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# REPORT

## On the right-inversion of partially minimum-phase systems: case study

Mohamed Elobaid, Mattia Mattioni, Salvatore Monaco, Dorothee Normand Cyrot

### ARTICLE HISTORY

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### ABSTRACT

This technical manuscript reports the detailed calculations, and simulations carried on the case study of the 4-tanks system based on the control procedure proposed in [2]. The case study illustrates how, starting from a nonminimum phase nonlinear system with a linear output, one goes about identifying the minimum phase zeros of the LTM model of the system, calculating a related dummy output, with respect to which the original nonlinear system is minimum phase. Regulation of the original output is thus possible using standard static feedback controls.

### KEYWORDS

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## 1. Case study: The bench-marking 4-tanks system

Consider the case of a 4-tanks system given by

$$\dot{h} = f(h) + Bu \quad (1a)$$

$$y = Ch \quad (1b)$$

with  $h = \text{col}\{h_1, h_2, h_3, h_4\}$ ,  $f(h) = 2F(h)h$

$$F(h) = \begin{pmatrix} -p_1(h_1) & 0 & \frac{A_3}{A_1}p_3(h_3) & 0 \\ 0 & -p_2(h_2) & 0 & \frac{A_4}{A_2}p_4(h_4) \\ 0 & 0 & -p_3(h_3) & 0 \\ 0 & 0 & 0 & -p_4(h_4) \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{pmatrix}, \quad C = \kappa_t \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}^\top$$

$p_i(h_i) = \frac{c_i \sqrt{2gh_i}}{2A_i h_i}$ . For the sake of compactness, let  $b_{ij}$  correspond to the element in position  $(i, j)$  of the input-state matrix  $B$ . In particular,  $h_i$ ,  $A_i$  and  $c_i$  are, respectively, the level of water in the  $i^{\text{th}}$ -tank, its cross-section area and the cross-section of the outlet hole for  $i = 1, 2, 3, 4$ . The control signals  $u_j$  with  $j = 1, 2$  correspond to the voltage applied to  $j^{\text{th}}$ -pump with  $k_j u_j$  being the corresponding flow. We consider the problem of locally asymptotically tracking the output of (1) to a desired  $y_\star = (h_1^\star, h_2^\star)$  corresponding to make  $h^\star = (h_1^\star, h_2^\star, h_3^\star, h_4^\star)^\top$  with

$$h_3^\star = \frac{(c_1 \gamma_2 \sqrt{h_1^\star} - c_2 (1 - \gamma_2) \sqrt{h_2^\star})^2}{c_3^2 a_3^2 \gamma_2^2}$$

$$h_4^\star = \frac{(c_2 \gamma_1 \sqrt{h_2^\star} - c_1 (1 - \gamma_1) \sqrt{h_1^\star})^2}{c_4^2 a_4^2 \gamma_1^2}$$

for  $a_3 = \frac{\gamma_2}{1-\gamma_2} - \frac{1-\gamma_1}{\gamma_1}$  and  $a_4 = \frac{\gamma_1}{1-\gamma_1} - \frac{1-\gamma_2}{\gamma_2}$  a locally asymptotically stable equilibrium for the closed-loop system under nonlinear feedback.

### 1.0.1. Analysis of the zero-dynamics

The vector relative degree of (1) is well defined and given by  $r = (1 \ 1)$  so that it exhibits a two-dimensional zero-dynamics. Accordingly, for investigating minimum-phasesness of (1), one computes the linear tangent model (LTM) at  $h^\star$  with  $x = h - h^\star$  and  $A = 2F(h^\star)$  with corresponding transfer function matrix

$$G(s) = \kappa_t \begin{pmatrix} \frac{b_{11}}{s+p_1} & \frac{b_{32} p_3}{(s+p_1)(s+p_3)} \\ \frac{b_{41} p_4}{(s+p_2)(s+p_4)} & \frac{b_{22}}{s+p_2} \end{pmatrix} \quad (2)$$

$p_i = p_i(h_i^\star) > 0$  for  $i = 1, 2, 3, 4$ , Smith form as  $M(s) = \text{diag}\{\frac{1}{d(s)}, z(s)\}$ , with pole-polynomial  $d(s) = (s + p_1)(s + p_2)(s + p_3)(s + p_4)$  and zero-polynomial  $z(s) = s^2 + (p_3 + p_4)s + \frac{p_3 p_4}{b_{11} b_{22}}(b_{11} b_{22} - b_{32} b_{41})$ . Thus, (1) is nonminimum-phase if  $b_{11} b_{22} - b_{32} b_{41} < 0$

so that one can factorize  $z(s) = (s - z_u)(s - z_s)$  for  $z_u \in \mathbb{R}^+$  and  $z_s \in \mathbb{R}^-$ . As a consequence, if  $b_{11}b_{22} - b_{32}b_{41} < 0$ , output regulation to  $y_*$  cannot be achieved through classical right-inversion even if the relative degree is well-defined.

