Navier-Stokes equations with Different Physical Boundary Conditions: Existence and Regularity in L^p-Theory

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I. Introduction and motivation

Let us consider in a bounded cylindrical domain $\Omega \times (0,T)$ the following Navier-Stokes equations

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = \mathbf{0} \quad \text{in} \quad \Omega \times (0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega, \end{cases}$$

where the unknowns $\boldsymbol{u}$ and π stand respectively for the velocity field and the pressure of a fluid occupying a domain Ω. Given data are the external force $\boldsymbol{f}$ and the initial velocity $\boldsymbol{u}_0$. To study Navier-Stokes equations it is necessary to add appropriate boundary conditions.

- The Navier-Stokes equations are more studied, since the famous works of Jean Leray (1933, 1934), with a Dirichlet boundary condition which is not always realistic since it does not reflect the behavior of the fluid on or near the boundary.
- In the applications and in particular in the electromagnetism problems, it is possible to find problems where it is necessary to consider others boundary conditions. This boundary conditions (BC) are also used to simulate flows near rough walls, such as in aerodynamics, in weather forecasts and in hemodynamics, as well as perforated walls. BC involving the pressure, such as for example in cases of pipes, hydraulic gears using pomps, containers, etc ...

H. Navier has suggested in 1824 a type of boundary conditions based on a proportionality between the tangential components of the normal dynamic tensor and the velocity

$$\boldsymbol{u}\cdot\boldsymbol{n}=0, \qquad 2\,\nu\,[\mathbb{D}\boldsymbol{u}\cdot\boldsymbol{n}]_{\boldsymbol{\tau}}+\alpha\,\boldsymbol{u}_{\boldsymbol{\tau}}=0 \qquad \text{on } \Gamma\times(0,T) \quad (1)$$

where ν is the viscosity and $\alpha\geq 0$ is the coefficient of friction and

$$\mathbb{D}\boldsymbol{u}=\frac{1}{2}(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T)$$

denotes the deformation tensor associated to the velocity field $\boldsymbol{u}$. Observe that if α tends to infinity, we get formally

$$\boldsymbol{u}=\boldsymbol{0} \qquad \text{on} \qquad \Gamma.$$

The Navier boundary conditions are often used to simulate flows near rough walls as well as perforated walls.
Such slip boundary conditions are used in the Large Eddy Simulations (LES) of turbulent flows.

Using the following relation

$$[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = -\mathbf{curl}\,\boldsymbol{v} \times \boldsymbol{n} - 2\boldsymbol{\Lambda}\boldsymbol{v} \quad \text{on}\,\Gamma,$$

where Λ is an operator of order 0 defined by

$$\boldsymbol{\Lambda}\boldsymbol{v} = \sum_{k=1}^{2} \left(\boldsymbol{v}_{\boldsymbol{\tau}} \cdot \frac{\partial \boldsymbol{n}}{\partial s_k}\right)\boldsymbol{\tau}_k,$$

one can observe that in the case of a flat boundary and when $\alpha = 0$ the conditions (37) may be replaced by

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \ \text{ on } \qquad \Gamma \times (0,T). \qquad (2)$$

We call them Navier-type boundary conditions.

We are interested here, in a first part, by the stationary case for the Stokes equations:

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla \pi &= \boldsymbol{f} \qquad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 \qquad \text{in } \Omega, \end{aligned}$$

with one of the following boundary conditions:

$$\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} \quad \text{and} \quad \pi = \pi_0 \quad \text{on } \Gamma, \tag{3}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = g \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} \quad \text{on } \Gamma, \tag{4}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = g \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h} \quad \text{on } \Gamma. \tag{5}$$

We will study, in a second part, the case of stationary Navier-Stokes equations:

$$\text{Find } \boldsymbol{u}, \pi, \alpha_1, \ldots, \alpha_I,, \text{ with } \alpha_i \in \mathbb{R}$$

$$\begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on } \Gamma, \\ \pi = \pi_0 \text{ on } \Gamma_0 \text{ and } \pi = \pi_0 + \alpha_i & \text{on } \Gamma_i, \ i = 1, \ldots, I, \end{cases}$$

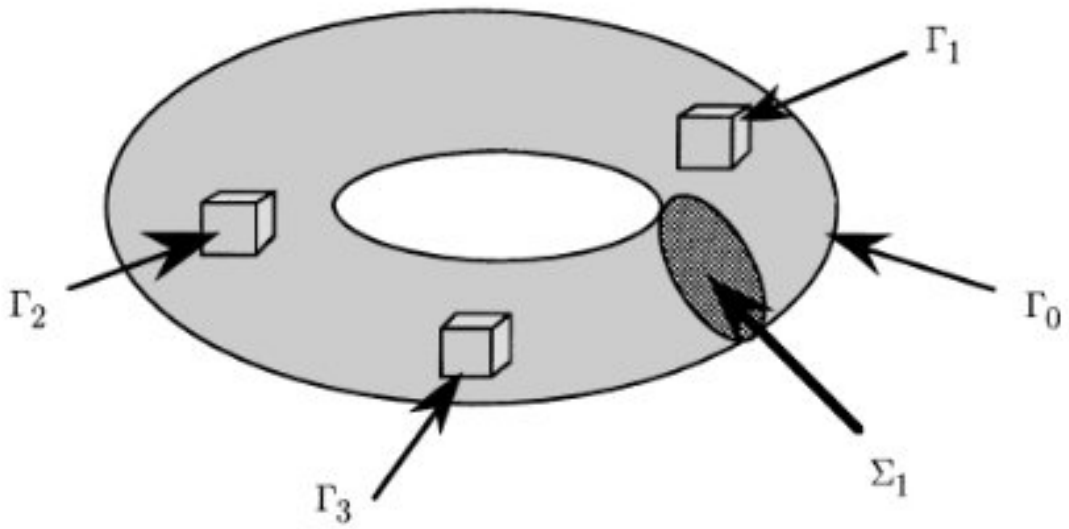
where we suppose that Ω is an open set possibly multiply connected sufficiently regular with a boundary Γ possibly non connected. We denote

$$\Gamma = \bigcup_{i=0}^{I} \Gamma_i, \qquad \Sigma = \bigcup_{j=1}^{J} \Sigma_j$$

with Γ_i the connected components of Γ and Σ_j a finite number of cuts such that

$$\Omega^\circ = \Omega \setminus \bigcup_{j=1}^{J} \Sigma_j$$

is simply connected.


Γ_1
Γ_2
Γ_0
Γ_3
Σ_1

Some works concerning the stationary Stokes and Navier-Stokes equations with non Dirichlet boundary conditions:

- Scadilov-Solonnikov, 1973
- Conca-Murat-Pironneau, 1994
- Neustupa-Penel, 2007
- Maruzic, 2007
- Beirao da Veiga, 2009
- Berselli, 2009
- Mitrea-Monniaux, 2009
- Amrouche-Seloula, 2011-2013
- Amrouche-Rejaiba, 2014

...

Considering for example the case of the Stokes equations with the homogeneous boundary conditions

$$(\mathcal{S}_T^0) \quad \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma. \end{cases}$$

Because these BC, we write

$$-\Delta \boldsymbol{u} = \mathbf{curl}\, \mathbf{curl} \boldsymbol{u} - \nabla \operatorname{div} \boldsymbol{u}$$

For the variational formulation, we will consider the following spaces:

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \ \operatorname{div} \boldsymbol{v} = 0, \ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \ \text{on } \Gamma\},$$

$$\boldsymbol{H}_0^2(\operatorname{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \ \operatorname{div} \boldsymbol{v} \in L^2(\Omega), \ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \ \text{on } \Gamma\}.$$

The Stokes problem $(\mathcal{S}_T^0)$ is then equivalent to

$$\begin{cases} \text{Find } \boldsymbol{u} \in \boldsymbol{V} \ \text{ such that} \\ \forall\, \boldsymbol{v} \in \boldsymbol{V}, \quad \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_\Omega. \end{cases}$$

We note that $[\boldsymbol{H}_0^2(\operatorname{div}, \Omega)]' \hookrightarrow \boldsymbol{H}^{-1}(\Omega)$.

Questions:

- The bilinear form is it coercive to apply the Lax-Milgram lemma ?

  We will see that if Ω is simply connected, we have:

$$\forall \boldsymbol{v} \in \boldsymbol{V}, \quad \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}.$$

- What happens if Ω is not simply connected ?
- Can we find generalized solution in $\boldsymbol{W}^{1,p}(\Omega)$ with $1 < p < \infty$?
- Can we find strong solution in $\boldsymbol{W}^{2,p}(\Omega)$ with $1 < p < \infty$?
- Can we find very weak solution in $\boldsymbol{L}^p(\Omega)$ with $1 < p < \infty$?

We need the following spaces, for $1 < p < \infty$:

$$\boldsymbol{H}^{\,p}(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathbf{curl}\ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\}\,,\ \ \boldsymbol{H}^{\,p}(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\ \boldsymbol{v} \in L^p(\Omega)\}$$

$$\boldsymbol{X}^{\,p}(\Omega) = \boldsymbol{H}^{\,p}(\mathbf{curl}, \Omega) \cap \boldsymbol{H}^{\,p}(\mathrm{div}, \Omega),$$

and their subspaces:

$$\boldsymbol{H}_0^{\,p}(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^{\,p}(\mathbf{curl}, \Omega);\ \ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ sur } \Gamma\}\,,$$

$$\boldsymbol{H}_0^{\,p}(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^{\,p}(\mathrm{div}, \Omega);\ \ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ sur } \Gamma\}\,,$$

$$\boldsymbol{X}_N^{\,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}^{\,p}(\Omega);\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ sur } \Gamma\}\,, \quad \boldsymbol{X}_T^{\,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}^{\,p}(\Omega); \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ sur } \Gamma\}$$

II. Sobolev's inequalities for vector fields

Lemma (Sobolev's inequalities)

i) Any function $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ satisfies:

$$\|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \leq C\Big(\|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i}|\Big). \tag{6}$$

ii) Any function $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ satisfies:

$$\|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \leq C\Big(\|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{j=1}^{J} |\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j}|\Big). \tag{7}$$

W. Von Wahl (1992) ($I = 0$, *i.e* Γ is connected and $J = 0$, *i.e* Ω is simply connected).

Idea of the proof of (6):

- We consider the following double layer potential:

$$T\,\lambda(\boldsymbol{x}) = -\frac{1}{2\pi}\int_{\Gamma}\lambda(\boldsymbol{\xi})\,\frac{\partial}{\partial \boldsymbol{n}}\Big(\frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}\Big)\,\mathrm{d}\sigma_{\boldsymbol{\xi}},$$

where $T:\ L^p(\Gamma)\longrightarrow W^{\,1,p}(\Gamma)$ is continuous and then compact from $L^p(\Gamma)$ into $L^p(\Gamma)$.
The Fredholm alternative implies that the dimension of $\mathrm{Ker}(Id+T)$ is finite. In fact, we have the following characterization of the kernel $\mathrm{Ker}\,(Id+T)$ which is spanned by the traces of the functions $\nabla\, q_i^N\cdot\boldsymbol{n}$ for $1\le i\le I$, where q_i^N is solution of:

$$(\mathcal{P}_N)\begin{cases} -\Delta q_i^N = 0 \quad \text{in } \Omega, \quad q_i^N = 0 \ \text{ on } \ \Gamma_0, \ \ q_i^N = \text{constant} \ \text{ on } \ \Gamma_k, \\ \big\langle \partial_n\, q_i^N,\, 1\big\rangle_{\Gamma_0} = -1 \ \text{ and } \ \ \big\langle \partial_n\, q_i^N,\, 1\big\rangle_{\Gamma_k} = \delta_{ik},\ 1\le k\le I. \end{cases}$$

Note that

$$\boldsymbol{K}_N^{\,p}(\Omega) = \{\boldsymbol{w}\in \boldsymbol{X}_N^{\,p}(\Omega);\ \mathbf{curl}\,\boldsymbol{w}=\mathbf{0},\ \mathrm{div}\,\boldsymbol{w}=0 \text{ in } \Omega\}$$

is spanned by $\nabla\, q_i^N$ for $1\le i\le I$

and $\boldsymbol{K}_N^{\,p}(\Omega)=\{0\}$ if $\Gamma=\Gamma_0$, that means that Γ is connected.

