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► **To cite this version:**

Antonin Chambolle, Matteo Novaga. Anisotropic and crystalline mean curvature flow of mean-convex sets. 2020. hal-02525796v1

**HAL Id: hal-02525796**

**<https://hal.science/hal-02525796v1>**

Preprint submitted on 31 Mar 2020 (v1), last revised 27 Oct 2020 (v2)

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# Anisotropic and crystalline mean curvature flow of mean-convex sets

Antonin Chambolle <sup>\*</sup>, Matteo Novaga <sup>†</sup>

We consider a variational scheme for the anisotropic (including crystalline) mean curvature flow of sets with strictly positive anisotropic mean curvature. We show that such condition is preserved by the scheme, and we prove the strict convergence in  $BV$  of the time-integrated perimeters of the approximating evolutions, extending a recent result of De Philippis and Laux to the anisotropic setting. We also prove uniqueness of the flat flow obtained in the limit.

**Keywords:** Anisotropic mean curvature flow, crystal growth, minimizing movements, mean convexity, arrival time, 1-superharmonic functions.

**MSC (2020):** 53E10, 49Q20, 58E12, 35A15, 74E10.

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# 1. Introduction

We are interested in the anisotropic mean curvature flow of sets with positive anisotropic mean curvature. More precisely, following [11, 9] we consider a family of sets  $t \mapsto E(t)$  governed by the geometric evolution law

$$V(x, t) = -\psi(\nu_{E(t)}) \kappa_{E(t)}^\phi(x), \quad (1)$$

where  $V(x, t)$  denotes the normal velocity of the boundary  $\partial E(t)$  at  $x$ ,  $\phi$  is a given norm on  $\mathbb{R}^d$ ,  $\kappa_{E(t)}^\phi$  is the *anisotropic mean curvature* of  $\partial E(t)$  associated with the anisotropy  $\phi$ , and  $\psi$  is another norm (usually called mobility) evaluated at the (outer) unit normal  $\nu_{E(t)}$  to  $\partial E(t)$ . We recall that when  $\phi$  is differentiable in  $\mathbb{R}^N \setminus \{0\}$ , then  $\kappa_E^\phi$  is given by the tangential divergence of the so-called *Cahn-Hoffman vector field* [6]

$$\kappa_E^\phi = \operatorname{div}_\tau (\nabla \phi(\nu_E)), \quad (2)$$

while in general (2) should be replaced with the differential inclusion

$$\kappa_E^\phi = \operatorname{div}_\tau \left( n_E^\phi \right), \quad n_E^\phi \in \partial \phi(\nu_E).$$

It is well-known that (1) can be interpreted as gradient flow of the anisotropic perimeter

$$P_\phi(E) = \int_{\partial E} \phi(\nu_E) d\mathcal{H}^{d-1},$$

and one can construct global-in-time weak solutions by means of the variational scheme introduced by Almgren, Taylor and Wang [1] and, independently, by Luckhaus and Sturzenhecker [14]. Such scheme consists in building a family of discrete-in-time evolutions by an iterative minimization procedure and in considering any limit of these discrete evolutions, as the time step  $h > 0$  vanishes, as an admissible solution to the geometric motion, usually referred to as a *flat flow*. The problem which is solved at each step takes the form [1, §2.6]  $E_h^n := T_h E_h^{n-1}$ , where  $T_h E$  is the solution of

$$\min_F P_\phi(F) + \frac{1}{h} \int_F d_E^{\psi^\circ}(x) dx, \quad (3)$$

where  $d_E^{\psi^\circ}$  is the signed distance function of  $E$ , with respect to the (non necessarily symmetric) “norm”  $\psi^\circ$ , which is defined as

$$d_E^{\psi^\circ}(x) := \inf_{y \in E} \psi^\circ(x - y) - \inf_{y \notin E} \psi^\circ(y - x). \quad (4)$$

In [1] it is proved that the discrete solution  $E_h(t) := E_h^{\lfloor \frac{t}{h} \rfloor}$ , with  $\psi = 1$  and  $\phi$  smooth, converges to a limit flat flow which is contained in the zero-level set of the (unique) viscosity solution of (1). Such a result has been extended in [11, 9] to general anisotropies  $\psi, \phi$ . In the isotropic case  $\phi = \psi = |\cdot|$  it is shown in [14] that  $E_h(t)$  converges to a distributional solution  $E(t)$  of (1), under the assumption that the perimeter is continuous in the limit, that is,

$$\lim_{h \rightarrow 0} \int_0^T P(E_h(t)) dt = \int_0^T P(E(t)) dt \quad \text{for } T > 0. \quad (5)$$

Recently, in [12] the authors proved that the continuity of the perimeter holds if the initial set is *outward minimizing* for the perimeter (see Section 2.1), a condition which implies the mean convexity and which is preserved by the variational scheme 3, as already shown in [17].

In this paper we generalize the result in [12] to the general anisotropic case, where the continuity of the perimeter was previously known only in the convex case [5], as a consequence of the convexity

preserving property of the scheme. Such result is obtained under a stronger condition of strong outward minimality of the initial set, which is also preserved by the scheme and implies the strict positivity of the anisotropic mean curvature. As a corollary, we obtain the continuity of the volume and of the (anisotropic) perimeter of the limit flat flow.

The plan of the paper is the following: In Section 2 we introduce the notion of outward minimizing set, and we recall the variational scheme proposed by Almgren, Taylor and Wang in [1]. We also show that the scheme preserves the strict outward minimality. In section 3 we show the strict  $BV$ -convergence of the discrete arrival time functions, we prove the uniqueness of the limit flow, and we show continuity in time of volume and perimeter, and in Section 4 we give some examples. Eventually, in Appendix A we recall some results on 1-superharmonic functions, adapted to the anisotropic setting.

## 2. Preliminary definitions

### 2.1. Outward minimizing sets

**Definition 2.1.** Let  $\Omega$  be an open subset of  $\mathbb{R}^d$  and let  $E \subset\subset \Omega$  be a finite perimeter set. We say that  $E$  is outward minimizing in  $\Omega$  (see [12, 17]) if

$$P_\phi(E) \leq P_\phi(F) \quad \forall F \supset E, F \subset\subset \Omega. \quad (MC)$$

Note that, if  $E, \phi$  are regular,  $(MC)$  implies that the  $\phi$ -mean curvature of  $\partial E$  is non-negative.

We observe that such a set (or rather its complement) satisfies the following density bound: there exists  $\gamma > 0$  such that for all  $x \in E$  such that  $|B(x, \rho) \setminus E| > 0$  for all  $\rho > 0$ , one has:

$$\frac{|B(x, \rho) \setminus E|}{|B(x, \rho)|} \geq \gamma \quad (6)$$

whenever  $B(x, \rho) \subset \Omega$ . A consequence is that whenever  $x \in E$  is a point of Lebesgue density one, then there is  $\rho$  small such that (6) does not hold, and it follows that for a smaller radius  $\rho'$ ,  $|B(x, \rho') \setminus E| = 0$ . (This is even quantitative in the following sense: if  $|B(x, \rho) \setminus E| < \gamma 2^{-d} |B(x, \rho)|$  then  $|B(y, \rho/2) \setminus E| < \gamma |B(y, \rho/2)|$  for all  $y \in B(x, \rho/2)$  so that  $|B(x, \rho/2) \setminus E| = 0$ .) Identifying thus  $E$  with its points of Lebesgue density one, one will always assume that  $E$  is an open subset of  $\mathbb{R}^d$ .

