left(1 + \frac{1}{t'}\right) \right] + 2 \log(t'+1) = \mathcal{O}(\log t')$$

integers. Let \mathbf{P}_t^f be the two-dimensional random variable denoting the node a Pareto flight which has started in the origin visits at time t , and let $H_{t'}$ be the event that in any time before $2t' \log t'$ the Pareto flight has displacement at least $\Theta(\sqrt{t})$. Observe that

$$p_{o,k} = \mathbb{P}\left(\mathbf{P}_t^f = o \mid H_{t'}\right) \mathbb{P}(H_{t'}) + \mathbb{P}\left(\mathbf{P}_t^f = o \mid H_{t'}^C\right) \mathbb{P}(H_{t'}^C),$$

by the law of total probability. Thus, if $I_{t'} = [2t' \log t', 2(t'+1) \log(t'+1))$, we have

$$\begin{aligned} \sum_{k=1}^t p_{o,k} &\leq \sum_{t'=0}^t \sum_{k \in I_{t'}} p_{o,k} \\ &\leq \sum_{t'=0}^t \left[\mathbb{P}\left(\mathbf{P}_t^f = o \mid H_{t'}\right) \mathbb{P}(H_{t'}) + \mathbb{P}\left(\mathbf{P}_t^f = o \mid H_{t'}^C\right) \mathbb{P}(H_{t'}^C) \right] \mathcal{O}(\log t') \\ &\leq \sum_{t'=0}^t \left[\mathcal{O}\left(\frac{1}{t'}\right) + \mathcal{O}\left(\frac{1}{(t')^2}\right) \right] \mathcal{O}(\log t') = \mathcal{O}(\log^2 t). \end{aligned}$$

□

Then, the following easily holds (steps 1 and 5 in the scheme).

Lemma 23. *Let $u \in \mathbb{Z}^2$ be any node. It holds that*

- (i) $\mathbb{E}[Z_u(t)] \leq c_t$
- (ii) $1 \leq \mathbb{E}[Z_u(t) \mid Z_u(t) > 0] \leq c_t$;
- (iii) $\mathbb{E}[Z_u(t)] / c_t \leq \mathbb{P}(Z_u(t) > 0) \leq \mathbb{E}[Z_u(t)]$.

Proof of Lemma 23. The proof is exactly the same as that of Lemma 16 in Section 5. □

Furthermore, from Lemma 21 in Subsection 7.1, the following holds (Step 2 in the scheme).

Corollary 5. *For any u in \mathbb{Z}^2 , we have $\mathbb{E}[Z_u(t)] \geq \mathbb{E}[Z_v(t)]$ for all $v \notin Q(d_u)$ (see Figure 10 for geometrical details).*

Namely, almost all the nodes that are “further” than u from the origin are less likely to be visited at any given future time. We also give a bound to the average number of visits to nodes that are further roughly $\sqrt{t} \cdot \log t$ from the origin (Step 3 in the scheme).

Lemma 24. *There exists a constant $\delta > 0$ such that*

$$\sum_{v \in \mathbb{Z}^2 : 4\sqrt{2(1+\delta)t \log t}} \mathbb{E}[Z_v(t)] = \mathcal{O}\left(t^{1-\frac{\alpha-3}{2}}\right).$$

Proof of Step 3. Consider a single agent moving according the Pareto walk with parameter $\alpha \in (3, +\infty)$. Then, by Equations (1) and (2) in the Preliminaries (Section 3), the expectation and the variance of a single jump-length is finite and the variance is finite. By Equation (3) in the Preliminaries (Section 3), the probability a jump length is at least \sqrt{t} is $\Theta\left(1/t^{\frac{\alpha-1}{2}}\right)$. Let us call A_j the event that the j -th jump-length is less than \sqrt{t} . Let us also define \mathbf{P}_j^f the random variable denoting the coordinates of the nodes the corresponding Pareto flight visits at the j -th jump. We can see this random variable as a couple (X_j, Y_j) , where X_j is x -coordinate of the ballistic Pareto walk after the j -th jump, and Y_j is the y -coordinate. Then, X_j can be seen as the sum $\sum_{i=1}^j S'_i$ of j random variables representing the projections of the jumps along the x -axis. Trivially, $\mathbb{E}[X_j] = 0$ for each j , while $\text{Var}(X_j) = j\text{Var}(S'_1) = \Theta(j)$ since S'_1 has finite variance. This comes by observing that $S'_1 \leq S_1$. Then, conditioning on $A = \cap_{i=1}^t A_i$, we can apply the Chernoff bound (Lemma 31 in Appendix A) on the sum of the first j jumps, for $j \leq t$. We have

$$\begin{aligned} \mathbb{P}\left(|X_t| \geq 2\sqrt{2(1+\delta)t \log t} \mid A\right) &\leq 2 \exp\left(-\frac{8(1+\delta)t \cdot \log^2 t}{\Theta(t) + \Theta\left(\sqrt{(1+\delta)t \cdot \log t}\right) \sqrt{t}}\right) \\ &= 2 \exp\left(-\Theta\left(\sqrt{1+\delta} \cdot \log t\right)\right) \\ &\leq \frac{2}{t^{\Theta(\sqrt{1+\delta})}}, \end{aligned}$$

which is less than $1/t^{\frac{\alpha-1}{2}}$ if we choose δ big enough. The same result holds for the random variable X_j for each $j < t$, since the variance of X_j is smaller than the variance of X_t . Notice that

$$\begin{aligned} \mathbb{P}\left(\cap_{j=1}^t \{|X_j| < 2\sqrt{2(1+\delta)t \log t}\} \mid A\right) &= 1 - \mathbb{P}\left(\cup_{j=1}^t \{|X_t| \geq 2\sqrt{2(1+\delta)t \log t}\} \mid A\right) \\ &\geq 1 - \frac{t}{t^{\frac{\alpha-1}{2}}} = 1 - \frac{1}{t^{\frac{\alpha-3}{2}}}, \end{aligned}$$

and that

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^C) = 1 - \mathbb{P}\left(\cup_{j=1}^t A_j^C\right) \geq 1 - \mathcal{O}\left(\frac{t}{t^{\frac{\alpha-1}{2}}}\right) = 1 - \mathcal{O}\left(\frac{1}{t^{\frac{\alpha-3}{2}}}\right).$$

An analogous argument holds for the random variable Y_t conditioned to the event A . Then,

$$\begin{aligned} &\mathbb{P}\left(\cap_{j=1}^t \{|X_j|_1 < 2\sqrt{2(1+\delta)t \cdot \log t}\}, \cap_{j=1}^t \{|Y_j|_1 < 2\sqrt{2(1+\delta)t \cdot \log t}\}\right) \\ &\geq \mathbb{P}\left(\cap_{j=1}^t \{|X_j|_1 < 2\sqrt{2(1+\delta)t \cdot \log t}\}, \cap_{j=1}^t \{|Y_j|_1 < 2\sqrt{2(1+\delta)t \cdot \log t}\} \mid A\right) \mathbb{P}(A) \\ &\geq \left(2\mathbb{P}\left(\cap_{j=1}^t \{|X_j|_1 < 2\sqrt{2(1+\delta)t \cdot \log t}\} \mid A\right) - 1\right) \mathbb{P}(A) \\ &\stackrel{(5)}{\geq} \left[2\left(1 - \frac{1}{t^{\frac{\alpha-3}{2}}}\right) - 1\right] \left(1 - \mathcal{O}\left(\frac{1}{t^{\frac{\alpha-3}{2}}}\right)\right) \\ &\geq \left(1 - \frac{1}{t^{\frac{\alpha-3}{2}}}\right) \left(1 - \mathcal{O}\left(\frac{1}{t^{\frac{\alpha-3}{2}}}\right)\right) \\ &\geq 1 - \mathcal{O}\left(\frac{1}{t^{\frac{\alpha-3}{2}}}\right), \end{aligned}$$

where (a) holds for symmetry (the distribution of Y_t is the same as the one of X_t) and for the union bound. Thus, in t jumps (which take at least time t), the walk has never reached distance $4\sqrt{2(1+\delta)t \cdot \log t}$, w.h.p. We denote this event as E . The average number of visits until time t to nodes at distance at least $4\sqrt{2(1+\delta)t \cdot \log t}$ is then less than $t \cdot \mathcal{O}\left(1/t^{\frac{\alpha-3}{2}}\right) = \mathcal{O}\left(t^{1-\frac{\alpha-3}{2}}\right)$. \square

The following puts together the previous estimations in order to get a lower bound on the average number of visits to some node u (Step 4 in the scheme).

Lemma 25. *Let T be the node in which the treasure is located. For any $t = \Omega(\ell^2)$, the following holds:*

$$m_T c_t + \mathbb{E}[Z_T(t)] \cdot 32(1+\delta)(t \log^2 t) + \mathcal{O}\left(t^{1-\frac{\alpha-3}{2}}\right) \geq t.$$

Proof of Lemma 25. Suppose the agent has made t jumps, thus visiting t nodes. Then,

$$\mathbb{E}\left[\sum_{v \in \mathbb{Z}^2} Z_v(t)\right] = t.$$

We divide the plane in different zones, and we bound the number of visits over each zone in expectation. First, we focus on item (1). From Lemma 23, the number of visits inside $\mathcal{A}_1 = Q(\ell)$ until time t is at most $m_T c_t$, where $m_T = |Q(\ell)| = 4\ell^2$. From Lemma 24, the number of visits \mathcal{A}_3 is at most $\mathcal{O}\left(t^{1-\frac{\alpha-3}{2}}\right)$. Each of the remaining nodes, i.e. the nodes in \mathcal{A}_2 , which are at most $32(1+\delta)(t \log^2 t)$ in total, is visited by the agent at most $\mathbb{E}[Z_u(t)]$ times, for Corollary 5. Then, we have that

$$m_T c_t + \mathbb{E}[Z_T(t)] \cdot 32(1+\delta)(t \log^2 t) + \mathcal{O}\left(t^{1-\frac{\alpha-3}{2}}\right) \geq t. \quad \square$$

Finally, we prove (Step 6 in the scheme) the following.

