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# The fundamental group of quotients of products of some topological spaces by a finite group — A generalization of a theorem of Bauer–Catanese–Grunewald–Pignatelli

Rodolfo Aguilar Aguilar

**Abstract.** We provide a description of the fundamental group of the quotient of a product of topological spaces  $X_i$ , each admitting a universal cover, by a finite group  $G$ , provided that there is only a finite number of path-connected components in  $X_i^g$  for every  $g \in G$ . This generalizes previous work of Bauer–Catanese–Grunewald–Pignatelli and Dedieu–Perroni.

**Keywords.** fundamental group; quotients by finite group; orbifolds

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[Français]

**Le groupe fondamental de quotients de produits de certains espaces topologiques par un groupe fini — Généralisation d’un théorème de Bauer–Catanese–Grunewald–Pignatelli**

**Résumé.** Nous fournissons une description du groupe fondamental du quotient d’un produit d’espaces topologiques  $X_i$ , chacun admettant un revêtement universel, par un groupe fini  $G$ , pourvu qu’il n’existe qu’un nombre fini de composantes connexes par arcs dans  $X_i^g$  pour chaque  $g \in G$ . Cela généralise des résultats antérieurs de Bauer–Catanese–Grunewald–Pignatelli et de Dedieu–Perroni.

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## 1. Introduction

The fundamental group of a quotient of a Hausdorff space  $X$  by a finite group  $G$  acting freely can be computed noticing that  $X \rightarrow X/G$  is a covering map, and then using the long exact sequence of homotopy groups associated to a fibration. When  $X = X_1 \times \cdots \times X_k$  and  $G$  acts on each  $X_i$  freely and diagonally on  $X$ , the fundamental group of  $X_1 \times \cdots \times X_k$  sits as a finite-index normal subgroup of  $\pi_1(X/G)$ .

In the case where each  $X_i$  is a projective smooth curve and the action of  $G$  is only *faithful*, the following Theorem was shown in [BCGP12].

**Theorem 1.1.** [BCGP12, Theorems 0.10 and 4.1] *Let  $C_1, \dots, C_k$  be smooth projective curves and let  $G$  be a finite group acting faithfully by automorphisms on each of them. Consider the diagonal action of  $G$  on the product  $C_1 \times \cdots \times C_k$ , then the fundamental group of  $(C_1 \times \cdots \times C_k)/G$  admits a normal finite index subgroup  $\mathcal{N}$  isomorphic to a product of fundamental groups of smooth projective curves.*

It was later extended in [DP12] to the case when the action of  $G$  is non-necessarily faithful. There, they quotient  $G$  to obtain a group acting faithfully, follow the subsequent arguments and then extend again to  $G$ .

Let us explain briefly the method of proof in [BCGP12]. First, they consider the orbifold surface groups  $T_i$  of  $C_i/G$ , which are an extension of  $G$  by  $\pi_1(C_i)$  and hence come with a surjective morphism to  $G$  (see Subsection 2.2.1). They show that the fundamental group  $\pi_1((C_1 \times \cdots \times C_k)/G)$  is isomorphic to the quotient of the fiber product  $\mathbb{H} := T_1 \times_G \cdots \times_G T_k$  by the normal subgroup  $\text{Tors}(\mathbb{H})$  generated by the elements of torsion.

The second part relies on the following proposition whose proof uses abstract group theoretic arguments.

**Proposition 1.2.** [DP12, Proposition 3.5] *There exists a short exact sequence of groups*

$$1 \rightarrow E \rightarrow \mathbb{H}/\text{Tors} \mathbb{H} \rightarrow T \rightarrow 1$$

where  $E$  is finite and  $T$  is a group of finite index in a product of orbifold surfaces groups.

They finally use Proposition 1.2 and properties of the orbifold surface groups such as residually finiteness and cohomological goodness to construct a subgroup of  $\mathbb{H}/\text{Tors} \mathbb{H}$  intersecting  $E$  trivially and satisfying the required properties.

Here, a more geometric approach is used via fundamental groups of stacks or orbispaces [Noo05, Che01]. This theory permits to see  $X \rightarrow [X/G]$  as a covering map under some conditions on  $X$ , where  $[X/G]$  denotes the quotient stack, and a long exact sequence of homotopy groups is available. We will denote the fundamental group of the stack  $[X/G]$  by  $\pi_1([X/G])$ .

For  $i = 1, \dots, k$ , let  $X_i$  be a connected, locally path-connected and semi-locally simply connected topological space with an action of a finite group  $G$ , consider the diagonal action of  $G$  in  $X := X_1 \times \cdots \times X_k$

and denote by  $I$  the subgroup generated by the elements having a fixed point in every  $X_i$  for  $i = 1, \dots, k$ . We can formulate now our first main Theorem.

**Theorem 1.3.** *Let  $X, X_1, \dots, X_k$  and  $G$  as above. Suppose that the number of path connected components in the fixed locus set  $X_i^g$  of an element  $g \in G$  is finite for every  $g \in G$  and  $i = 1, \dots, k$ . Then there exists a homomorphism*

$$\pi_1(X/G) \rightarrow \prod_{i=1}^k \pi_1((X_i/I)/(G/I))$$

*which has finite kernel and its image is a finite-index subgroup.*

This can be seen as a generalization of Proposition 1.2 (Bauer–Catanese–Grunewald–Pignatelli) by the remarks preceding the statement of the Proposition.

The action of  $G/I$  on  $X_i/I$  is induced by the action of  $G$  in  $X_i$ . Note that if  $k = 1$  then  $G/I$  can be seen to act freely on  $X_1/I$  and  $\pi_1((X_1/I)/(G/I)) = \pi_1((X_1/I)/(G/I))$  but  $(X_1/I)/(G/I) \cong X_1/G$ . The same argument works if we make the product of the same topological space, which gives the following corollary.

**Corollary 1.4.** *Let  $X_i = X_1$  for  $i = 2, \dots, k$  and  $G$  satisfy the hypothesis of the above theorem. Then the homomorphism  $\pi_1(X/G) \rightarrow \pi_1(X_1/G)^k$  has finite kernel and its image is a finite-index subgroup.*

An important case of Theorem 1.3 and Corollary 1.4 is when  $X_i$  is a smooth complex algebraic variety for  $i = 1, \dots, k$ . Indeed, the fundamental group of a variety with quotient singularities is the fundamental group of a smooth variety.

Our second main Theorem generalizes Theorem 1.1 (Bauer–Catanese–Grunewald–Pignatelli). It can be stated without using the language of stacks or orbispaces.

**Theorem 1.5.** *Let  $X, X_1, \dots, X_k$  and  $G$  satisfy the hypothesis of Theorem 1.3. Suppose that  $\pi_1(X/G)$  is residually finite. Then there exists a normal finite-index subgroup  $\mathcal{N} \triangleleft \pi_1(X/G)$  isomorphic to a product:*

$$\mathcal{N} \cong \prod_{i=1}^k H_i.$$

*with  $H_i \triangleleft \pi_1(X_i/I)$  normal finite-index subgroups.*

As a corollary, following closely the arguments used in [BCGP12], we show that for smooth curves  $C_1, \dots, C_k$  the group  $\pi_1((C_1 \times \dots \times C_k)/G)$  is residually finite. Hence, we have that Theorem 1.1 (Bauer–Catanese–Grunewald–Pignatelli, Dedieu–Perroni) is valid in the case when the curve is non-necessarily compact.

**Corollary 1.6.** *Let  $C_1, \dots, C_k$  be smooth algebraic curves and let  $G$  be a finite group acting on each of them. Then there exists a normal subgroup  $\mathcal{N} \triangleleft \pi_1((C_1 \times \dots \times C_k)/G)$  of finite index, isomorphic to a product  $\Pi_1 \times \dots \times \Pi_k$ , where  $\Pi_j$  is either the fundamental group of a smooth projective curve or a free group of finite rank.*

The paper is organized as follows: in Section 2 preliminary results are given. Then the first main Theorem is proved in Section 3 and the proof of the second main Theorem together with some applications are given in Section 4.

## 2. Preliminaries

### 2.1. Properties of fundamental group of topological stacks

Let  $X$  be a connected, semi-locally simply connected and locally path-connected topological space and  $G$  a finite group acting continuously on it.

