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**THE FUNDAMENTAL GROUP OF QUOTIENTS OF PRODUCTS
OF SOME TOPOLOGICAL SPACES BY A FINITE GROUP - A
GENERALIZATION OF A THEOREM OF
BAUER-CATANESE-GRUNEWALD-PIGNATELLI**

RODOLFO AGUILAR AGUILAR

ABSTRACT. We provide a description of the fundamental group of the quotient of a product of topological spaces X_i , each admitting a universal cover, by a finite group G , provided that there is only a finite number of path-connected components in X_i^g for every $g \in G$. This generalizes previous work of Bauer-Catanese-Grunewald-Pignatelli and Dedieu-Perroni.

1. INTRODUCTION

The fundamental group of a quotient of a Hausdorff space X by a finite group G acting freely can be computed noticing that $X \rightarrow X/G$ is a covering map, and then using the long exact sequence of homotopy groups associated to a fibration. When $X = X_1 \times \dots \times X_k$ and G acts over each X_i freely and diagonally in X , the fundamental group of $X_1 \times \dots \times X_k$ sits as a finite-index normal subgroup of $\pi_1(X/G)$.

In the case where each X_i is a projective smooth curve and the action of G is only *faithful*, the following Theorem was shown in [BCGP12].

Theorem 1.1. [BCGP12, Thm. 0.10 and Thm. 4.1] *Let C_1, \dots, C_k be smooth projective curves and let G be a finite group acting faithfully by automorphisms over each of them. Consider the diagonal action of G over the product $C_1 \times \dots \times C_k$, then the fundamental group of $(C_1 \times \dots \times C_k)/G$ admits a normal finite index subgroup \mathcal{N} isomorphic to a product of fundamental groups of smooth projective curves.*

It was later extended in [DP12] to the case when the action of G is non-necessarily faithful. There, they quotient G to obtain a group acting faithfully, follow the subsequent arguments and then extend again to G .

Let us explain briefly the method of proof in [BCGP12]. First, they consider the orbifold surface groups T_i of C_i/G , which are an extension of G by $\pi_1(C_i)$ and hence come with a surjective morphism to G (see Subsection 2.2.1). They show that the fundamental group $\pi_1((C_1 \times \dots \times C_k)/G)$ is isomorphic to the quotient of the fiber product $\mathbb{H} := T_1 \times_G \dots \times_G T_k$ by the normal subgroup $\text{Tors}(\mathbb{H})$ generated by the elements of torsion.

The second part relies in the following proposition whose proof uses abstract group theoretic arguments.

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Proposition 1.2. [DP12, Prop. 3.5] *There exists a short exact sequence of groups*

$$1 \rightarrow E \rightarrow \mathbb{H}/\text{Tors } \mathbb{H} \rightarrow T \rightarrow 1$$

where E is finite and T is a group of finite index in a product of orbifold surfaces groups.

They finally use the Proposition 1.2 and properties of the orbifold surface groups such as residually finiteness and cohomological goodness to construct a subgroup of $\mathbb{H}/\text{Tors } \mathbb{H}$ intersecting E trivially and satisfying the required properties.

Here, a more geometric approach is used via fundamental groups of stacks or orbispaces [Noo05],[Che01]. This theory permits to see $X \rightarrow [X/G]$ as a covering map under some conditions on X , where $[X/G]$ denotes the quotient stack, and a long exact sequence of homotopy groups is available. We will denote the fundamental group of the stack $[X/G]$ by $\pi_1([X/G])$.

For $i = 1, \dots, k$, let X_i be a connected, locally path-connected and semi-locally simply connected topological space with an action of a finite group G , consider the diagonal action of G in $X := X_1 \times \dots \times X_k$ and denote by I the subgroup generated by the elements having a fixed point in every X_i for $i = 1, \dots, k$. We can formulate now our first main Theorem.

Theorem 1.3. *Let X, X_1, \dots, X_k and G as above. Suppose that the number of path connected components in the fixed locus set X_i^g of an element $g \in G$ is finite for every $g \in G$ and $i = 1, \dots, k$. Then there exists a homomorphism $\pi_1(X/G) \rightarrow \prod \pi_1((X_i/I)/(G/I))$ which has finite kernel and its image is a finite-index subgroup.*

This can be seen as a generalization of Proposition 1.2 (Bauer-Catanese-Grunewald-Pignatelli) by the remarks preceding the statement of the Proposition.

The action of G/I over X_i/I is induced by the action of G in X_i . Note that if $k = 1$ then G/I can be seen to act freely on X_1/I and $\pi_1([(X_1/I)/(G/I)]) = \pi_1((X_1/I)/(G/I))$ but $(X_1/I)/(G/I) \cong X_1/G$. The same argument works if we make the product of the same topological space, which gives the following Corollary.

Corollary 1.4. *Let $X_i = X_1$ for $i = 2, \dots, k$ and G satisfy the hypothesis of the above theorem. Then the homomorphism $\pi_1(X/G) \rightarrow \pi_1(X_1/G)^n$ has finite kernel and its image is a finite-index subgroup.*

An important case of Theorem 1.3 and Corollary 1.4 is when X_i is a smooth complex algebraic variety for $i = 1, \dots, k$. Indeed, the fundamental group of a variety with quotient singularities is the fundamental group of a smooth variety.

Our second main Theorem generalizes Theorem 1.1 (Bauer-Catanese-Grunewald-Pignatelli). It can be stated without using the language of stacks or orbispaces.

Theorem 1.5. *Let X, X_1, \dots, X_k and G satisfy the hypothesis of Theorem 1.3. Suppose that $\pi_1(X/G)$ is residually finite. Then there exists a normal finite-index subgroup \mathcal{N} isomorphic to a product of normal finite-index subgroups $H_i \triangleleft_{f,i} \pi_1(X_i/I)$.*

As a Corollary, following closely the arguments used in [BCGP12], we show that for smooth curves C_1, \dots, C_k the group $\pi_1((C_1 \times \dots \times C_k)/G)$ is residually finite. Hence, we have that Theorem 1.1 (Bauer-Catanese-Grunewald-Pignatelli, Dedeiu-Perroni) is valid in the case when the curve is non-necessarily compact.

Corollary 1.6. *Let C_1, \dots, C_k be smooth algebraic curves and let G be a finite group acting on each of them. Then there exists a normal subgroup $\mathcal{N} < \pi_1((C_1 \times \dots \times C_k)/G)$ of finite index, isomorphic to a product $\Pi_1 \times \dots \times \Pi_k$, where Π_j is either the fundamental group of a smooth projective curve or a free group of finite rank.*

The paper is organized as follows: in Section 2 preliminary results are given. Then the first main Theorem is proved in Section 3 and the second main Theorem together with some applications are given in Section 4.

2. PRELIMINARIES

2.1. Properties of fundamental group of topological stacks. Let X be a connected, semi-locally simply connected and locally path-connected topological space and G a finite group acting continuously on it.

2.1.1. Fiber homotopy exact sequence. There exists a homotopy theory for stacks and the existence of the long exact sequence of homotopy, see [Noo14], is more general than what follows, however we only need the following case: consider the topological stack $\mathcal{X} = [X/G]$, a point $x \in X$ and denote by $\bar{x} \in \mathcal{X}$ the image of x . We have an associated fibration $G \rightarrow X \rightarrow \mathcal{X}$ and a long exact sequence of homotopy groups,

$$\dots \rightarrow \pi_{n+1}(\mathcal{X}, \bar{x}) \rightarrow \pi_n(G, \text{Id}) \rightarrow \pi_n(X, x) \rightarrow \pi_n(\mathcal{X}, \bar{x}) \rightarrow \pi_{n-1}(G, \text{Id}) \dots$$

the map $\pi_n(G, \text{Id}) \rightarrow \pi_n(X, x)$ is induced by the orbit $G \cdot x \hookrightarrow X$.

