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REPORT

On the right-inversion of partially minimum-phase systems: case study

Mohamed Elobaid, Mattia Mattioni, Salvatore Monaco, Dorothee Normand Cyrot

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ABSTRACT

This technical manuscript reports the detailed calculations, and simulations carried on the case study of the 4-tanks system based on the control procedure proposed in [2]. The case study illustrates how, starting from a nonminimum phase nonlinear system with a linear output, one goes about identifying the minimum phase zeros of the LTM model of the system, calculating a related dummy output, with respect to which the original nonlinear system is minimum phase. Regulation of the original output is thus possible using standard static feedback controls.

KEYWORDS

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1. Case study: The bench-marking 4-tanks system

Consider the case of a 4-tanks system given by

$$\dot{h} = f(h) + Bu \quad (1a)$$

$$y = Ch \quad (1b)$$

with $h = \text{col}\{h_1, h_2, h_3, h_4\}$, $f(h) = 2F(h)h$

$$F(h) = \begin{pmatrix} -p_1(h_1) & 0 & \frac{A_3}{A_1}p_3(h_3) & 0 \\ 0 & -p_2(h_2) & 0 & \frac{A_4}{A_2}p_4(h_4) \\ 0 & 0 & -p_3(h_3) & 0 \\ 0 & 0 & 0 & -p_4(h_4) \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{pmatrix}, \quad C = \kappa_t \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}^\top$$

$p_i(h_i) = \frac{c_i \sqrt{2gh_i}}{2A_i h_i}$. For the sake of compactness, let b_{ij} correspond to the element in position (i, j) of the input-state matrix B . In particular, h_i , A_i and c_i are, respectively, the level of water in the i^{th} -tank, its cross-section area and the cross-section of the outlet hole for $i = 1, 2, 3, 4$. The control signals u_j with $j = 1, 2$ correspond to the voltage applied to j^{th} -pump with $k_j u_j$ being the corresponding flow. We consider the problem of locally asymptotically tracking the output of (1) to a desired $y_\star = (h_1^\star, h_2^\star)$ corresponding to make $h^\star = (h_1^\star, h_2^\star, h_3^\star, h_4^\star)^\top$ with

$$h_3^\star = \frac{(c_1 \gamma_2 \sqrt{h_1^\star} - c_2 (1 - \gamma_2) \sqrt{h_2^\star})^2}{c_3^2 a_3^2 \gamma_2^2}$$

$$h_4^\star = \frac{(c_2 \gamma_1 \sqrt{h_2^\star} - c_1 (1 - \gamma_1) \sqrt{h_1^\star})^2}{c_4^2 a_4^2 \gamma_1^2}$$

for $a_3 = \frac{\gamma_2}{1-\gamma_2} - \frac{1-\gamma_1}{\gamma_1}$ and $a_4 = \frac{\gamma_1}{1-\gamma_1} - \frac{1-\gamma_2}{\gamma_2}$ a locally asymptotically stable equilibrium for the closed-loop system under nonlinear feedback.

1.0.1. Analysis of the zero-dynamics

The vector relative degree of (1) is well defined and given by $r = (1 \ 1)$ so that it exhibits a two-dimensional zero-dynamics. Accordingly, for investigating minimum-phasesness of (1), one computes the linear tangent model (LTM) at h^\star of the form

$$\dot{x} = Ax + Bu, \quad y = Cx \quad (2)$$

with $x = h - h^\star$ and $A = 2F(h^\star)$ with corresponding transfer function matrix

$$P(s) = \kappa_t \begin{pmatrix} \frac{b_{11}}{s+p_1} & \frac{b_{32} p_3}{(s+p_1)(s+p_3)} \\ \frac{b_{41} p_4}{(s+p_2)(s+p_4)} & \frac{b_{22}}{s+p_2} \end{pmatrix} \quad (3)$$