**2.1.1. Fiber homotopy exact sequence.**— There exists a homotopy theory for stacks and the existence of the long exact sequence of homotopy, see [Noo14], is more general than what follows, however we only need the following case: consider the topological stack  $\mathcal{X} = [X/G]$ , a point  $x \in X$  and denote by  $\bar{x} \in \mathcal{X}$  the image of  $x$ . We have an associated fibration  $G \rightarrow X \rightarrow \mathcal{X}$  and a long exact sequence of homotopy groups,

$$\cdots \rightarrow \pi_{n+1}(\mathcal{X}, \bar{x}) \rightarrow \pi_n(G, Id) \rightarrow \pi_n(X, x) \rightarrow \pi_n(\mathcal{X}, \bar{x}) \rightarrow \pi_{n-1}(G, Id) \rightarrow \cdots$$

the map  $\pi_n(G, Id) \rightarrow \pi_n(X, x)$  is induced by the orbit  $G \cdot x \hookrightarrow X$ .

**2.1.2. Action on the universal cover.**— The hypothesis made on  $X$  ensures that there exists an universal cover  $\tilde{X}$  and moreover, if we let  $\mathcal{X} = [X/G]$  as in §2.1.1, we have an action of  $\pi_1(\mathcal{X}, \bar{x})$  on  $\tilde{X}$  (see §3.2.1). We will use several times the following lemma in what follows.

**Lemma 2.1.** *Consider the action of  $\pi_1(\mathcal{X}, \bar{x})$  in  $\tilde{X}$ , let  $y \in \tilde{X}$  and denote by  $I_y$  the isotropy group of the action. Then there exists a monomorphism  $I_y \rightarrow G$ .*

*Proof.* By §2.1.1 we obtain a short exact sequence

$$1 \rightarrow \pi_1(X, x) \rightarrow \pi_1(\mathcal{X}, \bar{x}) \rightarrow G \rightarrow 1,$$

as the action of  $\pi_1(X, x)$  on  $\tilde{X}$  is free, we obtain that the restriction of  $\pi_1(\mathcal{X}, \bar{x}) \rightarrow G$  to  $I_y$  is injective.  $\square$

## 2.2. Product of topological spaces

**2.2.1. Fundamental group of the quotient of a product.**— For  $i = 1, \dots, k$  let  $X_i$  as in §2.1 be a connected, semi-locally simply connected and locally path-connected topological space and  $G$  a finite group acting on each of them.

By §2.1.1 we have  $k$  exact sequences

$$(2.1) \quad 1 \rightarrow \pi_1(X_i, x_i) \rightarrow \pi_1(\mathcal{X}_i, \bar{x}_i) \xrightarrow{\varphi_i} (G, Id) \rightarrow 1$$

where  $\mathcal{X}_i = [X_i/G]$ ,  $x_i \in X_i$  and its image in  $\mathcal{X}_i$  is denoted by  $\bar{x}_i$ .

Denote by  $\mathbb{H} := \pi_1(\mathcal{X}_1, x_1) \times_G \cdots \times_G \pi_1(\mathcal{X}_k, x_k)$ . The exact sequences in (2.1) can be assembled as follows

$$(2.2) \quad 1 \rightarrow \pi_1(X_1 \times \cdots \times X_k, x) \rightarrow \mathbb{H} \rightarrow G \rightarrow 1$$

with  $x = (x_1, \dots, x_k)$ . The geometric nature of  $\mathbb{H}$  is shown in the following lemma.

**Lemma 2.2.** *Let  $G$  act diagonally on  $X = X_1 \times \cdots \times X_k$ . Consider the stack  $\mathcal{X} = [X/G]$  then  $\pi_1(\mathcal{X}, \bar{x}) \cong \mathbb{H}$ .*

*Proof.* We have natural projection maps  $\mathcal{X} \rightarrow \mathcal{X}_i$  for  $i = 1, \dots, k$ , which together with the morphisms  $\varphi_i : \pi_1(\mathcal{X}_i, \bar{x}_i) \rightarrow G$  and the universal property of the fiber product give us a morphism  $\pi_1(\mathcal{X}, \bar{x}) \rightarrow \mathbb{H}$ . By the exact sequence of a fibration §2.1.1 applied to the action of  $G$  to  $X_1 \times \cdots \times X_k$  and by (2.2) we obtain

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(X_1 \times \cdots \times X_k, x) & \longrightarrow & \pi_1(\mathcal{X}, \bar{x}) & \longrightarrow & G \longrightarrow 1 \\ & & \downarrow \text{id} & & \downarrow & & \downarrow \text{id} \\ 1 & \longrightarrow & \pi_1(X_1 \times \cdots \times X_k, x) & \longrightarrow & \mathbb{H} & \longrightarrow & G \longrightarrow 1 \end{array}$$

which implies the result.  $\square$

**Lemma 2.3.** *Let  $X, X_i$  and  $G$  be as above. Then*

$$\pi_1(X/G, [x]) \cong \pi_1(\mathcal{X}, \bar{x})/N \cong \pi_1(\mathcal{X}, \bar{x})/\mathbf{I}$$

where  $N$  is the normal subgroup generated by the image of the inertia groups  $I_x$  and  $\mathbf{I}$  is the subgroup generated by the elements of  $\pi_1(\mathcal{X})$  having fixed points in the universal cover of  $X_1 \times \cdots \times X_k$ .

*Proof.* By [Noo08, Theorem 8.3(i)] we have that  $\pi_1(X/G, [x]) \cong \pi_1(\mathcal{X}, \bar{x})/N$ . The group  $\pi_1(\mathcal{X}, \bar{x})$  acts on  $\tilde{X} \cong \tilde{X}_1 \times \cdots \times \tilde{X}_k$  the universal cover of  $X_1 \times \cdots \times X_k$  in such a way that  $[(\tilde{X}_1 \times \cdots \times \tilde{X}_k)/\pi_1(\mathcal{X}, \bar{x})] \cong \mathcal{X}$ . As  $G$  is finite, by Lemma 2.1 any stabilizer  $I_x$  for the action of  $\pi_1(\mathcal{X})$  on  $\tilde{X}$  is finite, therefore it has the slice property and by [Noo08, Theorem 9.1] we obtain that  $\pi_1(X/G, [x]) \cong \pi_1(\mathcal{X}, \bar{x})/I$ .  $\square$

### 3. The fundamental group of the product of topological spaces

#### 3.1. Constructing the homomorphism

**3.1.1. Finite index of the group in the product.**— Let  $I_y$  denote the isotropy at the point  $y$  in  $\tilde{X}$  for the action of  $\pi_1(\mathcal{X}, \bar{x})$ . By Lemma 2.1 the map  $\pi_1(\mathcal{X}, \bar{x}) \rightarrow G$  restricted to  $I_y$  is injective, therefore we can identify  $I_y$  with a subgroup of  $G$ . When we do such identification we will denote it by  $I'_y < G$ .

Now as  $\pi_1(\mathcal{X}, \bar{x}) \cong \pi_1(\mathcal{X}_1, \bar{x}_1) \times_G \cdots \times_G \pi_1(\mathcal{X}_k, \bar{x}_k)$ , if  $y = (y_1, \dots, y_k)$  we define  $I_i < \pi_1(\mathcal{X}_i, \bar{x}_i)$  as the image of  $I_y$  via the morphism  $\pi_1(\mathcal{X}, \bar{x}) \rightarrow \pi_1(\mathcal{X}_i, \bar{x}_i)$ .