Thanks to the theorem of the open map we have for all $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$:

$$\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{L^p(\Gamma)} \leq C\Big(\|(Id+T)(\boldsymbol{v}\cdot\boldsymbol{n})\|_{L^p(\Gamma)} + \sum_{i=1}^{I} |\langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i}|\Big) \tag{8}$$

We write the following integral representation: for any $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ:

$$\begin{aligned}(Id+T)(\boldsymbol{v}\cdot\boldsymbol{n}) \quad &= \quad -\frac{1}{2\pi}\left(\mathbf{grad}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\cdot\boldsymbol{n} \\ &- \quad \frac{1}{2\pi}\left(\mathbf{rot}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{rot}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\cdot\boldsymbol{n}\end{aligned} \tag{9}$$

- Using the traces theorem, we prove that:

$$\begin{aligned}\|(Id+T)(\boldsymbol{v}\cdot\boldsymbol{n})\|_{\boldsymbol{L}^p(\Gamma)} \quad &\leq \quad C\Big(\Big\|\mathbf{grad}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1,p}(\Omega)}\\ &+ \quad \Big\|\,\mathbf{rot}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{rot}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathbf{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1,p}(\Omega)}\Big).\end{aligned}$$

- Using the Calderón-Zygmund inequalities, we get:

$$\|(Id+T)(\boldsymbol{v}\cdot\boldsymbol{n})\|_{L^p(\Gamma)} \leq C\left(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{rot}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\right).$$

- From (8), we deduce:

$$\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{L^p(\Gamma)} \leq C\Big(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{rot}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I}|\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}|\Big). \tag{10}$$

- Moreover, from (9) we have:

$$\begin{aligned}\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)} \leq & \Big(\|T(\boldsymbol{v}\cdot\boldsymbol{n})\|_{W^{1-\frac{1}{p},p}(\Gamma)}+\Big\|\mathbf{grad}\int_\Omega\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\\ & + \Big\|\mathbf{rot}\int_\Omega\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathbf{rot}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\Big),\end{aligned}$$

and by the traces theorem:

$$\begin{aligned}\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)} \leq & \; C\Big(\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{L^p(\Gamma)}+\Big\|\mathbf{grad}\int_\Omega\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1,p}(\Omega)}\\ & + \Big\|\mathbf{rot}\int_\Omega\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathbf{rot}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1,p}(\Omega)}\Big).\end{aligned}$$

- Using again the Calderón-Zygmund inequalities and (10), we obtain:

$$\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)} \leq C\Big(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)}+\|\mathbf{rot}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\sum_{i=1}^{I}|\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}|\Big). \qquad (11)$$

- As $\boldsymbol{v} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$, by the traces theorem, there exists $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ such that:

$$\boldsymbol{v} = \boldsymbol{u} \quad \text{sur } \Gamma \qquad \text{and} \quad \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{v}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}.$$

Because $\boldsymbol{v} \times \boldsymbol{n} = 0$ on Γ, we have $\boldsymbol{v}|_\Gamma = (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$. Using then (11), we get

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} &\leq C\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} \\ &\leq C\Big(\|\operatorname{div}\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{rot}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i}|\Big). \qquad (12)\end{aligned}$$

- As $\boldsymbol{u} - \boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)$, we know that for any function $\boldsymbol{w}$ belonging to $\boldsymbol{W}_0^{1,p}(\Omega)$, we have the following integral representaion integrale:

$$\boldsymbol{w} = -\mathbf{grad}\,\frac{1}{4\pi}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\operatorname{div}_{\boldsymbol{y}}\boldsymbol{w}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \mathbf{rot}\,\frac{1}{4\pi}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{rot}_{\boldsymbol{y}}\,\boldsymbol{w}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$

Thanks to the Calderón-Zygmund inequalities, we have

$$\|\nabla\,\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} \leq C\big(\|\operatorname{div}\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{rot}\,\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)}\big). \qquad (13)$$

- Applying (13) to $\boldsymbol{v} - \boldsymbol{u} \in \boldsymbol{W}_0^{1,p}(\Omega)$, we obtain:

$$\|\nabla\,(\boldsymbol{v}-\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)} \leq C\Big(\|\operatorname{div}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{rot}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{rot}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\Big).$$

Finally, we deduce the required estimate by using (12)

Idea of the proof of (7):

We use the same ideas with the following linear integral operator

$$R\,\boldsymbol{\lambda}(\boldsymbol{x}) = \frac{1}{2\pi}\int_{\Gamma} \mathbf{curl}\,\Big(\frac{\boldsymbol{\lambda}(\boldsymbol{\xi})}{|\boldsymbol{x}-\boldsymbol{\xi}|}\Big) \times \boldsymbol{n}\,\mathrm{d}\sigma_{\boldsymbol{\xi}}.$$

Because the embeddings $\boldsymbol{X}_N^p(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{X}_T^p(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$ are compact, by using Peetre-Tartar Theorem and the previous inequality, we deduce the following Poincaré's type inequalities:

Corollary (Poincaré's Inequalities)

i) We have $\boldsymbol{v} \in \boldsymbol{X}_N^p(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$ and for any $\boldsymbol{v} \in \boldsymbol{X}_N^p(\Omega)$

$$\| \boldsymbol{v} \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\Big(\|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i}| \Big).$$

ii) We have $\boldsymbol{v} \in \boldsymbol{X}_T^p(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$ and for any $\boldsymbol{v} \in \boldsymbol{X}_T^p(\Omega)$

$$\| \boldsymbol{v} \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\Big(\|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{j=1}^{J} |\langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j}| \Big).$$

Regularity properties

The previous results can be extended to the case of inhomogeneous boundary conditions :

- Let Ω of class $\mathcal{C}^{1,1}$ and

$$\boldsymbol{v} \in \boldsymbol{L}^p(\Omega), \quad \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^p(\Omega), \quad \operatorname{div} \boldsymbol{v} \in L^p(\Omega)$$

with

$$\boldsymbol{v} \cdot \boldsymbol{n} \in W^{1-\frac{1}{p},p}(\Gamma) \quad \text{or} \quad \boldsymbol{v} \times \boldsymbol{n} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma).$$

Then

$$\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega).$$

Moreover

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\Big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{L^p(\Omega)} + \|\boldsymbol{v}\cdot\boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)}\Big)$$

- Let Ω of class $\mathcal{C}^{2,1}$ and

$$\boldsymbol{v} \in \boldsymbol{L}^p(\Omega), \quad \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega), \quad \operatorname{div} \boldsymbol{v} \in W^{1,p}(\Omega)$$

with

$$\boldsymbol{v} \cdot \boldsymbol{n} \in W^{2-\frac{1}{p},p}(\Gamma) \quad \text{or} \quad \boldsymbol{v} \times \boldsymbol{n} \in \boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma).$$

Then

$$\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega),$$

with the corresponding estimates.

III. L^p-Theory for Vector Potentials

To study the Stokes problems, we need some results concerning the vector potentials and the inf-sup conditions.

Theorem (Tangential vector potentials)

$\boldsymbol{u} \in \boldsymbol{H}^{\,p}(\mathrm{div},\,\Omega)$ satisfies: $\mathrm{div}\,\boldsymbol{u} = 0 \quad \text{in}\,\Omega, \quad \langle \boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0, \quad 0 \le i \le I,$
iff there exists a vector potential $\boldsymbol{\psi}$ dans $\boldsymbol{X}^{\,p}(\Omega)$ such that

$$\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{\psi} = 0 \text{ in } \Omega, \tag{14}$$

$$\boldsymbol{\psi}\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma, \tag{15}$$

$$\langle \boldsymbol{\psi}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0,\ 1 \le j \le J. \tag{16}$$

This function is unique and moreover, we have

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{\,1,p}(\Omega)} \le C\|\boldsymbol{u}\,\|_{\boldsymbol{L}^p(\Omega)}.$$

Remark: If

$$\boldsymbol{u} \in \boldsymbol{L}^p(\Omega) \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0,$$

the condition

$$\langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I,$$

is necessary and sufficient for the existence of vector potential $\boldsymbol{\psi}$ satisfying (14) and (15). While the condition (16)

$$\langle \boldsymbol{\psi} \cdot \boldsymbol{n}, \quad 1 \rangle_{\Sigma_j} = 0,\ 1 \leq j \leq J$$

ensures the uniqueness of $\boldsymbol{\psi}$. When this last condition is not satisfied, the vector potential $\boldsymbol{\psi}$ exits up an additive vector function of the the following corresponding kernel:

$$\boldsymbol{K}_T^p(\Omega) = \{\boldsymbol{w} \in \boldsymbol{X}_T^p(\Omega);\ \mathbf{curl}\, \boldsymbol{w} = \mathbf{0},\ \operatorname{div} \boldsymbol{w} = 0 \text{ in } \Omega\}.$$

To characterize this kernel, let us introduce the following space

$$\Theta = \left\{ r \in H^1(\Omega^\circ);\ [\, r \,]_j = \text{constant},\ 1 \leq j \leq J \right\}.$$

For each $1 \leq j \leq J$, there exists a unique solution $q_j^T \in H^1(\Omega^\circ)$, up to an additive constant, satisfying

$$\forall r \in \Theta, \quad \int_{\Omega^\circ} \nabla q_j^T \cdot \nabla r = [r]_j$$

where $[\chi]_j$ is the jump of χ through Σ_j.