Conversely if  $E \subset \mathbb{R}^d$  is bounded and  $C^2$ ,  $\phi$  is  $C^2(\mathbb{R}^d \setminus \{0\})$ , and its mean curvature is positive, then one can find  $\Omega \supset\supset E$  such that  $E$  is outward minimizing in  $\Omega$ . More precisely, if  $E$  is of class  $C^2$  then, in a neighborhood of  $\partial E$ ,  $d_E^{\phi^\circ}$  is  $C^2$ , while in a smaller neighborhood we even have  $\operatorname{div} \nabla \phi(d_E^{\phi^\circ}) \geq \delta$  for some  $\delta > 0$ . Let  $\Omega$  be the union of  $E$  and this neighborhood, and set  $n_E^\phi := \nabla \phi(d_E^{\phi^\circ})$ : then if  $E \subset F \subset\subset \Omega$ ,

$$P_\phi(F) \geq \int_{\partial^* F} n_E^\phi \cdot \nu_F d\mathcal{H}^{d-1} = - \int_\Omega n_E^\phi \cdot D\chi_F$$

while by construction  $P_\phi(E) = - \int_\Omega n_E^\phi \cdot D\chi_E$ . Hence,

$$P_\phi(F) \geq P_\phi(E) - \int_\Omega n_E^\phi \cdot D(\chi_F - \chi_E) = P_\phi(E) + \int_{F \setminus E} \operatorname{div} n_E^\phi \geq P_\phi(E) + \delta |F \setminus E|.$$

Observe (see [12, Lemma 2.5]) that equivalently, one can express this as:

$$P_\phi(E \cap F) \leq P_\phi(F) - \delta |F \setminus E| \quad \forall F \subset\subset \Omega. \quad (MC_\delta)$$

Clearly, condition  $(MC_\delta)$  is stronger and reduces to  $(MC)$  whenever  $\delta = 0$ .

*Remark 2.2* (Non symmetric distances). As in the standard case (that is when  $\psi^\circ$  is smooth and even), the signed “distance” function defined in (4) is easily seen to satisfy the usual properties of a signed distance function. First, it is Lipschitz continuous, hence differentiable almost everywhere. Then, if  $x$  is a point of differentiability,  $d_E^{\psi^\circ}(x) > 0$  and  $y \in \partial E$  is such that  $\psi^\circ(x - y) = d_E^{\psi^\circ}(x)$ , then for  $s > 0$  small and  $h \in \mathbb{R}^d$   $d_E^{\psi^\circ}(x + sh) \geq \psi^\circ(x + sh - y) \geq \psi^\circ(x - y) + sz \cdot h$  for any  $z \in \partial\psi^\circ(x - y)$  and one deduces that  $\partial\psi^\circ(x - y) = \{\nabla d_E^{\psi^\circ}(x)\}$ . If  $d_E^{\psi^\circ}(x) < 0$ , one writes that  $\psi^\circ(y - x) = -d_E^{\psi^\circ}(x)$  for some  $y \in \partial E$  and uses  $\psi^\circ(y - x - sh) \geq \psi^\circ(y - x) - sz \cdot h$  for some  $z \in \partial\psi^\circ(y - x)$ , hence  $d_E^{\psi^\circ}(x + sh) - d_E^{\psi^\circ}(x) \leq sz \cdot h$  to deduce now that  $\partial\psi^\circ(y - x) = \{\nabla d_E^{\psi^\circ}(x)\}$ . In all cases, one has  $\phi(\nabla d_E^{\psi^\circ}(x)) = 1$  a.e. in  $\{d_E^{\psi^\circ} \neq 0\}$  (while of course  $\nabla d_E^{\psi^\circ}(x) = 0$  a.e. in  $\{d_E^{\psi^\circ} = 0\}$ ), and  $\nabla d_E^{\psi^\circ}(x) \cdot (x - y) = d_E^{\psi^\circ}(x)$ , which shows that  $y \in x - d_E^{\psi^\circ}(x)\partial\phi(\nabla d_E^{\psi^\circ}(x))$ .

## 2.2. The discrete scheme

We now consider here the discrete scheme introduced in [14, 1] and its generalisation in [8, 5, 11, 10]. It is based on the following process: given  $h > 0$ , and  $E$  a (bounded) finite perimeter set, we define  $T_h E$  as a minimizer of

$$\min_F P_\phi(F) + \frac{1}{h} \int_F d_E^{\psi^\circ}(x) dx \quad (ATW)$$

where  $d_E^{\psi^\circ}$  is defined in (4). If  $E$  satisfies  $(MC)$  in  $\Omega$ , it is clear that for  $h > 0$  small enough, one has  $T_h E \subset E$ . Indeed, for  $h$  small enough one has  $\overline{T_h E} \subset \Omega$ , and it follows from  $(MC)$  that

$$P_h(T_h E \cap E) + \frac{1}{h} \int_{T_h E \cap E} d_E^{\psi^\circ}(x) dx \leq P_h(T_h E) + \frac{1}{h} \int_{T_h E} d_E^{\psi^\circ}(x) dx - \frac{1}{h} \int_{T_h E \setminus E} d_E^{\psi^\circ}(x) dx,$$

showing that  $|T_h E \setminus E| = 0$ . We recall in addition that in this case,  $T_h E$  is also  $\phi$ -mean convex in  $\Omega$ , see the proof of [12, Lemma 2.7]. If  $E$  satisfies  $(MC_\delta)$  for  $\delta > 0$ , we can improve the inclusion  $T_h E \subset E$ :

**Lemma 2.3.** *Assume that  $E$  satisfies  $(MC_\delta)$ ,  $\delta > 0$ . Then for  $h > 0$  small enough,  $T_h E + \{\phi^\circ \leq \delta h\} \subset E$ . In particular,  $d_{T_h E}^{\psi^\circ} \geq d_E^{\psi^\circ} + \delta h$  and  $T_h E \subset \{d_E^{\psi^\circ} \leq -\delta h\}$ .*

*Proof.* Assume  $h$  is small enough so that  $T_h E \subset E$  and  $E + \{\psi^\circ \leq \delta h\} \subset \Omega$ . Choose  $\tau$  with  $\psi^\circ(\tau) < \delta h$  and consider  $F := T_h E + \tau$ . We show that also  $F \subset E$ . The set  $F \subset \subset \Omega$  is a minimizer of

$$P_\phi(F) + \frac{1}{h} \int_F d_E^{\psi^\circ}(x - \tau) dx.$$

In particular, we have

$$\begin{aligned} P_\phi(F) + \frac{1}{h} \int_F d_E^{\psi^\circ}(x - \tau) dx &\leq P_\phi(F \cap E) + \frac{1}{h} \int_{F \cap E} d_E^{\psi^\circ}(x - \tau) dx \\ &\leq P_\phi(F) + \frac{1}{h} \int_F d_E^{\psi^\circ}(x - \tau) dx - \int_{F \setminus E} \frac{1}{h} d_E^{\psi^\circ}(x - \tau) + \delta dx. \end{aligned}$$

By definition of the signed distance function, for  $x \notin E$ ,  $d_E^{\psi^\circ}(x - \tau) \geq -\psi^\circ(x - (x - \tau)) = -\psi^\circ(\tau) > -\delta h$  so that if  $|F \setminus E| > 0$  we have a contradiction. We deduce that  $T_h E + \{\psi^\circ \leq \delta h\} \subset E$ .