**Lemma 3.1.** *We have that  $I_y \cong I_i$  for all  $i = 1, \dots, k$  and  $I_y = I_1 \times_{I'_y} \cdots \times_{I'_y} I_k$ .*

*Proof.* For  $\gamma = (\gamma_1, \dots, \gamma_k) \in I_y$  note that  $\gamma_i \in \pi_1(\mathcal{X}_i, \bar{x}_i)$  fixes  $y_i \in \tilde{X}_i$ , otherwise  $\gamma$  can not fix a point in  $\tilde{X}$ . As above, the restriction of  $\pi_1(\mathcal{X}_i, \bar{x}_i) \rightarrow G$  to  $I_{y_i}$  is injective and as  $I_i \subset I_{y_i}$  we have that  $\gamma_i \neq \beta_i$  for  $\gamma, \beta \in I_y \subset \pi_1(\mathcal{X}_1, \bar{x}_1) \times_G \cdots \times_G \pi_1(\mathcal{X}_k, \bar{x}_k)$  with  $\gamma \neq \beta$ . Therefore we can construct an inverse to the projection. The result follows.  $\square$

Note that we obtain that  $I_i < I_{y_i}$ , but in general  $I_{y_i}$  can be bigger. Let us define the homomorphism  $I_y \rightarrow \prod I_{y_i}$  given by decomposing an element in its components. By Lemma 3.1 it is injective. Denote by  $N$  the subgroup in  $\pi_1(\mathcal{X}, \bar{x})$  generated by all the  $I_y$  and by  $N'_i$  the subgroup in  $\pi_1(\mathcal{X}_i, \bar{x}_i)$  generated by  $I_i$ .

**Lemma 3.2.** *The subgroup  $N'_i$  is normal in  $\pi_1(\mathcal{X}_i, \bar{x}_i)$ .*

*Proof.* Let  $\gamma_i \in N'_i$  and  $t_i \in \pi_1(\mathcal{X}_i, \bar{x}_i)$ . We can write  $\gamma_i = \gamma_{i_1} \cdots \gamma_{i_j}$  with each  $\gamma_{i_l} \in I_{i_l}$  coming from  $\gamma_l = (\gamma_{l_1}, \dots, \gamma_{l_i}, \dots, \gamma_{l_k}) \in I_{\gamma_l} \subset \pi_1(\mathcal{X}, \bar{x})$  and the point  $\gamma_l = (y_{l_1}, \dots, y_{l_i}, \dots, y_{l_k}) \in \tilde{X}$  for  $l = 1, \dots, j$ . As every  $\pi_1(\mathcal{X}_j, \bar{x}_j) \rightarrow G$  is surjective, for  $j = 1, \dots, i-1, i+1, \dots, k$ , there exists  $t_j \in \pi_1(\mathcal{X}_j, \bar{x}_j)$  such that  $t = (t_1, \dots, t_k) \in \pi_1(\mathcal{X}, \bar{x})$ .

As  $t \cdot \gamma_l \cdot t^{-1} \in I_{t\gamma_l}$  it follows that  $t_i \gamma_i t_i^{-1} \in N'_i$  and therefore

$$t_i \gamma_i t_i^{-1} = (t_i \gamma_{i_1} t_i^{-1}) \cdot t_i \cdots t_i^{-1} \cdot (t_i \gamma_{i_j} t_i^{-1}) \in N'_i.$$

$\square$

**Proposition 3.3.** *There is an homomorphism*

$$\pi_1(X/G, [x]) \rightarrow \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i$$

*such that the image has finite index.*

*Proof.* By Lemma 2.2 we have that  $\pi_1(\mathcal{X}, \bar{x}) \cong \pi_1(\mathcal{X}_1, \bar{x}_1) \times_G \cdots \times_G \pi_1(\mathcal{X}_k, \bar{x}_k)$ . Therefore there is an injective homomorphism  $\pi_1(\mathcal{X}, \bar{x}) \rightarrow \prod \pi_1(\mathcal{X}_i, \bar{x}_i)$ . By Lemma 3.2 we obtain the exact sequence

$$(3.1) \quad 1 \rightarrow \prod_{i=1}^k N'_i \rightarrow \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i) \rightarrow \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i \rightarrow 1,$$

and together with Lemma 2.3 we obtain a commutative diagram

$$(3.2) \quad \begin{array}{ccc} & 1 & \\ & \downarrow & \\ 1 & \longrightarrow & N & \longrightarrow & \prod_{i=1}^k N'_i & \\ & & \downarrow & & \downarrow & \\ & & \pi_1(\mathcal{X}, \bar{x}) & \longrightarrow & \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i) & \\ & & \downarrow & & \downarrow & \\ & & \pi_1(X/G, [x]) & \longrightarrow & \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i & \\ & & \downarrow & & \downarrow & \\ & & 1 & & 1. & \end{array}$$

This diagram provides a homomorphism

$$\pi_1(X/G, [x]) \rightarrow \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i$$

and shows that it is well defined. We can not complete (3.2) to a commutative diagram of groups with short exact sequence in the rows because usually  $\pi_1(\mathcal{X}, \bar{x})$  is not normal in  $\prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)$ . It will be normal, for example, if  $G$  is abelian.

As  $G$  is finite we obtain that  $\pi_1(\mathcal{X}, \bar{x})$  has finite index in  $\prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)$ . In fact the upper-bound

$$\left[ \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i) : \pi_1(\mathcal{X}, \bar{x}) \right] \leq |G|^{k-1}$$

can be seen as follows. For each surjection  $\varphi_i : \pi_1(\mathcal{X}_i, \bar{x}_i) \rightarrow G$  consider a lift  $G_i \subset \pi_1(\mathcal{X}_i, \bar{x}_i)$  of  $G$  with  $|G_i| = |G|$ . In  $\prod_{i=1}^k G_i$ , let us consider the equivalence relation

$$(g_1, \dots, g_k) \sim (g'_1, \dots, g'_k) \Leftrightarrow (\varphi_1(g_1), \dots, \varphi_k(g_k)) = (g\varphi_1(g'_1), \dots, g\varphi_k(g'_k)) \text{ with } g \in G.$$

It is easily seen that the quotient  $(\prod_{i=1}^k G_i) / \sim \cong (G \times \dots \times G) / \Delta_G$  is a set of representatives of left cosets  $(\prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i) / \pi_1(\mathcal{X}, \bar{x}))$ . By considering as coset representatives in  $\prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i) / N'_i$  the image of  $\prod_{i=1}^k G_i$  and using the diagram (3.2) we have that  $\pi_1(X/G, [x])$  has finite index in  $\prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i) / N'_i$ .  $\square$

## 3.2. The homomorphism has finite kernel

**3.2.1. The subgroup  $N'_i$  is finitely normally generated.**— Let  $X$  be a connected, semi-locally simply connected and locally path connected topological space. Let  $G$  be a discrete finite group acting on  $X$ ,  $x \in X$  and denote by  $\bar{x} \in \mathcal{X} = [X/G]$  the image of the point  $x$  and by  $p : X \rightarrow [X/G]$  the quotient map.

Let us briefly recall the description of  $\pi_1(\mathcal{X}, \bar{x})$  as given in [Che01]. It can be defined as  $\pi_0(\Omega(\mathcal{X}, \bar{x}))$  where  $\Omega(\mathcal{X}, \bar{x})$  denote the space loop of  $\mathcal{X}$  pointed at the constant loop of value  $\bar{x}$ . Every loop is given locally as a map from an open subset of  $S^1$  to a given uniformization of an open subset of  $\mathcal{X}_{\text{top}}$  and plus some gluing conditions. In our case of a global quotient, a more explicit description of  $\Omega(\mathcal{X}, \bar{x})$  can be given as follows.

Let  $P(X, x)$  consist of paths in  $X$  starting at  $x$ . As a subspace of  $\Lambda(X)$ , the free loop space of  $X$ , it inherits a structure of a topological space. By considering the constant loop  $x$  of value  $x \in X$ , we obtain  $(P(X, x), x)$  a pointed topological space. Define  $P(X, G, x)$  as the subspace of  $P(X, x) \times G$  consisting of the elements  $(\gamma, g)$  satisfying  $\gamma(1) = g \cdot \gamma(0) = g \cdot x$ . As a topological space it is pointed at  $(x, 1_G)$

**Lemma 3.4.** [Che01, Lemma 3.4.2] *There exists a natural homeomorphism between the pointed topological spaces  $(\Omega(\mathcal{X}, \bar{x}), x)$  and  $(P(X, G, x), (x, 1_G))$ .*

*Remark 3.5.* When  $(\mathcal{X}, \bar{x})$  is a pointed topological stack there exists  $(B[R \rightrightarrows X], x')$  a pointed topological space, where  $B[R \rightrightarrows X]$  is the classifying space of the topological groupoid  $[R \rightrightarrows X]$ , such that we can take  $\pi_1(\mathcal{X}, \bar{x}) := \pi_1(B[R \rightrightarrows X], x')$ . In the case of a global quotient  $\mathcal{X} = [X/G]$  it happens that  $B[R \rightrightarrows X]$  equals the Borel construction  $X \times_G EG$ , see [Nool2].