In fact, we have

$$\begin{cases} -\Delta q_j^T = 0 \qquad \text{in } \Omega^\circ, \\ \partial_n\, q_j^T = 0 \qquad \text{on } \Gamma, \\ \left[\, q_j^T \,\right]_k = \text{constant} \quad \text{and} \quad [\, \partial_n\, q_j^T \,]_k = 0,\ 1 \le k \le J, \\ \left\langle \partial_n\, q_j^T,\, 1 \right\rangle_{\Sigma_k} = \delta_{j\,k},\ 1 \le k \le J. \end{cases}$$

Let us consider any extension to $\boldsymbol{L}^2(\Omega)$ of ∇q_j^T which is denoted by $\widetilde{\mathbf{grad}}\, q_j^T$. It is easy to verify that

$$\operatorname{div} \widetilde{\mathbf{grad}}\, q_j^T = 0, \quad \mathbf{curl}\, \widetilde{\mathbf{grad}}\, q_j^T = \mathbf{0}$$

and thanks to the regularity results of vector fields, we deduce first that

$$\widetilde{\mathbf{grad}}\, q_j^T \in \boldsymbol{H}^1(\Omega),$$

and then by boostrap argument that

$$\widetilde{\mathbf{grad}}\, q_j^T \in \boldsymbol{W}^{1,p}(\Omega),$$

Moreover, $\boldsymbol{K}_T^p(\Omega)$ is spanned by

$$\widetilde{\mathbf{grad}}\, q_j^T, \qquad j = 1, \ldots, J.$$

Remark that $\boldsymbol{K}_T^p(\Omega) = \{0\}$ if $J = 0$ (i.e Ω is simply connected).

Theorem (Inf-Sup condition in Banach spaces)

Let X and M be two reflexive Banach spaces and X' and M' their dual spaces. Let a be the continuous bilinear form defined on $X \times M$, let $A \in \mathcal{L}(X;\ M')$ and $A' \in \mathcal{L}(M;\ X')$ be the operators defined by

$$\forall v \in X,\ \forall w \in M,\ a(v,w) = < Av, w > = < v, A'w >$$

and $V = \operatorname{Ker} A$. The following statements are equivalent:
i) There exist $\beta > 0$ such that

$$\inf_{\substack{w \in M \\ w \neq 0}} \sup_{\substack{v \in X \\ v \neq 0}} \frac{a(v,w)}{\|v\|_X \, \|w\|_M} \geq \beta. \tag{17}$$

ii) The operator $A :\ X/V \mapsto M'$ is an isomophism and $1/\beta$ is the continuity constant of A^{-1}.
iii) The operator $A' :\ M \mapsto X' \perp V$ is an isomophism and $1/\beta$ is the continuity constant of $(A')^{-1}$.

Remark

As consequence, if the Inf-Sup condition (17) is satisfied, then we have the following properties:

i) Because $A' : \ M \mapsto X' \perp V$ is an isomophism, then for any $f \in X'$, satisfying the compatibility condition

$$\forall v \in V, \ < f, v > = 0,$$

there exists a unique $w \in M$ such that

$$\forall v \in X, \ a(v, w) = < f, v > \quad \text{and} \quad \|w\|_M \leq \frac{1}{\beta} \|f\|_{X'}. \tag{18}$$

ii) Because $A : \ X/V \mapsto M'$ is an isomophism, then for any $g \in M', \ \exists v \in X$, unique up an additive element of V, such that:

$$\forall w \in M, \ a(v, w) = < g, w > \quad \text{and} \quad \|v\|_{X/V} \leq \frac{1}{\beta} \|g\|_{M'}$$

.

We define the space

$$\boldsymbol{V}_T^p(\Omega) = \{\boldsymbol{w} \in \boldsymbol{X}_T^p(\Omega);\ \mathrm{div}\,\boldsymbol{w} = 0 \text{ in } \Omega \text{ and } \langle \boldsymbol{w}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0,\ 1 \le j \le J\},$$

which is a Banach space for the norm $\|\cdot\|_{\boldsymbol{X}^p(\Omega)}$.

Using some classical Helmholtz decomposition and the previous tangential vector potential, we can deduce:

Lemma (Inf Sup Condition)

The following Inf-Sup Condition holds: there exists a constant $\beta > 0$, such that

$$\inf_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}_T^{p'}(\Omega)\\ \boldsymbol{\varphi}\neq\boldsymbol{0}}} \sup_{\substack{\boldsymbol{\xi}\in \boldsymbol{V}_T^{p}(\Omega)\\ \boldsymbol{\xi}\neq\boldsymbol{0}}} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\xi}\|_{\boldsymbol{X}_T^p(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_T^{p'}(\Omega)}} \ge \beta. \tag{19}$$

Theorem (Normal vector potentials)

$\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ satisfies:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \quad \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Sigma_j} = 0, \quad 0 \leq j \leq J,$$

iff there exists a vector potential $\boldsymbol{\psi}$ in $\boldsymbol{X}^p(\Omega)$ such that

$$\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi} \quad \text{and} \quad \operatorname{div} \boldsymbol{\psi} = 0 \text{ in } \Omega, \tag{20}$$

$$\boldsymbol{\psi} \times \boldsymbol{n} = 0 \qquad \text{on } \Gamma, \tag{21}$$

$$\langle \boldsymbol{\psi} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \ 1 \leq i \leq I. \tag{22}$$

This function is unique and moreover, we have:

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Remark : For

$$\boldsymbol{u} \in \boldsymbol{L}^p(\Omega), \quad \text{with} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in}\, \Omega \quad \text{and} \quad \boldsymbol{u} \cdot \boldsymbol{n} = 0 \ \text{on}\ \Gamma,$$

the condition

$$\langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_j} = 0, \quad \text{for}\, 1 \leq j \leq J,$$

is necessary and sufficient for the existence of the vector potential $\boldsymbol{\psi}$ satisfying (20) and (21). The condition

$$\langle \boldsymbol{\psi} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \quad \text{for}\, 0 \leq i \leq I$$

ensures the uniqueness of $\boldsymbol{\psi}$.

Idea of the proof

- We use the Inf-Sup condition (19) to solve the problem:

  Find $\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$ such that:

$$(\mathcal{P}) \quad \begin{cases} -\Delta\,\boldsymbol{\xi} = 0 \quad \text{and} \quad \operatorname{div}\boldsymbol{\xi} = 0 & \text{in } \Omega, \\ \boldsymbol{\xi}\cdot\boldsymbol{n} = 0, \ \mathbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{\psi}_0\times\boldsymbol{n} & \text{on } \Gamma, \\ \langle \boldsymbol{\xi}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0, \end{cases}$$

  where $\boldsymbol{\psi}_0$ is the tangential vector potential.

- The required vector potential is given by:

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 - \mathbf{curl}\,\boldsymbol{\xi} - \sum_{i=1}^{I} \langle (\boldsymbol{\psi}_0 - \mathbf{curl}\,\boldsymbol{\xi})\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i}\, \mathbf{grad}\, q_i^N$$

We define now the space

$$\boldsymbol{V}_N^p(\Omega) = \{\boldsymbol{w} \in \boldsymbol{X}_N^p(\Omega);\ \mathrm{div}\,\boldsymbol{w} = 0 \text{ in } \Omega \text{ and } \langle \boldsymbol{w}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0,\ 1 \le i \le I\},$$

which is a Banach space for the norm $\|\cdot\|_{\boldsymbol{X}^p(\Omega)}$.

Using a Helmholtz decomposition and the previous normal vector potential, we can deduce:

Lemma (Inf Sup Condition)

The following Inf-Sup Condition holds: there exists a constant $\beta > 0$, such that

$$\inf_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}_N^{p'}(\Omega)\\ \boldsymbol{\varphi}\neq\boldsymbol{0}}} \sup_{\substack{\boldsymbol{\xi}\in \boldsymbol{V}_N^{p}(\Omega)\\ \boldsymbol{\xi}\neq\boldsymbol{0}}} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\xi}\|_{\boldsymbol{X}_N^p(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N^{p'}(\Omega)}} \ge \beta. \tag{23}$$

IV. Stokes stationary problem with normal boundary conditions

$$(\mathcal{S}_T) \quad \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = g, \ \ \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Sigma_j} = 0, & 1 \leq j \leq J. \end{cases}$$

- We solve the pressure π : because $\operatorname{div}_\Gamma(\boldsymbol{z} \times \boldsymbol{n}) = \mathbf{curl}\, \boldsymbol{z} \cdot \boldsymbol{n}$, we have

  $\Delta \pi = \operatorname{div} \boldsymbol{f}$ in Ω, $\quad \frac{\partial \pi}{\partial n} = \boldsymbol{f} \cdot \boldsymbol{n} + \operatorname{div}_\Gamma(\boldsymbol{h} \times \boldsymbol{n})$ on Γ

- We set $\boldsymbol{F} = \boldsymbol{f} - \nabla \pi$. Then, $\boldsymbol{u}$ is a solution of the following elliptic problem:

$$(E_T) \quad \begin{cases} -\Delta \boldsymbol{u} = \boldsymbol{F} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = g, \ \ \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Sigma_j} = 0, & 1 \leq j \leq J. \end{cases}$$

- To solve the Stokes problem $(\mathcal{S}_T)$, we are reduced to solve the problem (E_T).

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IV. Stokes stationary problem with normal boundary conditions

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- To solve the Stokes problem $(\mathcal{S}_T)$, we are reduced to solve the problem (E_T).

- Contrarly to the Stokes problem with Dirichlet boundary condition, it appears that when

$$\operatorname{div} \boldsymbol{f} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{f} \cdot \boldsymbol{n} = \operatorname{div}_\Gamma(\boldsymbol{h} \times \boldsymbol{n}) = 0 \quad \text{on } \Gamma,$$

  the pressure π can be constant.

- In the case of the Stokes problem with Dirichlet, even if

$$\operatorname{div} \boldsymbol{f} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{f} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma,$$

  we have $\Delta \boldsymbol{u} \cdot \boldsymbol{n} \neq 0$ on Γ and then π is non constant.

