

L^p -Theory for the Stokes and Navier-Stokes equations with different boundary conditions

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I. Introduction and motivation

Let us consider in a bounded cylindrical domain $\Omega \times (0, T)$ the following **Navier-Stokes** equations

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi = \mathbf{f}, & \operatorname{div} \mathbf{u} = \mathbf{0} & \text{in } \Omega \times (0, T), \\ \mathbf{u}(0) = \mathbf{u}_0 & & \text{in } \Omega, \end{cases}$$

where the unknowns \mathbf{u} and π stand respectively for the velocity field and the pressure of a fluid occupying a domain Ω . Given data are the external force \mathbf{f} and the initial velocity \mathbf{u}_0 . To study Navier-Stokes equations it is necessary to add appropriate boundary conditions.

- The Navier-Stokes equations are more studied, since the famous works of Jean Leray (1933, 1934), with a **Dirichlet boundary condition** which is not always realistic since it does not reflect the behavior of the fluid on or near the boundary.
- In the applications and in particular in the **electromagnetism** problems, it is possible to find problems where it is necessary to consider others boundary conditions. This boundary conditions (BC) are also used to **simulate flows near rough walls**, such as in aerodynamics, in weather forecasts and in hemodynamics, as well as perforated walls. **BC involving the pressure**, such as for example in cases of pipes, hydraulic gears using pomps, containers, etc ...

H. Navier has suggested in 1824 a type of boundary conditions based on a **proportionality** between the **tangential components** of the **normal dynamic tensor** and the **velocity**

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad 2\nu [\mathbb{D}\mathbf{u} \cdot \mathbf{n}]_{\tau} + \alpha \mathbf{u}_{\tau} = 0 \quad \text{on } \Gamma \times (0, T) \quad (1)$$

where ν is the viscosity and $\alpha \geq 0$ is the **coefficient of friction** and

$$\mathbb{D}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$$

denotes the deformation tensor associated to the velocity field \mathbf{u} . Observe that if α tends to **infinity**, we get formally

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma.$$

The Navier boundary conditions are often used to **simulate flows near rough walls** as well as perforated walls.

Such slip boundary conditions are used in the **Large Eddy Simulations** (LES) of **turbulent flows**.

Using the following relation

$$[2\mathbf{D}(\mathbf{v})\mathbf{n}]_{\tau} = -\mathbf{curl} \mathbf{v} \times \mathbf{n} - 2\Lambda \mathbf{v} \quad \text{on } \Gamma,$$

where Λ is an operator of order 0 defined by

$$\Lambda \mathbf{v} = \sum_{k=1}^2 \left(\mathbf{v}_{\tau} \cdot \frac{\partial \mathbf{n}}{\partial s_k} \right) \boldsymbol{\tau}_k,$$

one can observe that in the case of a **flat boundary** and when $\alpha = 0$ the conditions (1) may be replaced by

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \mathbf{curl} \mathbf{u} \times \mathbf{n} = \mathbf{0} \quad \text{on} \quad \Gamma \times (0, T). \quad (2)$$

We call them **Navier-type boundary conditions**.

We are interested here, in a first part, by the **stationary case for the Stokes equations**:

$$\begin{aligned} -\Delta \mathbf{u} + \nabla \pi &= \mathbf{f} && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, \end{aligned}$$

with one of the following boundary conditions:

$$\mathbf{u} \times \mathbf{n} = \mathbf{g} \times \mathbf{n} \quad \text{and} \quad \pi = \pi_0 \quad \text{on } \Gamma, \quad (3)$$

$$\mathbf{u} \cdot \mathbf{n} = g \quad \text{and} \quad \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma, \quad (4)$$

$$\mathbf{u} \cdot \mathbf{n} = g \quad \text{and} \quad 2[\mathbf{D}(\mathbf{u})\mathbf{n}]_{\tau} + \alpha \mathbf{u}_{\tau} = \mathbf{h} \quad \text{on } \Gamma. \quad (5)$$

We will study, in a second part, the case of **stationary Navier-Stokes equations**:

Find $\mathbf{u}, \pi, \alpha_1, \dots, \alpha_I$, with $\alpha_i \in \mathbb{R}$

$$\begin{cases} -\Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi = \mathbf{f} & \text{and } \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = \mathbf{g} \times \mathbf{n} & & \text{on } \Gamma, \\ \pi = \pi_0 & \text{on } \Gamma_0 \text{ and } \pi = \pi_0 + \alpha_i & \text{on } \Gamma_i, \quad i = 1, \dots, I, \end{cases}$$

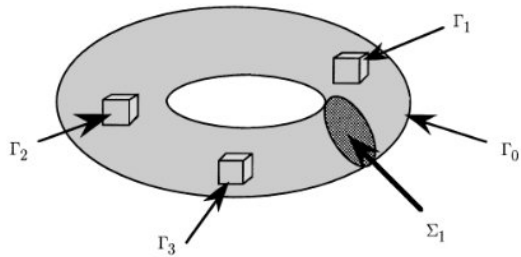
where we suppose that Ω is an open set possibly multiply connected sufficiently regular with a boundary Γ possibly non connected. We denote

$$\Gamma = \bigcup_{i=0}^I \Gamma_i, \quad \Sigma = \bigcup_{j=1}^J \Sigma_j$$

with Γ_i the connected components of Γ and Σ_j a finite number of cuts such that

$$\Omega^\circ = \Omega \setminus \bigcup_{j=1}^J \Sigma_j$$

is simply connected.



Some works concerning the **stationary Stokes** and **Navier-Stokes equations** with non Dirichlet boundary conditions:

- Scadilov-Solonnikov, 1973
- Conca-Murat-Pironneau, 1994
- Neustupa-Penel, 2007
- Maruzic, 2007
- Beirao da Veiga, 2009
- Berselli, 2009
- Mitrea-Monniaux, 2009
- Amrouche-Seloula, 2011-2013
- Amrouche-Rejaiba, 2014
- ...

Considering for example the case of the Stokes equations with the homogeneous boundary conditions

$$(\mathcal{S}_T^0) \quad \begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f} & \text{and } \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0, & \text{and } \mathbf{curl} \mathbf{u} \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma. \end{cases}$$

Because these BC, we write

$$-\Delta \mathbf{u} = \mathbf{curl} \mathbf{curl} \mathbf{u} - \nabla \operatorname{div} \mathbf{u}$$

For the variational formulation, we will consider the following spaces:

$$\mathbf{V} = \{ \mathbf{v} \in \mathbf{L}^2(\Omega), \mathbf{curl} \mathbf{v} \in \mathbf{L}^2(\Omega), \operatorname{div} \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma \},$$

$$\mathbf{H}_0^2(\operatorname{div}, \Omega) = \{ \mathbf{v} \in \mathbf{L}^2(\Omega), \operatorname{div} \mathbf{v} \in L^2(\Omega), \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma \}.$$

The Stokes problem (\mathcal{S}_T^0) is then equivalent to

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{V} \text{ such that} \\ \forall \mathbf{v} \in \mathbf{V}, \quad \int_{\Omega} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \mathbf{v} \, d\mathbf{x} = \langle \mathbf{f}, \mathbf{v} \rangle_{\Omega}. \end{cases}$$

We note that $[\mathbf{H}_0^2(\operatorname{div}, \Omega)]' \hookrightarrow \mathbf{H}^{-1}(\Omega)$.

Questions:

- The bilinear form is it coercive to apply the Lax-Milgram lemma ?

We will see that if Ω is simply connected, we have:

$$\forall \mathbf{v} \in \mathbf{V}, \quad \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \leq C \|\mathbf{curl} \mathbf{v}\|_{L^2(\Omega)}.$$

- What happens if Ω is not simply connected ?
- Can we find **generalized solution** in $\mathbf{W}^{1,p}(\Omega)$ with $1 < p < \infty$?
- Can we find **strong solution** in $\mathbf{W}^{2,p}(\Omega)$ with $1 < p < \infty$?
- Can we find **very weak solution** in $\mathbf{L}^p(\Omega)$ with $1 < p < \infty$?