Now, the construction of Chen also gives a natural isomorphism between  $\pi_1(\mathcal{X}, \bar{x})$  and  $\pi_1(X \times_G EG, x')$  [Che01, Theorem 3.4.1] linking both definitions.

There exists a canonical projection  $(P(X, G, x), (x, 1_G)) \rightarrow (G, 1_G)$  given by sending  $(\gamma, g)$  to  $g$ . This map can be seen to be a fibration [Che01, Lemma 3.4.3] having as fiber at  $1_G$  the space loop  $\Omega(X, x)$  via the embedding  $\Omega(X, x) \hookrightarrow P(X, G, x)$  where  $\gamma$  maps to  $(\gamma, 1_G)$ .

With this description at hand, suppose there is  $y \in X$  such that it is fixed by an element  $g$ , that is,  $y \in X^g$ . Denote by  $\gamma_y$  a path starting at  $x$  and finishing at  $y$ , then  $\gamma_y(g\gamma_y^{-1}) \in P(X, G, x)$ , where  $g\gamma_y^{-1}$  denotes the action of  $g$  applied to each point of the path.

**Lemma 3.6.** *Let  $I_y < G$  denote the inertia (stabilizer) of the action of  $G$  at  $y \in X$ . Every homotopy class of a path  $[\gamma_y] \in \pi_1(X, x, y)$  induces an injective morphism  $I_y \rightarrow \pi_1(\mathcal{X}, x)$ .*

*Proof.* As  $G$  is discrete  $g \mapsto \gamma_y(g\gamma_y^{-1})$  is continuous, with  $g \in I_y$ . Then by taking the functor  $\pi_0$  we got a morphism of groups  $\pi_0(I_y) \rightarrow \pi_0(P(X, G, x)) = \pi_1(\mathcal{X}, \bar{x})$ . Finally, by composing with the projection  $(\pi_0(P(X, G, x), x)) \rightarrow \pi_0((G, 1_G))$  we obtain that different points under  $\pi_0(I_y) \rightarrow \pi_1(\mathcal{X}, \bar{x}) \rightarrow \pi_0(G)$  have different images, thus the morphism is injective.  $\square$

**Lemma 3.7.** *Let  $Y \in \pi_0(X^g)$ ,  $y_1, y_2 \in Y$  and let  $\gamma_{y_1}, \gamma_{y_2}$  be paths starting at  $x \in X$  and finishing at  $y_1$  and  $y_2$  respectively, then  $\gamma_{y_1}(g\gamma_{y_1}^{-1})$  is a conjugate of  $\gamma_{y_2}(g\gamma_{y_2}^{-1})$  in  $\pi_1(\mathcal{X}, \bar{x})$  by elements of  $\pi_1(X, x)$ .*

*Proof.* There exists a path  $\beta \subset Y$  connecting  $y_1$  and  $y_2$ , therefore  $\gamma_{y_1}\beta(g\beta^{-1}\gamma_{y_1}^{-1}) \in P(X, G, x)$  but as  $g\beta = \beta$  passing to  $\pi_0(P(X, G, x), x)$  it equals  $[\gamma_{y_1}(g\gamma_{y_1}^{-1})]$ .

Now consider the path  $\gamma_{y_2}$ . Note that  $\theta := \gamma_{y_1}\beta\gamma_{y_2}^{-1} \in \Omega(X, x)$ . There exists a continuous map

$$\#: P(X, G, x) \times P(X, G, x) \rightarrow P(X, G, x)$$

which induces the multiplication in the fundamental group (see [Che01, Section 3.1]). The element  $\theta\#(\gamma_{y_2}(g\gamma_{y_2}^{-1})\#\theta^{-1})$  can be seen to be  $\theta(\gamma_{y_2} \cdot (g(\theta \cdot \gamma_{y_2})^{-1})) \in P(X, G, x)$ . By passing to the group  $\pi_1(\mathcal{X}, \bar{x}) = \pi_0(P(X, G, x), x)$  we have that  $[\theta][\gamma_{y_2}(g\gamma_{y_2}^{-1})][\theta^{-1}] = [\gamma_{y_1}(g\gamma_{y_1}^{-1})]$ .  $\square$

Recall that given  $(X, x)$  as above, we have a pointed universal cover map  $p : (\tilde{X}, \tilde{x}) \rightarrow (X, x)$  where  $\tilde{x}$  represents the constant loop of value  $x$ . Every element in  $\gamma \in \pi_1(X)$  corresponds to a point in  $p^{-1}(x)$ . So given a pointed map  $p_\gamma : (\tilde{X}, \gamma) \rightarrow (X, x)$  it induces a deck transformation of  $\tilde{X}$  in the following way: given  $y \in \tilde{X}$  take a path  $\alpha_y \subset \tilde{X}$  starting at  $\gamma$  and finishing at  $y$ . Consider the unique lift  $p_\gamma(\tilde{\alpha}_y) \subset \tilde{X}$  starting at  $x$  and assign to  $y$  the point  $p_\gamma(\tilde{\alpha}_y)(1)$ . It can be seen to be a well-defined map (see [Hat00]).

Now, by the description given above of  $\pi_1(\mathcal{X}, \bar{x})$ , any  $\gamma \in \pi_1(\mathcal{X}, \bar{x})$  such that  $\varphi(\gamma) = g$  (recall that  $\varphi : \pi_1(\mathcal{X}, \bar{x}) \rightarrow G$ ) have as a representative an element in  $P(X, G, x)$  which we still denote by  $\gamma$ . So  $\gamma$  starts at  $x$  and finishes at  $gx$ . Denote by  $\tilde{\pi} : (\tilde{X}, \tilde{x}) \rightarrow (\mathcal{X}, \bar{x})$  the universal cover morphism, note that  $\tilde{\pi}_\gamma : (\tilde{X}, \gamma) \rightarrow (\mathcal{X}, \bar{x})$  is also a cover morphism. By [Che01, Theorem 4.1.6] we obtain a deck transformation in the following way: given  $y \in \tilde{X}$  take a path  $\alpha_y \subset \tilde{X}$  starting at  $\gamma$  and ending at  $y$ . Using the notation of the precedent paragraph, the path  $p_\gamma(\alpha_y)$  starts at  $gx$ . Then the path  $g^{-1}p_\gamma(\alpha_y)$  starts at  $x$  so we can lift it to  $g^{-1}p_\gamma(\tilde{\alpha}_y)$  in  $(\tilde{X}, \tilde{x})$ , the end point of this lift is then defined as the image of  $y$ . It is shown that it is a well defined map and does not depend on the path chosen.



**Lemma 3.8.** *Let  $y \in X$  be fixed by  $g \in G$ , consider a path  $\gamma_y$  connecting  $x$  and  $y$ . Consider the action of  $\pi_1(\mathcal{X}, \bar{x})$  on  $\tilde{X}$  given by deck transformations  $\text{Deck}(\tilde{X}, \mathcal{X})$ , then the element  $\gamma_y(g\gamma_y^{-1}) \in \pi_1(\mathcal{X}, \bar{x})$  fixes a point in  $\tilde{X}$ . Moreover, any element of  $\pi_1(\mathcal{X}, \bar{x})$  fixing a point in  $\tilde{X}$  is of this form.*

*Proof.* As the endpoint of  $\gamma_y(g\gamma_y^{-1})$  is  $gx$  we have a pointed covering morphism

$$\tilde{\pi}_{\gamma_y(g\gamma_y^{-1})} : (\tilde{X}, \gamma_y(g\gamma_y^{-1})) \rightarrow (\mathcal{X}, \bar{x})$$

and we can consider  $g\gamma_y$  as a path in  $\tilde{X}$  connecting  $\gamma_y(g\gamma_y^{-1})$  and  $\gamma_y$  as follows: let us define  $f(t) = \gamma(g\gamma_y^{-1}) \cdot (g\gamma_y|_t)$  where  $g\gamma_y|_t(t') := g\gamma_y(t'/t)$  denote the path starting at  $gx$  and finishing at  $g\gamma_y(t)$  in time  $t$  for  $t \neq 0$  and being the constant path with value  $gx$  if  $t = 0$ . We project then  $f(t)$  to  $X$  and obtain  $g\gamma_y$  which starts at  $gx$  and finishes at  $\bar{y}$ . By the discussion before the lemma, we obtain that it lifts to  $\gamma_y$  in  $(\tilde{X}, \bar{x})$ , as  $g$  fixes  $y$  we obtain that the point  $\gamma_y \in \tilde{X}$  is fixed by the induced deck transformation.