Study of the elliptic problem :

$$(E_T) \quad \begin{cases} -\Delta \boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = g, \ \ \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Sigma_j} = 0, \qquad 1 \leq j \leq J. \end{cases}$$

Proposition (Weak and Strong solutions of (E_T))

i) Let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ with $\operatorname{div} \boldsymbol{f} = 0$ in Ω, $g \in W^{1-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ satisfying the following compatibility condition:

$$\forall \boldsymbol{v} \in \boldsymbol{K}_T^{p'}(\Omega), \quad \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{v} \rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)} = 0,$$

$$\boldsymbol{f} \cdot \boldsymbol{n} + \operatorname{div}_\Gamma (\boldsymbol{h} \times \boldsymbol{n}) = 0 \ \text{ on } \Gamma,$$

$$\int_\Gamma g \, \mathrm{d}\boldsymbol{\sigma} = 0.$$

Then, the problem (E_T) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ satisying the estimate:

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C \Big(\| \boldsymbol{f} \|_{\boldsymbol{L}^p(\Omega)} + \| g \|_{W^{1-1/p,p}(\Gamma)} + \| \boldsymbol{h} \times \boldsymbol{n} \|_{\boldsymbol{W}^{-1/p,p}(\Gamma)} \Big).$$

ii) If $g \in W^{2-1/p,p}(\Gamma)$ and $\boldsymbol{h} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$, then the solution $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$.

Theorem (Weak and strong solution of $(\mathcal{S}_T)$)

i) Let $\boldsymbol{f}$, g, $\boldsymbol{h}$ with:

$$\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))', \quad g \in W^{1-\frac{1}{p},p}(\Gamma), \quad \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma),$$

verifying the compatibility conditions: for any $\boldsymbol{v}$ in $\boldsymbol{K}_T^{p'}(\Omega)$:

$$\begin{aligned} \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{(\boldsymbol{H}_0^{p'}(\mathbf{div}, \Omega))' \times \boldsymbol{H}_0^{p'}(\mathbf{div}, \Omega)} \quad + \quad & \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{v} \rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)} = 0. \\ \int_\Gamma g \, \mathrm{d}\boldsymbol{\sigma} \quad = \quad & 0. \end{aligned}$$

Then, the Stokes problem $(\mathcal{S}_T)$ has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq \left(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))'} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right).$$

ii) If $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $g \in W^{2-1/p,p}(\Gamma)$, $\boldsymbol{h} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$ then the solution $(\boldsymbol{u}, \pi)$ of the problem $(\mathcal{S}_T)$ given previously belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.

Remark: We can consider the case of the problem $(\mathcal{S}_T)$ with $\mathrm{div}\, \boldsymbol{u} = \chi$ in Ω,

Very weak solution for $(\mathcal{S}_T)$

Let $\boldsymbol{f}$, χ, g, and $\boldsymbol{h}$ with

$$\boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))',\ \chi \in L^p(\Omega),\ g \in W^{-1/p,p}(\Gamma),\ \boldsymbol{h} \in \boldsymbol{W}^{-1-1/p,p}(\Gamma),$$

with $\boldsymbol{T}^{p'}(\Omega) = \left\{\boldsymbol{\varphi} \in \boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega);\ \mathrm{div}\,\boldsymbol{\varphi} \in W_0^{1,p'}(\Omega)\right\}$ and satisfying the compatibility conditions:

$$\forall\, \boldsymbol{\varphi} \in \boldsymbol{K}_T^{p'}(\Omega), \quad \langle \boldsymbol{f},\ \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times \boldsymbol{T}^{p'}(\Omega)} + \langle \boldsymbol{h}\times\boldsymbol{n},\ \boldsymbol{\varphi}\rangle_\Gamma = 0.$$

$$\int_\Omega \chi \,\mathrm{d}\,\boldsymbol{x} = \langle g,\ 1\rangle_\Gamma.$$

Then, the Stokes problem $(\mathcal{S}_T)$ has exactly one solution $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ and $\pi \in W^{-1,p}(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C > 0$ depending only on p and Ω such that:

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C\Big(\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\quad\quad\quad\quad}\Big)$$

Helmholtz Decomposition for vector fields in $L^p(\Omega)$

For any vector field $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$, we have the first following decomposition:

$$\boldsymbol{v} = \boldsymbol{z} + \nabla \chi + \mathbf{curl}\, \boldsymbol{u},$$

- $\boldsymbol{z} \in \boldsymbol{K}_N^p(\Omega)$ is unique,
- $\chi \in W_0^{1,p}(\Omega)$ is unique,
- $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ is the unique solution, up to an additive element of the kernel $\boldsymbol{K}_T^p(\Omega)$, of the problem :

$$\begin{cases} -\Delta \boldsymbol{u} = \mathbf{curl}\, \boldsymbol{v} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad (\mathbf{curl}\, \boldsymbol{u} - \boldsymbol{v}) \times \boldsymbol{n} = 0 & \text{on } \Gamma. \end{cases}$$

Helmholtz Decomposition for vector fields in $\boldsymbol{L}^p(\Omega)$

For any vector field $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$, we have the second following decomposition:

$$\boldsymbol{v} = \boldsymbol{z} + \nabla \chi + \mathbf{curl}\, \boldsymbol{u},$$

- $\boldsymbol{z} \in \boldsymbol{K}^p_T(\Omega)$ is unique,
- $\chi \in W^{1,p}(\Omega)$ is unique up an additive constant,
- $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ is the unique solution, up to an additive element of the kernel $\boldsymbol{K}^p_N(\Omega)$, of the problem :

$$\begin{cases} -\Delta\, \boldsymbol{u} = \mathbf{curl}\, \boldsymbol{v} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = 0, & \text{on } \Gamma. \end{cases}$$

V. Stokes Equations with Pressure Boundary Condition

$$(\mathcal{S}_N) \quad \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n}, \quad \pi = \pi_0 & \text{on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \end{cases}$$

- The pressure can be found independently of the velocity as a solution of the Dirichlet problem:

$$\Delta \pi = \operatorname{div} \boldsymbol{f} \ \text{ in } \ \Omega, \qquad \pi = \pi_0 \ \text{ on } \ \Gamma$$

- The velocity is a solution of the following system:

$$(E_N) \quad \begin{cases} -\Delta \boldsymbol{u} = \boldsymbol{F} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \quad \text{for all } 1 \leq i \leq I, \end{cases}$$

where $\boldsymbol{F} = \boldsymbol{f} - \nabla \pi$

Remarks:

- The condition $\operatorname{div} \boldsymbol{F} = 0$ in Ω is necessary to solve (E_N).
- The condition $\operatorname{div} \boldsymbol{u} = 0$ in $\Omega \Longleftrightarrow \operatorname{div} \boldsymbol{u} = 0$ on Γ on the one hand. On the other hand, since

$$\operatorname{div} \boldsymbol{u} = \operatorname{div}_\Gamma \boldsymbol{u_\tau} + K \boldsymbol{u} \cdot \boldsymbol{n} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} \quad \text{sur } \Gamma,$$

where K denotes the mean curvature of Γ, the condition $\operatorname{div} \boldsymbol{u} = 0$ on Γ is itself equivalent, if $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ, to the Fourier-Robin condition:

$$K \boldsymbol{u} \cdot \boldsymbol{n} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma.$$

That means that the problem (E_N) is equivalent to the following:

$$\begin{cases} -\Delta \boldsymbol{u} = \boldsymbol{F} & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma \\ K \boldsymbol{u} \cdot \boldsymbol{n} + \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} = 0 & \text{on } \Gamma. \end{cases}$$

Using the Inf-Sup condition, we deduce the following theorem

Theorem (Weak and Strong solutions for $(\mathcal{S}_N)$)

i) Let $\boldsymbol{f}$, $\boldsymbol{g}$, π_0 with

$$\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega))', \quad \boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma), \quad \pi_0 \in W^{1-1/p,p}(\Gamma),$$

satisfying the compatibility condition:

$$\forall \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \qquad \langle \boldsymbol{f}\,,\, \boldsymbol{v}\,\rangle_\Omega - \int_\Gamma \pi_0\, \boldsymbol{v}\cdot\boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0, \tag{24}$$

with $$\langle \cdot,\cdot\rangle_\Omega = \langle \cdot,\cdot\rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'\times \boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)}.$$

Then, the Stokes problem $(\mathcal{S}_N)$ has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,p}(\Omega)$ and satisfies the estimate:

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \quad &\leq \quad C\big(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega))'} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \\ &+ \quad \|\pi_0\|_{W^{1-1/p,p}(\Gamma)}\big). \end{aligned}$$

ii) If $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$, then $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$.

Question:

What happens if the previous compatibility condition is not satisfied?

Variant of the system $(\mathcal{S}_N)$:

Find $(\boldsymbol{u},\, \pi,\, \boldsymbol{c})$ such that:

$$(\mathcal{S}'_N) \quad \begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \;\text{ and }\; \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on } \Gamma, \\ \pi = \pi_0 \;\text{ on } \Gamma_0 \quad \text{and} \quad \pi = \pi_0 + c_i & \text{on } \Gamma_i,\;\; 1 \leq i \leq I \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I, & \end{cases}$$

where $\boldsymbol{c} = (c_i)_{1 \leq i \leq I}$.

Theorem (Weak and Strong solutions for $(\mathcal{S}'_N)$)

Let $\boldsymbol{f}$, $\boldsymbol{g}$ and π_0 such that:

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]', \quad \boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma), \quad \pi_0 \in W^{1-1/p,p}(\Gamma).$$

Then, the problem $(\mathcal{S}'_N)$ has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, $\pi \in W^{1,p}(\Omega)$ and constants $c_1, \ldots, c_I$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}} + \|\pi_0\|_{W^{1-1/p,p}}\big),$$

and where $c_1, \ldots, c_I$ are given by

$$c_i = \langle \boldsymbol{f}, \nabla q_i^N \rangle_\Omega - \langle \pi_0, \nabla q_i^N \cdot \boldsymbol{n} \rangle_\Gamma. \tag{25}$$

In particular, if $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$, then $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$.

Remark :

- Observe that if we suppose that the compatibility condition (24) is verified, we have that: $c_i = 0$ for all $i = 1, \ldots, I$. Then, we have reduced to solve the problem $(\mathcal{S}'_N)$ without the constant c_i and $(\mathcal{S}'_N)$ is anything other then $(\mathcal{S}_N)$.