In the following we show how the factorization procedure detailed in the previous section allows to deduce a new output  $y_s = C_s h$  and a nonlinear feedback locally solving the regulation problem with stability for (1).

### 1.0.2. The new dummy output using matrix transfer function factorization

Because  $(A, B, C)$  possesses three distinct poles in general, one gets that the matrix  $G_s(s) = \text{diag}\{1, s - z_s\} \text{diag}\{d(s), 1\} R(s)$  is improper for all choices of  $(L(s), R(s))$ . However, for the pair

$$L(s) = \begin{pmatrix} -\frac{\psi_1(s)}{b_{32} b_{41}^2 p_3 p_4^2 (p_2 - p_3) (p_3 - p_4)} & -\frac{p_2 - p_1 + p_4 + s}{(p_1 - p_2) (p_1 - p_4)} \\ -\frac{\psi_2(s)}{b_{32} b_{41} p_3 p_4 (p_2 - p_3) (p_3 - p_4)} & -\frac{b_{41} p_4}{b_{11} (p_1 - p_2) (p_1 - p_4)} \end{pmatrix}$$

$$R(s) = \begin{pmatrix} \frac{b_{41} p_4 (p_3 + s)}{b_{32} b_{41} p_3 p_4 (p_2 - p_3) (p_3 - p_4)} & -\frac{\psi_3(s)}{b_{11} (p_1 - p_2) (p_1 - p_4)} \\ -\frac{b_{11}^2 b_{22} (p_1 - p_2) (p_1 - p_4)}{b_{32} b_{41} p_3 p_4 (p_2 - p_3) (p_3 - p_4)} & \frac{\psi_4(s)}{b_{32} b_{41}^2 p_3 p_4^2 (p_2 - p_3) (p_3 - p_4)} \end{pmatrix}$$

where

$$\begin{aligned} \psi_1(s) &= b_{11} (p_2 + s) (p_4 + s) \left( b_{11} b_{22} s^4 + b_{32} b_{41} p_3^3 p_4 + b_{11} b_{22} p_2 s^3 + b_{11} b_{22} p_3 s^3 + 2 b_{11} b_{22} p_4 s^3 \right. \\ &\quad - b_{32} b_{41} p_3^2 p_4^2 - b_{11} b_{22} p_1^2 s^2 + b_{11} b_{22} p_4^2 s^2 + b_{11} b_{22} p_1 p_3 p_4^2 - b_{11} b_{22} p_1^2 p_3 p_4 - b_{32} b_{41} p_1 p_3 p_4^2 \\ &\quad + b_{32} b_{41} p_1^2 p_3 p_4 + b_{32} b_{41} p_2 p_3 p_4^2 - b_{32} b_{41} p_2 p_3^2 p_4 + b_{11} b_{22} p_1 p_2 s^2 - b_{11} b_{22} p_1^2 p_3 s + b_{11} b_{22} p_1 p_4 s^2 \\ &\quad + b_{11} b_{22} p_1 p_4^2 s + b_{11} b_{22} p_2 p_3 s^2 - b_{11} b_{22} p_1^2 p_4 s + b_{11} b_{22} p_2 p_4 s^2 + 2 b_{11} b_{22} p_3 p_4 s^2 + b_{11} b_{22} p_3 p_4^2 s \\ &\quad \left. - b_{32} b_{41} p_3 p_4 s^2 - b_{32} b_{41} p_3 p_4^2 s + b_{11} b_{22} p_1 p_2 p_3 p_4 - b_{32} b_{41} p_1 p_2 p_3 p_4 + b_{11} b_{22} p_1 p_2 p_3 s \right. \\ &\quad \left. + b_{11} b_{22} p_1 p_2 p_4 s + b_{11} b_{22} p_1 p_3 p_4 s + b_{11} b_{22} p_2 p_3 p_4 s - b_{32} b_{41} p_2 p_3 p_4 s \right) \\ \psi_2(s) &= (p_1 + s) (p_3 + s) \left( b_{32} b_{41} p_3^2 p_4 - b_{32} b_{41} p_3 p_4^2 - b_{32} b_{41} p_3 p_4 s - b_{32} b_{41} p_2 p_3 p_4 + b_{11} b_{22} p_4^2 s \right. \\ &\quad \left. + b_{11} b_{22} p_2 p_4^2 + 2 b_{11} b_{22} p_4 s^2 + 2 b_{11} b_{22} p_2 p_4 s + b_{11} b_{22} s^3 + b_{11} b_{22} p_2 s^2 \right) \\ \psi_3(s) &= (p_4 + s) \left( b_{11} b_{22} s^3 - b_{11} b_{22} p_1^2 p_3 - b_{11} b_{22} p_1^2 s + b_{11} b_{22} p_2 s^2 + b_{11} b_{22} p_3 s^2 + b_{11} b_{22} p_4 s^2 \right. \\ &\quad + b_{11} b_{22} p_1 p_2 p_3 + b_{11} b_{22} p_1 p_3 p_4 - b_{32} b_{41} p_2 p_3 p_4 \\ &\quad \left. + b_{11} b_{22} p_1 p_2 s + b_{11} b_{22} p_1 p_4 s + b_{11} b_{22} p_2 p_3 s + b_{11} b_{22} p_3 p_4 s - b_{32} b_{41} p_3 p_4 s \right) \\ \psi_4(s) &= b_{11} b_{22} \left( -b_{11} b_{22} p_1^2 p_4 - b_{11} b_{22} p_1^2 s + b_{11} b_{22} p_1 p_4^2 + b_{11} b_{22} p_1 p_4 s + b_{11} b_{22} p_2 p_1 p_4 + b_{11} b_{22} p_2 p_1 s \right. \\ &\quad + b_{32} b_{41} p_3^2 p_4 - b_{32} b_{41} p_3 p_4^2 - b_{32} b_{41} p_3 p_4 s - b_{32} b_{41} p_2 p_3 p_4 + b_{11} b_{22} p_4^2 s + 2 b_{11} b_{22} p_4 s^2 \\ &\quad \left. + b_{11} b_{22} p_2 p_4 s + b_{11} b_{22} s^3 + b_{11} b_{22} p_2 s^2 \right) \end{aligned}$$