In particular, if  $x \in T_h E$  and  $y \notin E$  is such that  $d_E^{\psi^\circ}(x) = -\psi^\circ(y - x)$ , then  $y' = y - \delta h(y - x)/\psi^\circ(y - x) \notin T_h E$  hence  $d_{T_h E}^{\psi^\circ} \geq -\psi^\circ(y' - x) = d_E^{\psi^\circ}(x) - \delta h$ . If  $x \in E \setminus T_h E$ ,  $d_E^{\psi^\circ}(x) = -\psi(y - x)$  for some  $y \in \overline{\Omega \setminus E}$ , and  $d_{T_h E}^{\psi^\circ}(x) = \psi(x - y')$  for some  $y' \in T_h E$ . Since  $\psi(x - y') + \psi(y - x) \geq \psi(y - y') \geq \delta h$  we conclude. Eventually if  $x \notin E$ , for  $y \in T_h E$  with  $d_{T_h E}^{\psi^\circ}(x) = \phi^\circ(x - y)$  we have

$y + \delta h(x - y)/\phi^\circ(x - y) \in E$ , so that  $d_E^{\psi^\circ}(x) \leq \phi^\circ(x - y) - \delta h = d_{T_h E}^{\psi^\circ}(x) - \delta h$ . This shows that  $d_{T_h E}^{\psi^\circ} \geq d_E^{\psi^\circ} + \delta h$ .  $\square$

**Corollary 2.4.** *For any  $n \geq 1$ ,  $T_h^{n+1}E + \{\psi^\circ \leq \delta h\} \subset T_h^n E$ , and  $d_{T_h^n E}^{\psi^\circ} \geq d_E^{\psi^\circ} - \delta n h$ .*

*Proof.* The first statement is obvious by induction: assuming that for  $\tau$  with  $\psi^\circ(\tau) \leq \delta h$  one has  $T_h^n E + \tau \subset T_h^{n-1} E$  (which is true for  $n = 1$ ), one has applying  $T_h$  again and using the translational covariance that  $T_h^{n+1} E + \tau \subset T_h^n E$ . The second statement is obviously deduced, as in the previous proof.  $\square$

*Remark 2.5 (Density estimates).* There exists  $\gamma > 0$ , depending only on  $\phi$  and the dimension, and  $r_0 > 0$ , depending also on  $\psi$ , such that the following holds: for  $x$  such that  $|B(x, r) \cap T_h E| > 0$  for all  $r > 0$  one has  $|B(x, r) \cap T_h E| \geq \gamma r^d$  if  $r < r_0 h$ . For the complement, as  $T_h E$  is  $\phi$ -mean convex in  $\Omega$ , we have as before that for  $x$  such that  $|B(x, r) \setminus T_h E| > 0$  for all  $r > 0$ , one has  $|B(x, r) \setminus T_h E| \geq \gamma r^d$  for all  $r$  with  $B(x, r) \subset \Omega$ , cf (6).

### 2.3. Preservation of the outward minimality

In the sequel, we show some further properties of the discrete evolutions and their limit. An interesting result in [12] is that the  $(MC_\delta)$ -condition is preserved during the evolution. We prove that it is also the case in the anisotropic setting.

We first show the following result:

**Lemma 2.6.** *Assume  $E$  satisfies  $(MC_\delta)$  in  $\Omega$ , with  $\delta > 0$ . Then for any  $F \subset \subset \Omega$ ,  $\delta|F| \leq P_\phi(F)$ .*

*Proof.* Since  $\delta|F| = \delta|F \cap E| + \delta|F \setminus E| \leq \delta|F \cap E| + (P_\phi(F) - P_\phi(F \cap E))$  (using  $(MC_\delta)$ ), it is enough to show the result for  $F \subset E$ . We introduce for any  $s > 0$   $E_s$  the largest minimizer of

$$P_\phi(E_s) + \frac{1}{s} \int_{E_s} d_E^{\psi^\circ} dx. \quad (7)$$

which is obtained as the level set  $\{w_s \leq 0\}$  of the (Lipschitz continuous) solution  $w_s$  of the equation

$$-\operatorname{sdiv} z_s + w_s = d_{E_s}^{\psi^\circ}, \quad z_s \in \partial\phi(\nabla w_s),$$

see for instance [8] for details. It turns out that there exists some  $s_0 > 0$  such that for  $s < s_0$ ,  $E_s \subset \Omega$ . This is true because by comparison it is easy to show that  $E_s \subset E + B(0, C\sqrt{s})$  for some constant  $C$  depending only on  $\phi, \psi$ , but in fact, as soon as the minimizer of the same problem (7) with  $P_\phi$  replaced with  $P_\phi(\cdot; \Omega)$  is strictly in  $\Omega$  (hence in  $E$ , cf Lemma 2.3), then it coincides with  $E_s$  and it even follows that  $E_s \subset \subset E$ . In addition,  $\psi(\nabla w_s) \leq 1 = \psi(\nabla d_E^{\psi^\circ})$  a.e. in  $\mathbb{R}^d$ . For  $s < s_0$ , we denote also  $E'_s = \{w_s < 0\}$  the smallest minimizer of (7).

As  $P_\phi(E_s) \leq P_\phi(E)$ ,  $\lim_{s \rightarrow 0} P_\phi(E_s) = P_\phi(E)$  and one deduces that for any  $F \subset E$ ,  $\lim_{s \rightarrow 0} P_\phi(F \cap E_s) = P_\phi(F)$ . Indeed,

$$P_\phi(F \cap E_s) + P_\phi(F \cup E_s) \leq P_\phi(F) + P_\phi(E_s),$$

and  $F \cup E_s \rightarrow E$ , so that

$$P_\phi(E) + \limsup_s P_\phi(F \cap E_s) \leq P_\phi(F) + P_\phi(E),$$

which shows the claim.

Thanks to Lemma 2.3, one has that  $d_E^{\psi^\circ} \leq -s\delta$  on  $\partial E_s = \{w_s \leq 0\}$ . Now if  $x \in E_s$  and  $y \in \partial E_s$ ,  $w_s(x) \geq w_s(y) - \psi^\circ(y - x) = -\psi^\circ(y - x)$  (using  $\psi^\circ(\nabla w_s) \leq 1$ ). If  $z \notin E$  and  $y \in [x, z] \cap \partial E_s$ , one has  $\psi^\circ(z - x) = \psi^\circ(z - y) + \psi^\circ(y - x)$  (by one-homogeneity) so that  $0 \leq w_s(x) + \psi^\circ(y - x) = w_s(x) + \psi^\circ(z - x) - \psi^\circ(z - y) \leq w_s(x) + \psi^\circ(z - x) - s\delta$ . Taking the infimum over  $z$ , we see that  $s\delta \leq w_s(x) - d_E^{\psi^\circ}(x)$ . Hence  $\operatorname{div} z_s \geq \delta$  a.e. in  $E$ .

Let now  $F \subset E$ , one has  $P_\phi(F \cap E_s) \geq \int_\Omega \operatorname{div} z_s \chi_{F \cap E_s} \geq \delta |F \cap E_s|$ . The lemma follows letting  $s \rightarrow 0$ .  $\square$

We can then deduce the following:

**Lemma 2.7.** *Let  $\delta > 0$ ,  $E$  (open) satisfy  $(MC_\delta)$  in  $\Omega$ ,  $h > 0$  small and  $T_h E \subset E$  the solution of  $(ATW)$ . Then  $T_h E$  satisfies  $(MC_\delta)$ .*

*Proof.* We remark that the sets  $E_s, E'_s$  built in the previous proof satisfy  $E_s \subset E'_{s'}$  for  $s > s'$ . This follows from the fact that the term  $s \mapsto d_E^{\psi^\circ}(x)/s < 0$  is increasing for  $x \in E$ . In particular, if  $s_n \rightarrow s$ ,  $s_n < s$ , then  $E_{s_n} \rightarrow E_s$ , while if  $s_n > s$ ,  $\Omega \setminus E'_{s_n}$  converges to  $\Omega \setminus E'_s$ . Moreover, as the sets satisfy uniform density estimates (for  $n$  large enough), these convergences are also in the Hausdorff sense. In particular, we deduce that  $E \setminus E'_s = \bigcup_{0 < s' \leq s} (E_{s'} \setminus E'_{s'})$  (we recall  $E_{s'} \setminus E'_{s'} = \{w_{s'} = 0\}$ ).