with $p_i = p_i(h_i^*) > 0$ for $i = 1, 2, 3, 4$ with Smith form as $M(s) = \text{diag}\{\frac{1}{d(s)}, z(s)\}$ with pole-polynomial $d(s) = (s + p_1)(s + p_2)(s + p_3)(s + p_4)$ and zero-polynomial $z(s) = s^2 + (p_3 + p_4)s + \frac{p_3 p_4}{b_{11} b_{22}}(b_{11} b_{22} - b_{32} b_{41})$. Thus, (1) is nonminimum-phase if $b_{11} b_{22} - b_{32} b_{41} < 0$ so that one can factorize $z(s) = (s - z_u)(s - z_s)$ for $z_u \in \mathbb{R}^+$ and $z_s \in \mathbb{R}^-$. As a consequence, if $b_{11} b_{22} - b_{32} b_{41} < 0$, output regulation to y_* cannot be achieved through classical right-inversion even if the relative degree is well-defined.

In the following we show how the procedure detailed in Section III of [2] allows to deduce a new output $y_s = C_s h$ and a nonlinear feedback locally solving the regulation problem with stability for (1).

1.0.2. The new dummy output

By virtue of Remark 3.1 in [2], because (A, B, C) possesses three distinct poles in general, one gets that the matrix $P_s(s) = \text{diag}\{1, s - z_s\} \text{diag}\{d(s), 1\} R(s)$ is improper for all choices of $(L(s), R(s))$. However, for the pair (The terms $\psi_i(s)$ are reported at the end of the manuscript for space reasons).

$$L(s) = \begin{pmatrix} -\frac{\psi_1(s)}{b_{32} b_{41}^2 p_3 p_4^2 (p_2 - p_3) (p_3 - p_4)} & -\frac{p_2 - p_1 + p_4 + s}{(p_1 - p_2) (p_1 - p_4)} \\ -\frac{\psi_2(s)}{b_{32} b_{41} p_3 p_4 (p_2 - p_3) (p_3 - p_4)} & -\frac{b_{41} p_4}{b_{11} (p_1 - p_2) (p_1 - p_4)} \end{pmatrix}$$

$$R(s) = \begin{pmatrix} b_{41} p_4 (p_3 + s) & -\frac{\psi_3(s)}{b_{11} (p_1 - p_2) (p_1 - p_4)} \\ -\frac{b_{11}^2 b_{22} (p_1 - p_2) (p_1 - p_4)}{b_{32} b_{41} p_3 p_4 (p_2 - p_3) (p_3 - p_4)} & \frac{\psi_4(s)}{b_{32} b_{41}^2 p_3 p_4^2 (p_2 - p_3) (p_3 - p_4)} \end{pmatrix}$$

One can obtain a matrix $K(s)$ such that $P_s(s) = (K(s))^{-1} P_s(s)$ is strictly proper transfer function matrix having as poles the original system poles and as zeros $z_s(s)$. Namely

$$K(s) = \begin{pmatrix} K_{1,1}(s) & K_{1,2}(s) \\ K_{2,1}(s) & K_{2,2}(s) \end{pmatrix}$$

with

$$K_{1,1}(s) = \frac{b_{41} p_4}{b_{11} (p_1 - p_2) (p_1 - p_4)} - \frac{b_{22} b_{41} p_4 (p_2 - p_1 + p_4 + s) (\gamma p_1 - \gamma p_3 + b_{11} b_{22} p_3^2 - b_{11} b_{22} p_1 p_3 + b_{11} b_{22} p_1 p_4 - b_{11} b_{22} p_3 p_4 + 2 b_{32} b_{41} p_3 p_4)}{\psi_5(s)}$$

$$K_{1,2}(s) = \frac{(p_2 - p_1 + p_4 + s)}{(p_1 - p_2) (p_1 - p_4)} \times \left(\frac{2 \gamma s}{b_{32} (p_2 - p_1 + p_4 + s) (\gamma p_1 - \gamma p_3 + b_{11} b_{22} p_3^2 - b_{11} b_{22} p_1 p_3 + b_{11} b_{22} p_1 p_4 - b_{11} b_{22} p_3 p_4 + 2 b_{32} b_{41} p_3 p_4)} + \frac{\gamma p_3 (\gamma - b_{11} b_{22} p_3 + b_{11} b_{22} p_4 - 2 b_{32} b_{41} p_4)}{\psi_6(s)} \right)$$