2.1.2. Action on the universal cover. The hypothesis made on X ensures that there exists an universal cover \tilde{X} and moreover, if we let $\mathcal{X} = [X/G]$ as in 2.1.1, we have an action of $\pi_1(\mathcal{X}, \bar{x})$ over \tilde{X} (see 3.2.1). We will use several times the following lemma in what follows:

Lemma 2.1. *Consider the action of $\pi_1(\mathcal{X}, \bar{x})$ in \tilde{X} , let $y \in \tilde{X}$ and denote by I_y the isotropy group of the action. Then there exists a monomorphism $I_y \rightarrow G$.*

Proof. By 2.1.1 we obtain a short exact sequence

$$1 \rightarrow \pi_1(X, x) \rightarrow \pi_1(\mathcal{X}, \bar{x}) \rightarrow G \rightarrow 1,$$

as the action of $\pi_1(X, x)$ over \tilde{X} is free, we obtain that the restriction of $\pi_1(\mathcal{X}, \bar{x}) \rightarrow G$ to I_y is injective. \square

2.2. Product of topological spaces.

2.2.1. Fundamental group of the quotient of a product. For $i = 1, \dots, k$ let X_i as in 2.1 be a connected, semi-locally simply connected and locally path-connected topological space and G a finite group acting on each of them.

By 2.1.1 we have k exact sequences

$$(1) \quad 1 \rightarrow \pi_1(X_i, x_i) \rightarrow \pi_1(\mathcal{X}_i, \bar{x}_i) \xrightarrow{\varphi_i} (G, \text{Id}) \rightarrow 1$$

where $\mathcal{X}_i = [X_i/G]$, $x_i \in X_i$ and its image in \mathcal{X}_i is denoted by \bar{x}_i .

Denote by $\mathbb{H} := \pi_1(\mathcal{X}_1, x_1) \times_G \dots \times_G \pi_1(\mathcal{X}_k, x_k)$. The exact sequences in (1) can be assembled as follows

$$(2) \quad 1 \rightarrow \pi_1(X_1 \times \dots \times X_k, x) \rightarrow \mathbb{H} \rightarrow G \rightarrow 1$$

with $x = (x_1, \dots, x_k)$. The geometric nature of \mathbb{H} is shown in the following Lemma.

Lemma 2.2. *Let G act diagonally over $X = X_1 \times \dots \times X_k$. Consider the stack $\mathcal{X} = [X/G]$ then $\pi_1(\mathcal{X}, \bar{x}) \cong \mathbb{H}$.*

Proof. We have natural projection maps $\mathcal{X} \rightarrow \mathcal{X}_i$ for $i = 1, \dots, k$, which together with the morphisms $\varphi_i : \pi_1(\mathcal{X}_i, \bar{x}_i) \rightarrow G$ and the universal property of the fiber product give us a morphism $\pi_1(\mathcal{X}, \bar{x}) \rightarrow \mathbb{H}$. By the exact sequence of a fibration 2.1.1 applied to the action of G to $X_1 \times \dots \times X_k$ and by (2) we obtain

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(X_1 \times \dots \times X_k, x) & \longrightarrow & \pi_1(\mathcal{X}, \bar{x}) & \longrightarrow & G \longrightarrow 1 \\ & & \downarrow = & & \downarrow & & \downarrow = \\ 1 & \longrightarrow & \pi_1(X_1 \times \dots \times X_k, x) & \longrightarrow & \mathbb{H} & \longrightarrow & G \longrightarrow 1 \end{array}$$

which implies the result. \square

Lemma 2.3. *Let X, X_i and G be as above. Then*

$$\pi_1(X/G, [x]) \cong \pi_1(\mathcal{X}, \bar{x})/N \cong \pi_1(\mathcal{X}, \bar{x})/\mathbf{I}$$

where N is the normal subgroup generated by the image of the inertia groups I_x and \mathbf{I} is the subgroup generated by the elements of $\pi_1(\mathcal{X})$ having fixed points in the universal cover of $X_1 \times \dots \times X_k$.

Proof. By [Noo08, Thm 8.3 i)] we have that $\pi_1(X/G, [x]) \cong \pi_1(\mathcal{X}, \bar{x})/N$.

The group $\pi_1(\mathcal{X}, \bar{x})$ acts over $\tilde{X} \cong \tilde{X}_1 \times \dots \times \tilde{X}_k$ the universal cover of $X_1 \times \dots \times X_k$ in such a way that $[(\tilde{X}_1 \times \dots \times \tilde{X}_k)/\pi_1(\mathcal{X}, \bar{x})] \cong \mathcal{X}$. As G is finite, by Lemma 2.1 any stabilizer I_x for the action of $\pi_1(\mathcal{X})$ over \tilde{X} is finite, therefore it has the slice property and by [Noo08, Thm 9.1] we obtain that $\pi_1(X/G, [x]) \cong \pi_1(\mathcal{X}, \bar{x})/\mathbf{I}$. \square

3. THE FUNDAMENTAL GROUP OF THE PRODUCT OF TOPOLOGICAL SPACES

3.1. Constructing the homomorphism.

3.1.1. *Finite index of the group in the product.* Let I_y denote the isotropy at the point y in \tilde{X} for the action of $\pi_1(\mathcal{X}, \bar{x})$. By Lemma 2.1 the map $\pi_1(\mathcal{X}, \bar{x}) \rightarrow G$ restricted to I_y is injective, therefore we can identify I_y with a subgroup of G . When we do such identification we will denote it by $I'_y < G$.

Now as $\pi_1(\mathcal{X}, \bar{x}) \cong \pi_1(\mathcal{X}_1, \bar{x}_1) \times_G \dots \times_G \pi_1(\mathcal{X}_k, \bar{x}_k)$, if $y = (y_1, \dots, y_k)$ we define $I_i < \pi_1(\mathcal{X}_i, \bar{x}_i)$ as the image of I_y via the morphism $\pi_1(\mathcal{X}, \bar{x}) \rightarrow \pi_1(\mathcal{X}_i, \bar{x}_i)$.

Lemma 3.1. *We have that $I_y \cong I_i$ for all $i = 1, \dots, k$ and $I_y = I_1 \times_{I'_y} \dots \times_{I'_y} I_k$.*

Proof. For $\gamma = (\gamma_1, \dots, \gamma_k) \in I_y$ note that $\gamma_i \in \pi_1(\mathcal{X}_i, \bar{x}_i)$ fixes $y_i \in \tilde{X}_i$, otherwise γ can not fix a point in \tilde{X} . As above, the restriction of $\pi_1(\mathcal{X}_i, \bar{x}_i) \rightarrow G$ to I_{y_i} is injective and as $I_i \subset I_{y_i}$ we have that $\gamma_i \neq \beta_i$ for $\gamma, \beta \in I_y \subset \pi_1(\mathcal{X}_1, \bar{x}_1) \times_G \dots \times_G \pi_1(\mathcal{X}_k, \bar{x}_k)$ with $\gamma \neq \beta$. Therefore we can construct an inverse to the projection. The result follows. \square

Note that we obtain that $I_i < I_{y_i}$, but in general I_{y_i} can be bigger.