Let  $\varepsilon > 0$ . The previous analysis also shows that there exists  $\eta > 0$  (depending only on  $\phi$ ) such that for any  $s \in (0, s_0)$ , in  $N_s = \{x : \operatorname{dist}(x, E_s) < s\eta\}$ , one has  $\operatorname{div} z_s \geq \delta - \varepsilon$  (we use here that  $w_s - d_E^{\psi^\circ}$  is Lipschitz continuous).

Let  $h > \bar{s} > \underline{s} > 0$ . The set  $E_{\underline{s}} \setminus E'_{\bar{s}}$  is covered by the open sets  $N_s \setminus \overline{E'_s}$ ,  $0 < s < h$ . Hence one can extract a finite covering indexed by  $s_1 > s_2 > \dots > s_{N-1}$ . We observe that necessarily,  $h > s_1 > \bar{s}$  and we let  $s_N := \underline{s}$ . Let  $F \subset\subset \Omega$  and up to an infinitesimal translation, assume  $\mathcal{H}^{d-1}(\partial^* F \cap \partial E_{s_i}) = 0$  for  $i = 1, \dots, N$ . One has for  $i \in \{1, \dots, N\}$

$$\begin{aligned} P_\phi(E_{s_{i+1}} \cap F) - P_\phi(E_{s_i} \cap F) &\geq \int_{\partial^*(F \cap E_{s_{i+1}} \setminus E_{s_i})} z_{s_i} \cdot \nu d\mathcal{H}^{d-1} \\ &= \int_{F \cap E_{s_{i+1}} \setminus E_{s_i}} \operatorname{div} z_{s_i} dx \geq (\delta - \varepsilon) |F \cap E_{s_{i+1}} \setminus E_{s_i}| \end{aligned}$$

so that, summing from  $i = 1$  to  $N$ , we find that

$$P_\phi(E_{s_1} \cap F) \leq P_\phi(E_{\underline{s}} \cap F) - (\delta - \varepsilon) |(E_{\underline{s}} \setminus E_{s_1}) \cap F|.$$

Since  $E_{\underline{s}}$  is outward minimizing,  $P_\phi(E_{\underline{s}} \cap F) \leq P_\phi(E \cap F) \leq P_\phi(F) - (\delta - \varepsilon) |F \setminus E|$ , so that:

$$P_\phi(E_{s_1} \cap F) \leq P_\phi(F) - (\delta - \varepsilon) (|F \setminus E| + |(E_{\underline{s}} \setminus E_{s_1}) \cap F|).$$

Sending  $\bar{s} < s_1$  to  $h$  and  $\underline{s}$  to 0, we deduce that  $P_\phi(E_h \cap F) \leq P_\phi(F) - (\delta - \varepsilon) |F \setminus E_h|$  hence the thesis holds, since  $\varepsilon$  is arbitrary.  $\square$

### 3. The arrival time function

Consider an open set  $\Omega \subset \mathbb{R}^d$  and a set  $E^0 \subset\subset \Omega$  such that  $(MC_\delta)$  holds for some  $\delta > 0$ . As usual [14, 1] we let  $E_h(t) := T_h^{\lfloor t/h \rfloor}(E^0)$ , here  $\lfloor \cdot \rfloor$  denotes the integer part. Being the sets  $T_h^n(E^0)$  mean-convex, we can choose an open representative. We can define the *discrete arrival time function* as

$$u_h(x) := \max\{t \chi_{E_h(t)}(x), t \geq 0\},$$

which is a l.s.c. function<sup>1</sup> which, thanks to the co-area formula, satisfies

$$\int_{\Omega} \phi(-Du_h) \leq \int_{\Omega} \phi(-Dv) \quad (8)$$

for any  $v \in BV(\mathbb{R}^d)$  with  $v \geq u_h$ ,  $v = 0$  in  $\mathbb{R}^d \setminus \Omega$  — hence a  $(\phi)$ -1-superharmonic function in the sense of [16], see Sec. A. One easily sees that  $(u_h)_h$  is uniformly bounded in  $BV(\Omega)$  so that a subsequence  $u_{h_k}$  converges in  $L^1(\Omega)$  to some  $u$  (which again is 1-superharmonic).

In addition, as  $E^0$  satisfies  $(MC_{\delta})$ , we have thanks to Corollary 2.4 that  $u_h$  has a sort of global Lipschitz bound. More precisely, for  $x, y \in \Omega$  there holds

$$u_h(x) - u_h(y) \leq h + \frac{\phi^{\circ}(y-x)}{\delta}$$

Indeed, one has  $u_h(x) = t \Rightarrow u_h(x+\tau) \geq t-h$  for any  $t \geq 0$  and  $\tau$  with  $\phi^{\circ}(\tau) \leq \delta h$ . The claim follows by induction.

As a consequence we obtain that  $u_h$  converges uniformly, up to a subsequence, to a limit function  $u$ , which is also Lipschitz continuous, and satisfies

$$u(x) - u(y) \leq \frac{\phi^{\circ}(y-x)}{\delta} \quad (9)$$

for any  $x, y \in \Omega$ . Moreover, recalling Lemma 2.7, we have that the functions  $u_h$  and  $u$  are  $(\phi, \delta)$ -1-superharmonic, in the sense of Definition A.1 below.

We will show that the function  $u$  is unique, and is the arrival time function of the anisotropic curvature flow starting from  $E^0$ , in the sense of [9].

**Theorem 3.1.** *Under the previous assumption on  $E^0$ , the arrival time function  $u_h$  converge to a unique limit  $u$ , as  $h \rightarrow 0$ , which is such that  $(x, t) \mapsto \{u(x) \leq t\}$  is a solution of (MCF) starting from  $E$ . Moreover, there holds*

$$\lim_{h \rightarrow 0} \int_{\Omega} \phi(-Du_h) = \int_{\Omega} \phi(-Du).$$

*Proof.* Let us denote  $E^s$  the (open) sets  $\{u > s\}$ . Since given  $x, \rho$  with  $B(x, \rho) \subset E$ , one knows that the curvature flow starting from  $B(x, \rho)$  will contain  $x$  for a time of order  $\rho^2$ , one has  $u(x) \gtrsim \rho^2$ . It follows that  $x \in E^s$  for some  $s \sim \rho^2$  so that  $\bigcup_{s>0} E^s = E$ .

As a consequence of the existence and uniqueness result in [11, 9]), for a.e.  $s > 0$  the arrival time functions  $u_h^s \leq u_h$  of the discrete flows  $T_h^{[t/h]} E^s$  converge uniformly to a unique limit  $u^s$ . In particular, considering the subsequence  $u_{h_k}$ , one has  $u^s \leq u$ . On the other hand, thanks to Corollary 2.4 and the Remark 2.5, given  $s > 0$  there is  $\tau_s > 0$  such that  $T_h^{[\tau_s/h]} E^0 \subset E^s$ . Then,  $T_h^{[\tau_s/h]+n} E^0 \subset T_h^n E^s$  by induction so that  $u_h - \tau_s - h \leq u_h^s$ . If  $v$  is the limit of a converging subsequence of  $(u_h)$ , we deduce  $v - \tau_s \leq u^s \leq u$ . Sending  $s \rightarrow 0$  we deduce  $v \leq u$ . Since this is true for any pair  $(u, v)$  of limits of converging subsequences of  $(u_h)$ , this limit is unique and  $u_h \rightarrow u$ .

The last statement is already proved in [12] in a simple way: clearly, one just needs to show that  $\limsup_h \int_{\Omega} \phi(-Du_h) \leq \int_{\Omega} \phi(-Du)$ . As  $(u_h)_h$  converges uniformly to  $u$ , given  $\varepsilon > 0$ , one has  $u_h \leq u + \varepsilon$  for  $h$  small enough. On the other hand, since all these functions vanish out of  $E$ , it follows  $u_h \leq u + \varepsilon \chi_E$ . Hence, being  $u_h$   $\phi$ -1-superharmonic,

$$\int_{\Omega} \phi(-Du_h) \leq \int_{\Omega} \phi(-D(u + \varepsilon \chi_E)) = \int_{\Omega} \phi(-Du) + \varepsilon P_{\phi}(E)$$

for  $h$  small enough, and the thesis follows.  $\square$

<sup>1</sup>We can say that  $u_h$  is a function in  $BV(\Omega)$  with compact support and such that its approximate lower limit  $u_h^-$  is lower semicontinuous.