We need the following spaces, for $1 < p < \infty$:

$$\mathbf{H}^p(\mathbf{curl}, \Omega) = \{v \in L^p(\Omega); \mathbf{curl} v \in L^p(\Omega)\}, \quad \mathbf{H}^p(\mathbf{div}, \Omega) = \{v \in L^p(\Omega); \mathbf{div} v \in L^p(\Omega)\}$$

$$\mathbf{X}^p(\Omega) = \mathbf{H}^p(\mathbf{curl}, \Omega) \cap \mathbf{H}^p(\mathbf{div}, \Omega),$$

and their subspaces:

$$\mathbf{H}_0^p(\mathbf{curl}, \Omega) = \{v \in \mathbf{H}^p(\mathbf{curl}, \Omega); v \times n = \mathbf{0} \text{ sur } \Gamma\},$$

$$\mathbf{H}_0^p(\mathbf{div}, \Omega) = \{v \in \mathbf{H}^p(\mathbf{div}, \Omega); v \cdot n = 0 \text{ sur } \Gamma\},$$

$$\mathbf{X}_N^p(\Omega) = \{v \in \mathbf{X}^p(\Omega); v \times n = \mathbf{0} \text{ sur } \Gamma\}, \quad \mathbf{X}_T^p(\Omega) = \{v \in \mathbf{X}^p(\Omega); v \cdot n = 0 \text{ sur } \Gamma\}$$

II. Sobolev's inequalities for vector fields

Lemma (Sobolev's inequalities)

i) Any function $\mathbf{v} \in \mathbf{W}^{1,p}(\Omega)$ with $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ on Γ satisfies:

$$\|\nabla \mathbf{v}\|_{L^p(\Omega)} \leq C \left(\|\operatorname{div} \mathbf{v}\|_{L^p(\Omega)} + \|\operatorname{curl} \mathbf{v}\|_{L^p(\Omega)} + \sum_{i=1}^I |\langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i}| \right). \quad (6)$$

ii) Any function $\mathbf{v} \in \mathbf{W}^{1,p}(\Omega)$ with $\mathbf{v} \cdot \mathbf{n} = 0$ on Γ satisfies:

$$\|\nabla \mathbf{v}\|_{L^p(\Omega)} \leq C \left(\|\operatorname{div} \mathbf{v}\|_{L^p(\Omega)} + \|\operatorname{curl} \mathbf{v}\|_{L^p(\Omega)} + \sum_{j=1}^J |\langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j}| \right). \quad (7)$$

W. Von Wahl (1992) ($I = 0$, i.e. Γ is connected and $J = 0$, i.e. Ω is simply connected).

Idea of the proof of (6):

- We consider the following double layer potential:

$$T \lambda(\mathbf{x}) = -\frac{1}{2\pi} \int_{\Gamma} \lambda(\boldsymbol{\xi}) \frac{\partial}{\partial \mathbf{n}} |\mathbf{x} - \boldsymbol{\xi}|^{-1} d\sigma_{\boldsymbol{\xi}},$$

$T : L^p(\Gamma) \rightarrow W^{1,p}(\Gamma)$ is continuous and then compact from $L^p(\Gamma)$ into $L^p(\Gamma)$. The Fredholm alternative implies that the dimension of $\text{Ker}(Id + T)$ is finite (equal to I) and thanks to the theorem of the open map we have for all $\mathbf{v} \in \mathbf{W}^{1,p}(\Omega)$:

$$\|\mathbf{v} \cdot \mathbf{n}\|_{L^p(\Gamma)} \leq C(\|(Id + T)(\mathbf{v} \cdot \mathbf{n})\|_{L^p(\Gamma)} + \sum_{i=1}^I |\langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i}|) \quad (8)$$

- We write the following integral representation: for any $\mathbf{v} \in \mathbf{W}^{1,p}(\Omega)$ with $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ on Γ :

$$\begin{aligned} (Id + T)(\mathbf{v} \cdot \mathbf{n}) &= -\frac{1}{2\pi} \left(\mathbf{grad} \int_{\Omega} \frac{1}{|\mathbf{x} - \mathbf{y}|} \text{div}_{\mathbf{y}} \mathbf{v}(\mathbf{y}) d\mathbf{y} \right) \cdot \mathbf{n} \\ &\quad - \frac{1}{2\pi} \left(\mathbf{rot} \int_{\Omega} \frac{1}{|\mathbf{x} - \mathbf{y}|} \mathbf{rot}_{\mathbf{y}} \mathbf{v}(\mathbf{y}) d\mathbf{y} \right) \cdot \mathbf{n} \end{aligned} \quad (9)$$