Corollary 6. *For any $t = \Omega(\ell^2)$, the probability to have visited T within time t is*

$$\mathbb{P}(Z_T(t) > 0) \geq \frac{t - m_T c_t - \mathcal{O}\left(t^{1-\frac{\alpha-3}{2}}\right)}{(t \log t - m_T) c_t}.$$

Proof of Corollary 6. The proof follows from the combination of Lemma 25 and Lemma 23. \square

We just have to show a coupling result in the following subsection (Step 6 in the scheme).

8.2.3 Coupling result

Here we show the link between the Pareto flight and the Pareto walk, showing that, essentially, the Pareto walk is at least as efficient as the Pareto flight. Indeed, an agent who performs t jumps of the Pareto walk (thus it has moved t steps according to the Pareto flight) with $\alpha \in (3, +\infty)$, has walked for time $\Theta(t)$, w.h.p.

Lemma 26. *Consider an agent performing the Pareto flight with $\alpha \in (3, +\infty)$, making t jumps. Then, the sum of all jump lengths is $\Theta(t)$, w.h.p.*

Proof of Lemma 26. The proof is easy. If S_i is the random variable yielding the i -th jump length, then it has finite expectation and variance. This means that the sum $\bar{S}_t = \sum_{i=1}^t S_i$ has expectation $\Theta(t)$ and variance $\Theta(t)$. Then, by Chebyshev's inequality

$$\mathbb{P}(\bar{S}_t \geq \Theta(t) + t) \leq \frac{\text{Var}(\bar{S}_t)}{t^2} = \mathcal{O}\left(\frac{1}{t}\right).$$

Thus, w.h.p., the sum of all jump lengths is $\Theta(t)$, w.h.p. \square

8.2.4 Wrap-up: proof of Proposition 7

The proof is now easy. Let T be the node in which the treasure is located ($d_T = \ell$). For t being some function in $\Theta(\ell^2 \log^2 \ell)$, by Corollary 6 we have that the probability the Pareto flight visits the treasure is

$$\Omega\left(\frac{1}{\log^3 \ell}\right).$$

For Lemma 26, and considering also the jumps in which the agent keeps still (which are at most $\Theta(t)$), the same holds for an agent performing the Pareto walk.

8.3 Analysis of the case $\alpha \in (3, +\infty)$: proof of propositions 8 and 9

In this subsection, we aim at proving the two results giving lower bounds on the hitting time of the treasure. First, we prove a result saying that $\Theta(\ell^\epsilon)$ agents need time at least $\tilde{\Omega}(\ell^2)$ to find the treasure, w.h.p., for $\epsilon \in [0, 3\alpha)$, letting the work to be $\tilde{\Omega}(\ell^{2+\epsilon})$.

Proposition 8. *Let k be any integer such that $k = \mathcal{O}(\ell^\epsilon)$ for some constant $\epsilon \in [0, \alpha - 3)$. Then, k agents performing the Pareto walk with parameter $\alpha > 3$ find the treasure in time at least $\Omega(\ell^2/(\log \ell))$, w.h.p.*

Proof of Proposition 8. Consider a single agent moving according the Pareto walk with parameter $\alpha \in (3, +\infty)$. Then, by eqs. (1) and (2) in the Preliminaries (Section 3), the expectation and the variance of a single jump-length is finite and the variance is finite. By Equation (3) in the Preliminaries (Section 3), the probability a jump length is less than $\ell/(\log \ell)$ is $1 - \Theta((\log \ell)^{\alpha-1}/\ell^{\alpha-1})$. Let us call A_j the event that the j -th jump-length is less than $\ell/(\log \ell)$. Let us also define \mathbf{P}_t^f the random variable denoting the coordinates of the nodes the corresponding Pareto flight visits at the t -th jump. We can see this random variable as a couple (X_t, Y_t) , where X_t is x -coordinate of the ballistic Pareto walk after the t -th jump, and Y_t is the y -coordinate. Then, X_t can be seen as the sum $\sum_{j=1}^t S'_j$ of t random variables representing the projections of the jumps along the x -axis. Trivially, $\mathbb{E}[X_t] = 0$ for each t , while $\text{Var}(X_t) = t\text{Var}(S'_1) = \Theta(t)$ since S'_1 has finite variance. This comes by observing that $S'_1 \leq S_1$. Then, conditioning on $A = \bigcap_{j=1}^{\delta \ell^2/(\log \ell)} A_j$, we can apply the Chernoff bound (Lemma 31 in A) on the sum of the first $\delta \ell^2/(\log \ell)$ jumps, for some δ accurately chosen. We have

$$\mathbb{P}\left(|X_{\delta \ell^2/(\log \ell)}| \geq \ell \mid A\right) \leq 2 \exp\left(-\frac{\ell^2}{\Theta(\delta \ell^2/(\log \ell)) + (\ell/(\log \ell))\ell}\right) = 2 \exp\left(-\Theta\left(\frac{\log \ell}{\delta}\right)\right) \leq \frac{2}{\ell^{\Theta(1/\delta)}},$$

which is less than $1/\ell^{\alpha-1}$ if we choose δ small enough. The same result holds for the random variable X_t for each $t < \delta \ell^2/(\log \ell)$, since the variance of X_t is smaller than the variance of $X_{\delta \ell^2/(\log \ell)}$. Notice that

$$\mathbb{P}\left(\bigcap_{t=1}^{\delta \ell^2/(\log \ell)} \{|X_t| < \ell\} \mid A\right) = 1 - \mathbb{P}\left(\bigcup_{t=1}^{\delta \ell^2/(\log \ell)} \{|X_t| \geq \ell\}\right) \geq 1 - \frac{\delta \ell^2/(\log \ell)}{\ell^{\alpha-1}} = 1 - \frac{\delta}{\ell^{\alpha-3} \log \ell},$$

and that

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^C) = 1 - \mathbb{P}\left(\bigcup_{j=1}^{\delta \ell^2/(\log \ell)} A_j^C\right) \geq 1 - \frac{\delta \ell^2 (\log \ell)^{\alpha-1}}{\ell^{\alpha-1} \log \ell} = 1 - \frac{\delta (\log \ell)^{\alpha-2}}{\ell^{\alpha-3}}.$$

An analogous argument holds for the random variable Y_t conditioned to the event A . Then,

$$\mathbb{P}\left(\bigcap_{t=1}^{\delta \ell^2/(\log \ell)} \{X_t < \ell\}, \bigcap_{t=1}^{\delta \ell^2/(\log \ell)} \{Y_t < \ell\}\right) \geq \mathbb{P}\left(\bigcap_{t=1}^{\delta \ell^2/(\log \ell)} \{X_t < \ell\}, \bigcap_{t=1}^{\delta \ell^2/(\log \ell)} \{Y_t < \ell\} \mid A\right) \mathbb{P}(A)$$

$$\begin{aligned}
&\geq \left(2\mathbb{P}\left(\bigcap_{t=1}^{\delta\ell^2/(\log\ell)}\{X_t < \ell\} \mid A\right) - 1\right)\mathbb{P}(A) \\
&\stackrel{(a)}{\geq} \left[2\left(1 - \frac{\delta}{\ell^{\alpha-3}\log\ell}\right) - 1\right]\left(1 - \frac{\delta(\log\ell)^{\alpha-2}}{\ell^{\alpha-3}}\right) \\
&\geq \left(1 - \frac{\delta}{\ell^{\alpha-3}\log\ell}\right)\left(1 - \frac{\delta(\log\ell)^{\alpha-2}}{\ell^{\alpha-3}}\right) \\
&\geq 1 - \frac{\delta(1 + (\log\ell)^{\alpha-1})}{\ell^{\alpha-3}\log\ell},
\end{aligned}$$

where (a) holds for symmetry (the distribution of Y_t is the same as the one of X_t) and for the union bound. Thus, in $\delta\ell^2/(\log\ell)$ jumps (which take at least time $\delta\ell^2/(\log\ell)$), the walk has never reached distance ℓ , w.h.p. If the number of agents is $\mathcal{O}(\ell^\epsilon)$ for some $\epsilon \in [0, 3 - \alpha)$, there is probability $1 - \tilde{\mathcal{O}}(1/\ell^{\alpha-3-\epsilon})$ that the treasure is not found within time $\delta\ell^2/(\log\ell)$. \square

The lower bound estimates the probability to find the treasure within time $t = \Theta(\ell^{2-\epsilon})$, for any $\epsilon > 0$.