The assumption on $\boldsymbol{f}$ in the previous theorem can be weakened by considering the space defined for all $1 < r,\, p < \infty$:

$$\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{L}^r(\Omega);\;\; \mathbf{curl}\,\boldsymbol{\varphi} \in \boldsymbol{L}^p(\Omega),\;\; \boldsymbol{\varphi}\times\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\}.$$

which is a Banach space for the norm

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega)} = \|\boldsymbol{\varphi}\|_{\boldsymbol{L}^r(\Omega)} + \|\mathbf{curl}\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^p(\Omega)}.$$

We can prove that

$$\boldsymbol{\mathcal{D}}(\Omega) \quad \text{is dense in} \quad \boldsymbol{H}_0^{r',\,p'}(\mathbf{curl},\,\Omega)$$

and its dual space can be characterized as:

$$[\boldsymbol{H}_0^{r',\,p'}(\mathbf{curl},\,\Omega)]' = \{\boldsymbol{F} + \mathbf{curl}\,\boldsymbol{\psi};\;\; \boldsymbol{F} \in \boldsymbol{L}^r(\Omega),\;\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)\}.$$

Theorem (Second Version for Weak solutions for $(\mathcal{S}'_N)$)

Let $\boldsymbol{f}$, $\boldsymbol{g}$ and π_0 such that

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]', \quad \boldsymbol{g} \times \boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma), \quad \pi_0 \in W^{1-1/r,r}(\Gamma),$$

with

$$r \leq p \quad \text{and} \quad \frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}.$$

Then, the problem $(\mathcal{S}'_N)$ has a unique solution

$$\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega), \quad \pi \in W^{1,r}(\Omega)$$

and constants $c_1, \ldots, c_I$ satisfying the estimate:

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \quad + \quad & \|\pi\|_{W^{1,r}(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} \\ + \quad & \|\boldsymbol{g} \times \boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\big), \end{aligned}$$

and $c_1, \ldots, c_I$ are given by (25), where we replace the duality brackets on Ω by

$$\langle \cdot, \cdot \rangle_\Omega = \langle \cdot, \cdot \rangle_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]' \times \boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)}.$$

Theorem (Very weak solutions for $(\mathcal{S}_N)$)

Let $\boldsymbol{f}$, $\boldsymbol{g}$, and π_0 with

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]', \ \boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}(\Gamma), \ \pi_0 \in W^{-1/p,p}(\Gamma),$$

and satisfying the compatibility conditions (24). Then, the Stokes problem $(\mathcal{S}_N)$ has exactly one solution

$$\boldsymbol{u} \in \boldsymbol{L}^p(\Omega) \quad \text{and} \quad \pi \in L^p(\Omega)/\mathbb{R}.$$

Moreover, there exists a constant $C > 0$ depending only on p and Ω such that:

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)} + \\ + \|\pi_0\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}\Big).$$

Remark. To study the case of Navier boundary conditions:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h},$$

it suffices to observe that

$$[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = -\mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n} - 2\boldsymbol{\Lambda}\boldsymbol{v} \quad \text{on}\, \Gamma,$$

where Λ is an operator of order 0 defined by

$$\boldsymbol{\Lambda}\boldsymbol{w} = \sum_{k=1}^{2} \left(\boldsymbol{w}_{\boldsymbol{\tau}} \cdot \frac{\partial \boldsymbol{n}}{\partial s_k}\right) \boldsymbol{\tau}_k,$$

and to use the results obtained with the Navier type boundary conditions.

VI. Oseen and Navier-Stokes Problem with Pressure Boundary Condition

We are interested to study the following problem:
Find $\boldsymbol{u}$, q and $\boldsymbol{\alpha} \in \mathbb{R}^I$ satisfying:

$$(\mathcal{NS}) \qquad \begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla q = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \Gamma, \\ q = q_0 \ \text{ on } \Gamma_0 \text{ and } q = q_0 + \alpha_i & \text{on } \Gamma_i, \ i = 1, \ldots, I, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}\boldsymbol{\sigma} = 0, \ i = 1, \ldots, I, \end{cases}$$

- Note that $\boldsymbol{\alpha}$ is a supplementary unknown Stokes which depends in fact on $\boldsymbol{u}$
- If we take $\chi = 0$ and $\boldsymbol{g} = \boldsymbol{0}$, unlike the Navier-Stokes problem with Dirichlet boundary conditions de Dirichlet, the property: $\int_\Omega (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} = 0$ does not hold.
- But, we have

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} + \frac{1}{2} \nabla |\boldsymbol{u}|^2$$

We rewrite then $(\mathcal{NS})$ under the following form:

$$(\mathcal{NS}_N) \qquad \begin{cases} -\Delta \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \Gamma, \\ \pi = \pi_0 \text{ sur } \Gamma_0 \text{ et } \pi = \pi_0 + \alpha_i & \text{on } \Gamma_i,\ i = 1, \ldots, I, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, & \end{cases}$$

Remarks.

- We can search directely weak solutions $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and $\pi \in L^2(\Omega)$ of the system $(\mathcal{NS}_N)$ by using a fixed point method.
- We can then obtain solutions $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ for $p > 2$ thanks to the Stokes problem theory.
- The case $p < 2$ to study the $(\mathcal{NS}_N)$ system is more complicated.
- For this reason, we will study the Oseen problem $(\mathcal{OS}_N)$.

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- We can search directely weak solutions $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and $\pi \in L^2(\Omega)$ of the system $(\mathcal{NS}_N)$ by using a fixed point method.
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- The case $p < 2$ to study the $(\mathcal{NS}_N)$ system is more complicated.
- For this reason, we will study the Oseen problem $(\mathcal{OS}_N)$.

Study of problem $(\mathcal{OS}_N)$

$$(\mathcal{OS}_N) \qquad \begin{cases} -\Delta \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \pi = \pi_0 + c_i & \text{on } \Gamma_i,\ 0 = 1, \ldots, I, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, \end{cases}$$

where we have take $\chi = 0$ and $\boldsymbol{g} = \boldsymbol{0}$. We suppose also that

$$\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$$

We introduce the following Hilbert space:

$$\boldsymbol{V}_N = \Big\{ \boldsymbol{v} \in H^1(\Omega);\ \operatorname{div} \boldsymbol{v} = 0 \ \text{ in } \Omega,\ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma$$
$$\text{and } \int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} = 0,\ 1 \le i \le I \Big\}$$

and recall that

$$\boldsymbol{v} \mapsto \Big(\int_\Omega |\mathbf{curl}\, \boldsymbol{v}|^2 \Big)^{1/2}$$

is a norm on $\boldsymbol{V}_N$ equivalent to the full norm of $\boldsymbol{H}^1(\Omega)$.

Before establishing the result of existence of a weak solution for the problem $(\mathcal{OS}_N)$, we will see in what functional space it is reasonable to take π_0 and to find the pressure π appearing in $(\mathcal{OS}_N)$, knowing that we are first interesting to velocity fields in

$$\boldsymbol{u} \in \boldsymbol{H}^{\,1}(\Omega) \quad \text{with} \quad \boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega).$$

With a such vector $\boldsymbol{u}$, we have

$$\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} \in \boldsymbol{L}^{\,6/5}(\Omega) \hookrightarrow \boldsymbol{H}^{\,-1}(\Omega).$$

Since $\Delta\, \boldsymbol{u} \in \boldsymbol{H}^{\,-1}(\Omega)$, we deduce from the first equation in $(\mathcal{OS}_N)$ that $\nabla\, \pi \in \boldsymbol{H}^{\,-1}(\Omega)$. Then the pressure π belongs to $L^2(\Omega)$. Furthermore,

$$-\Delta\, \pi = \operatorname{div} \boldsymbol{f} - \operatorname{div} (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \qquad \text{in } \Omega,$$

so that $\Delta\, \pi \in \boldsymbol{W}^{\,-1,6/5}(\Omega)$ and the trace of π on Γ belongs to $H^{\,-1/2}(\Gamma)$ so that we must assume that

$$\pi_0 \in H^{\,-1/2}(\Gamma).$$

Theorem

Let

$$\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega), \quad \pi_0 \in H^{-1/2}(\Gamma) \quad \text{and} \quad \boldsymbol{a} \in \boldsymbol{\mathcal{D}}'(\Omega) \quad \text{such that} \quad \mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega).$$

Then, the problem:

$$\text{Find } (\boldsymbol{u}, \pi, \boldsymbol{c}) \in \boldsymbol{V}_N \times L^2(\Omega) \times \mathbb{R}^{I+1} \text{ satisfying } (\mathcal{OS}_N) \text{ with } \langle \pi, 1 \rangle_\Gamma = 0 \tag{26}$$

is equivalent to the problem: Find $\boldsymbol{u} \in \boldsymbol{V}_N$ *such that*

$$\forall\, \boldsymbol{v} \in \boldsymbol{V}_N, \quad \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} + \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\,\boldsymbol{x} - \langle \pi_0, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_\Gamma \tag{27}$$

and find constants $c_0, \ldots, c_I$ *satisfying* $\sum_{i=0}^{I} c_i \operatorname{mes} \Gamma_i + \langle \pi_0, 1 \rangle_\Gamma = 0$ *and such that for any* $i = 1, \ldots, I$*:*

$$c_i - c_0 = \int_\Omega \boldsymbol{f} \cdot \nabla\, q_i^N\, \mathrm{d}\boldsymbol{x} - \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \nabla\, q_i^N\, \mathrm{d}\boldsymbol{x} - \langle \pi_0, \nabla\, q_i^N \cdot \boldsymbol{n} \rangle_\Gamma. \tag{28}$$

Using the Lax Milgram theorem and some regularity result of the Laplacian, we prove the following theorem.

Theorem

Let $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$, $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$ *and* $\pi_0 \in H^{-1/2}(\Gamma)$, *then the problem* (57) *has a unique solution* $(\boldsymbol{u}, \pi, \boldsymbol{c}) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^{I+1}$ *with* $\langle \pi, 1 \rangle_\Gamma = 0$ *and we have the following estimates:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)} + \|\pi_0\|_{H^{-1/2}(\Gamma)}\big), \tag{29}$$

$$\|\pi\|_{L^2(\Omega)} \leq C\big(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big)\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)} + \|\pi_0\|_{H^{-1/2}(\Gamma)}\big), \tag{30}$$

where $\boldsymbol{c} = (c_0, \ldots, c_I)$. *Moreover, if* $\pi_0 \in W^{1/6,6/5}(\Gamma)$ *and* Ω *is* $\mathcal{C}^{2,1}$, *then* $\boldsymbol{u} \in \boldsymbol{W}^{2,6/5}(\Omega)$ *and* $\pi \in W^{1,6/5}(\Omega)$.

Remark

Even if the pressure π *change in* $\pi - c_0$, *the system* $(\mathcal{OS}_N)$ *is equivalent to the following type-Oseen problem:*

$$\begin{cases} -\Delta \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \pi = \pi_0 \quad \text{on } \Gamma_0, \quad \text{and} \quad \pi = \pi_0 + \alpha_i,\ i = 1, \ldots, I, & \text{on } \Gamma_i, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, \end{cases}$$

where the unknowns constants satisfy for any $i = 1, \ldots, I$:

$$\alpha_i = \int_\Omega \boldsymbol{f} \cdot \nabla\, q_i^N \, \mathrm{d}\boldsymbol{x} - \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \nabla\, q_i^N \, \mathrm{d}\boldsymbol{x} - \langle \pi_0,\, \nabla\, q_i^N \cdot \boldsymbol{n} \rangle_\Gamma .$$

But, it is clear that the new pressure π *does not satisfy the condition* $\langle \pi,\, 1 \rangle_\Gamma = 0$.