$$K_{1,3}(s) = \frac{\psi_7(s)}{\psi_8(s)} - \frac{\psi_9(s)}{b_{32} b_{41} p_3 p_4 (p_2 - p_3) (p_3 - p_4) (b_{11} b_{22} p_3 - \gamma + b_{11} b_{22} p_4 + 2 b_{11} b_{22} s)}$$

$$K_{1,4} = \frac{1}{b_{32} b_{41}^2 p_3 p_4^2 (p_2 - p_3) (p_3 - p_4) (b_{11} b_{22} p_3 - \gamma + b_{11} b_{22} p_4 + 2 b_{11} b_{22} s)} \times$$

$$\left(\frac{-4 \gamma s b_{11}^2 b_{22} (p_2 + s) (p_4 + s)}{b_{32} (\gamma p_2 - \gamma p_3 + b_{11} b_{22} p_3^2 - b_{11} b_{22} p_2 p_3 + b_{11} b_{22} p_2 p_4 - b_{11} b_{22} p_3 p_4 + 2 b_{32} b_{41} p_3 p_4)} \right.$$

$$\left. + \frac{-2 b_{11}^2 b_{22} (p_2 + s) (p_4 + s) \gamma p_3 (\gamma - b_{11} b_{22} p_3 + b_{11} b_{22} p_4 - 2 b_{32} b_{41} p_4) \psi_{11}(s)}{\psi_{10}(s)} \right)$$

and

$$\gamma = \sqrt{b_{11} b_{22} (b_{11} b_{22} p_3^2 + b_{11} b_{22} p_4^2 - 2 b_{11} b_{22} p_3 p_4 + 4 b_{32} b_{41} p_3 p_4)}$$

From which one obtains

$$P_s(s) = \begin{pmatrix} \frac{b_{11}}{p_1+s} & \frac{b_{32} p_3}{(p_1+s)(p_3+s)} \\ P_{s,3}(s) & P_{s,4}(s) \end{pmatrix}$$

with

$$P_{s,3}(s) = \frac{b_{32} b_{41} p_4 (\gamma + 2 b_{11} b_{22} p_2 - b_{11} b_{22} p_3 + b_{11} b_{22} p_4 + 2 b_{11} b_{22} s)}{2 \gamma (p_2 + s) (p_4 + s)} - \frac{b_{11} b_{22} b_{32} b_{41} p_4}{\gamma (p_1 + s)}$$

$$P_{s,4}(s) = \frac{b_{22} b_{32} (\gamma - b_{11} b_{22} p_3 + b_{11} b_{22} p_4)}{2 \gamma (p_2 + s)}$$

$$- \left(\frac{b_{32} \gamma (p_1 + s) b_{22} (\gamma - b_{11} b_{22} p_3 + b_{11} b_{22} p_4) - 2 b_{32} b_{22} b_{32} b_{41} p_3 p_4}{2 \gamma (p_1 + s) (p_3 + s)} \right)$$

Consequently, utilizing Remark 3.2 of [2], One obtains

$$y_s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{b_{32} b_{41} p_4}{2 b_{11} \beta} & \frac{b_{32}}{2} \frac{b_{32} (p_3 + p_4)}{2 \beta} & -\frac{b_{22}}{2} \frac{b_{22} (p_3 + p_4)}{2 \beta} & \frac{b_{32} p_4}{2 \beta} \end{pmatrix} h \quad (4)$$

with $\beta = \sqrt{(p_3 + p_4)^2 - 4 \frac{p_3 p_4}{b_{11} b_{22}} (b_{11} b_{22} - b_{32} b_{41})}$ making the LTM model of (1) minimum-phase.

