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WEIGHTED DISCRETE HARDY'S INEQUALITIES

PASCAL LEFÈVRE

ABSTRACT. The purpose of this note is to give a short proof of a weighted version of the Hardy's inequality in the sequence case. This includes the known case of classical polynomial weights, with optimal constant. The proof relies on the ideas of the short direct proof given recently in [6].

In the sequel, we work with $p > 1$ and $p' = \frac{p}{p-1}$ denotes its conjugate exponent.

The notation \mathbb{N}_0 stands for the set of non negative integral numbers: $0, 1, 2, \dots$

For $y \in \mathbb{R}^+$, we write $[y] = \max\{k \in \mathbb{N}_0 \mid k \leq y\}$ its integral part.

As usual, given a sequence $(w_n)_{n \geq 0}$ of positive numbers, $\ell^p(w)$ is the space of sequences of complex numbers $a = (a_n)_{n \geq 0}$ such that $\sum_{n \geq 0} |a_n|^p w_n < \infty$, equipped with the norm $\|a\|_p = \left(\sum_{n=0}^{+\infty} |a_n|^p w_n \right)^{\frac{1}{p}}$.

When $w_n = 1$ for every $n \in \mathbb{N}_0$, we simply write ℓ^p .

Given a sequence $a = (a_k)_{k \geq 0}$ of complex numbers, we associate the sequence

$$A_n = \frac{1}{n+1} \sum_{k=0}^n a_k$$

We recall the famous **discrete Hardy's inequality**.

Let $p > 1$. For every $a \in \ell^p$, the sequence $A = (A_n)_{n \geq 0}$ belongs to ℓ^p and $\|A\|_p \leq p' \|a\|_p$ i.e.

$$(HI) \quad \left(\sum_{n=0}^{+\infty} \left| \frac{1}{n+1} \sum_{k=0}^n a_k \right|^p \right)^{\frac{1}{p}} \leq p' \left(\sum_{n=0}^{+\infty} |a_n|^p \right)^{\frac{1}{p}}.$$

This inequality is equivalent to the boundedness of the Cesàro operator defined by $\Gamma(a) = (A_n)_{n \in \mathbb{N}}$, with $\|\Gamma\| \leq p'$, viewed as an operator on ℓ^p . Actually the constant p' is optimal and $\|\Gamma\| = p'$. See [2] for the original result, [5] for a very interesting historical survey on the subject, [4] for a very recent nice extension, and [6] for a short proof.

The aim of this note is to give a short proof of the boundedness of Γ as an operator on weighted $\ell^p(w)$ spaces, under an homogeneity type assumption, following the same ideas than [6]. In particular, this includes the case of classical weights $(n^{\alpha p})_{n \geq 0}$ (with exact norm), so that we recover easily some results of [1] and [3].

Main Theorem 1. Let $w = (w_n)_{n \geq 0}$ and $w' = (w'_n)_{n \geq 0}$ be sequences of non-negative real numbers and we assume that $(w_n)_{n \geq 0}$ is non-decreasing.

We assume that there exists a measurable positive function f on $(0, 1)$ such that

- we have a sub-homogeneity property: for every $n \in (m/s, (m+1)/s) \cap \mathbb{N}$, $w'_{n-1} \leq f(s)w_m$, where $s \in (0, 1)$ and $m \in \mathbb{N}_0$.
- $K = \int_0^1 \left(\frac{f(s)}{s} \right)^{\frac{1}{p}} ds < \infty$.