Proposition 9. *Consider a single agent performing a Pareto walk for $\alpha > 3$. Let $\epsilon \geq 0$ be any arbitrary small constant, and let $t = \Theta(\ell^{2-\epsilon})$. Then, the probability to find the treasure within time t is $\mathcal{O}(1/\ell^{\alpha-3+\min(2\epsilon, 1+\epsilon)})$.*

Proof of Proposition 9. First consider an agent performing the Pareto run for the same α . Let X_i be the x -coordinate of the agent at the end of the i -th jump. For any $i \leq t$, we bound the probability that $X_i > \ell/4$. The probability that there is a jump whose length is at least ℓ among the first i jumps is $\mathcal{O}(i/\ell^{\alpha-1})$. Conditioning on the event that the first i jump-length are all smaller than ℓ (event C_i), the expectation of X_i is zero and its variance is

$$i \cdot \sum_{d=0}^{\ell/4} \Theta(d^2/(1+d)^\alpha) = \Theta(i\ell^{3-\alpha}),$$

for the integral test (Fact 1 in Appendix A). Chebyshev's inequality implies that

$$\mathbb{P}(|X_i| \geq \ell/4 \mid C_i) \leq \frac{\Theta(i\ell^{3-\alpha})}{\Theta(\ell^2)} = \Theta(i/\ell^{\alpha-1}).$$

Since the conditioning event has probability $1 - \mathcal{O}(i/\ell^{\alpha-1})$, then the “unconditional” probability that of the event $|X_i| \leq \ell/4$ is

$$[1 - \mathcal{O}(i/\ell^{\alpha-1})]^2 = 1 - \mathcal{O}(1/\ell^{\alpha-3+\epsilon}),$$

since $i \leq t = \Theta(\ell^{2-\epsilon})$. The same result holds analogously for Y_i (the y -coordinate of the agent after the i -th jump), obtaining that $|X_i| + |Y_i| \leq \ell/2$ with probability $1 - \mathcal{O}(1/\ell^{\alpha-3+\epsilon})$ by the union bound.

Consider the first jump. The probability it leads the agent to visit the treasure is $\mathcal{O}(1/\ell^\alpha)$ for Lemma 20 in Subsection 7.1. Now, let $2 \leq i \leq t$ and consider the i -th jump. We want to estimate the probability the jump leads the agent to visit the treasure. We call u the node in which the treasure is located, and we consider the rhombus centered in u that contains the nodes at distance at most $\frac{\ell}{4}$ from u , namely

$$R_{\ell/4}^*(u) = \{w \in \mathbb{Z}^2 : d(w, u) \leq \ell/4\}.$$

We call J_i the event that the i -th jump leads the agent to visit the treasure, R_{i-1} the event that the $(i-1)$ -th jump ends in $R_{\ell/4}^*(u)$, and F_{i-1} the event that the $(i-1)$ -th jump ends at distance farther than $\ell/2$ from the origin. Then, by the law of total probabilities, we have

$$\begin{aligned}
\mathbb{P}(J_i) &= \mathbb{P}(J_i | F_{i-1}) \mathbb{P}(F_{i-1}) + \mathbb{P}(J_i | F_{i-1}^C) \mathbb{P}(F_{i-1}^C) \\
&= [\mathbb{P}(J_i | F_{i-1}, R_{i-1}) \mathbb{P}(R_{i-1} | F_{i-1}) + \mathbb{P}(J_i | F_{i-1}, R_{i-1}^C) \mathbb{P}(R_{i-1}^C | F_{i-1})] \mathbb{P}(F_{i-1}) \\
&\quad + \mathbb{P}(J_i | F_{i-1}^C) \mathbb{P}(F_{i-1}^C) \\
&\stackrel{(a)}{\leq} [\mathbb{P}(J_i | R_{i-1}) \mathbb{P}(R_{i-1} | F_{i-1}) + \mathbb{P}(J_i | F_{i-1}, R_{i-1}^C)] \mathbb{P}(F_{i-1}) + \mathbb{P}(J_i | F_{i-1}^C) \mathbb{P}(F_{i-1}^C) \\
&\stackrel{(b)}{\leq} \left[\mathcal{O}\left(\frac{1}{\ell^2}\right) + \mathcal{O}\left(\frac{1}{\ell^\alpha}\right) \right] \mathcal{O}\left(\frac{1}{\ell^{\alpha-3+\epsilon}}\right) + \mathcal{O}\left(\frac{1}{\ell^\alpha}\right) \left[1 - \mathcal{O}\left(\frac{1}{\ell^{\alpha-3+\epsilon}}\right) \right] = \mathcal{O}\left(\frac{1}{\ell^{\alpha-1+\epsilon}} + \frac{1}{\ell^\alpha}\right)
\end{aligned} \tag{15}$$

where in (a) we used that $R_{i-1} \subset F_{i-1}$ and that $\mathbb{P}(R_{i-1}^C | F_{i-1}) \leq 1$, while in (b) we used that

$$\mathbb{P}(J_i | R_{i-1}) \mathbb{P}(R_{i-1} | F_{i-1}) = \mathcal{O}\left(\frac{1}{\ell^2}\right), \text{ (the proof is below)}$$

that $\mathbb{P}(J_i | F_{i-1}, R_{i-1}^C) = \mathcal{O}(1/\ell^\alpha)$ because the jump starts in a node whose distance from the treasure is $\Omega(\ell)$, and that $\mathbb{P}(J_i | F_{i-1}^C) = \mathcal{O}(1/\ell^\alpha)$ for the same reason. As for the term $\mathbb{P}(J_i | R_{i-1}) \mathbb{P}(R_{i-1} | F_{i-1})$ we observe the following. Let \mathbf{P}_i^r be the two-dimensional random variable denoting the coordinates of nodes the agent is located on after the i -th jump. Then

$$\begin{aligned}
\mathbb{P}(J_i | R_{i-1}) \mathbb{P}(R_{i-1} | F_{i-1}) &= \sum_{v \in R_{\ell/4}^*(u)} \mathbb{P}(J_i | \mathbf{P}_i^r = v) \mathbb{P}(\mathbf{P}_i^r = v | F_{i-1}) \\
&\leq \mathcal{O}\left(\frac{1}{\ell^2}\right) \sum_{v \in R_{\ell/4}^*(u)} \mathbb{P}(J_i | \mathbf{P}_i^r = v),
\end{aligned}$$

since Lemma 21 (Subsection 7.1) holds in a consequent way conditioning on F_{i-1} , and since, for each $v \in R_{\ell/4}^*(u)$, there are at least $\Theta(\ell^2)$ nodes at distance at least $\ell/2$ from the origin which are more probable to be visited than v . Then, we proceed like in the proof of Lemma 10 (Subsection 4.3) showing that $\sum_{v \in R_{\ell/4}^*(u)} \mathbb{P}(J_i | \mathbf{P}_i^r = v) = \mathcal{O}(1)$ and we obtain $\mathbb{P}(J_i | R_{i-1}) \mathbb{P}(R_{i-1} | F_{i-1}) = \mathcal{O}(1/\ell^2)$.

Thus, by the union bound and by the inequality (15), the probability that at least one between the t jumps leads the agent to find the treasure is

$$\begin{aligned}
\frac{1}{\ell^\alpha} + (t-1) \mathcal{O}\left(\frac{1}{\ell^{\alpha-1+\epsilon}} + \frac{1}{\ell^\alpha}\right) &= \mathcal{O}(\ell^{2-\epsilon}) \mathcal{O}\left(\frac{1}{\ell^{\alpha-1+\epsilon}} + \frac{1}{\ell^\alpha}\right) \\
&= \mathcal{O}\left(\frac{1}{\ell^{\alpha-3+2\epsilon}} + \frac{1}{\ell^{\alpha-2+\epsilon}}\right) \\
&= \mathcal{O}\left(\frac{1}{\ell^{\alpha-3+\min(2\epsilon, 1+\epsilon)}}\right).
\end{aligned}$$

We conclude observing that the Pareto run is at least as efficient as the Pareto walk, since the first takes just one time unit to perform a jump, while the latter takes a time equal to the jump length. \square

9 Pareto Walk Model: Case $\alpha \in (1, 2]$

In this section, we prove our main result on the performance of Pareto walks with parameter $\alpha \in (1, 2]$, showing that this model is “equivalent” to the ballistic walk model in terms of hitting

time and work efficiency. Indeed, the reader may compare the following result with that for the ballistic walk model (Theorem 5 in Section 6).

Theorem 7 (Ballistic regime). *Assume that the treasure is located in some node of the infinite grid at distance $\ell > 0$. Let k agents move performing mutually independent Pareto walks with $\alpha \in (1, 2]$. If $k = \tilde{\Theta}(\ell)$, then k agents find the treasure in time $\Theta(\ell)$, making a total work of $\tilde{\Theta}(\ell^2)$, w.h.p.²⁵ Furthermore, the result is almost-tight in a two-fold sense:*

- (i) *for all $k = \tilde{\Theta}(\ell^{1-\epsilon})$ for any arbitrary constant $\epsilon \in (0, 1]$, then the agents never find the treasure w.h.p., thus making an infinite work, w.h.p.;*
- (ii) *for all $k = \tilde{\Theta}(\ell^{1+\epsilon})$ for any arbitrary constant $\epsilon > 0$, then the agents need time $\Theta(\ell)$ to find the treasure and total work $k = \tilde{\Theta}(\ell^{2+\epsilon})$, w.h.p.*

Interesting enough, the average jump-length in this case is infinite, thus, in average, we have that the Pareto walk has moved to distance $\Theta(t)$ from the origin in time $\Theta(t)$, exactly as the ballistic walk. Next subsection is devoted to the proof of Theorem 7.