Theorem 3.1 shows that the scheme starting from a strict  $\phi$ -mean convex set always converges to a unique flow, with no loss of (anisotropic) perimeter. In particular, in the smooth and elliptic case, following [14] it allows to show that the limit satisfies a distributional formulation of the anisotropic curvature flow.

*Remark 3.2* (Continuity of volume and perimeter). As is well-known for general flat flows (see [14, 7]), the limit motion  $t \mapsto \{u \geq t\}$  is 1/2-Hölder in  $L^1(\Omega)$ , in the sense that, for  $s > t > 0$ ,

$$|\{s > u \geq t\} \cap \Omega| \leq C|t - s|^{1/2},$$

where  $C$  depends on the dimension and on the perimeter of the initial set. In particular,  $|\{u = t\}| = 0$  for all  $t > 0$ , so that up to a negligible set,  $\{u > t\} = \{u \geq t\}$ . (For  $t = 0$  it may happen that  $|\overline{\{u > 0\}}| > 0$ , as shown in the second example below.)

In addition, since each set  $\{u > t\}$  is  $\delta$ -superharmonic for  $t > 0$ , for  $s > t \geq 0$  one also has that

$$P_\phi(\{u > s\}) + \delta|\{s \geq u > t\}| = P_\phi(\{u > t\})$$

so that  $t \mapsto P_\phi(\{u > t\})$  is strictly decreasing, until extinction. Since  $\bigcup_{s>t} \{u > s\} = \{u > t\}$  one also sees that  $t \mapsto P_\phi(\{u > t\})$  is right-continuous. On the other hand, whether it could jump or not remains an open question.

## 4. Examples

### 4.1. The case $\delta = 0$

If the initial datum  $E^0$  satisfies (MC) we should distinguish two cases: If  $\phi$  is smooth and elliptic,  $\psi$  is smooth and  $\partial E^0$  is also smooth, then there exists a smooth solution to (1) on a time interval  $[0, \tau)$ , for some  $\tau > 0$  (see [15, Chapter 8]). Moreover, by the parabolic maximum principle, the solution  $E(t)$  become strictly mean-convex for  $t > 0$ . In particular, for any  $\varepsilon \in (0, \tau)$  there exist  $\delta_\varepsilon > 0$  and an open set  $\Omega_\varepsilon$  such that  $E(t_\varepsilon) \subset\subset \Omega_\varepsilon$ ,  $\delta_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , and  $E(t)$  satisfies  $(MC_{\delta_\varepsilon})$  in  $\Omega_\varepsilon$  for  $t \in (\varepsilon, \tau)$ . As a consequence, the previous results hold in all the time intervals  $[\varepsilon, +\infty)$ , so that and the limit function  $u$  it is still unique and continuous, and it is locally Lipschitz continuous in the interior of  $E^0$ .

On the other hand, for an arbitrary anisotropy  $\phi$ , the function  $u$  could be discontinuous on the boundary of  $E^0$ . As an example let us consider, in two dimensions, the case  $\psi(\xi, \eta) = \phi(\xi, \eta) = |\xi| + |\eta|$  (and  $\{\psi^\circ \leq 1\} = \{(x, y) : |x| \leq 1, |y| \leq 1\}$ ), and the cross-shaped initial datum

$$E^0 := ([-1, 1] \times [-2, 2]) \cup ([-2, 2] \times [-1, 1]) \subset \mathbb{R}^2.$$

It is easy to check that  $E^0$  is outward minimizing, so that  $E(t) \subset E^0$  is also outward minimizing for all  $t > 0$ . Moreover, the solution  $E(t) = \{(x, y) : u(x, y) \geq t\}$  is unique (see for instance [13]), and can be explicitly described as follows :

$$E(t) = \begin{cases} ([-1, 1] \times [-2 + t, 2 - t]) \cup ([-2 + t, 2 - t] \times [-1, 1]) & \text{for } t \in [0, 1] \\ \left[ -\sqrt{1 - 2(t - 1)}, \sqrt{1 - 2(t - 1)} \right] & \text{for } t \in \left[1, \frac{3}{2}\right] \\ \emptyset & \text{for } t \geq \frac{3}{2}. \end{cases} \quad (10)$$

In particular, the function  $u \in BV(\mathbb{R}^2)$  is *discontinuous* on  $\partial E^0 \setminus \partial([-2, 2] \times [-2, 2])$ .

We observe that formula (10) for  $E(t)$  can be easily obtained by finding explicit solutions to (ATW), starting from  $E_L = ([-1, 1] \times [-L, L]) \cup ([-L, L] \times [-1, 1])$ ,  $L > 1$ . An approach is as follows: a “calibration” is given by the vector field (which one defines only in  $E_L$ ):

$$z(x, y) = \begin{cases} (x, y) & \text{if } |x| \leq 1, |y| \leq 1, \\ (x, \pm 1) & \text{if } |x| \leq 1, 1 \leq \pm y \leq L, \\ (\pm 1, y) & \text{if } 1 \leq \pm x \leq L, |y| \leq 1. \end{cases}$$

One has  $\operatorname{div} z = 1 + \chi_{[-1, 1]^2}$  in  $E_L$ ,  $z(x, y) \in \{\psi^\circ \leq 1\}$ , and  $P_\phi(E_\ell) = \int_{\partial E_\ell} z \cdot \nu \, d\mathcal{H}^1$  for any  $1 \leq \ell \leq L$ . Hence, if  $L - h \geq 1$  and  $F \subset E_L$ , we have

$$\begin{aligned} P_\phi(F) + \int_F \frac{d_{E_L}^{\psi^\circ}}{h} dx &\geq \int_{\partial F} \nu \cdot z \, d\mathcal{H}^1 + \int_F \frac{d_{E_L}^{\psi^\circ}}{h} dx \\ &= \int_{\partial F} \nu \cdot z \, d\mathcal{H}^1 - \int_{\partial E_{L-h}} \nu \cdot z \, d\mathcal{H}^1 + P_\phi(E_{L-h}) + \int_F \frac{d_{E_L}^{\psi^\circ}}{h} dx \\ &= \int z \cdot (D\chi_{E_{L-h}} - D\chi_F) + P_\phi(E_{L-h}) + \int_{E_{L-h}} \frac{d_{E_L}^{\psi^\circ}}{h} dx + \int_{E_L} (\chi_F - \chi_{E_{L-h}}) \frac{d_{E_L}^{\psi^\circ}}{h} dx \\ &= P_\phi(E_{L-h}) + \int_{E_{L-h}} \frac{d_{E_L}^{\psi^\circ}}{h} dx + \int_{E_L} (\chi_F - \chi_{E_{L-h}}) \left( \frac{d_{E_L}^{\psi^\circ}}{h} + 1 + \chi_{[-1, 1]^2} \right) dx. \end{aligned}$$

Now, the last integral is nonnegative, since  $d_{E_L}^{\psi^\circ}/h + 1 \leq 0$  in  $E_{L-h}$ , and is positive outside. As a consequence,  $E_{L-h}$  solves (ATW) for  $E = E_L$ , and one deduces the first line in (10). The proof of the second line in (10) is a standard computation (see for instance [5]).

## 4.2. Continuity of the volume up to $t = 0$

We provide an example of an open set  $E$  satisfying  $(MC_\delta)$  for some  $\delta > 0$ , and such that  $|\partial E| = |\overline{\{u > 0\}}| > 0$ .