- Using the traces theorem, we prove that:

$$\begin{aligned} \|(Id + T)(\mathbf{v} \cdot \mathbf{n})\|_{L^p(\Gamma)} &\leq C \left(\left\| \mathbf{grad} \int_{\Omega} \frac{1}{|\mathbf{x} - \mathbf{y}|} \operatorname{div}_{\mathbf{y}} \mathbf{v}(\mathbf{y}) \, d\mathbf{y} \right\|_{W^{1,p}(\Omega)} \right. \\ &\quad \left. + \left\| \mathbf{rot} \int_{\Omega} \frac{1}{|\mathbf{x} - \mathbf{y}|} \mathbf{rot}_{\mathbf{y}} \mathbf{v}(\mathbf{y}) \, d\mathbf{y} \right\|_{W^{1,p}(\Omega)} \right). \end{aligned}$$

- Using the Calderón-Zygmund inequalities, we get:

$$\|(Id + T)(\mathbf{v} \cdot \mathbf{n})\|_{L^p(\Gamma)} \leq C (\|\operatorname{div} \mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{rot} \mathbf{v}\|_{L^p(\Omega)}).$$

- From (8), we deduce:

$$\|\mathbf{v} \cdot \mathbf{n}\|_{L^p(\Gamma)} \leq C (\|\operatorname{div} \mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{rot} \mathbf{v}\|_{L^p(\Omega)} + \sum_{i=1}^I |\langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i}|). \quad (10)$$

- Moreover, from (9) we have:

$$\begin{aligned} \|\mathbf{v} \cdot \mathbf{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)} &\leq \left(\|T(\mathbf{v} \cdot \mathbf{n})\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \left\| \mathbf{grad} \int_{\Omega} \frac{1}{|\mathbf{x} - \mathbf{y}|} \operatorname{div}_{\mathbf{y}} \mathbf{v}(\mathbf{y}) \, d\mathbf{y} \right\|_{W^{1-\frac{1}{p},p}(\Gamma)} \right. \\ &\quad \left. + \left\| \mathbf{rot} \int_{\Omega} \frac{1}{|\mathbf{x} - \mathbf{y}|} \mathbf{rot}_{\mathbf{y}} \mathbf{v}(\mathbf{y}) \, d\mathbf{y} \right\|_{W^{1-\frac{1}{p},p}(\Gamma)} \right), \end{aligned}$$

and by the traces theorem:

$$\begin{aligned} \|\mathbf{v} \cdot \mathbf{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)} &\leq C \left(\|\mathbf{v} \cdot \mathbf{n}\|_{L^p(\Gamma)} + \left\| \mathbf{grad} \int_{\Omega} \frac{1}{|\mathbf{x} - \mathbf{y}|} \operatorname{div}_{\mathbf{y}} \mathbf{v}(\mathbf{y}) \, d\mathbf{y} \right\|_{W^{1,p}(\Omega)} \right. \\ &\quad \left. + \left\| \mathbf{rot} \int_{\Omega} \frac{1}{|\mathbf{x} - \mathbf{y}|} \mathbf{rot}_{\mathbf{y}} \mathbf{v}(\mathbf{y}) \, d\mathbf{y} \right\|_{W^{1,p}(\Omega)} \right). \end{aligned}$$

- Using again the Calderón-Zygmund inequalities and (10), we obtain:

$$\|\mathbf{v} \cdot \mathbf{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)} \leq C \left(\|\operatorname{div} \mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{rot} \mathbf{v}\|_{L^p(\Omega)} + \sum_{i=1}^I |\langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i}| \right). \quad (11)$$

- As $\mathbf{v} \in \mathbf{W}^{1-\frac{1}{p},p}(\Gamma)$, by the traces theorem, there exists $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ such that:

$$\mathbf{v} = \mathbf{u} \quad \text{sur } \Gamma \quad \text{and} \quad \|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \|\mathbf{v}\|_{\mathbf{W}^{1-\frac{1}{p},p}(\Gamma)}.$$