Define a homomorphism $I_y \rightarrow \prod I_{y_i}$ given by decomposing an element in its components. By Lemma 3.1 it is injective. Denote by N the subgroup in $\pi_1(\mathcal{X}, \bar{x})$ generated by all the I_y and by N'_i the subgroup in $\pi_1(\mathcal{X}_i, \bar{x}_i)$ generated by I_i .

Lemma 3.2. *The subgroup N'_i is normal in $\pi_1(\mathcal{X}_i, \bar{x}_i)$.*

Proof. Let $\gamma_i \in N'_i$ and $t_i \in \pi_1(\mathcal{X}_i, \bar{x}_i)$. We can write $\gamma_i = \gamma_{i_1} \cdots \gamma_{i_j}$ with each $\gamma_{i_l} \in I_{i_l}$ coming from $\gamma_l = (\gamma_{1_l}, \dots, \gamma_{i_l}, \dots, \gamma_{k_l}) \in I_{y_l} \subset \pi_1(\mathcal{X}, \bar{x})$ and the point $y_l = (y_{1_l}, \dots, y_{i_l}, \dots, y_{k_l}) \in \tilde{X}$ for $l = 1, \dots, j$. As every $\pi_1(\mathcal{X}_j, \bar{x}_j) \rightarrow G$ is surjective, for $j = 1, \dots, i-1, i+1, \dots, k$, there exists $t_j \in \pi_1(\mathcal{X}_j, \bar{x}_j)$ such that $t = (t_1, \dots, t_k) \in \pi_1(\mathcal{X}, \bar{x})$.

As $t \cdot \gamma_l \cdot t^{-1} \in I_{t y_l}$ it follows that $t_i \gamma_{i_l} t_i^{-1} \in N'_i$ and therefore

$$t_i \gamma_i t_i^{-1} = (t_i \gamma_{i_1} t_i^{-1}) \cdot t_i \cdots t_i^{-1} \cdot (t_i \gamma_{i_j} t_i^{-1}) \in N'_i.$$

□

Proposition 3.3. *There is an homomorphism $\pi_1(X/G, [x]) \rightarrow \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i$ such that the image has finite index.*

Proof. By Lemma 2.2 we have that $\pi_1(\mathcal{X}, \bar{x}) \cong \pi_1(\mathcal{X}_1, \bar{x}_1) \times_G \dots \times_G \pi_1(\mathcal{X}_k, \bar{x}_k)$. Therefore there is an injective homomorphism $\pi_1(\mathcal{X}, \bar{x}) \rightarrow \prod \pi_1(\mathcal{X}_i, \bar{x}_i)$.

By Lemma 3.2 we obtain the exact sequence

$$(3) \quad 1 \rightarrow \prod N'_i \rightarrow \prod \pi_1(\mathcal{X}_i, \bar{x}_i) \rightarrow \prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i \rightarrow 1,$$

together with Lemma 2.3 we obtain a commutative diagram

$$(4) \quad \begin{array}{ccccc} & & 1 & & 1 \\ & & \downarrow & & \downarrow \\ 1 & \longrightarrow & N & \longrightarrow & \prod N'_i \\ & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \pi_1(\mathcal{X}, \bar{x}) & \longrightarrow & \prod \pi_1(\mathcal{X}_i, \bar{x}_i) \\ & & \downarrow & & \downarrow \\ & & \pi_1(X/G, [x]) & \longrightarrow & \prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i \\ & & \downarrow & & \downarrow \\ & & 1 & & 1 \end{array}$$

This diagram provides a homomorphism $\pi_1(X/G, [x]) \rightarrow \prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i$ and shows that it is well defined.

We can not complete (4) to a commutative diagram of groups with short exact sequence in the rows because usually $\pi_1(\mathcal{X}, \bar{x})$ is not normal in $\prod \pi_1(\mathcal{X}_i, \bar{x}_i)$. It will be normal, for example, if G is abelian.

As G is finite we obtain that $\pi_1(\mathcal{X}, \bar{x})$ has finite index in $\prod \pi_1(\mathcal{X}_i, \bar{x}_i)$. In fact $[\prod \pi_1(\mathcal{X}_i, \bar{x}_i) : \pi_1(\mathcal{X}, \bar{x})] \leq |G|^{k-1}$: for each surjection $\varphi_i : \pi_1(\mathcal{X}_i, \bar{x}_i) \rightarrow G$ consider a lift $G_i \subset \pi_1(\mathcal{X}_i, \bar{x}_i)$ of G with $|G_i| = |G|$. In $\prod G_i$ consider the equivalence relation $(g_1, \dots, g_k) \sim (g'_1, \dots, g'_k) \Leftrightarrow (\varphi_1(g_1), \dots, \varphi_k(g_k)) = (g\varphi_1(g'_1), \dots, g\varphi_k(g'_k))$ with $g \in G$. It is easily seen that $\prod G_i / \sim \cong (G \times \dots \times G) / \Delta_G$ is a set of representatives of left cosets $(\prod \pi_1(\mathcal{X}_i, \bar{x}_i)) / \pi_1(\mathcal{X}, \bar{x})$.

By considering as coset representatives in $\prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i$ the image of $\prod G_i$ and using the diagram (4) we have that $\pi_1(X/G, [x])$ has finite index in $\prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i$. □

3.2. The homomorphism has finite kernel.

3.2.1. N'_i is finitely normally generated. Let X be a connected, semi-locally simply connected and locally path connected topological space. Let G be a discrete finite group acting on X , $x \in X$ and denote by $\bar{x} \in \mathcal{X} = [X/G]$ the image of the point x and by $p : X \rightarrow [X/G]$ the quotient map.

Let us briefly recall the description of $\pi_1(\mathcal{X}, \bar{x})$ as given in [Che01]. It can be defined as $\pi_0(\Omega(\mathcal{X}, \bar{x}))$ where $\Omega(\mathcal{X}, \bar{x})$ denote the space loop of \mathcal{X} pointed at the constant loop of value \bar{x} . Every loop is given locally as a map from an open subset of S^1 to a given uniformization of an open subset of \mathcal{X}_{top} and plus some gluing conditions. In our case of a global quotient, a more explicit description of $\Omega(\mathcal{X}, \bar{x})$ can be given as follows:

Let $P(X, x)$ consist of paths in X starting at x . As subspace of $\Lambda(X)$, the free loop space of X , it inherits a structure of a topological space. By considering the constant loop x of value $x \in X$, we obtain $(P(X, x), x)$ a pointed topological space. Define $P(X, G, x)$ as the subspace of $P(X, x) \times G$ consisting of the elements (γ, g) satisfying $\gamma(1) = g \cdot \gamma(0) = g \cdot x$. As a topological space it is pointed at (x, Id_G)

Lemma 3.4. [Che01, Lemma 3.4.2] *There exists a natural homeomorphism between the pointed topological spaces $(\Omega(\mathcal{X}, \bar{x}), x)$ and $(P(X, G, x), (x, Id_G))$.*

Remark 1. *When (\mathcal{X}, \bar{x}) is a pointed topological stack there exists a pointed topological space $(B[R \rightrightarrows X], x')$, where $B[R \rightrightarrows X]$ is the classifying space of the topological grupoid $[R \rightrightarrows X]$, such that we can take $\pi_1(\mathcal{X}, \bar{x}) := \pi_1(B[R \rightrightarrows X], x')$. In the case of a global quotient $\mathcal{X} = [X/G]$ it happens that $B[R \rightrightarrows X]$ equals the Borel construction $X \times_G EG$, see [Noo12].*

Now, the construction of Chen also gives a natural isomorphism of $\pi_1(\mathcal{X}, \bar{x})$ and $\pi_1(X \times_G EG, x')$ [Che01, Theorem 3.4.1] linking the two definitions.