Consider the exact sequence

$$1 \rightarrow \pi_1(X, x) \rightarrow \pi_1(\mathcal{X}, \bar{x}) \xrightarrow{\varphi} G \rightarrow 1,$$

let  $\gamma \in \pi_1(\mathcal{X}, \bar{x})$  and  $z \in \tilde{X}$  such that  $\gamma$  fixes  $z$ . Let  $p : (\tilde{X}, \bar{x}) \rightarrow (X, x)$  be the projection, as it is  $\varphi$ -invariant we have that  $\varphi(\gamma)p(z) = p(z)$ . Then by considering the path in  $X$  corresponding to  $z$ , we can construct an element  $z\varphi(\gamma)z^{-1}$ , which fixes  $z \in \tilde{X}$ . As in the isotropy  $\varphi$  is injective by Lemma 2.1, we have that  $z\varphi(\gamma)z^{-1} = \gamma$ .  $\square$

**Proposition 3.9.** *Suppose that there are only a finite number of elements in  $\pi_0(X^g)$  for each  $g \in G$ , then there exists a finite set  $L \subset \pi_1(\mathcal{X}, \bar{x})$  consisting of elements having fixed points in  $\tilde{X}$  such that if  $\gamma \in \pi_1(\mathcal{X}, \bar{x})$  fixes a point in  $\tilde{X}$  then it is conjugate to an element of  $L$  by elements in  $\pi_1(X, x)$ .*

*Proof.* By Lemma 3.7 for every element in  $Y \in \pi_0(X^g)$  it suffices to fix an element  $\gamma_y(g\gamma_y^{-1})$  with  $y \in Y$ . For every  $g \in G$  and every element in  $\pi_0(X^g)$  we pick such an element. We define  $L$  as the set consisting of such elements. By Lemma 3.8 every such element fixes a point in  $\tilde{X}$  and any other fixing a point will be conjugate of the element in  $L$  corresponding to its connected component.  $\square$

**3.2.2. Proof that the homomorphism has finite kernel.**— Let us return to the case of  $k$ -topological spaces  $X_1, \dots, X_k$  and let  $G$  be a finite group acting on each one of them on the left as in 2.2.1. Proposition 3.9 gives us  $k$  subsets  $L(\mathcal{X}_i) \subset \pi_1(\mathcal{X}_i, \bar{x}_i)$  whose elements correspond to the element of  $\pi_0(X_i^g)$  with  $g \in G$ . Now consider the subsets  $L_i \subset L(\mathcal{X}_i)$  consisting of elements corresponding to  $\pi_0(X_i^g)$  where  $g$  fixes a point in  $X_i$  for  $i = 1, \dots, k$ .

Recall that  $N < \pi_1(\mathcal{X}, \bar{x})$  (with  $\mathcal{X} = [(X_1 \times \dots \times X_k)/G]$ ) is the subgroup generated by the inertia subgroups  $I_y$  given by the action of  $\pi_1(\mathcal{X}, \bar{x})$  in  $\tilde{X}$  and  $N'_i < \pi_1(\mathcal{X}_i, \bar{x}_i)$  is the image of the  $i$ -projection of  $N$ . The following lemma is immediate from Proposition 3.9

**Lemma 3.10.** *We have that  $N'_i = \langle \gamma_i l_i \gamma_i^{-1} \mid l_i \in L_i, \gamma_i \in \pi_1(X_i, x_i) \rangle$  in  $\pi_1(\mathcal{X}_i, \bar{x}_i)$  for  $i = 1, \dots, k$ .*

**Definition 3.11.** Let us define

$$C_i = C_i(\pi_1(X_i), L_i) := \left\langle \left\langle \gamma_i l_i \gamma_i^{-1} l_i^{-1} \mid \gamma_i \in \pi_1(X_i, x_i), l_i \in L_i \right\rangle \right\rangle_{\pi_1(\mathcal{X}_i, \bar{x}_i)},$$

to be the normal subgroup generated by the commutators of elements in  $\pi_1(X_i, x_i)$  and in  $L_i$ . Denote by  $\mathbb{T}_i := \pi_1(\mathcal{X}_i, \bar{x}_i)/C_i$  and by  $\hat{L}_i$  the image of  $L_i$  in  $\mathbb{T}_i$ .

**Lemma 3.12.** *It happens that  $C_i < N'_i$  and moreover we can consider  $C_i$  as a subgroup of  $N$  via  $\{e\} \times \dots \times C_i \times \dots \times \{e\}$  and  $C_1 \times \dots \times C_k < N$ .*

*Proof.* Let  $l_i \in L_i$  and  $\gamma_i \in \pi_1(X_i, x_i)$ , the elements of  $L_i$  were chosen such that there exists  $l_j \in L_j$  and  $y \in \tilde{X}$  such that  $l = (l_1, \dots, l_i, \dots, l_k) \in I_y < N$ . We have that  $\gamma'_i = (e, \dots, \gamma_i, \dots, e) \in \pi_1(\mathcal{X}, \bar{x})$  and as  $N$  is normal in  $\pi_1(\mathcal{X}, \bar{x})$  we have that  $\gamma'_i l \gamma'^{-1}_i \in N$ , so

$$\gamma'_i l \gamma'^{-1}_i = (e, \dots, \gamma_i l_i \gamma_i^{-1} l_i^{-1}, \dots, e) \in N$$

This element projects to  $[\gamma_i, l_i] \in C_i$ . Finally given  $\beta_i \in \pi_1(\mathcal{X}_i, \bar{x}_i)$ , as every  $\varphi_j$  is surjective, there exists  $\beta_j \in \pi_1(\mathcal{X}_j, \bar{x}_j)$  such that  $\varphi_i(\beta_i) = \varphi_j(\beta_j)$ , so  $\beta = (\beta_1, \dots, \beta_k) \in \pi_1(\mathcal{X}, \bar{x})$  and every conjugate of  $[\gamma_i, l_i]$  can be seen as an element of  $N$ .

Finally, by considering the product of the identification of the elements in  $C_j$  we have  $C_1 \times \dots \times C_k < N$ .  $\square$

Before stating the next lemma recall that  $N < N'_1 \times_G \dots \times_G N'_k$ .

**Lemma 3.13.** *The subgroup  $C_i$  has finite index in  $N'_i$ , in particular  $C_1 \times \dots \times C_k$  has finite index in  $N'_1 \times \dots \times N'_k$  hence also in  $N$ .*

*Proof.* First note that by Lemma 3.10 and by definition of  $\mathbb{T}_i$  we have

$$N'_i/C_i = \langle \langle L_i \rangle \rangle_{\pi_1(\mathcal{X}, x)} / C_i \cong \langle \langle \hat{L}_i \rangle \rangle_{R_i} = \langle \hat{L}_i \rangle,$$

with  $R_i$  the image of  $\pi_1(X_i, x_i)$  in  $\mathbb{T}_i$ .