Remark

If we suppose that

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega)]', \quad \mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega) \quad \text{and} \quad \pi_0 \in H^{-1/2}(\Gamma),$$

then the problems (26) *and* (27)-(28) *are again equivalent, with the difference that we use here the duality brackets between* $[\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega)]'$ *and* $\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega)$ *in place of the integral on* Ω *in the right hand side of* (27) *and the density of* $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ *in the space*

$$\boldsymbol{\mathcal{M}} = \left\{(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1_\sigma(\Omega) \times L^2(\Omega);\ -\Delta\, \boldsymbol{u} + \nabla\, \pi \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega)]'\right\}.$$

It is easy now to extend the previous theorem to the case where

- $\boldsymbol{f} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega)]'$,
- the divergence operator does not vanish

Theorem

Let $\boldsymbol{f} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega)]'$, $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$, $\chi \in W^{1,6/5}(\Omega)$, $\pi_0 \in H^{-1/2}(\Gamma)$ *and* $\boldsymbol{g} \in \boldsymbol{H}^{1/2}(\Gamma)$. *Then the problem*

$$\begin{cases} -\Delta \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on } \Gamma, \\ \pi = \pi_0 \quad \text{on } \Gamma_0, \quad \text{and} \quad \pi = \pi_0 + \alpha_i,\ i = 1, \ldots, I & \text{on } \Gamma_i, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, \end{cases} \tag{31}$$

has a unique solution $(\boldsymbol{u}, \pi, \boldsymbol{\alpha}) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^I$ *verifying the estimate:*

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \quad &\leq \quad C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)} + \big(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big) \times \\ &\times \quad \big(\|\chi\|_{W^{1,6/5}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{1/2}(\Gamma)}\big)\Big), \end{aligned}$$

$$\begin{aligned} \|\pi\|_{L^2(\Omega)} \quad &\leq C\big(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big)\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)} + \\ &+ \big(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big) \times \big(\|\chi\|_{W^{1,6/5}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{1/2}(\Gamma)}\big)\Big), \end{aligned}$$

In the rest of this talk, we suppose that

$$\Omega \quad \text{is} \quad \mathcal{C}^{2,1}$$

and we are interested in the study of strong solutions $\boldsymbol{u} \in W^{2,p}(\Omega)$ for the system $(\mathcal{OS}_N)$ when $p \geq 6/5$.

When $p < \frac{3}{2}$, because the embedding $W^{2,p}(\Omega) \hookrightarrow W^{1,p*}(\Omega)$, the term $\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} \in L^p(\Omega)$ and we can use the regularity results on the Stokes problem. But this is not more the case when $p \geq \frac{3}{2}$ and that $\mathbf{curl}\, \boldsymbol{a}$ belongs only to $\boldsymbol{L}^{3/2}(\Omega)$.

We give in the following theorem the good conditions to ensure the existence of strong solutions.

Theorem

Let $p \geq 6/5$,

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \quad \pi_0 \in W^{1-1/p,p}(\Gamma), \quad \mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$$

with

$$s = \frac{3}{2} \text{ if } p < \frac{3}{2}, \qquad s = p \text{ if } p > \frac{3}{2}, \qquad s = \frac{3}{2} + \varepsilon \text{ if } p = \frac{3}{2}, \tag{32}$$

for $\varepsilon > 0$ arbitrary. Then the solution $(\boldsymbol{u}, \pi)$ given by the previous theorem belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \leq C\big(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi_0\|_{W^{1-1/p,p}(\Gamma)}\big).$$

In the following theorem, we prove the existence of Generalized Solutions with $p > 2$.

Theorem

Let

$$p > 2, \quad \boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]', \quad \chi \in W^{1,r}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma).$$

We suppose that $\pi_0 \in W^{1-1/r,r}(\Gamma)$ *and* $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$ *with* $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$ *and* s *satisfies:*

$$s = \frac{3}{2} \text{ if } 2 < p < 3, \quad s = \frac{3}{2} + \varepsilon \text{ if } p = 3 \text{ and } s = r \text{ if } p > 3,$$

for some arbitrary $\varepsilon > 0$. *Then the problem* (31) *has a unique solution* $(\boldsymbol{u}, \pi, \boldsymbol{\alpha}) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ *satisfying the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,r}(\Omega)} \leq C\big(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)^2 \Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} + \|\chi\|_{W^{1,r}(\Omega)} \Big) \tag{33}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_I)$

- Generalized Solutions ($p < 2$):

Using a duality argument, we obtain the following result :

Theorem

We suppose that $p < 2$. Soit $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$ and $\pi_0 \in W^{1-1/r,r}(\Gamma)$ with

$$r = 1 + \epsilon' \text{ if } p < \frac{3}{2}, \quad r = \frac{9+6\epsilon}{9+2\epsilon} \text{ if } p = \frac{3}{2} \text{ and } r = \frac{3p}{3+p} \text{ if } \frac{3}{2} < p < 2, \tag{34}$$

$$s = (1+\epsilon')\frac{3p}{4p-3-\epsilon'(3-p)} \text{ if } p < \frac{3}{2}, \quad s = \frac{3}{2}+\epsilon \text{ if } p = \frac{3}{2} \text{ and } s = \frac{3}{2} \text{ if } \frac{3}{2} < p < 2, \tag{35}$$

where ϵ, $\epsilon' > 0$ are arbitrary. Problem $(\mathcal{OS}_N)$ has a unique solution $(\boldsymbol{u}, \pi, \boldsymbol{\alpha}) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)})^2(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)}),$$

$$\|\pi\|_{W^{1,r}(\Omega)} \leq C(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)})^3(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)})$$

The Navier-Stokes problem $(\mathcal{NS}_N)$

$$(\mathcal{NS}_N) \qquad \begin{cases} -\Delta \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \Gamma, \\ \pi = \pi_0 \ \text{ on } \Gamma_i \text{ and } \pi = \pi_0 + c_i & \text{on } \Gamma_i, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0, \ \ i = 1, \ldots, I, \end{cases}$$

In the search of a proof of the existence of generalized solution for the Navier-Stokes equations ($\mathcal{NS}_N$), we consider the case of small enough data.

Theorem

Let

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]', \quad \chi \in W^{1,r}(\Omega), \quad \boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma), \quad \pi_0 \in W^{1-1/r,r}(\Gamma)$$

with

$$p > \frac{3}{2} \quad \text{and} \quad r = \frac{3p}{3+p}.$$

i) There exists a constant $\alpha_1 > 0$ such that, if

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} + \|\chi\|_{W^{1,r}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} \leq \alpha_1,$$

then, there exists a solution

$$(\boldsymbol{u}, \pi, \boldsymbol{c}) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$$

to problem $(\mathcal{NS}_N)$ verifying the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} + \|\chi\|_{W^{1,r}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\big),$$

with $c_i = \langle \boldsymbol{f}, \nabla q_i^N \rangle_\Omega \quad + \int_\Gamma (\chi - \pi_0) \nabla q_i^N \cdot \boldsymbol{n} - \int_\Omega (\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u}) \cdot \nabla q_i^N$

ii) Moreover, there exists a constant $\alpha_2 \in]0, \alpha_1]$ such that this solution is unique, if

$$\| \boldsymbol{f} \|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} + \|\chi\|_{W^{1,r}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} \leq \alpha_2.$$

Let us consider in a bounded cylindrical domain $\Omega \times (0,T)$ the following linearised evolution Stokes problem

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = \boldsymbol{0} & \text{in} \quad \Omega \times (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in} \quad \Omega, \end{cases} \tag{36}$$

where the unknowns $\boldsymbol{u}$ and π stand respectively for the velocity field and the pressure of a fluid occupying a domain Ω. Given data are the external force $\boldsymbol{f}$ and the initial velocity $\boldsymbol{u}_0$.

To study Stokes Problem (36) it is necessary to add appropriate boundary conditions. Note that this problem is often studied with Dirichlet boundary condition, which is not always realistic since it does not reflect the behavior of the fluid on or near the boundary.
H. Navier has suggested in 1824 a type of boundary conditions based on a proportionality between the tangential components of the normal dynamic tensor and the velocity

$$\boldsymbol{u}\cdot\boldsymbol{n}=0, \qquad 2\,\nu\,[\mathbb{D}\boldsymbol{u}\cdot\boldsymbol{n}]_{\boldsymbol{\tau}}+\alpha\,\boldsymbol{u}_{\boldsymbol{\tau}}=0 \qquad \text{on } \Gamma\times(0,T) \quad (37)$$

where ν is the viscosity and $\alpha\geq 0$ is the coefficient of friction and $\mathbb{D}\boldsymbol{u}=\frac{1}{2}(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T)$ denotes the deformation tensor associated to the velocity field $\boldsymbol{u}$.

Using the following relation

$$[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = -\mathbf{curl}\,\boldsymbol{v} \times \boldsymbol{n} - 2\boldsymbol{\Lambda}\boldsymbol{v} \quad \text{on}\,\Gamma,$$

where Λ is an operator of order 0 defined by

$$\boldsymbol{\Lambda}\boldsymbol{v} = \sum_{k=1}^{2} \left(\boldsymbol{v}_{\boldsymbol{\tau}} \cdot \frac{\partial \boldsymbol{n}}{\partial s_k}\right)\boldsymbol{\tau}_k,$$

one can observe that in the case of a flat boundary and when $\alpha = 0$ the conditions (37) may be replaced by

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \ \text{ on } \qquad \Gamma \times (0, T). \qquad (38)$$

We call them Navier-type boundary conditions.

Some works concerning the evolutionary Stokes and Navier-Stokes equations with non Dirichlet boundary conditions:

- Yudovitch, 1964 (in 2D)
- Miyakawa, 1980 (Neumann BC)
- Saal, 2006 (Robin BC)
- Shimada, 2007 (Robin BC)
- Mitrea-Monniaux, 2009
- Al Baba-Amrouche, 2014

...

Our purpose is to obtain a semi-group theory for the Stokes operator with Navier or Navier type-boundary conditions on $\boldsymbol{L}^p$-spaces as it is well known for Dirichlet conditions.