There exists a canonical projection $(P(X, G, x), (x, Id_G)) \rightarrow (G, Id_G)$ given by sending (γ, g) to g . This map can be seen to be a fibration [Che01, Lemma 3.4.3] having as fiber at Id_G the space loop $\Omega(X, x)$ via the embedding $\Omega(X, x) \hookrightarrow P(X, G, x)$ where γ maps to (γ, Id_G) .

With this description at hand, suppose there is $y \in X$ such that it is fixed by an element g , this is, $y \in X^g$. Denote by γ_y a path starting at x and finishing at y , then $\gamma_y(g\gamma_y^{-1}) \in P(X, G, x)$, where $g\gamma_y^{-1}$ denotes the action of g applied to each point of the path.

Lemma 3.5. *Let $I_y < G$ denote the inertia (stabilizer) of the action of G at $y \in X$. Every homotopy class of a path $[\gamma_y] \in \pi_1(X, x, y)$ induces an injective morphism $I_y \rightarrow \pi_1(\mathcal{X}, x)$.*

Proof. As G is discrete $g \mapsto \gamma_y(g\gamma_y^{-1})$ is continuous, with $g \in I_y$. Then by taking the functor π_0 we got a morphism of groups $\pi_0(I_y) \rightarrow \pi_0(P(X, G, x)) = \pi_1(\mathcal{X}, \bar{x})$. Finally, by composing with the projection $(\pi_0(P(X, G, x), x)) \rightarrow \pi_0((G, Id_G))$ we obtain that different points under $\pi_0(I_y) \rightarrow \pi_1(\mathcal{X}, \bar{x}) \rightarrow \pi_0(G)$ have different images, thus the morphism is injective. \square

Lemma 3.6. *Let $Y \in \pi_0(X^g)$, $y_1, y_2 \in Y$ and let $\gamma_{y_1}, \gamma_{y_2}$ be paths starting at $x \in X$ and finishing at y_1 and y_2 respectively, then $\gamma_{y_1}(g\gamma_{y_1}^{-1})$ is a conjugate of $\gamma_{y_2}(g\gamma_{y_2}^{-1})$ in $\pi_1(\mathcal{X}, \bar{x})$ by elements of $\pi_1(X, x)$.*

Proof. There exists a path $\beta \subset Y$ connecting y_1 and y_2 , therefore $\gamma_{y_1}\beta(g\beta^{-1}\gamma_{y_1}^{-1}) \in P(X, G, x)$ but as $g\beta = \beta$ passing to $\pi_0(P(X, G, x), x)$ it equals $[\gamma_{y_1}(g\gamma_{y_1}^{-1})]$.

Now consider the path γ_{y_2} . Note that $\theta := \gamma_{y_1} \beta \gamma_{y_2}^{-1} \in \Omega(X, x)$. There exists a continuous map $\# : P(X, G, x) \times P(X, G, x) \rightarrow P(X, G, x)$ which induces the multiplication in the fundamental group (see [Che01, Section 3.1]). The element $\theta \# (\gamma_{y_2} (g \gamma_{y_2}^{-1}) \# \theta^{-1})$ can be seen to be $\theta (\gamma_{y_2} \cdot (g (\theta \cdot \gamma_{y_2})^{-1})) \in P(X, G, x)$. By passing to $\pi_1(\mathcal{X}, \bar{x}) = \pi_0(P(X, G, x), x)$ we have that $[\theta][\gamma_{y_2}(g\gamma_{y_2}^{-1})][\theta^{-1}] = [\gamma_{y_1}(g\gamma_{y_1}^{-1})]$. \square

Recall that given (X, x) as above, we have a pointed universal cover map $p : (\tilde{X}, \tilde{x}) \rightarrow (X, x)$ where \tilde{x} represents the constant loop of value x . Every element in $\gamma \in \pi_1(X)$ correspond to a point in $p^{-1}(x)$. So given a pointed map $p_\gamma : (\tilde{X}, \gamma) \rightarrow (X, x)$ it induces a deck transformation of \tilde{X} in the following way: given $y \in \tilde{X}$ take a path $\alpha_y \subset \tilde{X}$ starting at γ and finishing at y . Consider the unique lift $p_\gamma(\alpha_y) \subset \tilde{X}$ starting at x and assign to y the point $p_\gamma(\alpha_y)(1)$. It can be seen to be a well-defined map (See [Hat00]).

Now, by the description given above of $\pi_1(\mathcal{X}, \bar{x})$, any $\gamma \in \pi_1(\mathcal{X}, \bar{x})$ such that $\varphi(\gamma) = g$ (recall that $\varphi : \pi_1(\mathcal{X}, \bar{x}) \rightarrow G$) have as a representative an element in $P(X, G, x)$ which we still denote by γ . So γ starts at x and finishes at gx . Denote by $\tilde{\pi} : (\tilde{X}, \tilde{x}) \rightarrow (\mathcal{X}, \bar{x})$ the universal cover morphism, note that $\tilde{\pi}_\gamma : (\tilde{X}, \gamma) \rightarrow (\mathcal{X}, \bar{x})$ is also a cover morphism. By [Che01, Thm 4.1.6] we obtain a deck transformation in the following way: given $y \in \tilde{X}$ take a path $\alpha_y \subset \tilde{X}$ starting at γ and ending at y . Using the notation of the precedent paragraph, the path $p_\gamma(\alpha_y)$ starts at gx . Then the path $g^{-1}p_\gamma(\alpha_y)$ starts at x so we can lift it to $g^{-1}p_\gamma(\alpha_y)$ in (\tilde{X}, \tilde{x}) , the end point of this lift is then defined as the image of y . It is shown that it is a well defined map and does not depend on the path chosen.

Lemma 3.7. *Let $y \in X$ be fixed by $g \in G$, consider a path γ_y connecting x and y . Consider the action of $\pi_1(\mathcal{X}, \bar{x})$ on \tilde{X} given by deck transformations $\text{Deck}(\tilde{X}, \mathcal{X})$, then the element $\gamma_y(g\gamma_y^{-1}) \in \pi_1(\mathcal{X}, \bar{x})$ fixes a point in \tilde{X} . Moreover, any element of $\pi_1(\mathcal{X}, \bar{x})$ fixing a point in \tilde{X} is of this form.*

Proof. As the endpoint of $\gamma_y(g\gamma_y^{-1})$ is gx we have a pointed covering morphism $\tilde{\pi}_{\gamma_y(g\gamma_y^{-1})} : (\tilde{X}, \gamma_y(g\gamma_y^{-1})) \rightarrow (\mathcal{X}, \bar{x})$, we can consider $g\gamma_y$ as a path in \tilde{X} connecting $\gamma_y(g\gamma_y^{-1})$ and γ_y as follows: define $f(t) = \gamma(g\gamma_y^{-1}) \cdot (g\gamma_y|_t)$ where $g\gamma_y|_t(t') := g\gamma_y(t'/t)$ denote the path starting at gx and finishing at $g\gamma_y(t)$ in time 1 for $t \neq 0$ and being the constant path with value gx if $t = 0$. We project then $f(t)$ to X and obtain $g\gamma_y$ which starts at gx and finishes at \bar{y} . By the discussion before the lemma, we obtain that it lifts to γ_y in (\tilde{X}, \tilde{x}) , as g fixes y we obtain that the point $\gamma_y \in \tilde{X}$ is fixed by the induced deck transformation.