9.1 Analysis of the case $\alpha \in (1, 2]$: proof of Theorem 7

We need three lemmas. The first gives an upper bound on the hitting time of the Pareto walk.

Lemma 27. *Let $k \in \mathbb{N}$ be any integer such that $k = \Theta(\ell \log^2 \ell)$. Then, k agents performing independent Pareto walks with parameter $\alpha \in (1, 2]$ find the treasure in time $\Theta(\ell)$, w.h.p., letting the work to be $\Theta(\ell^2 \log^2 \ell)$, w.h.p.*

Proof of Lemma 27. Consider a single agent moving according the Pareto walk with parameter $\alpha \in (1, 2]$. By Equation (3) in the Preliminaries (Section 3), the probability the agent chooses a jump of length at least d is of the order of $\Theta(1/(1+d)^{\alpha-1})$. Thus, for some constant $c > 0$, we look at the probability of choosing a jump of length no less than $c\ell$, which is $\Theta(1/(1+c\ell)^{\alpha-1}) = \Theta(1/\ell^{\alpha-1})$. This means that an agent chooses such a distance at least once in $\ell^{\alpha-1}/\log(c\ell)$ jumps with probability

$$1 - \left(1 - \Theta\left(\frac{1}{\ell^{\alpha-1}}\right)\right)^{\frac{\ell^{\alpha-1}}{\log(c\ell)}} = o(1).$$

Let E_i be the event that the all the jumps until the i -th one (included) have length less than $c\ell$. By what we have said before, it is true that

$$\mathbb{P}(E_i) = 1 - o(1) \quad \text{for all } i \leq \ell^{\alpha-1}/\log(c\ell).$$

Then, we show that, conditional on the event E_i , the sum of the first i jumps is at most $\ell/2$ with constant probability. Indeed, if $j < i$, the expected value of S_j is

$$\mathbb{E}[S_j | E_i] = \sum_{d=0}^{c\ell-1} \frac{c_\alpha d}{(1+d)^\alpha} = \mathcal{O}(c\ell^{2-\alpha} \log(c\ell))$$

for the integral test (Fact 1 in Appendix A), where the $\log(c\ell)$ factor takes care of the case $\alpha = 2$. Thus,

$$\mathbb{E}\left[\sum_{j=1}^i S_j | E_i\right] \leq \sum_{j=1}^{\frac{\ell^{\alpha-1}}{\log(c\ell)}} \mathbb{E}[S_j | E_i] = \mathcal{O}(c\ell).$$

²⁵For a formal comprehension of this statement, we refer to Remark 1 in the Preliminaries (Section 3).

We choose c small enough so that this expression is less than $\ell/2$. Conditional on E_i , the $\{S_j\}_{j \leq i}$ random variables are non negative and bounded from above since all jump indexes are less than $\ell^{\alpha-1}/\log(c\ell)$. Then, we can use the Chernoff bound on their sum (normalized dividing by $c\ell$) to concentrate the probability around its normalized expectation (Lemma 30 in Appendix A). We have that

$$\mathbb{P} \left(\sum_{j=1}^i \frac{S_j}{c\ell} \geq \frac{\ell}{2} \cdot \frac{1+1/2}{c\ell} \mid E_i \right) \leq \exp \left(-\frac{1}{3 \cdot 4} \frac{1}{c} \right),$$

which is a constant. Then it is guaranteed that there is at least constant probability the agent has displacement at most $3\ell/4$ from the origin in time $\mathcal{O}(\ell)$ (since the sum of all jumps is at most linear in ℓ), without any conditional event. Indeed,

$$\begin{aligned} \mathbb{P} \left(\sum_{j \leq i} S_j \leq 3\ell/4 \right) &\geq \mathbb{P} \left(\sum_{j \leq i} S_j \leq 3\ell/4 \mid E_i \right) \mathbb{P}(E_i) \\ &\geq \Theta(1) (1 - o(1)) = \Theta(1), \end{aligned}$$

for each $i \leq \ell^{\alpha-1}/\log(c\ell)$.

Let $F_i = \{\sum_{j \leq i} S_j \leq 3\ell/4\}$. We now want to compute the probability that, given $i \leq \ell^{\alpha-1}/\log(c\ell)$, the i -th jump leads the agent finding the treasure. Let J_i be such an event. Then, since

$$\mathbb{P}(J_i, F_{i-1}) = \mathbb{P}(J_i \mid F_{i-1}) \mathbb{P}(F_{i-1})$$

we estimate $\mathbb{P}(J_i \mid F_{i-1})$. If t_i is the time the agent starts the i -th jump, we have

$$\mathbb{P}(J_i \mid F_{i-1}) \geq \sum_{v \in Q(\ell/4)} \mathbb{P}(J_i \mid \mathbf{P}_{t_i} = v, F_{i-1}) \mathbb{P}(\mathbf{P}_{t_i} = v \mid F_{i-1}).$$

By Lemma 20 in Subsection 7.1, the term $\mathbb{P}(J_i \mid \mathbf{P}_{t_i} = v, F_{i-1})$ is $\Theta(1/\ell^\alpha)$. At the same time, by Lemma 21 in Subsection 7.1, we have that $\mathbb{P}(\mathbf{P}_{t_i} = v \mid F_{i-1}) = \Omega(1/\ell^2)$ since $v \in Q(\ell/4)$. Then

$$\sum_{v \in Q(\ell/4)} \mathbb{P}(J_i \mid \mathbf{P}_{t_i} = v, F_{i-1}) \mathbb{P}(\mathbf{P}_{t_i} = v \mid F_{i-1}) \geq \Theta \left(\frac{1}{\ell^\alpha} \right) \cdot \sum_{v \in Q(\ell/4)} \Omega \left(\frac{1}{\ell^2} \right) = \Omega \left(\frac{1}{\ell^\alpha} \right),$$

implying $\mathbb{P}(J_i) = \Omega \left(\frac{1}{\ell^\alpha} \right)$ for all $i \leq \ell^{\alpha-1}/\log(c\ell)$. Then, for the chain rule, the probability that none of the events $F_i \cap F_{i-1}$ holds for each $i \leq \ell^{\alpha-1}/\log(c\ell)$ is

$$\begin{aligned} \mathbb{P} \left(\bigcup_{i \leq \frac{\ell^{\alpha-1}}{\log(c\ell)}} (J_i \cap F_{i-1}) \right) &= 1 - \mathbb{P} \left(\bigcap_{i \leq \frac{\ell^{\alpha-1}}{\log(c\ell)}} (J_i^C \cup F_{i-1}^C) \right) \\ &= 1 - \prod_{i \leq \frac{\ell^{\alpha-1}}{\log(c\ell)}} \mathbb{P} \left(J_i^C \cup F_{i-1}^C \mid \bigcap_{j \leq i-1} (J_j^C \cup F_{j-1}^C) \right) \\ &= 1 - \prod_{i \leq \frac{\ell^{\alpha-1}}{\log(c\ell)}} \left(1 - \mathbb{P} \left(J_i \cap F_{i-1} \mid \bigcap_{j \leq i-1} (J_j^C \cup F_{j-1}^C) \right) \right) \\ &= 1 - \prod_{i \leq \frac{\ell^{\alpha-1}}{\log(c\ell)}} \left(1 - \mathbb{P} \left(J_i \cap F_{i-1} \mid \bigcap_{j \leq i-1} (J_j^C \cup F_{j-1}^C) \right) \right) \end{aligned}$$

$$\begin{aligned}
&\stackrel{(a)}{\geq} 1 - \prod_{i \leq \frac{\ell^{\alpha-1}}{\log(c\ell)}} \left(1 - \mathbb{P} \left(J_i \cap F_{i-1}, \bigcap_{j \leq i-1} (J_j^C \cup F_{j-1}^C) \right) \right) \\
&\stackrel{(b)}{=} 1 - \prod_{i \leq \frac{\ell^{\alpha-1}}{\log(c\ell)}} (1 - \mathbb{P}(J_i \cap F_{i-1})) \\
&= 1 - \left(1 - \Omega \left(\frac{1}{\ell^\alpha} \right) \right)^{\frac{\ell^{\alpha-1}}{\log(c\ell)}} \\
&= e^{-\Omega \left(\frac{1}{\ell \log \ell} \right)} = 1 - \mathcal{O} \left(\frac{1}{\ell \log \ell} \right),
\end{aligned}$$

where, (a) holds since $\mathbb{P}(A | B) \geq \mathbb{P}(A, B)$, (b) holds since $F_{i-1} \subseteq (F_{j-1}^C \cup J_j^C)$ for $j \leq i-1$, and last equality holds for the Taylor decomposition of $f(x) = e^x$. Then, there is probability at least $\Omega(1/(\ell \log \ell))$ to find the treasure within time $\mathcal{O}(\ell)$. On the other hand, it is trivial that the process needs at least time t to get to the target. Thus, if $k = \Theta(\ell \log^2 \ell)$, then k agents find the treasure within time $\Theta(\ell)$, w.h.p., letting the work to be $\Theta(\ell^2 \log^2 \ell)$. \square

The next two lemmas aim at giving a lower bound on the hitting time of the treasure.