1.0.3. Asymptotic tracking with stability

It is easily checked that, the nonlinear dynamics (1a) with output as in (4) possesses a well-defined relative degree $r_s = (1, 2)$ at h^* . Also, it is a matter of computations to verify that (1a) with output as in (4) is locally minimum-phase with zero-dynamics $\dot{\eta}^s = q_s(0, \eta^s)$ verifying $\frac{\partial q_s}{\partial \eta_s}(0, \eta_s^*) = z_s < 0$. At this point, along the lines of Remark 4.3 in [2] and by exploiting the results in [1, Chapter 5], one gets that output tracking of (1) can be solved over the dummy output (4) by setting the constant $y_*^s = (y_{1,*}^s, y_{2,*}^s)^\top \in \mathbb{R}^2$ as solution to $y_* = Z_u(d) y_*^s$ which is given by construction as $y_*^s = C_s h^*$. Accordingly, for all $k_0, k_1 > 0$ the feedback

$$u = -M_s^{-1}(h) \begin{pmatrix} c_1^s f(h) + y_1^s - y_{1,*}^s \\ L_f c_2^s f(h) + k_1 c_2^s f(h) + k_0 (y_2^s - y_{2,*}^s) \end{pmatrix} \quad (5)$$

with decoupling matrix

$$M_s^{-1}(h) = \begin{pmatrix} c_1^s B \\ L_f^2 c_2^s f(h) B \end{pmatrix}$$

ensures local asymptotic tracking of $y(t)$ to the desired y^* while preserving internal stability.

1.0.4. Simulations

For completeness, simulations are reported in Figure 1 for the closed-loop system under the stabilizing feedback designed over the new dummy output highlighting the locally minimum-phase components of (1). Simulations are performed for the parameters fixed as in the Table below

A_1 [cm^2]	28	A_3 [cm^2]	28
A_2 [cm^2]	32	A_4 [cm^2]	32
c_1 [cm^2]	0.071	c_3 [cm^2]	0.071
c_2 [cm^2]	0.057	c_4 [cm^2]	0.057
k_t [V/cm]	1	g [cm/s^2]	981
γ_1	0.43	γ_2	0.34
k_1	65.12	k_2	94.12

and with $y_* = (7.1, 6.2)^\top$ corresponding to $h^* = (7.1, 6.2, 3.58, 1.632)^\top$. In particular, with this choice of parameters, the plant is nonminimum-phase with the zeros of LTM model at the desired equilibrium provided by $z_u = 0.018$ and $z_s = -0.0789$. The gains of the controller (5) are fixed as $(k_0, k_1) = (1, 2)$. Simulations report the story of the original and dummy outputs plus the internal dynamics (that is the water levels of the third and fourth tank) while proving the effectiveness of the proposed control design approach.

Appendix

The terms $\psi_i(s)$ used to simplify the expressions of $L(s)$, $R(s)$ and $K(s)$ are