Consider the exact sequence

$$1 \rightarrow \pi_1(X, x) \rightarrow \pi_1(\mathcal{X}, \bar{x}) \xrightarrow{\varphi} G \rightarrow 1,$$

let $\gamma \in \pi_1(\mathcal{X}, \bar{x})$ and $z \in \tilde{X}$ such that γ fixes z . Let $p : (\tilde{X}, \tilde{x}) \rightarrow (X, x)$ be the projection, as it is φ -invariant we have that $\varphi(\gamma)p(z) = p(z)$. Then by considering the path in X corresponding to z , we can construct an element $z\varphi(\gamma)z^{-1}$, which fixes $z \in \tilde{X}$. As in the isotropy φ is injective by Lemma 2.1, we have that $z\varphi(\gamma)z^{-1} = \gamma$. \square

Proposition 3.8. *Suppose that there are only a finite number of elements in $\pi_0(X^g)$ for each $g \in G$, then there exists a finite set $L \subset \pi_1(\mathcal{X}, \bar{x})$ consisting*

of elements having fixed points in \tilde{X} such that if $\gamma \in \pi_1(\mathcal{X}, \bar{x})$ fixes a point in \tilde{X} then it is conjugate to an element of L by elements in $\pi_1(X, x)$.

Proof. By Lemma 3.6 for every element in $Y \in \pi_0(X^g)$ it suffices to fix an element $\gamma_y(g\gamma_y^{-1})$ with $y \in Y$. For every $g \in G$ and every element in $\pi_0(X^g)$ we pick such an element. We define L the set consisting of such elements. By Lemma 3.7 every such element fixes a point in \tilde{X} and any other fixing a point will be conjugate of the element in L corresponding to its connected component. \square

3.2.2. Proof that the homomorphism has finite kernel. Let us return to the case of k -topological spaces X_1, \dots, X_k and let G be a finite group acting on each one of them on the left as in 2.2.1. The Proposition 3.8 gives us k subsets $L(\mathcal{X}_i) \subset \pi_1(\mathcal{X}_i, \bar{x}_i)$ whose elements correspond to the element of $\pi_0(X_i^g)$ with $g \in G$. Now consider the subsets $L_i \subset L(\mathcal{X}_i)$ consisting of elements corresponding to $\pi_0(X_i^g)$ where g fixes a point in X_i for $i = 1, \dots, k$.

Recall that $N < \pi_1(\mathcal{X}, \bar{x})$ (with $\mathcal{X} = [(X_1 \times \dots \times X_k)/G]$) is the subgroup generated by the inertia subgroups I_y given by the action of $\pi_1(\mathcal{X}, \bar{x})$ in \tilde{X} and $N'_i < \pi_1(\mathcal{X}_i, \bar{x}_i)$ is the image of the i -projection of N . The following Lemma is immediate from Proposition 3.8

Lemma 3.9. *We have that $N'_i = \langle \gamma_i l_i \gamma_i^{-1} \mid l_i \in L_i, \gamma_i \in \pi_1(X_i, x_i) \rangle$ in $\pi_1(\mathcal{X}_i, \bar{x}_i)$ for $i = 1, \dots, k$.*

Definition 3.1. *Let us define*

$$C_i = C_i(\pi_1(X_i), L_i) := \langle \langle \gamma_i l_i \gamma_i^{-1} l_i^{-1} \mid \gamma_i \in \pi_1(X_i, x_i), l_i \in L_i \rangle \rangle_{\pi_1(\mathcal{X}_i, \bar{x}_i)},$$

to be the normal subgroup generated by the commutators of elements in $\pi_1(X_i, x_i)$ and in L_i . Denote by $\mathbb{T}_i := \pi_1(\mathcal{X}_i, \bar{x}_i)/C_i$ and by \hat{L}_i the image of L_i in \mathbb{T}_i .

Lemma 3.10. *It happens that $C_i < N'_i$ and moreover we can consider C_i as a subgroup of N via $\{e\} \times \dots \times C_i \times \dots \times \{e\}$ and $C_1 \times \dots \times C_k < N$.*

Proof. Let $l_i \in L_i$ and $\gamma_i \in \pi_1(X_i, x_i)$, the elements of L_i were chosen such that there exists $l_j \in L_j$ and $y \in \tilde{X}$ such that $l = (l_1, \dots, l_i, \dots, l_k) \in I_y < N$. We have that $\gamma'_i = (e, \dots, \gamma_i, \dots, e) \in \pi_1(\mathcal{X}, \bar{x})$ and as N is normal in $\pi_1(\mathcal{X}, \bar{x})$ we have that $\gamma'_i l \gamma_i'^{-1} \in N$, so

$$\gamma'_i l \gamma_i'^{-1} l^{-1} = (e, \dots, \gamma_i l_i \gamma_i^{-1} l_i^{-1}, \dots, e) \in N$$

This element projects to $[\gamma_i, l_i] \in C_i$. Finally given $\beta_i \in \pi_1(\mathcal{X}_i, \bar{x}_i)$, as every φ_j is surjective, there exists $\beta_j \in \pi_1(\mathcal{X}_j, \bar{x}_j)$ such that $\varphi_i(\beta_i) = \varphi_j(\beta_j)$, so $\beta = (\beta_1, \dots, \beta_k) \in \pi_1(\mathcal{X}, \bar{x})$ and every conjugate of $[\gamma_i, l_i]$ can be seen as an element of N .

Finally, by considering the product of the identification of the elements in C_j we have that $C_1 \times \dots \times C_k < N$. \square

Before stating the next lemma recall that $N < N'_1 \times_G \dots \times_G N'_k$.

Lemma 3.11. *The subgroup C_i has finite index in N'_i , in particular $C_1 \times \dots \times C_k$ has finite index in $N'_1 \times \dots \times N'_k$ hence also in N .*

Proof. First note that by Lemma 3.9 and by definition of \mathbb{T}_i we have that $N'_i/C_i = \langle \langle L_i \rangle \rangle_{\pi_1(X_i, x_i)}/C_i \cong \langle \langle \hat{L}_i \rangle \rangle_{R_i} = \langle \hat{L}_i \rangle$, with R_i the image of $\pi_1(X_i, x_i)$ in \mathbb{T}_i .