Moreover as  $\varphi(C_i) = \{e\}$  we have that  $C_i < \ker \varphi \cong \pi_1(X_i, x_i)$ . As  $\pi_1(X_i, x_i)$  has finite index in  $\pi_1(\mathcal{X}_i, \bar{x}_i)$ , it follows that  $R_i$  has finite index in  $\mathbb{T}_i$ , which implies that  $R_i \cap \langle \hat{L}_i \rangle$  has finite index in  $\langle \hat{L}_i \rangle$ . Note that  $\langle \hat{L}_i \rangle$  is generated by a finite number of torsion elements and that by construction  $R_i \cap \langle \hat{L}_i \rangle$  is a central group in  $\langle \hat{L}_i \rangle$ . As any group generated by a finite number of torsion elements and such that the center has finite index is finite (see [BCGP12, Lemma 4.6]) the result follows.  $\square$

**Theorem 3.14.** *The homomorphism  $\pi_1(X/G, [x]) \rightarrow \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i$  has finite kernel.*

*Proof.* By composing the quotient map  $\prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i) \rightarrow \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i$  with the inclusion  $\pi_1(\mathcal{X}, \bar{x}) \rightarrow \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)$  we obtain  $\pi_1(\mathcal{X}, \bar{x}) \rightarrow \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i$  with kernel  $(N'_1 \times \dots \times N'_k) \cap \pi_1(\mathcal{X}, \bar{x}) = N'_1 \times_G \dots \times_G N'_k$  by the description of  $\pi_1(\mathcal{X}, \bar{x})$  as fiber product. We put this as a row in the following commutative diagram together with a vertical column given by Lemma 2.3 and complete to

$$(3.3) \quad \begin{array}{ccccc} & & 1 & & 1 \\ & & \downarrow & & \downarrow \\ & & N & \xrightarrow{\text{id}} & N \\ & & \downarrow & & \downarrow \\ 1 & \longrightarrow & N'_1 \times_G \dots \times_G N'_k & \longrightarrow & \pi_1(\mathcal{X}, \bar{x}) & \longrightarrow & \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i \\ & & \downarrow & & \downarrow & & \downarrow \text{id} \\ 1 & \longrightarrow & N'_1 \times_G \dots \times_G N'_k / N & \longrightarrow & \pi_1(X/G, [x]) & \longrightarrow & \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i \\ & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & \end{array}$$

By Lemma 3.13 both  $N'_1 \times_G \dots \times_G N'_k / C_1 \times \dots \times C_k$  and  $N / C_1 \times \dots \times C_k$  are finite hence

$$\frac{N'_1 \times_G \dots \times_G N'_k / C_1 \times \dots \times C_k}{N / C_1 \times \dots \times C_k} \cong N'_1 \times_G \dots \times_G N'_k / N$$

is finite.  $\square$

**3.2.3. Geometric interpretation of the groups  $\pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i$ .**— Let us denote by  $I$ , the subgroup of  $G$  generated by the elements having a fixed point in every  $X_i$  for  $i = 1, \dots, k$ . Note that  $I$  is a normal subgroup.

Let  $x'_i$  denote the class of  $x_i$  in  $X/I$  and  $\bar{x}'_i$  the image of  $x'_i$  in  $[(X_i/I)/(G/I)]$ .

**Proposition 3.15.** *There is an isomorphism*

$$\pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i \xrightarrow{\sim} \pi_1([(X_i/I)/(G/I)], \bar{x}'_i).$$

*Proof.* Observe that the action of  $G$  on  $X_i$  descends to an action of  $G/I$  on  $X_i/I$  and therefore we can define  $[(X_i/I)/(G/I)]$ . Recall by the previous subsection 3.2.1 that  $\pi_1(\mathcal{X}_i, \bar{x}_i)$  can be identified with the set of path-components of  $P(X_i, G, x)$ . Therefore an element  $[\gamma] \in \pi_1(\mathcal{X}_i, \bar{x}_i)$  can be represented by a path  $\gamma$  in  $X_i$  starting at  $x_i$  and finishing at  $gx_i$  for some  $g \in G$ . Denote by  $p_i : X_i \rightarrow X_i/I$  the quotient map. By considering  $p_i(\gamma)$ , we obtain a morphism between  $\pi_1(\mathcal{X}_i, \bar{x}_i)$  and  $\pi_1([(X_i/I)/(G/I)], \bar{x}'_i)$ .

It is immediate to see that the paths coming from the inertia of  $I$  in  $X_i$ , that is, the elements of the form  $\gamma_y(g\gamma_y^{-1})$  with  $g \in I$  and  $y \in X_i^g$ , are sent to the trivial element in  $\pi_1(X_i/I, x'_i)$ .

Now consider  $\gamma \in \ker(\pi_1(\mathcal{X}_i, \bar{x}_i) \rightarrow \pi_1([(X_i/I)/(G/I)], \bar{x}'_i))$ . Then  $\gamma$  is represented by a path in  $X_i$ , which we still denote by  $\gamma$ , starting at  $x_i$  and finishing at  $gx_i$  with  $g \in G$ . Note that moreover  $g \in I$ , otherwise by the projection  $\pi_1([(X_i/I)/(G/I)], \bar{x}'_i) \rightarrow G$  the element would be sent to a non-zero element. Hence the image of  $\gamma$  lies in  $\pi_1(X_i/I, x'_i)$  and it is trivial. By the exact sequence

$$1 \longrightarrow N_{[X_i/I]} \longrightarrow \pi_1([(X_i/I)/(G/I)], \bar{x}'_i) \longrightarrow \pi_1(X_i/I, x'_i) \longrightarrow 1$$

and noticing that  $N_{[X_i/I]} = N'_i$  we have that  $\gamma \in N'_i$  which proves the result.  $\square$

## 4. Applications

### 4.1. Product of the same topological space

Now let us describe a case where  $N'_i$  equals the whole subgroup  $N_i$  generated by the elements having a fixed point in the universal cover.

**Corollary 4.1.** *Let  $X_i = X_1$  for  $i = 2, \dots, k$  and  $G$  finite acting on  $X_1$ . Then the morphism*

$$\pi_1((X_1 \times \dots \times X_1)/G, [x]) \longrightarrow \prod_{i=1}^k \pi_1(X_1/G, [x_i])$$

*has finite kernel.*

*Proof.* We only have to show that  $N'_1 = N_1$  and then we obtain the result by applying Theorem 3.14. By construction we have that  $N'_1 \subset N_1$ . Let us show the inverse inclusion. Take  $\gamma_1 \in N_1$ , then we can write  $\gamma_1 = \gamma_{1_1} \dots \gamma_{1_l}$  such that there exists  $y_{1_j} \in \tilde{X}_1$  satisfying  $\gamma_{1_j} \in I_{y_{1_j}}$  for  $j = 1, \dots, l$ . As  $\tilde{X} = \tilde{X}_1 \times \dots \times \tilde{X}_k$  by taking  $y_j = (y_{1_j}, \dots, y_{1_j}) \in \tilde{X}$  we have that  $\gamma^j = (\gamma_{1_j}, \dots, \gamma_{1_j}) \in I_{y_j}$  and therefore  $\gamma = \gamma^1 \dots \gamma^l \in N$  and the image of  $\gamma$  in  $N_1$  equals  $\gamma_1$ .  $\square$

Another proof using Proposition 3.15 can be obtained as follows: the action of  $G/I$  is free in  $X_1/I$  and since  $X_1/G \cong (X_1/I)/(G/I)$  we have  $\pi_1([(X_1/I)/(G/I)]) \cong \pi_1(X_1/G)$ .

### 4.2. Second Main Theorem

**Theorem 4.2.** *Let  $X_1, \dots, X_k$  admit a universal cover and let  $G$  be a finite group acting on each of them such that  $|\pi_0(X_i^g)| < +\infty$  for every  $g \in G$  and  $i = 1, \dots, k$ . Denote  $X = X_1 \times \dots \times X_k$  and consider the diagonal action of  $G$  on it. Suppose  $\pi_1(X/G, [x])$  is residually finite, then  $\pi_1(X/G, [x])$  has a normal finite-index subgroup  $\mathcal{N} \cong H_1 \times \dots \times H_k$  isomorphic to a product of normal finite index subgroups  $H_i \triangleleft \pi_1(X_i/I, [x_i])$ .*

*Proof.* By Theorem 1.3 we get a morphism  $\Theta : \pi_1(X/G, [x]) \rightarrow \prod_{i=1}^k \pi_1([X_i/I/G/I])$  having finite kernel  $E$ . As  $\pi_1(X/G, [x])$  is residually finite we can construct a finite-index normal subgroup  $\Gamma \triangleleft \pi_1(X/G, [x])$  such that  $\Gamma \cap E = \{e\}$ .