Let  $(x_n)_{n \geq 1}$  be a dense sequence of rational points in  $\Omega := B(0, 1) \subset \mathbb{R}^2$ . We shall construct inductively a sequence  $(r_n)_{n \geq 1}$  of positive numbers with  $\sum_n r_n^{d-1} < +\infty$  such that the following property holds: Letting  $E_0 = \emptyset$  and  $E_n = E_{n-1} \cup B(x_n, r_n)$  for  $n \geq 1$ , the sets  $E_n$  all satisfy  $(MC_\delta)$  in  $\Omega$  for some  $\delta > 0$ .

Notice first that there exists  $\delta > 0$  such that each ball  $B(x, r) \subset \Omega$  satisfies  $(MC_{2\delta})$  in  $\Omega$ . Choose now  $r_1 > 0$  in such a way that  $E_1 = B(x_1, r_1) \subset \Omega$ , then  $E_1$  satisfies  $(MC_{2\delta})$ . Assume now by induction that  $E_n$  satisfies  $(MC_{(1+1/n)\delta})$ . Then, if  $d_n := \operatorname{dist}(x_{n+1}, E_n) = 0$  we let  $r_{n+1} = 0$ , so that  $E_{n+1} = E_n$ . Otherwise, if  $d_n > 0$  we choose  $r_{n+1} \in (0, 2^{-n})$  in such a way that

$$r_{n+1} \leq \min \left( \frac{1}{2} \left( \frac{1}{n} - \frac{1}{n+1} \right) \frac{\delta d_n^2}{2\pi C}, \frac{d_n}{6} \right), \quad (11)$$

where the constant  $C > 0$  will be chosen later in *Case 3*. Let also  $\mathcal{N} \subset \mathbb{N}$  be the (infinite) set of indices such that  $r_n > 0$ .

Let us check that  $E_{n+1}$  satisfies  $(MC_{\delta+\delta/(n+1)})$ . We consider a set  $F$  of finite perimeter such that  $E_{n+1} \subset F \subset \Omega$ , and we distinguish three cases:

Case 1.  $|F \cap B(x_n, d_n)| \geq d_n^2/C$ . In this case we have

$$\begin{aligned}
P(F) &\geq P(E_n) + \left(1 + \frac{1}{n}\right) \delta |F \setminus E_n| \\
&\geq P(E_{n+1}) - 2\pi r_{n+1} + \left(1 + \frac{1}{n+1}\right) \delta |F \setminus E_n| + \left(\frac{1}{n} - \frac{1}{n+1}\right) \delta |F \cap B(x_n, d_n)| \\
&\geq P(E_{n+1}) + \left(1 + \frac{1}{n+1}\right) \delta |F \setminus E_{n+1}| + \left(\frac{1}{n} - \frac{1}{n+1}\right) \frac{\delta d_n^2}{C} - 2\pi r_{n+1} \\
&\geq P(E_{n+1}) + \left(1 + \frac{1}{n+1}\right) \delta |F \setminus E_{n+1}|,
\end{aligned}$$

where in the last inequality we used (11).

Case 2.  $|F \cap B(x_n, d_n)| \leq d_n^2/C$  and  $\mathcal{H}^1(F \cap \partial B(x_n, r)) = 0$  for some  $r \in (r_{n+1}, d_n)$ . In this case, we write  $F = F_1 \cup F_2$ , with  $F_1 = F \cap B(x_n, r) \supset B(x_n, r_{n+1})$  and  $F_2 = F \setminus B(x_n, r) \supset E_n$ , and we have

$$\begin{aligned}
P(F_1) &\geq P(B(x_n, r_{n+1})) + 2\delta |F_1 \setminus B(x_n, r_{n+1})| \\
P(F_2) &\geq P(E_n) + \left(1 + \frac{1}{n}\right) \delta |F_2 \setminus E_n|.
\end{aligned}$$

Summing up the two inequalities above, we get

$$\begin{aligned}
P(F) &= P(F_1) + P(F_2) \geq P(E_{n+1}) + \left(1 + \frac{1}{n}\right) \delta (|F_1 \setminus B(x_n, r_{n+1})| + |F_2 \setminus E_n|) \\
&= P(E_{n+1}) + \left(1 + \frac{1}{n}\right) \delta |F \setminus E_{n+1}|.
\end{aligned}$$

Case 3.  $|F \cap B(x_n, d_n)| \leq d_n^2/C$  and  $\mathcal{H}^1(F \cap \partial B(x_n, r)) > 0$  for a.e.  $r \in (r_{n+1}, d_n)$ . In this case, by coarea formula we have

$$\left| F \cap \left( B\left(x_n, \frac{d_n}{3}\right) \setminus B\left(x_n, \frac{d_n}{6}\right) \right) \right| = \int_{\frac{d_n}{6}}^{\frac{d_n}{3}} \mathcal{H}^1(F \cap \partial B(x_n, r)) dr \leq \frac{d_n^2}{C}.$$

It follows that there exists  $r_1 \in (d_n/6, d_n/3)$  such that

$$\mathcal{H}^1(F \cap \partial B(x_n, r_1)) \leq \frac{6d_n}{C}.$$

Similarly we have

$$\left| F \cap \left( B(x_n, d_n) \setminus B\left(x_n, \frac{2d_n}{3}\right) \right) \right| = \int_{\frac{2d_n}{3}}^{d_n} \mathcal{H}^1(F \cap \partial B(x_n, r)) dr \leq \frac{d_n^2}{C},$$

and there exists  $r_2 \in (2d_n/3, d_n)$  such that

$$\mathcal{H}^1(F \cap \partial B(x_n, r_2)) \leq \frac{3d_n}{C}.$$

Using that  $\mathcal{H}^1(F \cap \partial B(x_n, r)) > 0$  for all  $r \in (r_{n+1}, d_n)$  we deduce that either for a.e.  $r \in (r_1, r_2)$ ,  $\mathcal{H}^0(\partial^* F \cap B(x_n, r)) \geq 2$  and it follows that  $P(F, B(x_n, r_2) \setminus B(x_n, r_1)) \geq 2(r_2 - r_1) \geq 2d_n/3$ , or for a set of positive measure of radii  $r \in (r_1, r_2)$  one has  $\mathcal{H}^1(F \cap \partial B(x_n, r)) = 2\pi r$ , however this implies  $P(F, B(x_n, r_2) \setminus B(x_n, r_1)) \geq 2\pi r_1 - 6d_n/C \geq d_n(\pi/3 - 6/C) \geq 2d_n/3$  provided we have chosen  $C \geq 18/(\pi - 2)$ .

Then, proceeding as in the previous case we let  $F_1 = F \cap B(x_n, r_1)$  and  $F_2 = F \setminus B(x_n, r_2)$ , and we have

$$\begin{aligned}
P(F) &= P(F_1) + P(F_2) - \mathcal{H}^1(F \cap \partial B(x_n, r_1)) - \mathcal{H}^1(F \cap \partial B(x_n, r_2)) + P(F, B(x_n, r_2) \setminus B(x_n, r_1)) \\
&\geq P(E_{n+1}) + \left(1 + \frac{1}{n}\right) \delta (|F_1 \setminus B(x_n, r_{n+1})| + |F_2 \setminus E_n|) - \frac{9d_n}{C} + \frac{2d_n}{3} \\
&\geq P(E_{n+1}) + \left(1 + \frac{1}{n}\right) \delta |F \setminus E_{n+1}| - \left(1 + \frac{1}{n}\right) \delta \frac{d_n^2}{C} - \frac{9d_n}{C} + \frac{2d_n}{3} \\
&\geq P(E_{n+1}) + \left(1 + \frac{1}{n}\right) \delta |F \setminus E_{n+1}| - \frac{2\delta + 9}{C} d_n + \frac{2d_n}{3} \\
&\geq P(E_{n+1}) + \left(1 + \frac{1}{n}\right) \delta |F \setminus E_{n+1}|,
\end{aligned}$$

as long as we choose  $C \geq 3(2\delta + 9)/2$ .