Lemma 28. *Let $i = 1, \dots, t$, denote the t jumps an agent performing a Pareto walk with $\alpha \in (1, 2]$ which starts at the origin takes. The probability that during the i -th jump the agent finds the treasure is $\mathcal{O}(\log \ell / \ell^\alpha)$.*

Proof of Lemma 28. Consider the starting point v of the i -th jump. We want to give probabilities to the distance at which v is from the origin. Call u the node in which the treasure is located, and consider the rhombus centered in u of nodes that are distant at most $\frac{\ell}{4}$ from u , namely

$$R_{\ell/4}^*(u) = \{w \in \mathbb{Z}^2 : d(w, u) \leq \ell/4\}.$$

For any $v \in R_{\ell/4}^*(u)$, the probability that the i -th jump starts in v is at most $\mathcal{O}(1/\ell^2)$ due to Lemma 21 in Subsection 7.1. At the same time, for any distance $1 \leq d \leq \ell/4$, there are at most $4d$ nodes in $R_{\ell/4}^*(u)$ at distance d from u . Then, for the expression of conditional probability and Lemma 20 in Subsection 7.1, the probability that the i -th jump starts from $R_{\ell/4}^*(u)$ and the agent visits the treasure during the jump is

$$\mathcal{O} \left(\frac{1}{\ell^2} \right) \sum_{d=1}^{\ell/4} 4d \cdot \mathcal{O} \left(\frac{1}{d^\alpha} \right) + \mathcal{O} \left(\frac{1}{\ell^2} \right) = \mathcal{O} \left(\frac{\log \ell}{\ell^\alpha} \right),$$

where, in the first expression, the last term $\mathcal{O}(1/\ell^2)$ is the contribution of u itself. If v is outside $R_{\ell/4}^*(u)$, then the probability that a jump that starts from v leads the agent to visit the treasure is at most $\mathcal{O}(1/\ell^\alpha)$.

Let J_i be the event that the i -th jump (which starts in v) leads the agent to visit the treasure, and V_i be the event that the starting point of the i -th jump is in $R_{\ell/4}^*(u)$. Then

$$\mathbb{P}(J_i) \leq \mathbb{P}(J_i | V_i) \mathbb{P}(V_i) + \mathbb{P}(J_i | V_i^C) \leq \mathcal{O} \left(\frac{\log \ell}{\ell^\alpha} \right) + \mathcal{O} \left(\frac{1}{\ell^\alpha} \right) = \mathcal{O} \left(\frac{\log \ell}{\ell^\alpha} \right),$$

which is the thesis. \square

Next lemma give us the probability an agent never finds the treasure.

Lemma 29. Consider a single agent performing a Pareto walk with $\alpha \in (1, 2]$. The probability that the agent never finds the treasure is $1 - \mathcal{O}(\log^2(\ell)/\ell)$.

Proof of Lemma 29. Consider the first time t_i the agent is at distance at least $\lambda_i = 2^i \ell$ from the origin, for each $i \geq 1$. Define, for $i \geq 1$, $\tau_i = 2\lambda_i^{\alpha-1} \log \lambda_i$. Then,

$$\mathbb{P}(t_i \leq \tau_i) \geq 1 - \left[1 - \mathcal{O}\left(\frac{1}{\lambda_i^{\alpha-1}}\right)\right]^{2\lambda_i^{\alpha-1} \log \lambda_i} = 1 - \mathcal{O}\left(\frac{1}{\lambda_i^2}\right) = 1 - \mathcal{O}\left(\frac{1}{2^{2i} \ell^2}\right).$$

Then, the expected number of visits to the treasure from time t_i until time t_{i+1} is then $\mathcal{O}(\tau_{i+1} \log(\ell)/\lambda_i^\alpha) = \mathcal{O}(\tau_i \log(\ell)/\lambda_i^\alpha)$ by Lemma 28, since the agent starts at distance $\Theta(\lambda_i)$ from the target. At the same time, the average number of visits to the treasure until time t_1 is $\mathcal{O}(t_1 \log(\ell)/\ell^\alpha) = \mathcal{O}(\tau_1 \log(\ell)/\ell^\alpha)$. Combining the above, we have that the expected total number of visits to the treasure is

$$\mathcal{O}\left(\frac{\tau_1 \log \ell}{\ell^\alpha}\right) + \sum_{i \geq 1} \mathcal{O}\left(\frac{\tau_i \log \ell}{\lambda_i^\alpha}\right) = \mathcal{O}\left(\frac{\log^2 \ell}{\ell}\right) + \sum_{i \geq 1} \mathcal{O}\left(\frac{\log(2^i) \log^2 \ell}{2^i \ell}\right) = \mathcal{O}\left(\frac{\log^2 \ell}{\ell}\right).$$

Thus, for the Markov property, the probability that the agent visits the treasure at least once is $\mathcal{O}(\log^2(\ell)/\ell)$. \square

We are ready to prove our main result.

9.1.1 Wrap-up: proof of Theorem 7

Proof of Theorem 7. As for the main result, Lemma 27 tells us that $k = \Theta(\ell \log^2 \ell)$ agents find the treasure in time $\Theta(\ell)$, making a total work of $\Theta(\ell^2 \log^2 \ell)$, w.h.p. Furthermore, if we increase the number of agents by multiplying by any polylogarithmic factor, the same upper bound on the hitting time holds, w.h.p., while if we decrease it by dividing by any polylogarithmic factor, the upper bound on the hitting time holds with non-negligible probability.

At the same time, for all k in the family $\tilde{\Theta}(\ell)$, we have that it is needed at least time $\Omega(\ell)$ to find the treasure, making a total work of $\tilde{\Omega}(\ell^2)$, almost surely.

Item (i) is a direct consequence of Lemma 29. Indeed, $\tilde{\Theta}(\ell^{1-\epsilon})$ agents eventually find the treasure with probability $\tilde{\mathcal{O}}(\log^2(\ell)/\ell^\epsilon)$ for the union bound. Thus, w.h.p., the work is infinite.

Item (ii) comes from the fact that the minimum time needed to find the treasure is ℓ almost surely. \square

A Tools

Fact 1. Let $\alpha > 0$ be a constant, $d > 0$ a positive integer, and let $d_{max} > d$ be another integer. The followings hold

$$\frac{1}{(\alpha-1)(d)^{\alpha-1}} \leq \sum_{k \geq d} \frac{1}{k^\alpha} \leq \frac{1}{(\alpha-1)(d)^{\alpha-1}} + \frac{1}{d^\alpha} \quad \text{and} \quad (16)$$

$$\frac{1}{(\alpha-1)} \left(\frac{1}{d^{\alpha-1}} - \frac{1}{d_{max}^{\alpha-1}} \right) \leq \sum_{k=d}^{d_{max}} \frac{1}{k^\alpha} \leq \frac{1}{(\alpha-1)} \left(\frac{1}{d^{\alpha-1}} - \frac{1}{d_{max}^{\alpha-1}} \right) + \frac{1}{d^\alpha} \quad \text{for } \alpha > 1, \quad (17)$$

and, at the same time,

$$\log\left(\frac{d_{max}}{d}\right) \leq \sum_{k=d}^{d_{max}} \frac{1}{k} \leq \log\left(\frac{d_{max}}{d}\right) + \frac{1}{d} \quad (\text{case } \alpha = 1), \quad \text{and} \quad (18)$$

$$\frac{(d_{\max})^{1-\alpha} - d^{1-\alpha}}{1-\alpha} \leq \sum_{k=d}^{d_{\max}} \frac{1}{k^\alpha} \leq \frac{(d_{\max})^{1-\alpha} - d^{1-\alpha}}{1-\alpha} + \frac{1}{d^\alpha} \quad \text{for } \alpha < 1. \quad (19)$$

Proof. By the integral test, it holds that

$$\int_d^{d_{\max}} \frac{1}{k^\alpha} dk \leq \sum_{k=d}^{d_{\max}} \frac{1}{k^\alpha} \leq \int_d^{d_{\max}} \frac{1}{k^\alpha} dk + \frac{1}{d^\alpha}.$$

Straightforward calculations give the result for eqs. (17) to (19). As for eq. (16), it comes from the integral test letting $d_{\max} \rightarrow \infty$. \square

Fact 2. Let $n \geq 0$ be any integer, and let $a, b \in [0, n-2]$ such that $n-a-b$ is even and non-negative, and $a \geq b$. Then, it holds that

$$\binom{n}{\frac{n+a+b}{2}} \geq \binom{n}{\frac{n+a+b}{2} + 1}, \quad \text{and} \quad (20)$$

$$\binom{n}{\frac{n+a+b}{2}} \binom{n}{\frac{n+a-b}{2}} \geq \binom{n}{\frac{n+a+b}{2} + 1} \binom{n}{\frac{n+a-b}{2} - 1}. \quad (21)$$