The morphism  $\Theta|_\Gamma : \Gamma \rightarrow \prod_{i=1}^k \pi_1([X_i/I/G/I], \bar{x}'_i)$  is therefore injective and moreover as the subgroup  $\Theta(\pi_1(X/G)) < \prod_{i=1}^k \pi_1([X_i/I/G/I], \bar{x}'_i)$  has finite index it follows that  $\Theta(\Gamma) < \prod_{i=1}^k \pi_1([X_i/I/G/I], \bar{x}'_i)$  has finite index.

For every  $i = 1, \dots, k$ , we have  $\pi_1(X_i/I, [x_i]) < \pi_1([X_i/I/G/I], \bar{x}'_i)$  as a normal finite-index subgroup. Define the subgroup

$$\Theta(\Gamma)_i := \Theta(\Gamma) \cap (\{e_1\} \times \dots \times \pi_1(X_i/I, [x_i]) \times \dots \times \{e_k\})$$

where  $e_k \in \pi_1(X_j/I, [x_j])$  is the identity element. As  $\Theta(\Gamma)_i$  has finite index in  $\pi_1(X_i/I, [x_i])$ , there exists a normal subgroup of finite index  $H_i$  of  $\pi_1([X_i/I/G/I])$  contained in  $\Theta(\Gamma)_i$ . Set  $H := H_1 \times \dots \times H_k$ , then  $H \triangleleft \Theta(\Gamma)$  and it is a finite-index normal subgroup of  $\prod_{i=1}^k \pi_1([X_i/I/G/I], \bar{x}'_i)$ . The subgroup  $\mathcal{N} := \Theta^{-1}(H) \cap \Gamma$  satisfies the stated properties.  $\square$

#### 4.2.1. Case of smooth curves.—

**Corollary 4.3.** *Let  $C_1, \dots, C_k$  be smooth algebraic curves and let  $G$  be a finite group acting on each  $C_i$ . Denote  $C = C_1 \times \dots \times C_k$ . Consider  $\mathcal{C} = [C/G]$  with  $G$  acting diagonally on  $C$ . Then  $\pi_1(C/G)$  has a normal subgroup  $\mathcal{N}$  of finite index isomorphic to  $\Pi_1 \times \dots \times \Pi_k$  where  $\Pi_i$  is either a surface group or a finitely generated free group for  $i = 1, \dots, k$ .*

By Theorem 1.3 we have a morphism  $\pi_1(C/G) \rightarrow \prod_{i=1}^k \pi_1([C_i/I/G/I])$  with finite kernel, however if the action of  $G/I$  is not faithful on  $C_i/I$  then  $\pi_1([C_i/I/G/I])$  is not necessarily an orbifold surface group. This can be overcome as follows: let  $K_i := \ker(G/I \rightarrow \text{Aut } C_i/I)$  and  $H_i := (G/I)/K_i$ . Denote by  $\mathcal{C}_i := [(C_i/I)/G/I]$  and by  $\mathcal{C}'_i := [(C_i/I)/H_i]$ , we have a canonical morphism  $\mathcal{C}_i \rightarrow \mathcal{C}'_i$ .

**Lemma 4.4.** *The induced homomorphism  $q_i : \pi_1(\mathcal{C}_i) \rightarrow \pi_1(\mathcal{C}'_i)$  is surjective and has finite kernel.*

*Proof.* By choosing a point  $x_i \in C_i$  and denoting by  $\bar{x}_i$  its image in both  $\mathcal{C}_i$  and  $\mathcal{C}'_i$  we obtain a fibration  $[\text{pt}/K, \text{pt}] \hookrightarrow (\mathcal{C}_i, \bar{x}_i) \rightarrow (\mathcal{C}'_i, \bar{x}_i)$ . By taking the long homotopy exact sequence

$$\dots \rightarrow \pi_2(\mathcal{C}'_i, \bar{x}_i) \rightarrow \pi_1(\text{pt}/K, \text{pt}) \rightarrow \pi_1(\mathcal{C}_i, \bar{x}_i) \rightarrow \pi_1(\mathcal{C}'_i, \bar{x}_i) \rightarrow 1,$$

as there is an isomorphism between  $\pi_1(\text{pt}/K, \text{pt})$  and  $\pi_0(K, 1_K)$ , the result follows.  $\square$

So by composing, we obtain a morphism  $\Theta : \pi_1(C/G) \rightarrow \prod_{i=1}^k \pi_1(\mathcal{C}_i) \rightarrow \prod_{i=1}^k \pi_1(\mathcal{C}'_i)$ , this allows us to prove the following lemma, which together with Theorem 4.2 will imply Corollary 4.3.

**Lemma 4.5.** *The group  $\pi_1(C/G)$  is residually finite.*

*Proof.* First note that, as  $\pi_1(\mathcal{C}'_i)$  is an orbifold surface group, it is in particular residually finite. Now, it follows that  $\Theta(\pi_1(C/G))$  is residually finite as it is a finite-index subgroup of a direct product of residually finite groups.

We need another property of these groups to continue. Let  $H$  be a group and let  $\hat{H}$  be its profinite completion. A group  $H$  is called *good* if for each  $k \geq 0$  and for each finite  $H$ -module  $M$  the natural homomorphism

$$H^k(\hat{H}, M) \rightarrow H^k(H, M)$$

is an isomorphism. In [GJZZ08, Lemmas 3.2 and 3.4, Proposition 3.6] it is shown that a finite-index subgroup of a good group is good, the product of good groups is good and that  $\pi_1(\mathcal{C})$  for  $\mathcal{C}$  an algebraic orbifold curve is good. We obtain therefore that  $\Theta(\pi_1(C/G))$  is good.

Finally, [GJZZ08, Proposition 6.1] asserts that if  $T$  is a residually finite good group and  $\varphi : H \rightarrow T$  is a surjective homomorphism with finite kernel then  $H$  is residually finite. Applying this to  $\Theta' : \pi_1(C/G) \rightarrow \Theta(\pi_1(C/G))$  we obtain the result.  $\square$

### 4.3. Partial compactifications of arrangement of lines

The original motivation of this work was to study the partial compactifications of the complement of an arrangement of lines in  $\mathbb{P}_{\mathbb{C}}^2$  which is the topic of my Ph.D. thesis. In [Agu19] a general method for computing a presentation of the fundamental group was given and some examples studied. A family of arrangements related to the studied in *op. cit.* is available for any  $n \in \mathbb{N}$ , however this will require a treatment one by one. The results obtained here can be used to study some of these partial compactifications in family.

**4.3.1. Partial compactification of the complement of an arrangement of lines.**— Consider the projective plane  $\mathbb{P}_{\mathbb{C}}^2$  with homogeneous coordinates  $(z_1 : z_2 : z_3)$ .

Let  $\mathcal{A} = \sum_{i=1}^k L_i$  be a divisor in  $\mathbb{P}^2$  such that the irreducible components  $L_i$  are copies of  $\mathbb{P}^1$  (lines). Then the singular set  $\text{Sing } \mathcal{A}$  of  $\mathcal{A}$  consists only of points. Consider  $\pi : \text{Bl}_{\text{Sing } \mathcal{A}} \mathbb{P}^2 \rightarrow \mathbb{P}^2$  the blow up of the projective plane at the points  $\text{Sing } \mathcal{A}$ . The divisor  $\pi^* \mathcal{A} = \sum_{i=1}^{k+|\text{Sing } \mathcal{A}|} D_i$  has as irreducible components copies of  $\mathbb{P}^1$ , with  $D_1, \dots, D_k$  being the strict transform of  $L_1, \dots, L_k$  respectively and  $D_{k+1}, \dots, D_{k+|\text{Sing } \mathcal{A}|}$  being the exceptional divisors. Take a subset  $J \subset \{1, 2, \dots, k + |\text{Sing } \mathcal{A}|\}$ . The surface  $\text{Bl}_{\text{Sing } \mathcal{A}} \mathbb{P}^2 \setminus (\cup_{j \in J} D_j)$  is called a *partial compactification of  $\mathbb{P}^2 \setminus (\cup L_i)$* . We are interested in how the fundamental group changes when we partially compactify the complement of such an arrangement  $\mathcal{A} \subset \mathbb{P}^2$ .