Because if

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma,$$

then

$$\Delta \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma.$$

This means that we will study the Laplacian operator with Navier type boundary conditions:

$$\left\{ \begin{array}{lll} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega \times (0,T), \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} & \text{on} \quad \Gamma \times (0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in} \quad \Omega. \end{array} \right. \tag{39}$$

To solve Stokes Problem (36) we use weak-Neumann problem:

$$\operatorname{div} (\mathbf{grad}\, \pi(t) - \boldsymbol{f}(t)) = 0 \ \text{ in } \Omega, \qquad (\mathbf{grad}\, \pi(t) - \boldsymbol{f}(t)) \cdot \boldsymbol{n} = 0 \ \text{ on } \Gamma. \tag{40}$$

We define the Laplacian operator with Navier-type boundary condition by

$$\Delta \,:\, \mathbf{D}_p(\Delta) \subset \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \longmapsto \boldsymbol{L}^p_{\sigma,\tau}(\Omega), \tag{41}$$

where

$$\boldsymbol{L}^p_{\sigma,\tau}(\Omega) \,=\, \Big\{ \boldsymbol{f} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div} \boldsymbol{f} = 0 \ \text{ in } \ \Omega, \ \ \boldsymbol{f} \cdot \boldsymbol{n} = 0 \ \text{ on } \ \Gamma \Big\} \tag{42}$$

and

$$\mathbf{D}_p(\Delta) \,=\, \Big\{ \boldsymbol{v} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \ ;\ \Delta\, \boldsymbol{v} \in \ \boldsymbol{L}^p_{\sigma,\tau}(\Omega), \\ \text{and } \ \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \ \text{ on } \ \Gamma \Big\} \tag{43}$$

with

$$\mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \ \text{ on } \ \Gamma \quad \text{in the sense } \ \boldsymbol{W}^{-1-1/p,p}(\Gamma).$$

- The Laplacian operator defined above is a densely defined operator:

$$\mathbf{D}_p(\Delta) \quad \text{is dense in} \quad \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$$

- Suppose that Ω is of class $C^{2,1}$, then

$$\mathbf{D}_p(\Delta) = \Big\{ \boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega);\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega, \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0,\ \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma \Big\} \quad (44)$$

Let $0 \leq \theta < \pi/2$ and let Σ_θ be the sector

$$\Sigma_\theta = \Big\{\lambda \in \mathbb{C}^*;\ \ |\arg \lambda| < \pi - \theta\Big\}.$$

Definition. Let X be a Banach space. We say that a linear densely defined operator $\mathcal{A} : D(\mathcal{A}) \subseteq X \longmapsto X$ is sectorial if there exists a constant $M > 0$ such that

$$\forall\, \lambda \in \Sigma_\theta, \qquad \|R(\lambda, \mathcal{A})\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda|}, \tag{45}$$

where $R(\lambda, \mathcal{A}) = (\lambda\, I - \mathcal{A})^{-1}$.

Theorem

An operator $\mathcal{A}$ generates a bounded analytic semi-group if and only if $\mathcal{A}$ is sectorial.

However, it is not always easy to prove that an operator is sectorial in the sense of the previous definition. However, it is suffices to prove in the half plane

$$\{\lambda \in \mathbb{C}^*;\, \mathrm{Re}\,\lambda \geq w\}, \quad \text{for some } w \geq 0.$$

Proposition

Let $\mathcal{A} : \mathrm{D}(\mathcal{A}) \subseteq X \longmapsto X$ be a linear densely defined operator, let $w \geq 0$ and $M > 0$ such that

$$\forall\, \lambda \in \mathbb{C}^*,\ \ \mathrm{Re}\,\lambda \geq w, \qquad \|R(\lambda,\, \mathcal{A})\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda|}.$$

Then $\mathcal{A}$ is sectorial.

Let us consider now the problem:

$$\left\{\begin{array}{ccc} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} & \text{on} \quad \Gamma, \end{array}\right. \tag{46}$$

where $\lambda \in \Sigma_\varepsilon$, with $\varepsilon \in]0, \pi[$ be fixed and $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.

Existence and Resolvent estimate

Theorem

Let $\lambda \in \Sigma_\varepsilon$ and let $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, with $1 < p < \infty$.

- *The Problem (46) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. Moreover if Ω is of class $C^{2,1}$ then $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$.*
- *In addition for $\operatorname{Re}\lambda \geq 0$, i.e. for $\varepsilon = \pi/2$, the solution $\boldsymbol{u}$ satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega,p)}{|\lambda|} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{47}$$

- *The Laplacian operator with Navier-type boundary conditions generates a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.*

Sketch of the proof

We first recall some properties:

- For all $\boldsymbol{v} \in \mathbf{D}_p(\Delta)$ and for any $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$ such that $\operatorname{div} \boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi} \cdot \boldsymbol{n} = 0$ on Γ, we have the following Green-Formula:

$$- \int_\Omega \Delta \boldsymbol{v} \cdot \overline{\boldsymbol{\varphi}} \, \mathrm{d}\, x \; = \; \int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \mathbf{curl}\, \overline{\boldsymbol{\varphi}} \, \mathrm{d}\, x, \tag{48}$$

where

$$- \Delta \boldsymbol{v} \; = - \, \mathbf{grad}(\operatorname{div} \boldsymbol{v}) \; + \; \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{v}. \tag{49}$$

Existence

The proof is done in three steps:

(i) For $p = 2$: Consider the space

$$\boldsymbol{V}^2_\tau(\Omega) = \Big\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \\ \operatorname{div} \boldsymbol{v} = 0 \ \text{ in } \Omega,\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \ \text{ on } \Gamma \Big\} \quad (50)$$

which is a Hilbert space for the norm:

$$\|\boldsymbol{v}\|_{\boldsymbol{V}^2_\tau(\Omega)} = (\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)})^{1/2} \quad (51)$$

Now consider the variational problem: find $\boldsymbol{u} \in \boldsymbol{V}^2_\tau(\Omega)$ such that for any $\boldsymbol{v} \in \boldsymbol{V}^2_\tau(\Omega)$

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}\, x, \tag{52}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = \lambda \int_\Omega \boldsymbol{u} \cdot \bar{\boldsymbol{v}} \, \mathrm{d}x + \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \bar{\boldsymbol{v}} \, \mathrm{d}x. \tag{53}$$

For the coercivity, observe that since $\lambda \in \Sigma_\varepsilon$ there exists a constant $C_\varepsilon > 0$ such that

$$\forall\, a,\, b > 0, \qquad |\lambda\, a \,+\, b| \geq C_\varepsilon(|\lambda| a \,+\, b). \tag{54}$$

As a result:

$$\begin{aligned} |\, a(\boldsymbol{u},\boldsymbol{u})| &= \left|\lambda \,\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}\right| \\ &\geq C_\varepsilon \left(|\lambda|\, \|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\ &\geq C_\varepsilon\, \min(|\lambda|\,, 1)\|\boldsymbol{u}\|^2_{\boldsymbol{V}^2_\tau(\Omega)}. \end{aligned}$$

Using Lax-Milgram lemma, we get the existence of a unique solution

$$\boldsymbol{u} \in \boldsymbol{V}^2_\tau(\Omega) \hookrightarrow \boldsymbol{H}^1(\Omega)$$

Existence

(ii) For $2 \leq p \leq 6$: We use the Sobolev imbedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$ and we write our problem in the form

$$\left\{ \begin{array}{lll} -\Delta \boldsymbol{u} = \boldsymbol{F}, & \nabla \cdot \boldsymbol{u} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} & \text{on} \quad \Gamma, \end{array} \right. \tag{55}$$

where

$$\boldsymbol{F} = \boldsymbol{f} - \lambda\, \boldsymbol{u} \in \boldsymbol{L}^p(\Omega).$$

Thanks to (Amrouche-Seloula, 2011), the solution $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$ (to $\boldsymbol{W}^{2,p}(\Omega)$ if Ω is of class $\mathcal{C}^{2,1}$)

Existence

(iii) For $p \geq 6$: Because the Sobolev imbedding

$$\boldsymbol{W}^{1,6}(\Omega) \hookrightarrow \boldsymbol{L}^{\infty}(\Omega)$$

we have again

$$\boldsymbol{F} = \boldsymbol{f} - \lambda \boldsymbol{u} \in \boldsymbol{L}^{p}(\Omega).$$

(iv) For $p \leq 2$: The existence of solution is proved by a duality argument.

Resolvent estimate

(i) For $p = 2$: The resolvent estimate is immediate, we multiply the first equation of Problem (46) by $\overline{\boldsymbol{u}}$ and we integrate both sides.

(ii) For $p \geq 2$: Multiplying the first equation of Problem (46) by $|\boldsymbol{u}|^{p-2}\,\overline{\boldsymbol{u}}$ and integrating both sides one gets

$$\lambda \int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}\,x + \int_\Omega |\boldsymbol{u}|^{p-2} \,|\nabla\, \boldsymbol{u}|^2 \,\mathrm{d}\,x + 4\,\frac{p-2}{p^2} \int_\Omega |\nabla\, |\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x$$

$$+ \,(p-2)\, i \sum_{k=1}^{3} \int_\Omega |\boldsymbol{u}|^{p-4}\, \mathrm{Re}\left(\frac{\partial\, \boldsymbol{u}}{\partial x_k} \cdot \overline{\boldsymbol{u}}\right) \mathrm{Im}\left(\frac{\partial\, \boldsymbol{u}}{\partial x_k} \cdot \overline{\boldsymbol{u}}\right) \mathrm{d}\,x$$

$$= \int_\Gamma |\boldsymbol{u}|^{p-2} \Big(\frac{\partial\, \boldsymbol{u}}{\partial\, \boldsymbol{n}}\Big)_{\boldsymbol{\tau}} \cdot \overline{\boldsymbol{u}} \,\mathrm{d}\,\sigma + \int_\Omega |\boldsymbol{u}|^{p-2}\, \boldsymbol{f} \cdot \bar{\boldsymbol{u}} \,\mathrm{d}\,x. \quad (56)$$

Next using the fact that Ω is of class $\mathcal{C}^{1,1}$ and

$$\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n} \;=\; \nabla_{\boldsymbol{\tau}}(\boldsymbol{v}\cdot\boldsymbol{n}) - \Big(\frac{\partial\,\boldsymbol{v}}{\partial\,\boldsymbol{n}}\Big)_{\boldsymbol{\tau}} - \sum_{j=1}^{2}\Big(\frac{\partial\,\boldsymbol{n}}{\partial\,s_j}\cdot\boldsymbol{v}_{\boldsymbol{\tau}}\Big)\boldsymbol{\tau}_j \qquad \text{on } \Gamma,$$

we can easily verify that

$$\Big|\int_{\Gamma}|\boldsymbol{u}|^{p-2}\Big(\frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\Big)_{\boldsymbol{\tau}}\cdot\overline{\boldsymbol{u}}_{\boldsymbol{\tau}}\,\mathrm{d}\,\sigma\Big| \;\leq\; C_1(\Omega)\int_{\Gamma}|\boldsymbol{u}|^{p}\,\mathrm{d}\sigma.$$

We split formula (56) in to parts: the real and the imaginary part and we study each part separately.