$$\begin{aligned}
\psi_1(s) &= b_{11} (p_2 + s) (p_4 + s) \left(b_{11} b_{22} s^4 + b_{32} b_{41} p_3^3 p_4 + b_{11} b_{22} p_2 s^3 + b_{11} b_{22} p_3 s^3 + 2 b_{11} b_{22} p_4 s^3 \right. \\
&\quad - b_{32} b_{41} p_3^2 p_4^2 - b_{11} b_{22} p_1^2 s^2 + b_{11} b_{22} p_4^2 s^2 + b_{11} b_{22} p_1 p_3 p_4^2 - b_{11} b_{22} p_1^2 p_3 p_4 - b_{32} b_{41} p_1 p_3 p_4^2 \\
&\quad + b_{32} b_{41} p_1^2 p_3 p_4 + b_{32} b_{41} p_2 p_3 p_4^2 - b_{32} b_{41} p_2 p_3^2 p_4 + b_{11} b_{22} p_1 p_2 s^2 - b_{11} b_{22} p_1^2 p_3 s + b_{11} b_{22} p_1 p_4 s^2 \\
&\quad + b_{11} b_{22} p_1 p_4^2 s + b_{11} b_{22} p_2 p_3 s^2 - b_{11} b_{22} p_1^2 p_4 s + b_{11} b_{22} p_2 p_4 s^2 + 2 b_{11} b_{22} p_3 p_4 s^2 + b_{11} b_{22} p_3 p_4^2 s \\
&\quad - b_{32} b_{41} p_3 p_4 s^2 - b_{32} b_{41} p_3 p_4^2 s + b_{11} b_{22} p_1 p_2 p_3 p_4 - b_{32} b_{41} p_1 p_2 p_3 p_4 + b_{11} b_{22} p_1 p_2 p_3 s \\
&\quad \left. + b_{11} b_{22} p_1 p_2 p_4 s + b_{11} b_{22} p_1 p_3 p_4 s + b_{11} b_{22} p_2 p_3 p_4 s - b_{32} b_{41} p_2 p_3 p_4 s \right) \\
\psi_2(s) &= (p_1 + s) (p_3 + s) \left(b_{32} b_{41} p_3^2 p_4 - b_{32} b_{41} p_3 p_4^2 - b_{32} b_{41} p_3 p_4 s - b_{32} b_{41} p_2 p_3 p_4 + b_{11} b_{22} p_4^2 s \right. \\
&\quad \left. + b_{11} b_{22} p_2 p_4^2 + 2 b_{11} b_{22} p_4 s^2 + 2 b_{11} b_{22} p_2 p_4 s + b_{11} b_{22} s^3 + b_{11} b_{22} p_2 s^2 \right) \\
\psi_3(s) &= (p_4 + s) \left(b_{11} b_{22} s^3 - b_{11} b_{22} p_1^2 p_3 - b_{11} b_{22} p_1^2 s + b_{11} b_{22} p_2 s^2 + b_{11} b_{22} p_3 s^2 + b_{11} b_{22} p_4 s^2 \right. \\
&\quad + b_{11} b_{22} p_1 p_2 p_3 + b_{11} b_{22} p_1 p_3 p_4 - b_{32} b_{41} p_2 p_3 p_4 \\
&\quad \left. + b_{11} b_{22} p_1 p_2 s + b_{11} b_{22} p_1 p_4 s + b_{11} b_{22} p_2 p_3 s + b_{11} b_{22} p_3 p_4 s - b_{32} b_{41} p_3 p_4 s \right) \\
\psi_4(s) &= b_{11} b_{22} \left(- b_{11} b_{22} p_1^2 p_4 - b_{11} b_{22} p_1^2 s + b_{11} b_{22} p_1 p_4^2 + b_{11} b_{22} p_1 p_4 s + b_{11} b_{22} p_2 p_1 p_4 + b_{11} b_{22} p_2 p_1 s \right. \\
&\quad + b_{32} b_{41} p_3^2 p_4 - b_{32} b_{41} p_3 p_4^2 - b_{32} b_{41} p_3 p_4 s - b_{32} b_{41} p_2 p_3 p_4 + b_{11} b_{22} p_4^2 s + 2 b_{11} b_{22} p_4 s^2 \\
&\quad \left. + b_{11} b_{22} p_2 p_4 s + b_{11} b_{22} s^3 + b_{11} b_{22} p_2 s^2 \right) \\
\psi_5(s) &= b_{32} \left(b_{11}^2 b_{22}^2 p_2 p_3^2 - b_{11}^2 b_{22}^2 p_3^3 + b_{11}^2 b_{22}^2 p_2 p_4^2 - b_{11}^2 b_{22}^2 p_3 p_4^2 + 2 b_{11}^2 b_{22}^2 p_3^2 p_4 + b_{11} b_{22} \gamma p_3^2 \right. \\
&\quad - 2 b_{11}^2 b_{22}^2 p_2 p_3 p_4 - b_{11} b_{22} \gamma p_2 p_3 + b_{11} b_{22} \gamma p_2 p_4 - b_{11} b_{22} \gamma p_3 p_4 + b_{32} b_{41} \gamma p_3 p_4 \\