Moreover as $\varphi(C_i) = \{e\}$ we have that $C_i < \ker \varphi \cong \pi_1(X_i, x_i)$. As $\pi_1(X_i, x_i)$ has finite index in $\pi_1(\mathcal{X}_i, \bar{x}_i)$, it follows that R_i has finite index in \mathbb{T}_i , which implies

that $R_i \cap \langle \hat{L}_i \rangle$ has finite index in $\langle \hat{L}_i \rangle$. Note that $\langle \hat{L}_i \rangle$ is generated by a finite number of torsion elements and that by construction $R_i \cap \langle \hat{L}_i \rangle$ is a central group in $\langle \hat{L}_i \rangle$. As any group generated by a finite number of torsion elements and such that the center has finite index is finite (see [BCGP12, Lem. 4.6]) the result follows. \square

Theorem 3.12. *The homomorphism $\pi_1(X/G, [x]) \rightarrow \prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i$ has finite kernel.*

Proof. By composing the quotient map $\prod \pi_1(\mathcal{X}_i, \bar{x}_i) \rightarrow \prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i$ with the inclusion $\pi_1(\mathcal{X}, \bar{x}) \rightarrow \prod \pi_1(\mathcal{X}_i, \bar{x}_i)$ we obtain $\pi_1(\mathcal{X}, \bar{x}) \rightarrow \prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i$ with kernel $N'_1 \times \dots \times N'_k \cap \pi_1(\mathcal{X}, \bar{x}) = N'_1 \times_G \dots \times_G N'_k$ by the description of $\pi_1(\mathcal{X}, \bar{x})$ as fiber product. We put this as a row in the following commutative diagram together with a vertical column given by Lemma 2.3 and complete to

$$(5) \quad \begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \\ & & N & \xrightarrow{=} & N & & \\ & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & N'_1 \times_G \dots \times_G N'_k & \longrightarrow & \pi_1(\mathcal{X}, \bar{x}) & \longrightarrow & \prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i \\ & & \downarrow & & \downarrow & & \downarrow = \\ & & N'_1 \times_G \dots \times_G N'_k/N & \longrightarrow & \pi_1(X/G, [x]) & \longrightarrow & \prod \pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i \\ & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & \end{array}$$

By Lemma 3.11 both $N'_1 \times_G \dots \times_G N'_k/C_1 \times \dots \times C_k$ and $N/C_1 \times \dots \times C_k$ are finite hence

$$\frac{N'_1 \times_G \dots \times_G N'_k/C_1 \times \dots \times C_k}{N/C_1 \times \dots \times C_k} \cong N'_1 \times_G \dots \times_G N'_k/N$$

is finite. \square

3.2.3. Geometric interpretation of the groups $\pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i$. Let us denote by I , the subgroup of G generated by the elements having a fixed point in every X_i for $i = 1, \dots, k$. Note that I is a normal subgroup.

Let x'_i denote the class of x_i in X/I and \bar{x}'_i the image of x'_i in $[(X_i/I)/(G/I)]$.

Proposition 3.13. *There is an isomorphism*

$$\pi_1(\mathcal{X}_i, \bar{x}_i)/N'_i \xrightarrow{\cong} \pi_1([(X_i/I)/(G/I)], \bar{x}'_i).$$

Proof. Observe that the action of G on X_i descends to an action of G/I on X_i/I and therefore we can define $[(X_i/I)/(G/I)]$. Recall by the previous subsection 3.2.1 that $\pi_1(\mathcal{X}_i, \bar{x}_i)$ can be identified with the set of path-components of $P(X_i, G, x)$. Therefore an element $[\gamma] \in \pi_1(\mathcal{X}_i, \bar{x}_i)$ can be represented by a path γ in X_i starting at x_i and finishing at gx_i for some $g \in G$. Denote by $p_i : X_i \rightarrow X_i/I$ the quotient map. By considering $p_i(\gamma)$, we obtain a morphism between $\pi_1(\mathcal{X}_i, \bar{x}_i)$ and $\pi_1([(X_i/I)/(G/I)], \bar{x}'_i)$.

It is immediate to see that the paths coming from the inertia of I in X_i , this is, the elements of the form $\gamma_y(g\gamma_y^{-1})$ with $g \in I$ and $y \in X_i^g$, are sent to the trivial element in $\pi_1(X_i/I, x'_i)$.

Now consider $\gamma \in \ker(\pi(\mathcal{X}_i, \bar{x}_i) \rightarrow \pi_1([(X_i/I)/(G/I)], \bar{x}'_i))$. Then γ is represented by a path in X_i , which we still denote by γ , starting at x_i and finishing at gx_i with $g \in G$. Note that moreover $g \in I$, otherwise by the projection $\pi_1([X_i/I/G/I], \bar{x}'_i) \rightarrow G$ the element would be sent to a non-zero element. Hence the image of γ lies in $\pi_1(X_i/I, x'_i)$ and it is trivial. By the exact sequence

$$1 \rightarrow N_{[X_i/I]} \rightarrow \pi_1([X_i/I], \bar{x}_i) \rightarrow \pi_1(X/I, x'_i) \rightarrow 1$$

and noticing that $N_{[X_i/I]} = N'_i$ we have that $\gamma \in N'_i$ which proves the result. \square

4. APPLICATIONS

4.1. Product of the same topological space. Now let us describe a case where N'_i equals the whole subgroup N_i generated by the elements having a fixed point in the universal cover.

Corollary 4.1. *Let $X_i = X_1$ for $i = 2, \dots, k$ and G finite acting on X_1 . Then the morphism $\pi_1((X_1 \times \dots \times X_1)/G, [x]) \rightarrow \prod \pi_1(X_1/G, [x_i])$ has finite kernel.*

Proof. We only have to show that $N'_1 = N_1$ and then we obtain the result by applying Theorem 3.12. By construction we have that $N'_1 \subset N_1$. Let us show the inverse inclusion. Take $\gamma_1 \in N_1$, then we can write $\gamma_1 = \gamma_{1_1} \cdots \gamma_{1_l}$ such that there exists $y_{1_j} \in \tilde{X}_1$ satisfying $\gamma_{1_j} \in I_{y_{1_j}}$ for $j = 1, \dots, l$. As $\tilde{X} = \tilde{X}_1 \times \dots \times \tilde{X}_k$ by taking $y_j = (y_{1_j}, \dots, y_{1_j}) \in \tilde{X}$ we have that $\gamma^j = (\gamma_{1_j}, \dots, \gamma_{1_j}) \in I_{y_j}$ and therefore $\gamma = \gamma^1 \cdots \gamma^l \in N$ and the image of γ in N_1 equals γ_1 . \square

Another proof using Proposition 3.13 is as follows: The action of G/I is free in X_1/I and $X_1/G \cong X_1/I/G/I$ so $\pi_1([X_1/I/G/I]) = \pi_1(X_1/G)$.

4.2. Second Main Theorem.

Theorem 4.2. *Let X_1, \dots, X_k admit a universal cover and let G be a finite group acting on each of them such that $|\pi_0(X_i^g)| < +\infty$ for every $g \in G$ and $i = 1, \dots, k$. Denote $X = X_1 \times \dots \times X_k$ and consider the diagonal action of G on it. Suppose $\pi_1(X/G, [x])$ is residually finite, then $\pi_1(X/G, [x])$ has a normal finite-index subgroup $\mathcal{N} \cong H_1 \times \dots \times H_k$ isomorphic to a product of normal finite index subgroups $H_i < \pi_1(X_i/I, [x_i])$.*

Proof. By Theorem 1.3 we get a morphism $\Theta : \pi_1(X/G, [x]) \rightarrow \prod \pi_1([X_i/I/G/I])$ having finite kernel E . As $\pi_1(X/G, [x])$ is residually finite we can construct a finite-index normal subgroup $\Gamma \triangleleft \pi_1(X/G, [x])$ such that $\Gamma \cap E = \{e\}$.

The morphism $\Theta|_{\Gamma} : \Gamma \rightarrow \prod \pi_1([X_i/I/G/I], \bar{x}'_i)$ is therefore injective and moreover as $\Theta(\pi_1(X/G)) < \prod \pi_1([X_i/I/G/I], \bar{x}'_i)$ has finite index it follows that $\Theta(\Gamma) < \prod \pi_1([X_i/I/G/I], \bar{x}'_i)$ has finite index.