Because $\mathbf{v} \times \mathbf{n} = 0$ on Γ , we have $\mathbf{v}|_{\Gamma} = (\mathbf{v} \cdot \mathbf{n})\mathbf{n}$. Using then (11), we get

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} &\leq C \|\mathbf{v} \cdot \mathbf{n}\|_{\mathbf{W}^{1-\frac{1}{p},p}(\Gamma)} \\ &\leq C \left(\|\operatorname{div} \mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{rot} \mathbf{v}\|_{L^p(\Omega)} + \sum_{i=1}^I |\langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i}| \right). \end{aligned} \quad (12)$$

- As $\mathbf{u} - \mathbf{v} \in \mathbf{W}_0^{1,p}(\Omega)$, we know that for any function w belonging to $\mathbf{W}_0^{1,p}(\Omega)$, we have the following integral representation integrale:


$$w = -\operatorname{grad} \frac{1}{4\pi} \int_{\Omega} \frac{1}{|\mathbf{x} - \mathbf{y}|} \operatorname{div}_{\mathbf{y}} w(\mathbf{y}) \, d\mathbf{y} + \mathbf{rot} \frac{1}{4\pi} \int_{\Omega} \frac{1}{|\mathbf{x} - \mathbf{y}|} \mathbf{rot}_{\mathbf{y}} w(\mathbf{y}) \, d\mathbf{y}.$$

Thanks to the Calderón-Zygmund inequalities, we have

$$\|\nabla w\|_{L^p(\Omega)} \leq C (\|\operatorname{div} w\|_{L^p(\Omega)} + \|\mathbf{rot} w\|_{L^p(\Omega)}). \quad (13)$$

- Applying (13) to $\mathbf{v} - \mathbf{u} \in \mathbf{W}_0^{1,p}(\Omega)$, we obtain:

$$\|\nabla(\mathbf{v} - \mathbf{u})\|_{L^p(\Omega)} \leq C \left(\|\operatorname{div} \mathbf{v}\|_{L^p(\Omega)} + \|\operatorname{div} \mathbf{u}\|_{L^p(\Omega)} + \|\mathbf{rot} \mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{rot} \mathbf{u}\|_{L^p(\Omega)} \right).$$

Finally, we deduce the required estimate by using (12), 

Idea of the proof of (7):

We use the same ideas with the following linear integral operator

$$R \boldsymbol{\lambda}(x) = \frac{1}{2\pi} \int_{\Gamma} \mathbf{curl} \left(\frac{\boldsymbol{\lambda}(\boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|} \right) \times \mathbf{n} \, d\sigma_{\boldsymbol{\xi}}.$$

Because the embeddings $\mathbf{X}_N^p(\Omega) \hookrightarrow \mathbf{L}^p(\Omega)$ and $\mathbf{X}_T^p(\Omega) \hookrightarrow \mathbf{L}^p(\Omega)$ are compact, by using Peetre-Tartar Theorem and the previous inequality, we deduce the following Poincaré's type inequalities:

Corollary (Poincaré's Inequalities)

i) We have $\mathbf{v} \in \mathbf{X}_N^p(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$ and for any $\mathbf{v} \in \mathbf{X}_N^p(\Omega)$

$$\|\mathbf{v}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \left(\|\operatorname{div} \mathbf{v}\|_{L^p(\Omega)} + \|\operatorname{curl} \mathbf{v}\|_{L^p(\Omega)} + \sum_{i=1}^I |\langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i}| \right).$$

ii) We have $\mathbf{v} \in \mathbf{X}_T^p(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$ and for any $\mathbf{v} \in \mathbf{X}_T^p(\Omega)$

$$\|\mathbf{v}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \left(\|\operatorname{div} \mathbf{v}\|_{L^p(\Omega)} + \|\operatorname{curl} \mathbf{v}\|_{L^p(\Omega)} + \sum_{j=1}^J |\langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j}| \right).$$

Regularity properties

The previous results can be extended to the case of inhomogeneous boundary conditions :