We proved that  $E_n$  satisfies  $(MC_\delta)$  for all  $n \in \mathbf{N}$ , therefore also the limit set

$$E = \bigcup_{n \in \mathbf{N}} E_n = \bigcup_{n \in \mathbf{N}} B(x_n, r_n)$$

satisfies  $(MC_\delta)$  in  $\Omega$ . In this case, the solution  $u$  in Theorem 3.1 is explicit and is given by

$$u(x) = \sum_{n \in \mathbf{N}} \frac{(r_n^2 - |x - x_n|^2)^+}{2(d-1)}.$$

Notice that we have

$$\partial\{u > 0\} = \partial E = \overline{B(0, 1)} \setminus E,$$

so that  $|\partial\{u > 0\}| = \pi - |E| > 0$ .

## A. 1-superharmonic functions

The goal of this appendix is to recall some results proved in [16] on 1-superharmonic functions, to give precise statements in the anisotropic case, and to propose some simple proofs, when possible.

**Definition A.1.** We say that  $u$  is  $(\phi)$ -1-superharmonic in  $\Omega$  if  $\{u \neq 0\} \subset\subset \Omega$  and for any  $v$  with  $v \geq u$ ,  $\{v \neq 0\} \subset\subset \Omega$ , one has

$$\int_{\Omega} \phi(-Du) \leq \int_{\Omega} \phi(-Dv),$$

or, equivalently, for any  $v$  with compact support in  $\Omega$ ,

$$\int_{\Omega} \phi(-D(u \wedge v)) \leq \int_{\Omega} \phi(-Dv). \quad (SH)$$

Given  $\delta > 0$ , we say that  $u$  is  $((\phi, \delta)$ -1-superharmonic in  $\Omega$  if  $\{u \neq 0\} \subset\subset \Omega$  and one has:

$$\int_{\Omega} \phi(-D(u \wedge v)) \leq \int_{\Omega} \phi(-Dv) - \delta \int_{\Omega} (v - u)^+ dx \quad \forall v, \{v \neq 0\} \subset\subset \Omega. \quad (SH_\delta)$$

Equivalently,  $u$  is a minimizer of

$$\int_{\Omega} \phi(-Du) - \delta \int_{\Omega} u dx,$$

with respect to larger competitors with the same boundary condition.

Obviously then,  $u \geq 0$  (using  $v = u^+$  in  $(SH)$ ). Notice that  $\chi_E$  is 1-superharmonic if and only if the set  $E$  is outward minimizing.

Observe that, in this case, the set  $E^0 = \{u > 0\}$  has finite perimeter and satisfies  $(MC_\delta)$ . Indeed, for  $E \subset F \subset \subset \Omega$ , letting  $v = \varepsilon \chi_F$  for  $\varepsilon > 0$ , we have

$$\begin{aligned} \int_{\Omega} \phi(-D(u \wedge \varepsilon \chi_F)) &= \int_0^\varepsilon P_\phi(\{u > s\} \cap F) ds \\ &\leq \varepsilon P_\phi(F) - \delta \int_{\Omega} (\varepsilon \chi_F - u)^+ dx = \varepsilon \left( P_\phi(F) - \delta \int_{\Omega} (\chi_F - u/\varepsilon)^+ dx \right). \end{aligned}$$

Hence:

$$\int_0^1 P_\phi(\{u > t\varepsilon\} \cap F) dt \leq P_\phi(F) - \delta \int_{\Omega} (\chi_F - u/\varepsilon)^+ dx.$$

Sending  $\varepsilon \rightarrow 0$ , we deduce  $(MC_\delta)$ .

In particular, it follows from Lemma 2.6 that for any  $v \in BV(\Omega)$  compactly supported,  $\delta \int_{\Omega} |v| dx \leq \int_{\Omega} \phi(-Dv)$ . We then deduce that if  $u$  satisfies  $(SH_\delta)$  also  $u \wedge T$  for any  $T > 0$ . Indeed,

$$\int_{\Omega} \phi(-D((u \wedge T) \wedge v)) \leq \int_{\Omega} \phi(-D(v \wedge T)) - \delta \int_{\Omega} ((v \wedge T) - u)^+ dx$$

On the other hand,

$$\int_{\Omega} \phi(-D(v \wedge T)) = \int_{\Omega} \phi(-Dv) - \int_{\Omega} \phi(-D(v - T)^+) \leq \int_{\Omega} \phi(-Dv) - \delta \int_{\Omega} (v - T)^+ dx,$$

and it follows

$$\int_{\Omega} \phi(-D((u \wedge T) \wedge v)) \leq \int_{\Omega} \phi(-Dv) - \delta \int_{\Omega} (v - (u \wedge T))^+ dx.$$

Then, the following characterization holds:

**Proposition A.2.** *Let  $u$  satisfy  $(SH_\delta)$ . Then there exists  $z \in L^\infty(\Omega; \{\phi^\circ \leq 1\})$  with  $\operatorname{div} z \geq \delta$ ,  $[z, Du^+] = |Du|$  in the sense of measures (equivalently,  $\int_{\Omega} u^+ \operatorname{div} z dx = \int \phi(-Du)$ ), and  $\operatorname{div} z = \delta$  on  $\{u = 0\}$ .*

**Corollary A.3.** *Let  $u$  satisfy  $(SH_\delta)$ . Then for any  $s > 0$ ,  $\{u^+ \geq s\}$  and  $\{u^+ > s\}$  satisfy  $(MC_\delta)$ .*

Here,  $u^+$  is as usual the superior approximate limit of  $u$  (defined  $\mathcal{H}^{d-1}$ -a.e.) and  $[z, Du^+]$  the pairing in the sense of Anzellotti [4].

*Proof.* For  $n \geq 1$ , let  $v_n$  be the unique minimizer of

$$\min_{v=0 \text{ on } \partial\Omega} \int_{\Omega} \phi(-Dv) + \int_{\Omega} \frac{n}{2} (v - u \wedge n)^2 - \delta v dx. \quad (12)$$

(the boundary condition is to be intended in a relaxed sense, adding a term  $\int_{\partial\Omega} |\operatorname{Tr} v| \phi(\nu_\Omega) d\mathcal{H}^{d-1}$  in the energy if the trace of  $v$  on the boundary does not vanish). The Euler-Lagrange equation for this problem asserts the existence of a field  $z_n \in L^\infty(\Omega; \{\phi^\circ \leq 1\})$  with bounded divergence such that

$$\operatorname{div} z_n + n v_n = n(u \wedge n) + \delta$$

a.e. in  $\Omega$ , and  $\int_{\Omega} \operatorname{div} z_n v_n dx = \int_{\Omega} \phi(-Dv_n)$ . On the other hand  $\int_{\Omega} \phi(-Dv_n) \leq \int_{\Omega} \phi(-D(u \wedge n)) \leq \int_{\Omega} \phi(-Du)$  and we have  $v_n \rightarrow u$ ,  $\int_{\Omega} \phi(-Dv_n) \rightarrow \int_{\Omega} \phi(-Du)$  as  $n \rightarrow \infty$ .