For every $i = 1 \dots, k$, we have $\pi_1(X_i/I, [x_i]) < \pi_1([X_i/I/G/I], \bar{x}'_i)$ as a normal finite-index subgroup. Define the subgroup

$$\Theta(\Gamma)_i := \Theta(\Gamma) \cap \{\{e_1\} \times \dots \times \pi_1(X_i/I, [x_i]) \times \dots \times \{e_k\}\}$$

where $e_k \in \pi_1(X_j/I, [x_j])$ is the identity element. As $\Theta(\Gamma)_i$ has finite index in $\pi_1(X_i/I, [x_i])$, there exists a normal subgroup of finite index H_i of $\pi_1([X_i/I/G/I])$

contained in $\Theta(\Gamma)_i$. Set $H := H_1 \times \dots \times H_k$, then $H \triangleleft \Theta(\Gamma)$ and it is a finite-index normal subgroup of $\prod \pi_1([X_i/I/G/I], \bar{x}'_i)$. The subgroup $\mathcal{N} := \Theta^{-1}(H) \cap \Gamma$ satisfies the stated properties. \square

4.2.1. *Case of smooth curves.*

Corollary 4.3. *Let C_1, \dots, C_k be smooth algebraic curves and let G be a finite group acting on each C_i . Denote $C = C_1 \times \dots \times C_k$. Consider $\mathcal{C} = [C/G]$ with G acting diagonally on C . Then $\pi_1(C/G)$ has a normal subgroup \mathcal{N} of finite index isomorphic to $\Pi_1 \times \dots \times \Pi_k$ where Π_i is either a surface group or a finitely generated free group for $i = 1, \dots, k$.*

By Theorem 1.3 we have a morphism $\pi_1(C/G) \rightarrow \prod \pi_1([C_i/I/G/I])$ with finite kernel, however if the action of G/I is not faithful on C_i/I then $\pi_1([C_i/I/G/I])$ is not necessarily an orbifold surface group. This can be overcome as follows: let $K_i := \ker(G/I \rightarrow \text{Aut } C_i/I)$ and $H_i := (G/I)/K_i$. Denote by $\mathcal{C}_i := [(C_i/I)/G/I]$ and by $\mathcal{C}'_i := [(C_i/I)/H_i]$, we have a canonical morphism $\mathcal{C}_i \rightarrow \mathcal{C}'_i$.

Lemma 4.4. *The induced homomorphism $q_i : \pi_1(\mathcal{C}_i) \rightarrow \pi_1(\mathcal{C}'_i)$ is surjective and has finite kernel.*

Proof. By choosing a point $x_i \in C_i$ and denoting by \bar{x}_i its image in both \mathcal{C}_i and \mathcal{C}'_i we obtain a fibration $[pt/K, pt] \hookrightarrow (\mathcal{C}_i, \bar{x}_i) \rightarrow (\mathcal{C}'_i, \bar{x}_i)$. By taking the long homotopy exact sequence

$$\dots \rightarrow \pi_2(\mathcal{C}'_i, \bar{x}_i) \rightarrow \pi_1(pt/K, pt) \rightarrow \pi_1(\mathcal{C}_i, \bar{x}_i) \rightarrow \pi_1(\mathcal{C}'_i, \bar{x}_i) \rightarrow 1,$$

as there is an isomorphism between $\pi_1(pt/K, pt)$ and $\pi_0(K, Id)$, the result follows. \square

So by composing, we obtain a morphism $\Theta : \pi_1(C/G) \rightarrow \prod \pi_1(\mathcal{C}_i) \rightarrow \prod \pi_1(\mathcal{C}'_i)$, this allows us to prove the following Lemma, which together with Theorem 4.2 will imply Corollary 4.3.

Lemma 4.5. *The group $\pi_1(C/G)$ is residually finite.*

Proof. First note that as $\pi_1(\mathcal{C}'_i)$ is an orbifold surface group. In particular it is residually finite.

Now, it follows that $\Theta(\pi_1(C/G))$ is residually finite as it is a finite-index subgroup of a direct product of residually finite groups.

We need another property of these groups to continue. Let H be a group and let \hat{H} be its profinite completion. A group H is called *good* if for each $k \geq 0$ and for each finite H -module M the natural homomorphism

$$H^k(\hat{H}, M) \rightarrow H^k(H, M)$$

is an isomorphism. In [GJZZ08, Lem. 3.2, 3.4, Prop. 3.6] it is shown that a finite-index subgroup of a good group is good, the product of good groups is good and that $\pi_1(\mathcal{C})$ for \mathcal{C} an algebraic orbifold curve is good. We obtain therefore that $\Theta(\pi_1(C/G))$ is good.

Finally, [GJZZ08, Proposition 6.1] asserts that if T is a residually finite good group and $\varphi : H \rightarrow T$ is a surjective homomorphism with finite kernel then H is residually finite. Applying this to $\Theta' : \pi_1(C/G) \rightarrow \Theta(\pi_1(C/G))$ we obtain the result. \square

4.3. Partial compactifications of arrangement of lines. The original motivation of this work was to study the partial compactifications of the complement of an arrangement of lines in $\mathbb{P}_{\mathbb{C}}^2$ which is the topic of my Ph.D. thesis. In [Agu19] a general method for computing a presentation of the fundamental group was given and some examples studied. A family of arrangements related to the studied in op. cit. is available for any $n \in \mathbb{N}$, however this will require a treatment one by one. The results obtained here can be used to study some of these partial compactifications in family.

4.3.1. Partial compactification of the complement of an arrangement of lines. Consider the projective plane $\mathbb{P}_{\mathbb{C}}^2$ with homogeneous coordinates $(z_1 : z_2 : z_3)$.

Let $\mathcal{A} = \sum_{i=1}^k L_i$ be a divisor in \mathbb{P}^2 such that the irreducible components L_i are copies of \mathbb{P}^1 (lines). Then the singular set $\text{Sing } \mathcal{A}$ of \mathcal{A} consists only of points. Consider $\pi : \text{Bl}_{\text{Sing } \mathcal{A}} \mathbb{P}^2 \rightarrow \mathbb{P}^2$ the blow up of the projective plane at the points $\text{Sing } \mathcal{A}$. The divisor $\pi^* \mathcal{A} = \sum_{i=1}^{k+|\text{Sing } \mathcal{A}|} D_i$ has as irreducible components copies of \mathbb{P}^1 , with D_1, \dots, D_k being the strict transform of L_1, \dots, L_k respectively and $D_{k+1}, \dots, D_{k+|\text{Sing } \mathcal{A}|}$ being the exceptional divisors. Take a subset $J \subset \{1, 2, \dots, k + |\text{Sing } \mathcal{A}|\}$. The surface $\text{Bl}_{\text{Sing } \mathcal{A}} \mathbb{P}^2 \setminus (\cup_{j \in J} D_j)$ is called a *partial compactification of $\mathbb{P}^2 \setminus (\cup L_i)$* . We are interested in how the fundamental group changes when we partially compactify the complement of such an arrangement $\mathcal{A} \subset \mathbb{P}^2$.