One can obtain a matrix  $K(s)$  so getting in the proper factorization for which the minimum phase proper component;

$$\tilde{G}_s(s) = \begin{pmatrix} \frac{b_{11}}{p_1 + s} & \frac{b_{32} p_3}{(p_1 + s) (p_3 + s)} \\ \psi_5(s) & \psi_6(s) \end{pmatrix}$$

where

$$\begin{aligned}\psi_5(s) &= \frac{b_{32} b_{41} p_4 (\beta + 2 b_{11} b_{22} p_2 - b_{11} b_{22} p_3 + b_{11} b_{22} p_4 + 2 b_{11} b_{22} s)}{2 \beta (p_2 + s) (p_4 + s)} - \frac{b_{11} b_{22} b_{32} b_{41} p_4}{\beta (p_1 + s)} \\ \psi_6(s) &= \frac{b_{22} b_{32} (\beta - b_{11} b_{22} p_3 + b_{11} b_{22} p_4)}{2 \beta (p_2 + s)} \\ &\quad - \left( \frac{b_{32} \beta (p_1 + s) b_{22} (\beta - b_{11} b_{22} p_3 + b_{11} b_{22} p_4) - 2 b_{32} b_{22} b_{32} b_{41} p_3 p_4}{2 \beta (p_1 + s) (p_3 + s)} \right)\end{aligned}$$

In the original state space coordinates, a suitable realization of  $G_s(s)$  yeids;

$$y_s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{b_{32} b_{41} p_4}{2 b_{11} \beta} & \frac{b_{32}}{2} - \frac{b_{32} (p_3 + p_4)}{2 \beta} & -\frac{b_{22}}{2} - \frac{b_{22} (p_3 + p_4)}{2 \beta} & \frac{b_{32} p_4}{2 \beta} \end{pmatrix} h \quad (3)$$

with  $\beta = \sqrt{(p_3 + p_4)^2 - 4 \frac{p_3 p_4}{b_{11} b_{22}} (b_{11} b_{22} - b_{32} b_{41})}$  making the LTM model of (1a) minimum-phase.