$$\operatorname{Re}\lambda \, \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2} \, |\nabla \boldsymbol{u}|^2 \, \mathrm{d}\, x + 4 \, \frac{p-2}{p^2} \int_\Omega |\nabla \, |\boldsymbol{u}|^{p/2}|^2 \, \mathrm{d}\, x$$
$$\leq \, C_1(\Omega) \int_\Gamma |\boldsymbol{u}|^p \, \mathrm{d}\, \sigma \, + \, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \quad (57)$$

and

$$|\operatorname{Im}\lambda| \, \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} \, \leq \, \frac{p-2}{2} \int_\Omega |\boldsymbol{u}|^{p-2} \, |\nabla \boldsymbol{u}|^2 \, \mathrm{d}\, x + C_1(\Omega) \int_\Gamma |\boldsymbol{u}|^p \, \mathrm{d}\, \sigma +$$
$$+ \, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \quad (58)$$

Now putting together (57) and (58) one has

$$|\lambda| \, \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2} \, |\nabla \boldsymbol{u}|^2 \, \mathrm{d}\, x + 4 \, \frac{p-2}{p^2} \int_\Omega |\nabla \, |\boldsymbol{u}|^{p/2}|^2 \, \mathrm{d}\, x$$
$$\leq \frac{p-2}{2} \int_\Omega |\boldsymbol{u}|^{p-2} \, |\nabla \boldsymbol{u}|^2 \, \mathrm{d}\, x + 2 \, C_1(\Omega) \int_\Gamma |\boldsymbol{u}|^p \, \mathrm{d}\, \sigma + 2 \, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega}$$

Using, for any $\varepsilon > 0$, the following inequality, with $w = |\boldsymbol{u}|^{p/2}$,

$$\int_\Gamma |w|^2 \, \mathrm{d}\, \sigma \leq \varepsilon \int_\Omega |\nabla w|^2 \, \mathrm{d}\, x + C_\varepsilon \int_\Omega |w|^2 \, \mathrm{d}\, x, \qquad (60)$$

Applying formula (60) to $w = |\boldsymbol{u}|^{p/2}$ and substituting in (59) one gets

$$|\lambda| \, \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2} \, |\nabla \, \boldsymbol{u}|^2 \, \mathrm{d}\, x + 4 \, \frac{p-2}{p^2} \int_\Omega |\nabla \, |\boldsymbol{u}|^{p/2}|^2 \, \mathrm{d}\, x$$

$$\leq \frac{p-2}{2} \int_\Omega |\boldsymbol{u}|^{p-2} \, |\nabla \, \boldsymbol{u}|^2 \, \mathrm{d}\, x + 2 \, C_1(\Omega) \Big[\, \varepsilon \int_\Omega |\nabla \, |\boldsymbol{u}|^{p/2}|^2 \, \mathrm{d}\, x + C_\varepsilon \int_\Omega |\boldsymbol{u}|^p$$

$$+ 2 \, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}.$$

We chose $\varepsilon > 0$ such that $\varepsilon \, C_1(\Omega) = \frac{p-2}{p^2}$. As a result the constant C_ε in (61) depends on p and Ω. Then by setting $C_\varepsilon = C_2(\Omega, p)$ one has

one gets after computation

$$|\lambda| \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}\, x + 2\,\frac{p-2}{p^2} \int_\Omega |\nabla |\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\, x$$

$$\leq C_3(\Omega, p)\, \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \frac{p-2}{2} \int_\Omega |\boldsymbol{u}|^{p-2} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}\, x + 2\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}, \quad (62)$$

where

$$C_3(\Omega, p) = 2\, C_1(\Omega) C_2(\Omega, p). \quad (63)$$

We define

$$\lambda_0 = 2\, C_3(\Omega, p), \quad (64)$$

and we consider only the case

$$|\lambda| \geq \lambda_0.$$

One has then

$$\frac{|\lambda|}{2} \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}\, x + 2\,\frac{p-2}{p^2} \int_\Omega |\nabla |\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\, x$$

$$\leq \frac{p-2}{2} \int_\Omega |\boldsymbol{u}|^{p-2} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}\, x + 2\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}.$$

To obtain estimates, we consider the case $2 \leq p \leq 4$ and then the case $p > 4$.

Remark: The Stokes operator with Navier-type boundary conditions generates also a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ for all $1 < p < \infty$.
We observe also the following estimates:

Corollary

Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re}\lambda \geq 0$, let $1 < p < \infty$, $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ be the unique solution of Problem (46). In addition, suppose that Ω is of class $C^{2,1}$ then $\boldsymbol{u}$ satisfies the estimates

$$\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega,p)}{\sqrt{|\lambda|}}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{65}$$

and

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C(\Omega,p)\,\frac{1+|\lambda|}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{66}$$

Sketch of the proof.

Let us introduce the following space

$$\Theta = \left\{ r \in H^1(\Omega^\circ);\ [\, r \,]_j = \text{constant},\ 1 \leq j \leq J \right\}.$$

For each $1 \leq j \leq J$, there exists a unique solution $q_j^\tau \in H^1(\Omega^\circ)$, up to an additive constant, satisfying

$$\forall r \in \Theta, \quad \int_{\Omega^\circ} \nabla q_j^\tau \cdot \nabla r = [r]_j$$

where $[\chi]_j$ is the jump of χ through Σ_j.

In fact, we have

$$\begin{cases} -\Delta q_j^\tau = 0 & \text{in } \Omega^\circ, \\ \partial_n q_j^\tau = 0 & \text{on } \Gamma, \\ \left[q_j^\tau \right]_k = \text{constant}, & 1 \le k \le J, \\ \left[\partial_n q_j^\tau \right]_k = 0; & 1 \le k \le J, \\ \langle \partial_n q_j^\tau \, , \, 1 \rangle_{\Sigma_k} = \delta_{jk}, & 1 \le k \le J. \end{cases} \tag{67}$$

Let us consider any extension to $\boldsymbol{L}^2(\Omega)$ of ∇q_j^τ which is denoted by $\widetilde{\mathbf{grad}}\, q_j^\tau$. It is easy to verify that

$$\operatorname{div} \widetilde{\mathbf{grad}}\, q_j^\tau = 0, \quad \mathbf{curl}\, \widetilde{\mathbf{grad}}\, q_j^\tau = \mathbf{0}$$

and thanks to the regularity results of vector fields, we deduce first that

$$\widetilde{\mathbf{grad}}\, q_j^\tau \in \boldsymbol{H}^1(\Omega),$$

and then by boostrap argument that

$$\widetilde{\mathbf{grad}}\, q_j^\tau \in \boldsymbol{W}^{1,p}(\Omega),$$

Next, set

$$\widetilde{\boldsymbol{u}} = \boldsymbol{u} - \sum_{j=1}^{J} \langle \boldsymbol{u} \cdot \boldsymbol{n}\,,\, 1 \rangle_{\Sigma_j} \widetilde{\mathbf{grad}}\, q_j^{\tau}\,.$$

Using the Gagliardo-Nirenberg inequality we have

$$\|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} = \|\mathbf{curl}\, \widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega, p)\, \|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{W}^{2,p}(\Omega)} \|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}$$

But thanks to Amrouche-Seloula we know that

$$\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C(\Omega, p) \|\Delta \widetilde{\boldsymbol{u}}\|_{\boldsymbol{W}^{2,p}(\Omega)}.$$

Then

$$\begin{aligned} \|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} &\leq C(\Omega, p)\, \|\Delta \widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} \|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} \\ &= C(\Omega, p)\, \|\Delta \boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} \|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}. \end{aligned}$$

Moreover we know that

$$\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega, p)\, \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Finally it is clear that

$$\|\Delta \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \|\boldsymbol{f} - \lambda\, \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq (C(\Omega, p) + 1)\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

As a consequence we deduce estimate (65).
Notice that, since

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \simeq \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\Delta \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$$

one has estimate (66).

We can also consider the Lapalce equations with tangential boundary conditions:

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}, & \text{on} \quad \Gamma \end{cases} \tag{68}$$

and we have the following result

Existence and Resolvent estimate

Theorem

Let $\lambda \in \Sigma_\varepsilon$ and let $\boldsymbol{f} \in \boldsymbol{L}^p_\sigma(\Omega)$, with $1 < p < \infty$.

- *The Problem (68) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. Moreover if Ω is of class $C^{2,1}$ then $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$.*
- *In addition for $\operatorname{Re}\lambda \geq 0$, i.e. for $\varepsilon = \pi/2$, the solution $\boldsymbol{u}$ satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega, p)}{|\lambda|} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{69}$$

- *The Laplacian operator with tangential boundary conditions generates a bounded analytic semi-group on $\boldsymbol{L}^p_\sigma(\Omega)$.*

For Further Reading

C. Amrouche, A. Rejaiba, *L^p-Theory for Stokes and Navier-Stokes Equations with Navier Boundary Condition*, Journal Diff. Equat., 256 (2014) 1515–1547.

C. Amrouche, C. Bernardi, M. Dauge, V. Girault, *Vector potentials in three-dimensional nonsmooth domains*, Math. Meth. Applied Sc., Vol. 21, (1998), 823–864.

C. Amrouche, N. Seloula, *L^p-theory for vector potentials and Sobolev's inequqlities for vector fields. Application to the Stokes problem's with pressure boundary conditions*, M3AS, Volume 23-1, (2013), 37–92.

C. Amrouche, N. Seloula, *On the Stokes equations with the Navier-type boundary conditions*, Differential Equations and Applications, Volume 3, Number 4, (2011), 581-607.


C. Conca, F. Murat, O. Pironneau, *The Stokes and Navier–Stokes equations with boundary conditions involving the pressure*, Japan. J. Math., Vol. 20, (1994), 263–318.


H. Kozono, T. Yanagisawa, *L^r-variational Inequality for Vector Fielss and the Helmholtz-Weyl Decomposition in Bounded domains*, Indiana University Mathematics Journal, Vol. 58, No. 4, (2009), 1853–1920..


V. A. Solonnikov and V. E. Ščadilov. *A certain boundary value problem for the stationary system of Navier-Stokes equations.* Trudy Mat. Inst. Steklov. **125**, 196–210, 1973.


W. Von Wahl, *Estimating ∇u by* div u, **curl** u, Math. Meth. Appl. Sci., Vol. 15, (1992), 123–143.


H. Al Baba, C. Amrouche, M. Escobedo: Analyticity of the semi-group generated by the Stokes operator with Navier-type boundary condition on L^p-spaces. *Work in progress*


H. Al Baba, C. Amrouche, M. Escobedo: Semi-group theory for the Stokes operator with Navier-type boundary condition on L^p-spaces. *Work in progress*


V. Barbu: *Nonlinear semigroups and differential equations in Banach spaces*, 1976.


G. Dore, A. Venni: On The Closedness of the sum of two closed operators. *Math. Z.* **196**, 189-201, (1987).


K. Engel, R. Nagel: *One parameter semigroups for linear evolution equations* Springer-verlag New York Berlin Heidelberg 1999.

A. Pazy: *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, Inc 1983.