4.3.2. Examples. The subvariety of \mathbb{P}^2

$$\text{Ceva}(n) := \{(z_1 : z_2 : z_3) \mid (z_1^n - z_2^n)(z_1^n - z_3^n)(z_2^n - z_3^n) = 0\}$$

can be seen as the union of the closure of the three singular fibers of the rational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ given by $(z_1 : z_2 : z_3) \mapsto ((z_1^n - z_2^n) : (z_2^n - z_3^n))$. The map f is not defined in a subset $S = \{p_1, \dots, p_{n^2}\} \subset \text{Sing } \text{Ceva}(n)$ consisting of n^2 points where $\mathcal{A}_1 := \{z_1^n - z_2^n = 0\} = \sum_{i=1}^n L_i$ intersects $\mathcal{A}_2 := \{z_1^n - z_3^n = 0\} = \sum_{i=n+1}^{2n} L_i$. In actually happens that $S \subset \mathcal{A}_3 := \{z_2^n - z_3^n = 0\} = \sum_{i=2n+1}^{3n} L_i$ and S consists of points where 3 lines of $\text{Ceva}(n)$ meet. We have another 3 points p_{n^2+i} in $\text{Sing}(\text{Ceva}(n))$ which correspond to each singular point of \mathcal{A}_i for $i = 1, 2, 3$ and hence of multiplicity n .

The rational map f will extend to a morphism in $\text{Bl}_S \mathbb{P}^2$ and moreover to a morphism $\tilde{f} : \text{Bl}_{\text{Sing}(\text{Ceva}(n))} \mathbb{P}^2 \rightarrow \mathbb{P}^1$ having as generic fiber the *Fermat curve of degree n* defined as $F(n) := \{z_1^n + z_2^n + z_3^n = 0\} \subset \mathbb{P}^2$. Therefore \tilde{f} is an isotrivial fibration.

Denote by $\mu(n)$ the group of roots of unity of degree n . By taking 3 copies of it we define $H(n) := \mu_1(n) \oplus \mu_2(n) \oplus \mu_3(n) / \langle \mu_1 \mu_2 \mu_3 = 1 \rangle$ where $\mu_i \in \mu_i(n)$. It acts over $F(n)$ via $(z_1 : z_2 : z_3) \mapsto (\mu_1 z_1 : \mu_2 z_2 : \mu_3 z_3)$. The proof of the following Theorem will appear elsewhere.

Theorem 4.6. *Consider the diagonal action of $H(n)$ in $F(n) \times F(n)$. Denote by S the minimal resolution of $F(n) \times F(n)/H(n)$.*

- (1) *The fibration $S \rightarrow (F(n) \times F(n))/H(n) \rightarrow F(n)/H(n) \cong \mathbb{P}^1$ is isomorphic to \tilde{f} .*
- (2) *Every singular point in $F(n) \times F(n)/H(n)$ corresponds to the contraction of the strict transform D_i of some line $L_i \in \text{Ceva}(n)$.*

- (3) The contraction of the n lines corresponding to \mathcal{A}_i lie in the line E_i which is the exceptional divisor corresponding to the unique singular point in \mathcal{A}_i .
- (4) E_i maps to a point via $F(n) \times F(n)/H(n) \rightarrow \mathbb{P}^1$.

The Fermat curve $F(n)$ of degree n can be seen as a branched covering of \mathbb{P}^1 of degree n^2 via the morphism in \mathbb{P}^2 given by $F(n) \ni (z_1 : z_2 : z_3) \rightarrow (z_1^n : z_2^n : z_3^n) \in \{w_0 + w_1 + w_2 = 0\}$ which branches at the points $(1 : -1 : 0), (1 : 0 : -1), (0 : 1 : -1)$. Over each branching point there are n points, we denote by X_1, \dots, X_n for those over $(1 : -1 : 0)$, by Y_1, \dots, Y_n over $(1 : 0 : -1)$ and Z_1, \dots, Z_n over $(0 : 1 : -1)$.

Recall that for $S \rightarrow S'$ be a resolution of singularities of S' , if S' has only quotient singularities, by [Kol93, Thm 7.8.1] we have that $\pi_1(S) \rightarrow \pi_1(S')$ is an isomorphism.

Example 4.1. Consider the surface $S_1 := (F(n) \times F(n) \setminus \{X_1, \dots, X_n\})/H(n)$. The subgroup I generated by the elements of $H(n)$ having fixed points both in $F(n)$ and in $F(n) \setminus \{X_1, \dots, X_n\}$ equals $H(n)$. As $F(n)/H(n) \cong \mathbb{P}^1$, $F(n) \setminus \{X_1, \dots, X_n\}/H(n) \cong \mathbb{C}$ and by Theorem 1.3 the morphism

$$\pi_1(S_1) \rightarrow \pi_1(\mathbb{P}^1) \times \pi_1(\mathbb{C})$$

has finite kernel, it follows that $\pi_1(S_1)$ is finite.

The minimal resolution of singularities $S'_1 \rightarrow S_1$ can be identified with the following partial compactification of Ceva(n). Consider

$$J := \{1, \dots, n, 3n + n^3 + 1\} \subset \{1, \dots, 3n + n^2 + 3\}$$

then following the construction given in 4.3.1 we have that

$$\text{Bl}_{\text{Sing Ceva}(n)} \mathbb{P}^2 \setminus \{\cup_{j \in J} D_j\} \cong S'_1.$$

This is from the surface $\text{Bl}_{\text{Sing Ceva}(n)} \mathbb{P}^2$ we remove only the strict transform of \mathcal{A}_1 and the exceptional divisor coming from the singular point of \mathcal{A}_1 . This can be identified with a singular fiber or \tilde{f} .

Example 4.2. Consider now $S_2 := (F(n) \times F(n) \setminus \{X_i, Y_i\})/H(n)$. In this case the subgroup I , defined as in the previous paragraph, is isomorphic to $\mu(n)$. As $F(n)/\mu(n) \cong \mathbb{P}^1$, $F(n) \setminus \{X_i, Y_i\}/\mu(n) \cong \mathbb{C}^*$ and by Theorem 1.3 the morphism

$$\pi_1(S_2) \rightarrow \pi_1([\mathbb{P}^1/\mu(n)]) \times \pi_1([\mathbb{C}^*/\mu(n)])$$

has finite kernel and the image is a finite-index subgroup.

By Theorem 4.2 and Corollary 4.3, we have that $\mathbb{Z} \triangleleft_{f.i.} \pi_1(S_2)$. As in 4.1 the minimal resolution of singularities $S'_2 \rightarrow S_2$ can be identified with $\text{Bl}_{\text{Sing Ceva}(n)} \mathbb{P}^2$ minus two singular fibers of \tilde{f} .

Example 4.3. If we consider $S_3 := (F(n) \times F(n) \setminus \{X_i, Y_i, Z_i\})/H(n)$ it can be identified with $\text{Bl}_{\text{Sing } \mathcal{A}} \mathbb{P}^2$ minus the three singular fibers of \tilde{f} . As $H(n)$ acts freely in $F(n) \times F(n) \setminus \{X_i, Y_i, Z_i\}$. By the long exact sequence of homotopy associated to the covering map $F(n) \times F(n) \setminus \{X_i, Y_i, Z_i\} \rightarrow S_3$ we have

$$1 \rightarrow \pi_1(F(n)) \times \pi_1(F(n) \setminus \{X_i, Y_i, Z_i\}) \rightarrow \pi_1(S_2) \rightarrow H(n) \rightarrow 1.$$

Remark 2. We can remove points also in the first component $F(n)$ of the product. However, we can not get more partial compactifications of Ceva(n) in this way. This can be shown by drawing the dual graph of the divisor $\pi^* \text{Ceva}(n)$ and noticing that the lines obtained by removing points does not satisfy the intersection pattern of the graph.

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