- Let Ω of class $\mathcal{C}^{1,1}$ and $\mathbf{v} \in \mathbf{X}^p(\Omega)$ with

$$\mathbf{v} \cdot \mathbf{n} \in W^{1-\frac{1}{p},p}(\Gamma) \quad \text{or} \quad \mathbf{v} \times \mathbf{n} \in \mathbf{W}^{1-\frac{1}{p},p}(\Gamma).$$

Then

$$\mathbf{v} \in \mathbf{W}^{1,p}(\Omega).$$

Moreover

$$\|\mathbf{v}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C(\|\mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{curl} \mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{div} \mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{v} \cdot \mathbf{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)})$$

or

$$\|\mathbf{v}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C(\|\mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{curl} \mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{div} \mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{v} \times \mathbf{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)})$$

- Let Ω of class $\mathcal{C}^{2,1}$ and

$$\mathbf{v} \in \mathbf{L}^p(\Omega), \quad \mathbf{curl} \mathbf{v} \in \mathbf{W}^{1,p}(\Omega), \quad \operatorname{div} \mathbf{v} \in W^{1,p}(\Omega)$$

with

$$\mathbf{v} \cdot \mathbf{n} \in W^{2-\frac{1}{p},p}(\Gamma) \quad \text{or} \quad \mathbf{v} \times \mathbf{n} \in \mathbf{W}^{2-\frac{1}{p},p}(\Gamma).$$

Then

$$\mathbf{v} \in \mathbf{W}^{2,p}(\Omega),$$

with the corresponding estimates.

III. L^p -Theory for Vector Potentials

To study the Stokes problems, we need some results concerning the vector potentials and the inf-sup conditions.

Theorem (Tangential vector potentials)

$\mathbf{u} \in \mathbf{H}^p(\text{div}, \Omega)$ satisfies: $\text{div } \mathbf{u} = 0$ in Ω , $\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0$, $0 \leq i \leq I$,
iff there exists a vector potential $\boldsymbol{\psi}$ in $\mathbf{X}^p(\Omega)$ such that

$$\mathbf{u} = \text{curl } \boldsymbol{\psi} \quad \text{and} \quad \text{div } \boldsymbol{\psi} = 0 \text{ in } \Omega, \quad (14)$$

$$\boldsymbol{\psi} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \quad (15)$$

$$\langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J. \quad (16)$$

This function is unique and moreover, we have

$$\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \|\mathbf{u}\|_{L^p(\Omega)}.$$

For $p = 2$, see : C. Amrouche, C. Bernardi, M. Dauge, V. Girault [1998].

Remark: If

$$\mathbf{u} \in \mathbf{L}^p(\Omega) \quad \text{and} \quad \operatorname{div} \mathbf{u} = 0,$$

the condition

$$\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I,$$

is necessary and sufficient for the existence of vector potential $\boldsymbol{\psi}$ satisfying (14) and (15). While the condition (16)

$$\langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J$$

ensures the uniqueness of $\boldsymbol{\psi}$. When this last condition is not satisfied, the vector potential $\boldsymbol{\psi}$ exits up an additive vector function of the the following corresponding kernel:

$$\mathbf{K}_T^p(\Omega) = \{ \mathbf{w} \in \mathbf{X}_T^p(\Omega); \operatorname{curl} \mathbf{w} = \mathbf{0}, \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega \}.$$

To characterize this kernel, let us introduce the following space

$$\Theta = \{r \in H^1(\Omega^\circ); [r]_j = \text{constant}, 1 \leq j \leq J\}.$$

For each $1 \leq j \leq J$, there exists a unique solution $q_j^T \in H^1(\Omega^\circ)$, up to an additive constant, satisfying

$$\forall r \in \Theta, \quad \int_{\Omega^\circ} \nabla q_j^T \cdot \nabla r = [r]_j$$

where $[\chi]_j$ is the **jump** of χ through Σ_j .