Proof of Fact 2. We first prove eq. (20). Expressing the binomial coefficients, we have

$$\begin{aligned} \binom{n}{\frac{n+a+b}{2}} &\geq \binom{n}{\frac{n+a+b}{2} + 1} \quad \text{iff} \\ \frac{n!}{\left(\frac{n+a+b}{2}\right)! \left(\frac{n-a-b}{2}\right)!} &\geq \frac{n!}{\left(\frac{n+a+b}{2} + 1\right)! \left(\frac{n-a-b}{2} - 1\right)!} \quad \text{iff} \\ \frac{1}{\left(\frac{n-a-b}{2}\right)} &\geq \frac{1}{\left(\frac{n+a+b}{2} + 1\right)} \quad \text{iff} \\ \frac{n+a+b}{2} + 1 &\geq \frac{n-a-b}{2}, \end{aligned}$$

which is true. As for eq. (21), we have

$$\begin{aligned} \binom{n}{\frac{n+a+b}{2}} \binom{n}{\frac{n+a-b}{2}} &\geq \binom{n}{\frac{n+a+b}{2} + 1} \binom{n}{\frac{n+a-b}{2} - 1} \quad \text{iff} \\ \frac{n!}{\left(\frac{n+a+b}{2}\right)! \left(\frac{n-a-b}{2}\right)!} \cdot \frac{n!}{\left(\frac{n+a-b}{2}\right)! \left(\frac{n-a+b}{2}\right)!} &\geq \frac{n!}{\left(\frac{n+a+b}{2} + 1\right)! \left(\frac{n-a-b}{2} - 1\right)!} \cdot \frac{n!}{\left(\frac{n+a-b}{2} - 1\right)! \left(\frac{n-a+b}{2} + 1\right)!} \quad \text{iff} \\ \frac{1}{\left(\frac{n-a-b}{2}\right)} \cdot \frac{1}{\left(\frac{n-a+b}{2}\right)} &\geq \frac{1}{\left(\frac{n+a+b}{2} + 1\right)} \cdot \frac{1}{\left(\frac{n-a+b}{2} + 1\right)} \quad \text{iff} \\ \left(\frac{n+a+b}{2} + 1\right) \left(\frac{n-a+b}{2} + 1\right) &\geq \left(\frac{n-a-b}{2}\right) \left(\frac{n+a-b}{2}\right), \end{aligned}$$

which is true (compare the second factor on the left with the first factor on the right, and the first factor on the left with the second factor on the right). \square

As for the probabilistic tools, first we state the well-known central limit theorem, which can be found in [Fel68] (Chapter X).

Theorem 8 (Central Limit Theorem). Let $\{X_k\}_{k \geq 1}$ be a sequence of i.i.d. random variables. Let $\mu = \mathbb{E}[X_1]$, $\sigma^2 = \text{Var}(X_1)$, and $S_n = \sum_{k=1}^n X_k$ for any $n \geq 1$. Let $\Phi : \mathbb{R} \rightarrow [0, 1]$ be the cumulative distribution function of a standard normal distribution. Then, for any $\beta \in \mathbb{R}$, it holds that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{S_n - n\mu}{\sigma\sqrt{n}} < \beta \right) = \Phi(\beta).$$

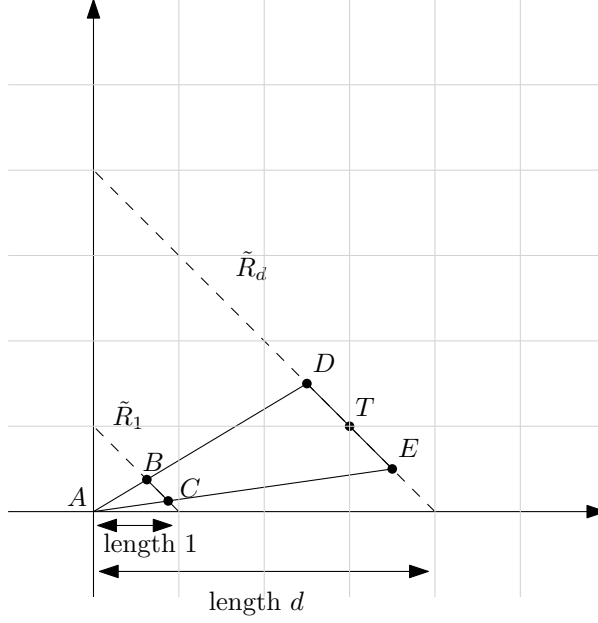


Figure 11: Mapping from $\tilde{R}_d(u)$ to $\tilde{R}_1(u)$

Furthermore, we give some forms of the Chernoff bounds. The first form can be found in the appendix of [DP09], and can be stated as follow.

Lemma 30 (Multiplicative forms of Chernoff bounds). *Let X_1, X_2, \dots, X_n be independent random variables taking values in $[0, 1]$. Let $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[X]$. Then:*

(i) *for any $\delta > 0$ and $\mu \leq \mu_+ \leq n$, it holds that*

$$P(X \geq (1 + \delta)\mu_+) \leq e^{-\frac{1}{3}\delta^2\mu_+}, \quad (22)$$

(ii) *for any $\delta \in (0, 1)$ and $0 \leq \mu_- \leq \mu$, it holds that*

$$P(X \leq (1 - \delta)\mu_-) \leq e^{-\frac{1}{2}\delta^2\mu_-}. \quad (23)$$

We also use the following form of Chernoff bound, which can be found in [CL06] (Theorem 3.4).

Lemma 31 (Additive form of Chernoff bound using variance). *Let X_1, \dots, X_n be independent random variables satisfying $X_i \leq \mathbb{E}[X_i] + M$ for some $M \geq 0$, for all $i = 1, \dots, n$. Let $X = \sum_{i=1}^n X_i$, $\mu = \mathbb{E}[X]$, and $\sigma^2 = \text{Var}(X)$. Then, for any $\lambda > 0$, it holds that*

$$\mathbb{P}(X \geq \mu + \lambda) \leq \exp\left(-\frac{\lambda^2}{\sigma^2 + \frac{M\lambda}{3}}\right). \quad (24)$$

B Proofs: Preliminaries

We first recall what a direction-approximating path is and what we mean by an agent that chooses a direction.

Definition 1 (*r*-approximating path). Let r be the unique ray identified by some unit vector \vec{v} applied to some node u . Consider, for $d \geq 0$, the sequence of rhombus centered at u

$$R_d(u) = \{v \in \mathbb{Z}^2 : |u - v|_1 = d\}.$$

For each $d \geq 0$, consider the “natural immersion” of the rhombus in the continuous plane, namely

$$\tilde{R}_d(u) = \{(x, y) \in \mathbb{R}^2 : (y + x + 1)(y + x - 1)(y - x + 1)(y - x - 1) = 0, |x| \leq d, |y| \leq d\},$$

as in Figure 1. Let v_d the intersection between r and $\tilde{R}_d(u)$. An *r*-approximating path is a simple path starting at u , whose d -th node is the node $w_d \in R_d(u)$ that minimize the distance $\min_{w \in R_d(u)} |w - v_d|_2$. Ties are broken uniformly at random.

Definition 2 (Direction choice procedure). An agent at some node u chooses a direction r in the following way: it samples uniformly at random one node v of $\tilde{R}_1(u)$ and takes r as the unique ray starting in u and crossing v .

Consider an agent starting at some node u and a direction r which itself starts at u chosen according to Definition 2. We first argue that there is probability equal to zero that Definition 1 leads to an ambiguity. Indeed, for any $d \geq 0$, consider the mapping $f_d : \tilde{R}_d(u) \rightarrow \tilde{R}_1(u)$ such that $f(w) = w/d$. This clearly is a homothetic transformation, which is a similarity and a bijection. For each $d \geq 0$, define

$$A_d = \{v \in \tilde{R}_d(u) : \exists w_1 \neq w_2 \in R_d(u) \text{ such that } |v - w_1|_2 = |v - w_2|_2\}.$$

The probability that there is an ambiguity in the determination of the *r*-approximating path is equal to the probability that the ray r crosses one point of A_d for some $d \geq 0$. But the set $\bigcup_{d \geq 0} A_d$ is countable (each A_d has cardinality equal to $4d$) and so is the set $\bigcup_{d \geq 0} f_d(A_d)$, thus there is probability equal to zero that a point chosen u.a.r. in $\tilde{R}_1(u)$ lies in $\bigcup_{d \geq 0} f_d(A_d)$ (i.e., it leads to ambiguity). Since we look at the mobility models for a countable amount of time and for a countable number of agents, during the whole process there is probability zero that the procedure to choose a direction leads to ambiguity.

Thus, Definition 1 is well-posed with probability one, and we are ready to prove Lemma 1.

Lemma 1. *Let u be any node of \mathbb{Z}^2 , $d \geq 1$, and $v \in R_d(u)$. Suppose an agent is on u and chooses a direction according to the procedure in Definition 2. Then, there is probability $1/(4d)$ that the corresponding direction-approximating path crosses v .*

Proof of Lemma 1. Consider an agent starting at some node u and a direction r which itself starts at u chosen according to the procedure in Definition 2. We argue that there is probability exactly $1/(4d)$ that a given node $v \in R_d(u)$ belongs to a corresponding *r*-approximating path. Indeed, for $d \geq 0$, consider the same mapping $f_d : \tilde{R}_d(u) \rightarrow \tilde{R}_1(u)$ such that $f(w) = w/d$. As we said, this is a homothetic transformation, which is a similarity, and the probability that the *r*-approximating path crosses v is the probability that a point chosen u.a.r. in $\tilde{R}_1(u)$ lies in the segment BC in Figure 11 (the boundaries have probability zero to be chosen). Since the rhombuses $\tilde{R}_1(u)$ and $\tilde{R}_d(u)$ are similar with a dilatation factor of d , and so are the triangles ABC and ADE , then the probability that a point chosen u.a.r. in $\tilde{R}_1(u)$ lies in BC is the same as the probability that a point chosen u.a.r. in $\tilde{R}_d(u)$ lies in DE , which is $\overline{DE}/(4d\sqrt{2}) = 1/(4d)$. \square

C Monotonicity property

In this part of the appendix we show a really interesting monotonicity property which holds for any mobility model such that the step-length distribution is non-increasing and does not depend on direction. Namely, mobility models that fulfill the following definition.