&\quad \left. + b_{11} b_{22} b_{32} b_{41} p_3 p_4^2 - 3 b_{11} b_{22} b_{32} b_{41} p_3^2 p_4 + 2 b_{11} b_{22} b_{32} b_{41} p_2 p_3 p_4 \right) \\
\psi_6(s) &= b_{32} \left(b_{11}^2 b_{22}^2 p_2 p_3^2 - b_{11}^2 b_{22}^2 p_3^3 + b_{11}^2 b_{22}^2 p_2 p_4^2 - b_{11}^2 b_{22}^2 p_3 p_4^2 + 2 b_{11}^2 b_{22}^2 p_3^2 p_4 + b_{11} b_{22} \gamma p_3^2 \right. \\
&\quad - 2 b_{11}^2 b_{22}^2 p_2 p_3 p_4 - b_{11} b_{22} \gamma p_2 p_3 + b_{11} b_{22} \gamma p_2 p_4 - b_{11} b_{22} \gamma p_3 p_4 + b_{32} b_{41} \gamma p_3 p_4 + b_{11} b_{22} b_{32} b_{41} p_3 p_4^2 \\
&\quad \left. - 3 b_{11} b_{22} b_{32} b_{41} p_3^2 p_4 + 2 b_{11} b_{22} b_{32} b_{41} p_2 p_3 p_4 \right) \\
\psi_7(s) &= 2 b_{11}^2 b_{22}^2 (p_2 + s) (p_4 + s) (\gamma p_1 - \gamma p_3 + b_{11} b_{22} p_3^2 - b_{11} b_{22} p_1 p_3 + b_{11} b_{22} p_1 p_4 - b_{11} b_{22} p_3 p_4 \\
&\quad + 2 b_{32} b_{41} p_3 p_4) \left(b_{11} b_{22} s^4 + b_{32} b_{41} p_3^3 p_4 + b_{11} b_{22} p_2 s^3 + b_{11} b_{22} p_3 s^3 + 2 b_{11} b_{22} p_4 s^3 - b_{32} b_{41} p_3^2 p_4^2 \right. \\
&\quad - b_{11} b_{22} p_1^2 s^2 + b_{11} b_{22} p_4^2 s^2 + b_{11} b_{22} p_1 p_3 p_4^2 - b_{11} b_{22} p_1^2 p_3 p_4 - b_{32} b_{41} p_1 p_3 p_4^2 + b_{32} b_{41} p_1^2 p_3 p_4 \\
&\quad + b_{32} b_{41} p_2 p_3 p_4^2 - b_{32} b_{41} p_2 p_3^2 p_4 + b_{11} b_{22} p_1 p_2 s^2 - b_{11} b_{22} p_1^2 p_3 s + b_{11} b_{22} p_1 p_4 s^2 + b_{11} b_{22} p_1 p_4^2 s \\
&\quad + b_{11} b_{22} p_2 p_3 s^2 - b_{11} b_{22} p_1^2 p_4 s + b_{11} b_{22} p_2 p_4 s^2 + 2 b_{11} b_{22} p_3 p_4 s^2 + b_{11} b_{22} p_3 p_4^2 s - b_{32} b_{41} p_3 p_4 s^2 \\
&\quad - b_{32} b_{41} p_3 p_4^2 s + b_{11} b_{22} p_1 p_2 p_3 p_4 - b_{32} b_{41} p_1 p_2 p_3 p_4 + b_{11} b_{22} p_1 p_2 p_3 s \\
&\quad \left. + b_{11} b_{22} p_1 p_2 p_4 s + b_{11} b_{22} p_1 p_3 p_4 s + b_{11} b_{22} p_2 p_3 p_4 s - b_{32} b_{41} p_2 p_3 p_4 s \right) \\
\psi_8(s) &= b_{32} b_{41} p_3 p_4 (p_2 - p_3) (p_3 - p_4) (b_{11} b_{22} p_3 - \gamma + b_{11} b_{22} p_4 + 2 b_{11} b_{22} s) \left(b_{11}^2 b_{22}^2 p_2 p_3^2 - b_{11}^2 b_{22}^2 p_3^3 \right. \\
&\quad + b_{11}^2 b_{22}^2 p_2 p_4^2 - b_{11}^2 b_{22}^2 p_3 p_4^2 + 2 b_{11}^2 b_{22}^2 p_3^2 p_4 + b_{11} b_{22} \gamma p_3^2 - 2 b_{11}^2 b_{22}^2 p_2 p_3 p_4 \\
&\quad - b_{11} b_{22} \gamma p_2 p_3 + b_{11} b_{22} \gamma p_2 p_4 - b_{11} b_{22} \gamma p_3 p_4 + b_{32} b_{41} \gamma p_3 p_4 + b_{11} b_{22} b_{32} b_{41} p_3 p_4^2 \\
&\quad \left. - 3 b_{11} b_{22} b_{32} b_{41} p_3^2 p_4 + 2 b_{11} b_{22} b_{32} b_{41} p_2 p_3 p_4 \right)
\end{aligned}$$