### 1.0.3. The new dummy output using the normal form

At this point, one can also verify that the dummy output can be obtained directly using the normal form through an iterative procedure. To this end, note that the coordinates change

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b_{41} & 0 & 0 & -b_{11} \\ 0 & b_{32} & -b_{22} & 0 \end{pmatrix}$$

puts the LTM at  $h^*$  of the 4-tanks system into the normal form;

$$\begin{aligned}\dot{z}_1 &= R_1 z + S_1 \eta_\star + a_{1,1} u_1 + a_{1,2} u_2 \\ \dot{z}_2 &= R_2 z + S_2 \eta_\star + a_{2,1} u_1 + a_{2,2} u_2 \\ \dot{\eta}_\star &= P z + Q_\star \eta_\star \\ y_1 &= z_1, \quad y_2 = z_2\end{aligned}$$

where  $a_{1,1} = b_{11}$ ,  $a_{2,2} = b_{22}$ ,  $a_{1,2} = a_{2,1} = 0$  and

$$\begin{aligned}R_1 &= \begin{pmatrix} -p_1 & \frac{b_{32} p_3}{b_{22}} \end{pmatrix}, \quad R_2 = \begin{pmatrix} \frac{b_{41} p_4}{b_{11}} & -p_2 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & -\frac{p_3}{b_{22}} \end{pmatrix}, \quad S_2 = \begin{pmatrix} -\frac{p_4}{b_{11}} & 0 \end{pmatrix} \\ P &= \begin{pmatrix} -b_{41} (p_1 - p_4) & \frac{b_{32} b_{41} p_3}{b_{22}} \\ \frac{b_{32} b_{41} p_4}{b_{11}} & -b_{32} (p_2 - p_3) \end{pmatrix}, \quad Q_\star = \begin{pmatrix} -p_4 & -\frac{b_{41} p_3}{b_{22}} \\ -\frac{b_{32} p_4}{b_{11}} & -p_3 \end{pmatrix}\end{aligned}$$

The transformation  $V$  putting  $Q_\star$  in Jordan canonical form is;

$$V = \begin{pmatrix} \frac{\beta - b_{11} b_{22} p_3 + b_{11} b_{22} p_4}{2 b_{22} b_{32} p_4} & -\frac{\beta + b_{11} b_{22} p_3 - b_{11} b_{22} p_4}{2 b_{22} b_{32} p_4} \\ 1 & 1 \end{pmatrix} \implies \tilde{Q} = V Q V^{-1} = \begin{pmatrix} z_s & 0 \\ 0 & z_u \end{pmatrix}$$

**Step 1:** Fix the first output as  $\tilde{y}_1^s = h_1$  and for the second output the  $\tilde{y}_2^s = \eta_\star^2$  corresponding to  $z_u$  in original  $h$  coordinates. Namely, let

$$\begin{aligned}\tilde{y}_2^s &= \tilde{C}_2^s h \\ \tilde{C}_2^s &= \left( 0 \quad 0 \quad -\frac{b_{22} b_{32} p_4}{\beta} \quad \frac{\beta - b_{11} b_{22} p_3 + b_{11} b_{22} p_4}{2\beta} \right) T^{-1} \\ &= \left( -\frac{b_{32} b_{41} p_4}{2b_{11}\beta} \quad \frac{b_{32}}{2} - \frac{b_{32}(p_3 + p_4)}{2\beta} \quad -\frac{b_{22}}{2} - \frac{b_{22}(p_3 + p_4)}{2\beta} \quad \frac{b_{32} p_4}{2\beta} \right)\end{aligned}$$

and testing this output, it has relative degree  $r = (1 \ 2)$  and with respect to which only  $z_s$  is the corresponding zero. In fact this output  $\tilde{y}^s = (\tilde{y}_1^s \ \tilde{y}_2^s)^\top$  is precisely that found in (3). Consequently, no need to reiterate and the procedure terminates.

#### 1.0.4. Asymptotic tracking with stability

It is easily checked that, the nonlinear dynamics (1a) with output as in (3) possesses a well-defined relative degree  $r_s = (1, 2)$  at  $h^\star$ . Also, it is a matter of computations to verify that (1a) with output (3) is locally minimum-phase with zero-dynamics  $\dot{\eta}^s = q_s(0, \eta^s)$  verifying  $\frac{\partial q_s}{\partial \eta^s}(0, \eta_\star^s) = z_s < 0$ . At this point, along the lines the previous section and by exploiting the results in [1, Chapter 5], one gets that output tracking of (1) can be solved over the dummy output (3) by setting the constant  $y_\star^s = (y_{1,\star}^s, y_{2,\star}^s)^\top \in \mathbb{R}^2$  as solution to  $y_\star = Z_u(d)y_\star^s$  which is given by construction as  $y_\star^s = C_s h^\star$ . Accordingly, for all  $k_0, k_1 > 0$  the feedback