Definition 9. Given two nodes u and v , the probability to jump in just *one* step from u to v is $p(u, v) = \rho(|u - v|_1)$, where $\rho : \mathbb{N} \rightarrow \mathbb{R}$ is a non-increasing distribution function.

This monotonicity is characterized by a useful geometric shape that can be “roughly” characterized as follows. For any node $u = (u_x, u_y)$, let $d_u = |u_x| + |u_y|$, and define the square

$$Q(d_u) = \{(x', y') \in \mathbb{Z}^2 : \max(|x'|, |y'|) \leq d_u\}$$

(See Figure 12 in the proof of Lemma 32). Furthermore, for any node $u \in \mathbb{Z}^2$, let $p_{u,t}$ be the probability that the agent is located in u at time t . Then, the following geometric property holds.

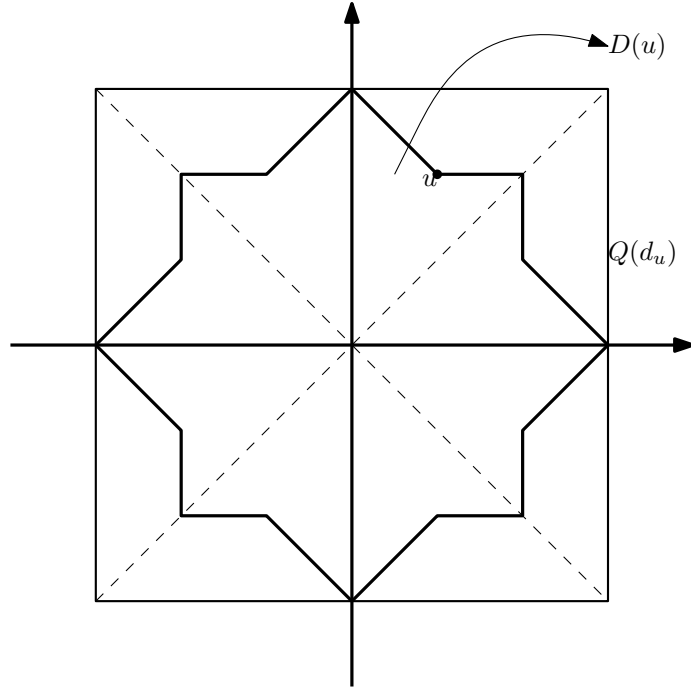


Figure 12: The set $D(u)$, consisting in all inner nodes of the “star”, and the square $Q(d_u)$.

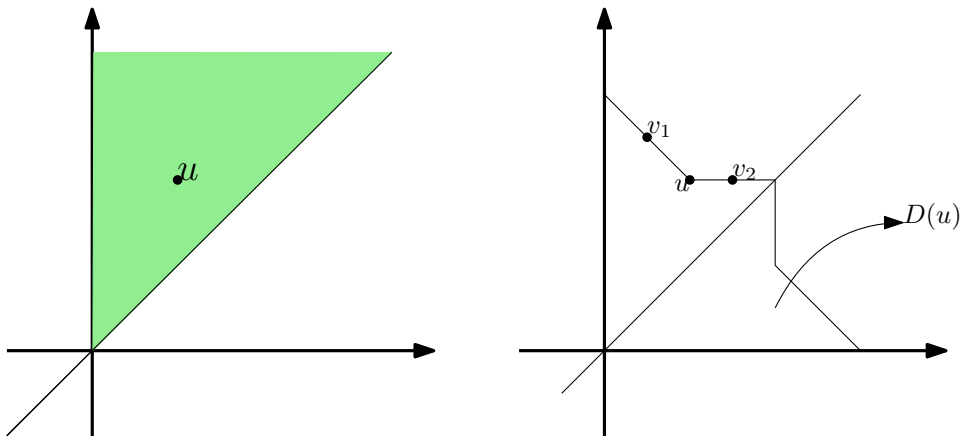


Figure 13: The “area” in which we take u , and the possible choices of v .

Lemma 32 (Monotonicity property). *Let $u \in \mathbb{Z}^2$ be an arbitrary node, and consider an agent performing any mobility model as in Definition 9 who starts at the origin. Then, for each node $v \notin Q(d_u)$ and each time t , it holds that $p_{u,t} \geq p_{v,t}$.*

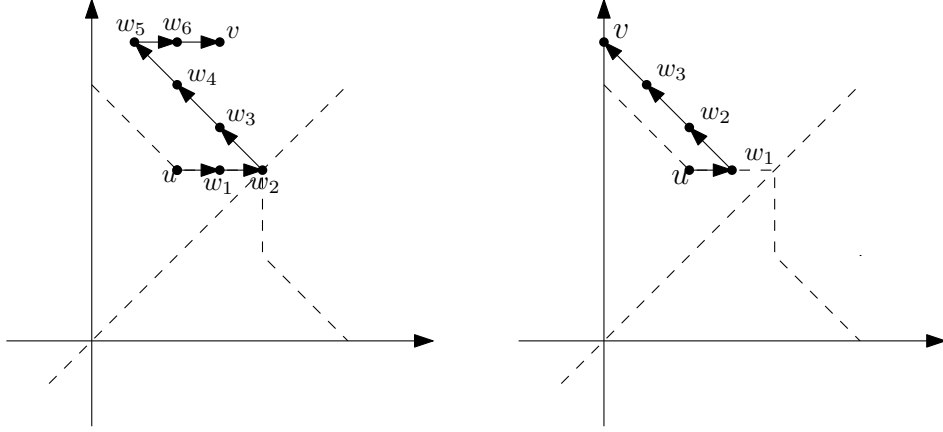


Figure 14: Two “path” examples.

Proof of Lemma 3. For a given distance $d \geq 0$, consider the rhombus $R_d^*(o) = \{(x, y) \in \mathbb{Z}^2 : |x| + |y| \leq d\}$. For any point (x, y) we also define a square $T(x, y) = \{(x', y') \in \mathbb{Z}^2 : \max(|x'|, |y'|) \leq \max(|x|, |y|)\}$.

Let $u = (u_x, u_y)$ be any point in \mathbb{Z}^2 , $d_u = |u_x| + |u_y|$ its distance from the origin, and $t \geq 1$ any time step. Let \mathbf{X}_t be a the random variable representing the coordinate of the node the agent is located on at time t . We are now going to prove a stronger result which will imply the thesis. In particular, we show that, for any $v \in \mathbb{Z}^2$ which is outside the set $D(u) = R_{d_u}^*(o) \cup T(u)$, or, at most, on its “border”²⁶, it holds that

$$\mathbb{P}(\mathbf{X}_t = u) \geq \mathbb{P}(\mathbf{X}_t = v), \quad (25)$$

for any $t \geq 1$. Let $Q(d_u) = \{(x', y') : \max(|x'|, |y'|) \leq d_u\}$, and note that $D(u) \subseteq Q(d_u)$. In Figure 12 such sets are plotted.

Without loss of generality, suppose u is in the first quadrant and not below the main bisector, namely in the set $\{(x, y) \in \mathbb{Z}^2 : y \geq 0, x \geq y\}$ (Figure 13). First, we argue that it is sufficient to show the statement for

$$v \in \{v_1 = (u_x - 1, u_y + 1), v_2 = (u_x + 1, u_y)\},$$

as in Figure 13. Indeed, for any $v \notin D(u)$ that “lives” in the highlighted area in Figure 13, there exists a sequence of nodes $u = w_0, w_1, \dots, w_k = v$ from u to v such that w_{i+1} belongs to the set

$$\{(x_{w_i} - 1, y_{w_i} + 1), (x_{w_i} + 1, y_{w_i})\},$$

where $w_i = (x_{w_i}, y_{w_i})$, as Figure 14 shows. Thus, if the thesis is true for $v \in \{v_1, v_2\}$, then it is true also for all $v \notin D(u)$ in the highlighted area in Figure 13. At the same time, for any other $v \notin D(u)$, outside the highlighted area in Figure 13, there exists a symmetrical argument explained in Figure 15. Thus, if the thesis is true for all $v \notin D(u)$ in the highlighted area in Figure 13, then it is also true for any $v \notin D(u)$. We now consider some geometric constructions which will be used in the proof, one for each choice of v . The following description is showed in Figure 16 .

- (i) $v = (u_x - 1, u_y + 1)$: consider the strict line defined by $r : y = x + (u_y - u_x) + 1$ (i.e. the line in \mathbb{R}^2 which is the set of points that are equidistant from u and v according to the Euclidean distance). Call $V \subset \mathbb{Z}^2$ the set of nodes that are “above” this line, namely the ones that are

²⁶By “border” of $D(u)$ we mean the set $R_{d_u}(o) \cup T'(u)$, where $R_{d_u}(o) = \{(x, y) \in \mathbb{Z}^2 : |x| + |y| = d_u\}$ and $T'(u) = \{(x, y) \in \mathbb{Z}^2 : \max(|x|, |y|) = \max(|u_x|, |u_y|)\}$.