In fact, we have

$$\left\{ \begin{array}{l} -\Delta q_j^T = 0 \quad \text{in } \Omega^\circ, \\ \partial_n q_j^T = 0 \quad \text{on } \Gamma, \\ \left[q_j^T \right]_k = \text{constant} \quad \text{and} \quad [\partial_n q_j^T]_k = 0, \quad 1 \leq k \leq J, \\ \left\langle \partial_n q_j^T, 1 \right\rangle_{\Sigma_k} = \delta_{jk}, \quad 1 \leq k \leq J. \end{array} \right.$$

Let us consider any extension to $L^2(\Omega)$ of ∇q_j^T which is denoted by $\widetilde{\mathbf{grad}} q_j^T$. It is easy to verify that

$$\operatorname{div} \widetilde{\mathbf{grad}} q_j^T = 0, \quad \operatorname{curl} \widetilde{\mathbf{grad}} q_j^T = \mathbf{0}$$

and thanks to the regularity results of vector fields, we deduce first that

$$\widetilde{\mathbf{grad}} q_j^T \in H^1(\Omega),$$

and then by bootstrap argument that

$$\widetilde{\mathbf{grad}} q_j^T \in W^{1,p}(\Omega),$$

Moreover, $\mathbf{K}_T^p(\Omega)$ is spanned by

$$\widetilde{\mathbf{grad}} q_j^T, \quad j = 1, \dots, J.$$

Remark that $\mathbf{K}_T^p(\Omega) = \{0\}$ if $J = 0$ (i.e Ω is **simply connected**).

Theorem (Inf-Sup condition in Banach spaces)

Let X and M be two reflexive Banach spaces and X' and M' their dual spaces. Let a be the continuous bilinear form defined on $X \times M$, let $A \in \mathcal{L}(X; M')$ and $A' \in \mathcal{L}(M; X')$ be the operators defined by

$$\forall v \in X, \forall w \in M, a(v, w) = \langle Av, w \rangle = \langle v, A'w \rangle$$

and $V = \text{Ker } A$. The following statements are equivalent:

i) There exist $\beta > 0$ such that

$$\inf_{\substack{w \in M \\ w \neq 0}} \sup_{\substack{v \in X \\ v \neq 0}} \frac{a(v, w)}{\|v\|_X \|w\|_M} \geq \beta. \quad (17)$$

ii) The operator $A : X/V \mapsto M'$ is an isomorphism and $1/\beta$ is the continuity constant of A^{-1} .

iii) The operator $A' : M \mapsto X' \perp V$ is an isomorphism and $1/\beta$ is the continuity constant of $(A')^{-1}$.

Remark

As consequence, if the Inf-Sup condition (17) is satisfied, then we have the following properties:

i) Because $A' : M \mapsto X' \perp V$ is an isomorphism, then for any $f \in X'$, satisfying the compatibility condition

$$\forall v \in V, \langle f, v \rangle = 0,$$

there exists a unique $w \in M$ such that

$$\forall v \in X, a(v, w) = \langle f, v \rangle \quad \text{and} \quad \|w\|_M \leq \frac{1}{\beta} \|f\|_{X'}. \quad (18)$$

ii) Because $A : X/V \mapsto M'$ is an isomorphism, then for any $g \in M'$, $\exists v \in X$, unique up an additive element of V , such that:

$$\forall w \in M, a(v, w) = \langle g, w \rangle \quad \text{and} \quad \|v\|_{X/V} \leq \frac{1}{\beta} \|g\|_{M'}$$

We define the space

$$\mathbf{V}_T^p(\Omega) = \{\mathbf{w} \in \mathbf{X}_T^p(\Omega); \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega \text{ and } \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, 1 \leq j \leq J\},$$

which is a Banach space for the norm $\|\cdot\|_{\mathbf{X}^p(\Omega)}$.

Using some classical Helmholtz decomposition and the previous tangential vector potential, we can deduce:

Lemma (Inf Sup Condition)

The following Inf-Sup Condition holds: there exists a constant $\beta > 0$, such that

$$\inf_{\substack{\varphi \in \mathbf{V}_T^{p'}(\Omega) \\ \varphi \neq 0}} \sup_{\substack{\boldsymbol{\xi} \in \mathbf{V}_T^p(\Omega) \\ \boldsymbol{\xi} \neq 0}} \frac{\int_{\Omega} \operatorname{curl} \boldsymbol{\xi} \cdot \operatorname{curl} \varphi \, dx}{\|\boldsymbol{\xi}\|_{\mathbf{X}_T^p(\Omega)} \|\varphi\|_{\mathbf{X}_T^{p'}(\Omega)}} \geq \beta. \quad (19)$$