Then Γ is bounded from $\ell^p(w)$ to $\ell^p(w')$ with $\|\Gamma\| \leq K$:

$$(WHI) \quad \left(\sum_{n=0}^{+\infty} w'_n \left| \frac{1}{n+1} \sum_{k=0}^n x_k \right|^p \right)^{\frac{1}{p}} \leq K \left(\sum_{n=0}^{+\infty} |x_n|^p w_n \right)^{\frac{1}{p}}.$$

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In particular, with $f(s) = s^{-\alpha p}$, we immediately get the classical weights version:

Theorem 2. *Let $\alpha \in [0, 1/p')$ and $w_n = (n+1)^{\alpha p}$ for $n \in \mathbb{N}_0$.*

Then Γ is bounded on $\ell^p(w)$ with $\|\Gamma\| = \left(\frac{1}{p'} - \alpha\right)^{-1}$, equivalently, for every $a \in \ell^p$:

$$(CWHI) \quad \left(\sum_{n=0}^{+\infty} \left| \frac{1}{(n+1)^{1-\alpha}} \sum_{k=0}^n \frac{a_k}{(k+1)^\alpha} \right|^p \right)^{\frac{1}{p}} \leq \left(\frac{1}{p'} - \alpha \right)^{-1} \left(\sum_{n=0}^{+\infty} |a_n|^p \right)^{\frac{1}{p}}.$$

It is well known and easy to check that this bound is sharp (see [6] for the case $\alpha = 0$). For instance, just test $a_n = (n+1)^{-\left(\frac{1}{p} + \varepsilon\right)}$ when $\varepsilon \rightarrow 0^+$.

Before giving the proof of the main theorem, let us mention a general remark about monotone rearrangements of sequences.

We recall that we can define the monotone rearrangement of a vanishing sequence of non negative numbers $(b_k)_{k \geq 0}$ as the following non-increasing sequence:

$$\forall N \in \mathbb{N}_0, \quad b_N^* = \inf_{|E|=N} \sup_{n \notin E} b_n.$$

Remark. The following consequence of the Abel transform principle is well known:

Let $(c_k)_{k \geq 0}$ a non-increasing sequence of non negative numbers. Let $(u_k)_{k \geq 0}$ and $(u'_k)_{k \geq 0}$ be two sequence such that for every $n \geq 0$, $\sum_{k=0}^n u'_k \geq \sum_{k=0}^n u_k$.

Then $\sum_{n=0}^N c_n u'_n \geq \sum_{n=0}^N c_n u_n$ for every $N \geq 0$.

Indeed write $U'_n = \sum_{k=0}^n u'_k$ and $U_n = \sum_{k=0}^n u_k$ and defines for convenience $U'_{-1} = U_{-1} = 0$. A simple Abel transform gives, for every $N \geq 0$,

$$\sum_{n=0}^N c_n u'_n = \sum_{n=0}^N c_n (U'_n - U'_{n-1}) = c_{N+1} U'_N + \sum_{n=0}^N (c_n - c_{n+1}) U'_n \geq c_{N+1} U_N + \sum_{n=0}^N (c_n - c_{n+1}) U_n$$

and another Abel transform gives the result.

In particular, we have the two following simple facts.

Fact 1. *Let $(c_k)_{k \geq 0}$ a non-increasing sequence of non negative real numbers. Let $(u_k)_{k \geq 0}$ be a vanishing sequence of non negative numbers and $(u_k^*)_{k \geq 0}$ its monotone rearrangement.*

Then

$$(RI) \quad \sum_{n=0}^{+\infty} c_n u_n \leq \sum_{n=0}^{+\infty} c_n u_n^*.$$

Indeed we just point out that, by definition, for every $n \geq 0$, we have $\sum_{k=0}^n u_k^* \geq \sum_{k=0}^n u_k$.

Fact 2. *Let $(\alpha_m)_{m \geq 0}$ be a non-increasing summable sequence of non-negative real numbers and $\lambda > 0$. Then $\sum_{m \geq 0} \alpha_m ([(m+1)\lambda] - [m\lambda]) \leq \lambda \sum_{m \geq 0} \alpha_m$.*

Indeed, we only have to point out that $[N\lambda] \leq N\lambda$ for every $N \geq 0$.