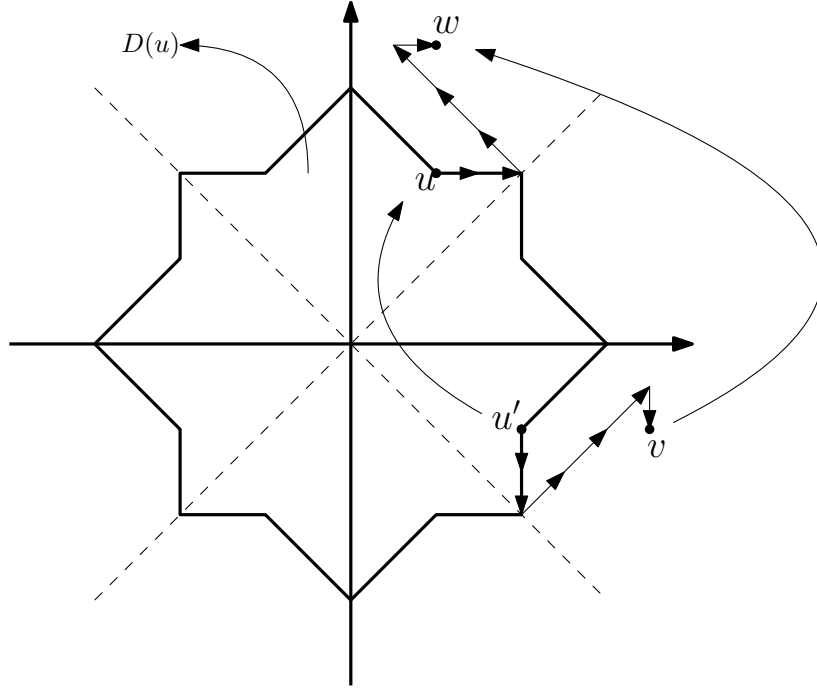


Figure 15: Symmetrical argument.

closer to v than u . Define $U = \mathbb{Z}^2 \setminus (V \cup r)$ the complementary set without line r . Consider the injective function $f : V \rightarrow U$ such that $f(x, y) = (y - (u_y - u_x) - 1, x + (u_y - u_x) + 1)$, which is the symmetry with respect to r . It trivially holds that for any $w \in V$, $|w - v|_1 = |f(w) - u|_1$ and $|w - u|_1 = |f(w) - v|_1$. Furthermore, it holds that for each $w \in V$, either $w \notin D(f(w))$, or w lies on the “border” of $D(f(w))$. All these properties are well-shown in Figure 17.

- (ii) $v = (u_x + 1, u_y)$: the same construction can be done in this case. Indeed, the strict line will be $x = u_x + \frac{1}{2}$, and the injective function $f(x, y) = (2u_x + 1 - x, y)$. The same properties we have seen in the previous case hold here too.

Now we go for the proof. For any time i , and any two nodes $u', v' \in \mathbb{Z}^2$, define

$$p^i(u', v') = \mathbb{P}(\mathbf{X}_i = v' \mid \mathbf{X}_0 = u').$$

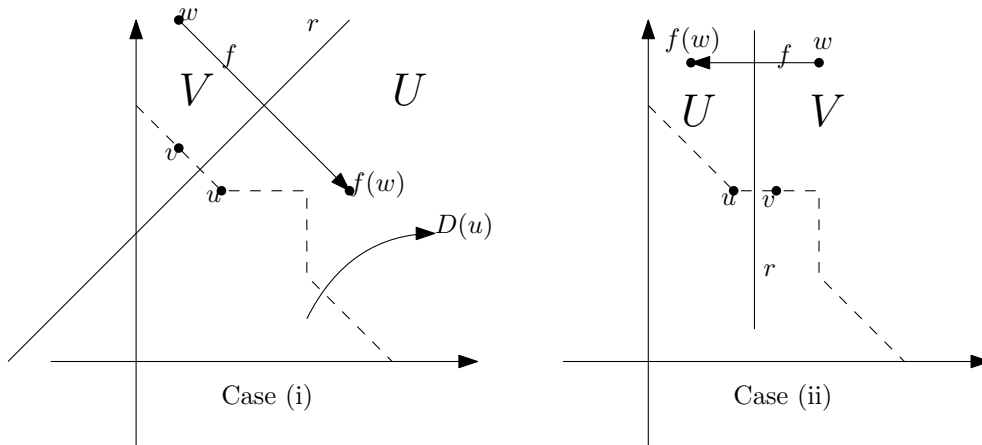


Figure 16: Geometric constructions in the three cases.

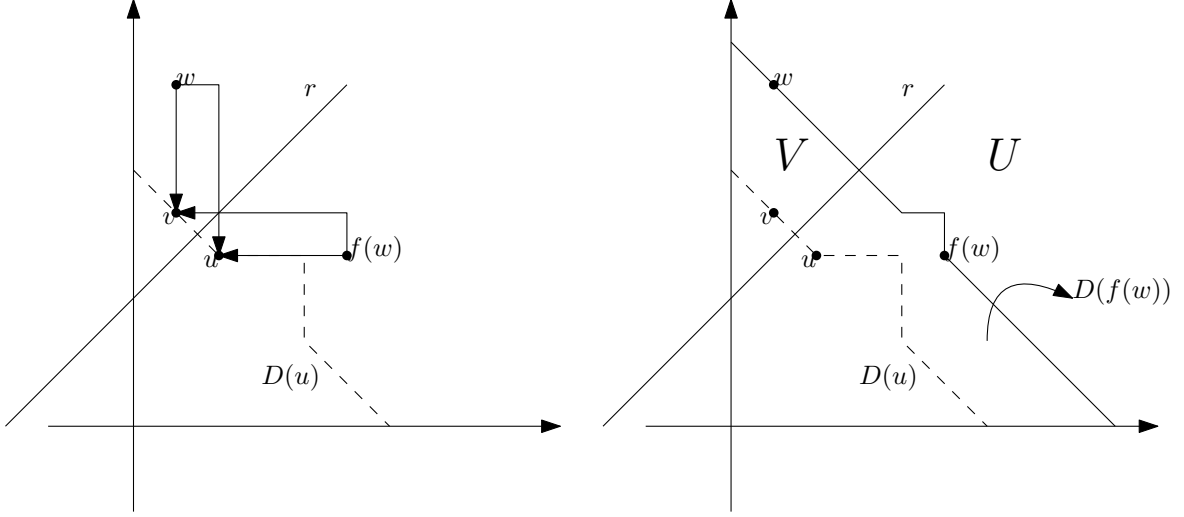


Figure 17: First, equivalence between distances. Second, $w \notin D(f(w))$.

Let o be the origin, u be any node, and $v \in \{v_1, v_2\}$. We show that $p^t(o, u) \geq p^t(o, v)$ by induction on t . The base case is $t = 1$. From the hypothesis on the mobility model, we know that

$$p^1(o, u) - p^1(o, v) \geq 0$$

for any u and v in \mathbb{Z}^2 such that $|u|_1 \leq |v|_1$. We now suppose $t \geq 2$ and the thesis true for $t - 1$. Fix u and v as in Figure 13; then, for the geometric construction we made above, it holds that

$$\begin{aligned} p^t(o, u) - p^t(o, v) &= \sum_{w \in \mathbb{Z}^2} p^{t-1}(o, w) (p^1(w, u) - p^1(w, v)) \\ &\geq \sum_{w \in U} p^{t-1}(o, w) (p^1(w, u) - p^1(w, v)) + \sum_{w \in V} p^{t-1}(o, w) (p^1(w, u) - p^1(w, v)) \end{aligned}$$

where last inequality is immediate for case (ii), indeed the line r does not contain elements of \mathbb{Z}^2 , while in case (i) the sum over nodes in line r is zero. Then, the previous value is equal to

$$\sum_{w \in V} p^{t-1}(o, f(w)) (p^1(f(w), u) - p^1(f(w), v)) + \sum_{w \in V} p^{t-1}(o, w) (p^1(w, u) - p^1(w, v))$$

because of the definition of $f : V \rightarrow U$, and, changing the sign of the second sum, we obtain

$$\sum_{w \in V} p^{t-1}(o, f(w)) (p^1(f(w), u) - p^1(f(w), v)) - \sum_{w \in V} p^{t-1}(o, w) (p^1(w, v) - p^1(w, u)).$$

Now, observe that the definition of f implies that for each $w \in V$, $|w - v|_1 = |f(w) - u|_1$ and $|f(w) - v|_1 = |w - u|_1$ (Figure 17). Thus we can group out the term $p^1(f(w), u) - p^1(f(w), v) = p^1(w, v) - p^1(w, u)$, and we have

$$\sum_{w \in V} (p^{t-1}(o, f(w)) - p^{t-1}(o, w)) (p^1(f(w), u) - p^1(f(w), v)). \quad (26)$$

We observe that $p^{t-1}(o, f(w)) - p^{t-1}(o, w) \geq 0$ by the inductive hypothesis, since either $w \notin D(f(w))$ or w lies on the “border” of $D(f(w))$ (Figure 17), and $p^1(f(w), u) - p^1(f(w), v) \geq 0$ by definition of f , since the distance between $f(w)$ and u is no more than the distance between $f(w)$ and v . It follows that (26) is non-negative, and, thus, the thesis. \square

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