**4.3.2. Examples.**— The subvariety of  $\mathbb{P}^2$

$$\text{Ceva}(n) := \{(z_1 : z_2 : z_3) \mid (z_1^n - z_2^n)(z_1^n - z_3^n)(z_2^n - z_3^n) = 0\}$$

can be seen as the union of the closure of the three singular fibers of the rational map  $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$  given by  $(z_1 : z_2 : z_3) \mapsto ((z_1^n - z_2^n) : (z_2^n - z_3^n))$ . The map  $f$  is not defined in a subset  $S = \{p_1, \dots, p_{n^2}\} \subset \text{Sing } \text{Ceva}(n)$  consisting of  $n^2$  points where  $\mathcal{A}_1 := \{z_1^n - z_2^n = 0\} = \sum_{i=1}^n L_i$  intersects  $\mathcal{A}_2 := \{z_1^n - z_3^n = 0\} = \sum_{i=n+1}^{2n} L_i$ . It actually happens that  $S \subset \mathcal{A}_3 := \{z_2^n - z_3^n = 0\} = \sum_{i=2n+1}^{3n} L_i$  and  $S$  consists of points where 3 lines of  $\text{Ceva}(n)$  meet. We have another 3 points  $p_{n^2+i}$  in  $\text{Sing}(\text{Ceva}(n))$  which correspond to each singular point of  $\mathcal{A}_i$  for  $i = 1, 2, 3$  and hence of multiplicity  $n$ .

The rational map  $f$  extends to a morphism  $\tilde{f} : \text{Bl}_{\text{Sing } \text{Ceva}(n)} \mathbb{P}^2 \rightarrow \mathbb{P}^1$  having as generic fiber the *Fermat curve of degree  $n$*  defined as  $F(n) := \{z_1^n + z_2^n + z_3^n = 0\} \subset \mathbb{P}^2$ . Therefore  $\tilde{f}$  is an isotrivial fibration.

Denote by  $\mu(n)$  the group of  $n^{\text{th}}$  roots of unity. By taking 3 copies of it we define  $H(n) := \mu_1(n) \oplus \mu_2(n) \oplus \mu_3(n) / \langle \mu_1 \mu_2 \mu_3 = 1 \rangle$  where  $\mu_i \in \mu_i(n)$ . It acts on  $F(n)$  via  $(z_1 : z_2 : z_3) \mapsto (\mu_1 z_1 : \mu_2 z_2 : \mu_3 z_3)$ . The proof of the following theorem will appear elsewhere.

**Theorem 4.6.** *Consider the diagonal action of  $H(n)$  in  $F(n) \times F(n)$ . Denote by  $S$  the minimal resolution of  $F(n) \times F(n) / H(n)$ .*

- (1) *The fibration  $S \rightarrow (F(n) \times F(n)) / H(n) \rightarrow F(n) / H(n) \cong \mathbb{P}^1$  is isomorphic to  $\tilde{f}$ .*
- (2) *Every singular point in  $F(n) \times F(n) / H(n)$  corresponds to the contraction of the strict transform  $D_i$  of some line  $L_i \in \text{Ceva}(n)$ .*
- (3) *The contraction of the  $n$  lines corresponding to  $\mathcal{A}_i$  lie in the line  $E_i$  which is the exceptional divisor corresponding to the unique singular point in  $\mathcal{A}_i$ .*
- (4)  *$E_i$  is mapped to a point via  $F(n) \times F(n) / H(n) \rightarrow \mathbb{P}^1$ .*

The Fermat curve  $F(n)$  of degree  $n$  can be seen as a branched covering of  $\mathbb{P}^1$  of degree  $n^2$  via the morphism in  $\mathbb{P}^2$  given by  $F(n) \ni (z_1 : z_2 : z_3) \rightarrow (z_1^n : z_2^n : z_3^n) \in \{w_0 + w_1 + w_2 = 0\}$  which branches at the points  $(1 : -1 : 0), (1 : 0 : -1), (0 : 1 : -1)$ . Over each branching point there are  $n$  points, we denote by  $X_1, \dots, X_n$  for those over  $(1 : -1 : 0)$ , by  $Y_1, \dots, Y_n$  over  $(1 : 0 : -1)$  and  $Z_1, \dots, Z_n$  over  $(0 : 1 : -1)$ .

Recall that for  $S \rightarrow S'$  be a resolution of singularities of  $S'$ , if  $S'$  has only quotient singularities, by [Kol93, Theorem 7.8.1] we have that  $\pi_1(S) \rightarrow \pi_1(S')$  is an isomorphism.

**Example 4.7.** Consider the surface  $S_1 := (F(n) \times (F(n) \setminus \{X_1, \dots, X_n\})) / H(n)$ . The subgroup  $I$  generated by the elements of  $H(n)$  having fixed points both in  $F(n)$  and in  $F(n) \setminus \{X_1, \dots, X_n\}$  equals  $H(n)$ . As  $F(n)/H(n) \cong \mathbb{P}^1$ ,  $F(n) \setminus \{X_1, \dots, X_n\}/H(n) \cong \mathbb{C}$  and by Theorem 1.3 the morphism

$$\pi_1(S_1) \rightarrow \pi_1(\mathbb{P}^1) \times \pi_1(\mathbb{C})$$

has finite kernel, it follows that  $\pi_1(S_1)$  is finite.

The minimal resolution of singularities  $S'_1 \rightarrow S_1$  can be identified with the following partial compactification of  $\text{Ceva}(n)$ . Consider

$$J := \{1, \dots, n, 3n + n^3 + 1\} \subset \{1, \dots, 3n + n^2 + 3\}$$

then following the construction given in §4.3.1 we have that

$$\text{Bl}_{\text{SingCeva}(n)} \mathbb{P}^2 \setminus \{\cup_{j \in J} D_j\} \cong S'_1.$$

That is, from the surface  $\text{Bl}_{\text{SingCeva}(n)} \mathbb{P}^2$  we remove only the strict transform of  $\mathcal{A}_1$  and the exceptional divisor coming from the singular point of  $\mathcal{A}_1$ . This can be identified with a singular fiber or  $\tilde{f}$ .

**Example 4.8.** Consider now  $S_2 := (F(n) \times F(n) \setminus \{X_i, Y_i\}) / H(n)$ . In this case the subgroup  $I$ , defined as in the previous paragraph, is isomorphic to  $\mu(n)$ . As  $F(n)/\mu(n) \cong \mathbb{P}^1$ ,  $F(n) \setminus \{X_i, Y_i\}/\mu(n) \cong \mathbb{C}^*$  and by Theorem 1.3 the morphism

$$\pi_1(S_2) \rightarrow \pi_1([\mathbb{P}^1/\mu(n)]) \times \pi_1([\mathbb{C}^*/\mu(n)])$$

has finite kernel and the image is a finite-index subgroup.

By Theorem 4.2 and Corollary 4.3, we have that  $\mathbb{Z} \triangleleft \pi_1(S_2)$  has finite index. As in Example 4.7 the minimal resolution of singularities  $S'_2 \rightarrow S_2$  can be identified with  $\text{Bl}_{\text{SingCeva}(n)} \mathbb{P}^2$  minus two singular fibers of  $\tilde{f}$ .

**Example 4.9.** If we consider  $S_3 := (F(n) \times F(n) \setminus \{X_i, Y_i, Z_i\}) / H(n)$  it can be identified with  $\text{Bl}_{\text{Sing}\mathcal{A}} \mathbb{P}^2$  minus the three singular fibers of  $\tilde{f}$ . As  $H(n)$  acts freely in  $F(n) \times F(n) \setminus \{X_i, Y_i, Z_i\}$ , the long exact sequence of homotopy associated to the covering map  $F(n) \times F(n) \setminus \{X_i, Y_i, Z_i\} \rightarrow S_3$  yields

$$1 \longrightarrow \pi_1(F(n)) \times \pi_1(F(n) \setminus \{X_i, Y_i, Z_i\}) \longrightarrow \pi_1(S_3) \longrightarrow H(n) \longrightarrow 1.$$

*Remark 4.10.* We can remove points also in the first component  $F(n)$  of the product. However, we can not get more partial compactifications of  $\text{Ceva}(n)$  in this way. This can be shown by drawing the dual graph of the divisor  $\pi^* \text{Ceva}(n)$  and noticing that the lines obtained by removing points does not satisfy the intersection pattern of the graph.

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