Proof of the main Theorem. Let $a \in \ell^p(w)$. We assume first that $(|a_k|^p w_k)_k$ is non-increasing.

Let us fix an arbitrary $N \in \mathbb{N}_0$. For every $n \in \mathbb{N}_0$, we write

$$A_n = \sum_{k=0}^n \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} a_k ds = \int_0^1 a_{[(n+1)s]} ds.$$

Thanks to the triangular inequality for integrals, we have

$$\left(\sum_{n=0}^N |A_n|^p w'_n \right)^{\frac{1}{p}} \leq \int_0^1 \left(\sum_{n=0}^N |a_{[(n+1)s]}|^p w'_n \right)^{\frac{1}{p}} ds \leq \int_0^1 \left(\sum_{n \geq 1} |a_{[ns]}|^p w'_{n-1} \right)^{\frac{1}{p}} ds.$$

For every $s \in (0, 1)$ and for $m \in \mathbb{N}_0$, in order to gather terms, we introduce

$$I_m(s) = \{n \geq 1 \mid [ns] = m\} = \left[\frac{m}{s}, \frac{m+1}{s} \right) \cap \mathbb{N}.$$

Clearly, $(I_m)_{m \in \mathbb{N}_0}$ is a partition of \mathbb{N} , so we have

$$(1) \quad \sum_{n \geq 1} |a_{[ns]}|^p w'_{n-1} = \sum_{m \geq 0} |a_m|^p \sum_{n \in I_m(s)} w'_{n-1} \leq f(s) \sum_{m \geq 0} |a_m|^p w_m(\text{card } I_m(s))$$

by hypothesis.

We point out that $\text{card}((0, A) \cap \mathbb{N}) = [A]$ when $A \notin \mathbb{N}$. Therefore, for every $s \in [0, 1] \setminus \mathbb{Q}$, we have for every $m \geq 0$,

$$(2) \quad \text{card } I_m(s) = \left[\frac{m+1}{s} \right] - \left[\frac{m}{s} \right].$$

From (1), (2) and Fact 2, we obtain $\left(\sum_{n \geq 1} |a_{[ns]}|^p w'_n \right)^{\frac{1}{p}} \leq f(s)^{\frac{1}{p}} s^{-\frac{1}{p}} \|a\|_p$ almost everywhere.

Integrating with respect to s , we get $\left(\sum_{n=0}^N |A_n|^p w'_n \right)^{\frac{1}{p}} \leq K \|a\|_p$.

Since $N \in \mathbb{N}_0$ is arbitrary, the result is proved in the particular case when $(|a_k|^p w_k)_k$ is non-increasing.

Now, in the general case, take $a \in \ell^p(w)$. The sequence $(u_k)_{k \geq 0} = (|a_k| w_k^{\frac{1}{p}})_k$ is vanishing so we can consider its monotone rearrangement $(u_k^*)_{k \geq 0}$. We define also $c_k = w_k^{-\frac{1}{p}}$ for $k \geq 0$.

We point out that for every $N \geq 0$, thanks to Fact 1, we have

$$\left| \sum_{n=0}^N a_n \right| \leq \sum_{n=0}^N |a_n| = \sum_{n=0}^N c_n u_n \leq \sum_{n=0}^N c_n u_n^*$$

Applying the first step to the sequence $(c_k u_k^*)_{k \geq 0}$, we get

$$\|\Gamma(a)\|_{\ell^p(w')} \leq K \left(\sum_{n=0}^{+\infty} w_n |c_n u_n^*|^p \right)^{\frac{1}{p}}$$

but

$$\left(\sum_{n=0}^{+\infty} w_n |c_n u_n^*|^p \right)^{\frac{1}{p}} = \left(\sum_{n=0}^{+\infty} |u_n^*|^p \right)^{\frac{1}{p}} = \left(\sum_{n=0}^{+\infty} |u_n|^p \right)^{\frac{1}{p}} = \|a\|_{\ell^p(w)}.$$

□

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