$$\begin{aligned}
\psi_9(s) &= 2 b_{11} b_{22} (p_1 + s) (p_3 + s) (b_{32} b_{41} p_3^2 p_4 - b_{32} b_{41} p_3 p_4^2 - b_{32} b_{41} p_3 p_4 s \\
&\quad - b_{32} b_{41} p_2 p_3 p_4 + b_{11} b_{22} p_4^2 s + b_{11} b_{22} p_2 p_4^2 + 2 b_{11} b_{22} p_4 s^2 + 2 b_{11} b_{22} p_2 p_4 s + b_{11} b_{22} s^3 + b_{11} b_{22} p_2 s^2) \\
\psi_{10} &= b_{32} (b_{11}^2 b_{22}^2 p_2 p_3^2 - b_{11}^2 b_{22}^2 p_3^3 + b_{11}^2 b_{22}^2 p_2 p_4^2 - b_{11}^2 b_{22}^2 p_3 p_4^2 + 2 b_{11}^2 b_{22}^2 p_3^2 p_4 + b_{11} b_{22} \gamma p_3^2 \\
&\quad - 2 b_{11}^2 b_{22}^2 p_2 p_3 p_4 - b_{11} b_{22} \gamma p_2 p_3 + b_{11} b_{22} \gamma p_2 p_4 - b_{11} b_{22} \gamma p_3 p_4 + b_{32} b_{41} \gamma p_3 p_4 + b_{11} b_{22} b_{32} b_{41} p_3 p_4^2 \\
&\quad - 3 b_{11} b_{22} b_{32} b_{41} p_3^2 p_4 + 2 b_{11} b_{22} b_{32} b_{41} p_2 p_3 p_4) \\
\psi_{11}(s) &= (b_{11} b_{22} s^4 + b_{32} b_{41} p_3^3 p_4 + b_{11} b_{22} p_2 s^3 + b_{11} b_{22} p_3 s^3 + 2 b_{11} b_{22} p_4 s^3 - b_{32} b_{41} p_3^2 p_4^2 - b_{11} b_{22} p_1^2 s^2 \\
&\quad + b_{11} b_{22} p_4^2 s^2 + b_{11} b_{22} p_1 p_3 p_4^2 - b_{11} b_{22} p_1^2 p_3 p_4 - b_{32} b_{41} p_1 p_3 p_4^2 + b_{32} b_{41} p_1^2 p_3 p_4 + b_{32} b_{41} p_2 p_3 p_4^2 \\
&\quad - b_{32} b_{41} p_2 p_3^2 p_4 + b_{11} b_{22} p_1 p_2 s^2 - b_{11} b_{22} p_1^2 p_3 s + b_{11} b_{22} p_1 p_4 s^2 + b_{11} b_{22} p_1 p_4^2 s + b_{11} b_{22} p_2 p_3 s^2 \\
&\quad - b_{11} b_{22} p_1^2 p_4 s + b_{11} b_{22} p_2 p_4 s^2 + 2 b_{11} b_{22} p_3 p_4 s^2 + b_{11} b_{22} p_3 p_4^2 s - b_{32} b_{41} p_3 p_4 s^2 - b_{32} b_{41} p_3 p_4^2 s \\
&\quad + b_{11} b_{22} p_1 p_2 p_3 p_4 - b_{32} b_{41} p_1 p_2 p_3 p_4 + b_{11} b_{22} p_1 p_2 p_3 s + b_{11} b_{22} p_1 p_2 p_4 s \\
&\quad + b_{11} b_{22} p_1 p_3 p_4 s + b_{11} b_{22} p_2 p_3 p_4 s - b_{32} b_{41} p_2 p_3 p_4 s)
\end{aligned}$$