We show that  $v_n \leq u \wedge n$ . Indeed,  $\int_{\Omega} \phi(-D(v_n \wedge u \wedge n)) \leq \int_{\Omega} \phi(-Dv_n) - \delta \int_{\Omega} (v_n - (u \wedge n))^+ dx$ , while  $\int_{\Omega} (v_n - (u \wedge n))^2 dx \geq \int_{\Omega} ((v_n \wedge u \wedge n) - (u \wedge n))^2$ . Hence,

$$\begin{aligned} & \int_{\Omega} \phi(-D(v_n \wedge u \wedge n)) + \frac{n}{2} \int_{\Omega} ((v_n \wedge u \wedge n) - (u \wedge n))^2 - \delta \int_{\Omega} (v_n \wedge u \wedge n) dx \\ & \leq \int_{\Omega} \phi(-Dv_n) + \frac{n}{2} \int_{\Omega} (v_n - (u \wedge n))^2 dx - \delta \int_{\Omega} v_n dx \\ & \quad + \delta \int_{\Omega} (v_n - (v_n \wedge u \wedge n)) - (v_n - (u \wedge n))^+ dx \\ & = \int_{\Omega} \phi(-Dv_n) + \frac{n}{2} \int_{\Omega} (v_n - (u \wedge n))^2 dx - \delta \int_{\Omega} v_n dx \end{aligned}$$

and as the minimizer  $v_n$  of (12) is unique, we deduce  $v_n = v_n \wedge u \wedge n$ . In particular, it follows  $\operatorname{div} z_n \geq \delta$ . (Observe that since  $v_n \geq 0$ , one also has  $\operatorname{div} z_n \leq \delta + n(u \wedge n)$ , in particular  $\operatorname{div} z_n = \delta$  a.e. in  $\{u = 0\}$ . Also,  $\int_{\{u > 0\}} \operatorname{div} z_n \leq P_{\phi}(E^0)$ , hence  $(\operatorname{div} z_n)_{n \geq 1}$  are uniformly bounded Radon measures. Hence, up to a subsequence, we may assume that  $z_n \overset{*}{\rightharpoonup} z$  weakly-\* in  $L^{\infty}(\Omega; \{\phi^{\circ} \leq 1\})$  while  $\operatorname{div} z_n \overset{*}{\rightharpoonup} \operatorname{div} z$  weakly-\* in  $\mathcal{M}^1(\Omega; \mathbb{R}_+)$ , that is, as positive measures.

We now write

$$\int_{\Omega} \phi(-Dv_n) = \int_{\Omega} v_n \operatorname{div} z_n dx \leq \int_{\Omega} (u \wedge n) \operatorname{div} z_n dx = \int_0^n \int_{\{u \geq s\}} \operatorname{div} z_n dx ds,$$

hence, since  $v_n \rightarrow u$ ,

$$\int_{\Omega} \phi(-Du) \leq \limsup_{n \rightarrow \infty} \int_0^n \int_{\{u \geq s\}} \operatorname{div} z_n dx ds \leq \int_0^{\infty} \left( \limsup_{n \rightarrow \infty} \int_{\{u \geq s\}} \operatorname{div} z_n dx \right) ds$$

thanks to Fatou's lemma (and the fact  $\int_{\{u \geq s\}} \operatorname{div} z_n dx \leq P_{\phi}(E^0)$  are uniformly bounded).

We now study the limit of  $\int_{\{u \geq s\}} \operatorname{div} z_n dx$ , for  $s > 0$  given, assuming  $\{u > s\}$  has finite perimeter (this is true for a.e.  $s$ , and in fact one could independently check that  $s \mapsto P_{\phi}(\{u \geq s\})$  is nonincreasing).

We consider a set  $F = \{u \geq s\}$  with finite perimeter, and we recall  $D\chi_F$  is supported on the reduced boundary  $\partial^* F$ . By inner regularity, given  $\varepsilon > 0$ , we find a compact set  $K \subset \partial^* F$  with  $|D\chi_F|(\Omega \setminus K) < \varepsilon$ . We observe that  $\mathcal{H}^{d-1}$ -a.e. on  $K$  (which is countably rectifiable),  $\chi_F$  has an upper an lower trace, respectively  $\chi_F^+ = 1$  and  $\chi_F^- = 0$ . By the Meyers-Serrin Theorem (or its  $BV$  version, cf [3] or [2, Theorem 3.9]), there exists  $\varphi_k$  a sequence of functions in  $C^{\infty}(\Omega \setminus K; [0, 1])$  with  $\varphi_k \rightarrow \chi_F$  and

$$\int_0^1 \mathcal{H}^{d-1}(\{x \in \Omega \setminus K : \varphi_k(x) = k\}) = \int_{\Omega \setminus K} |\nabla \varphi_k| dx \rightarrow |D\chi_F|(\Omega \setminus K) < \varepsilon.$$

Moreover, by construction the traces of  $\varphi_k$  in  $K$  coincide with the traces of  $\chi_F$  (see [2, Section 3.8]).

We choose for each  $k$   $s_k \in [1/4, 3/4]$  such that  $\mathcal{H}^{d-1}(\partial\{\varphi_k \geq s_k\} \setminus K) \leq 2\varepsilon$ . We then define the closed (compact) sets  $F_k := \{\varphi_k \geq s_k\} \cup K$ . One has  $\int_{\Omega} |D\chi_F - D\chi_{F_k}| = \int_{\Omega \setminus K} |D\chi_F - D\chi_{F_k}| \leq 3\varepsilon$ . (This shows that  $F$  can be approximated strongly in  $BV$  norm by closed sets.)

Then, one has  $\limsup_n \int_{F_k} \operatorname{div} z_n dx \leq \int_{F_k} \operatorname{div} z$  as the measures are nonnegative and  $\chi_{F_k}$  is scs. On the other hand,  $|\int_{\Omega} \operatorname{div} z_n (\chi_F - \chi_{F_k}) dx| \leq 3\varepsilon$ , so that

$$\limsup_{n \rightarrow \infty} \int_F \operatorname{div} z_n dx \leq 3\varepsilon + \int_F \operatorname{div} z + \int (\chi_{F_k} - \chi_F) \operatorname{div} z \leq 3\varepsilon + \int_F \operatorname{div} z + \int (\chi_{F_k} - \chi_F)^+ \operatorname{div} z.$$

Notice that it is important to specify precisely the set  $F$  that we consider in the last inequality: We pick for  $F$  the complement  $F^+$  of its points of density zero, equivalently  $F^+ = \{u^+ \geq s\}$ . In that case,

up to a set of zero  $\mathcal{H}^{d-1}$ -measure,  $\chi_G := (\chi_{F_k} - \chi_{F^+})^+ = \chi_{F_k \setminus F^+}$  vanishes on  $K$  pointwise, moreover at  $\mathcal{H}^{d-1}$ -a.e.  $x \in K$ ,  $G$  has Lebesgue density 0. Hence  $G$  coincides  $\mathcal{H}^{d-1}$ -a.e. with a Caccioppoli set strictly inside  $\Omega$  and with  $\int_{\Omega} |D\chi_G| \leq 3\varepsilon$ . Thanks to [18, Thm 5.12.4] it follows  $\operatorname{div} z(G) \leq C\varepsilon$  for  $C$  depending only on  $\phi$  and the dimension (see also [16, Prop. 3.5]). As a consequence, since  $\varepsilon > 0$  is arbitrary,

$$\limsup_{n \rightarrow \infty} \int_{\{u \geq s\}} \operatorname{div} z_n dx \leq \int_{\{u^+ \geq s\}} \operatorname{div} z.$$

We obtain that

$$\int_{\Omega} \phi(-Du) \leq \int_{\Omega} u^+ \operatorname{div} z.$$

The reverse inequality also holds thanks to [16, Prop. 3.5, (3.9)], and can be proved by localizing and smoothing with kernels depending on the local orientation of the jump. We also deduce that, for a.e.  $s > 0$ ,

$$\int_{\{u^+ \geq s\}} \operatorname{div} z = P_{\phi}(\{u \geq s\}).$$

Note that  $s \mapsto \operatorname{div} z(\{u^+ \geq s\})$  is left-continuous, and  $s \mapsto \operatorname{div} z(\{u^+ > s\})$  is right-continuous, whereas  $s \mapsto P_{\phi}(\{u^+ \geq s\})$  is left-semicontinuous, which implies the thesis.  $\square$

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