$$u = -M_s^{-1}(h) \begin{pmatrix} c_1^s f(h) + y_{1,\star}^s - y_{1,\star}^s \\ L_f c_2^s f(h) + k_1 c_2^s f(h) + k_0 (y_{2,\star}^s - y_{2,\star}^s) \end{pmatrix} \quad (4)$$

with decoupling matrix

$$M_s^{-1}(h) = \begin{pmatrix} c_1^s B \\ L_f^2 c_2^s f(h) B \end{pmatrix}$$

ensures local asymptotic regulation of  $y$  to the desired  $y^\star$  while preserving internal stability.

#### 1.0.5. Simulations

For completeness, simulations are reported in Figure 1 for the closed-loop system under the stabilizing feedback designed over the new dummy output highlighting the locally minimum-phase components of (1). Simulations are performed for the parameters fixed as in the Table below

$A_1$ [ $cm^2$ ]	28	$A_3$ [ $cm^2$ ]	28
$A_2$ [ $cm^2$ ]	32	$A_4$ [ $cm^2$ ]	32
$c_1$ [ $cm^2$ ]	0.071	$c_3$ [ $cm^2$ ]	0.071
$c_2$ [ $cm^2$ ]	0.057	$c_4$ [ $cm^2$ ]	0.057
$k_t$ [ $V/cm$ ]	1	$g$ [ $cm/s^2$ ]	981
$\gamma_1$	0.43	$\gamma_2$	0.34
$k_1$	65.12	$k_2$	94.12

and with  $y_\star = (7.1, 6.2)^\top$  corresponding to  $h^\star = (7.1, 6.2, 3.58, 1.632)^\top$ . In particular, with this choice of parameters, the plant is nonminimum-phase with the zeros of

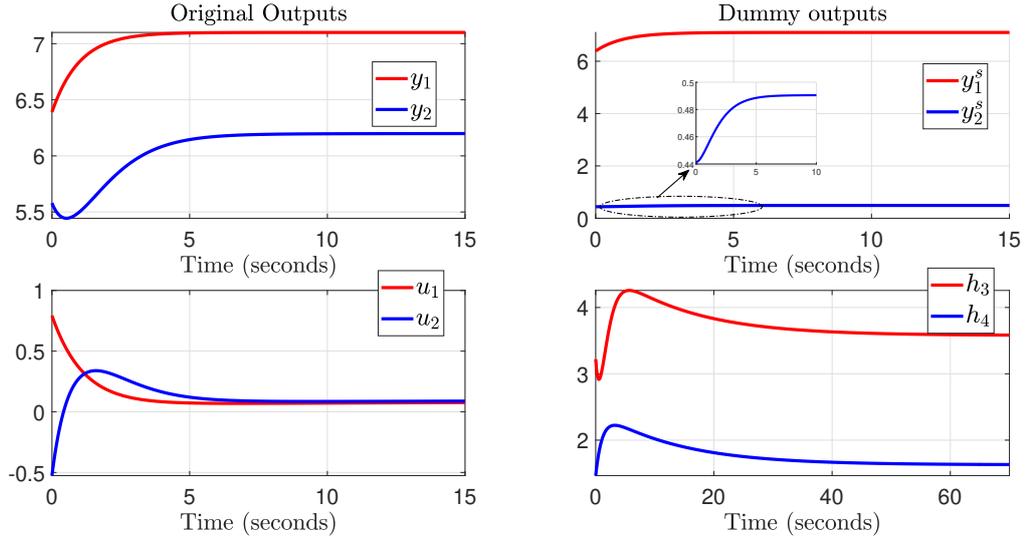


Figure 1.: The four tank model under stable dynamic inversion.

LTM model at the desired equilibrium provided by  $z_u = 0.018$  and  $z_s = -0.0789$ . The gains of the controller (4) are fixed as  $(k_0, k_1) = (1, 2)$ . Simulations report the story of the original and dummy outputs plus the internal dynamics (that is the water levels of the third and fourth tank) while proving the effectiveness of the proposed control design approach.

## References

- [1] Alberto Isidori, *Nonlinear Control Systems*, Springer-Verlag, 1995.
- [2] M. Elobaid, M. Mattioni, S. Monaco and D. Normand-Cyrot, *On the right-inversion of partially minimum phase systems*, submitted for review to the CDC 2020.