References

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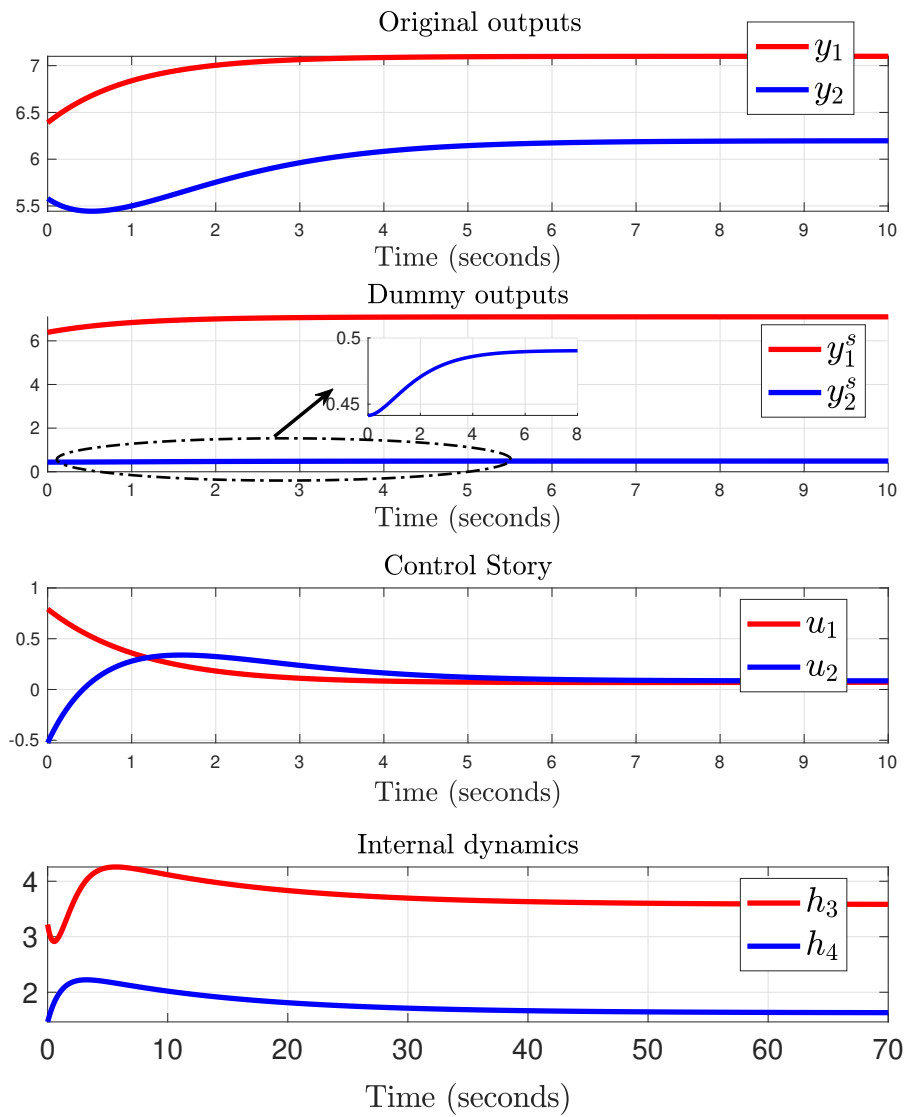


Figure 1.: The four tank model under stable dynamic inversion.