Theorem (Normal vector potentials)

$\mathbf{u} \in \mathbf{L}^p(\Omega)$ satisfies:

$$\mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma \quad \text{and} \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 0 \leq j \leq J,$$

iff there exists a vector potential $\boldsymbol{\psi}$ in $\mathbf{X}^p(\Omega)$ such that

$$\mathbf{u} = \operatorname{curl} \boldsymbol{\psi} \quad \text{and} \quad \operatorname{div} \boldsymbol{\psi} = 0 \text{ in } \Omega, \quad (20)$$

$$\boldsymbol{\psi} \times \mathbf{n} = 0 \quad \text{on } \Gamma, \quad (21)$$

$$\langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \quad (22)$$

This function is unique and moreover, we have:

$$\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \|\mathbf{u}\|_{L^p(\Omega)}.$$

Remark : For

$$\mathbf{u} \in \mathbf{L}^p(\Omega), \quad \text{with} \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega \quad \text{and} \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma,$$

the condition

$$\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad \text{for } 1 \leq j \leq J,$$

is necessary and sufficient for the existence of the vector potential $\boldsymbol{\psi}$ satisfying (20) and (21). The condition

$$\langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0, \quad \text{for } 0 \leq i \leq I$$

ensures the uniqueness of $\boldsymbol{\psi}$.

Idea of the proof

- We use the Inf-Sup condition (19) to solve the problem:

Find $\boldsymbol{\xi} \in \mathbf{W}^{1,p}(\Omega)$ such that:

$$(\mathcal{P}) \quad \begin{cases} -\Delta \boldsymbol{\xi} = 0 \quad \text{and} \quad \operatorname{div} \boldsymbol{\xi} = 0 & \text{in } \Omega, \\ \boldsymbol{\xi} \cdot \mathbf{n} = 0, \quad \mathbf{curl} \boldsymbol{\xi} \times \mathbf{n} = \boldsymbol{\psi}_0 \times \mathbf{n} & \text{on } \Gamma, \\ \langle \boldsymbol{\xi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \end{cases}$$

where $\boldsymbol{\psi}_0$ is the tangential vector potential.

- The required vector potential is given by:

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 - \mathbf{curl} \boldsymbol{\xi} - \sum_{i=1}^I \langle (\boldsymbol{\psi}_0 - \mathbf{curl} \boldsymbol{\xi}) \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} \mathbf{grad} q_i^N$$

We define now the space

$$\mathbf{V}_N^p(\Omega) = \{\mathbf{w} \in \mathbf{X}_N^p(\Omega); \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega \text{ and } \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0, 1 \leq i \leq I\},$$

which is a Banach space for the norm $\|\cdot\|_{\mathbf{X}^p(\Omega)}$.

Using a Helmholtz decomposition and the previous normal vector potential, we can deduce:

Lemma (Inf Sup Condition)

The following Inf-Sup Condition holds: there exists a constant $\beta > 0$, such that

$$\inf_{\substack{\varphi \in \mathbf{V}_N^{p'}(\Omega) \\ \varphi \neq \mathbf{0}}} \sup_{\substack{\boldsymbol{\xi} \in \mathbf{V}_N^p(\Omega) \\ \boldsymbol{\xi} \neq \mathbf{0}}} \frac{\int_{\Omega} \operatorname{curl} \boldsymbol{\xi} \cdot \operatorname{curl} \varphi \, dx}{\|\boldsymbol{\xi}\|_{\mathbf{X}_N^p(\Omega)} \|\varphi\|_{\mathbf{X}_N^{p'}(\Omega)}} \geq \beta. \quad (23)$$

IV. Stokes stationary problem with normal boundary conditions

$$(S_T) \quad \begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = g, \quad \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} & \text{on } \Gamma, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, & 1 \leq j \leq J. \end{cases}$$

- We solve the pressure π :

$$\Delta \pi = \operatorname{div} \mathbf{f} \text{ in } \Omega, \quad \frac{\partial \pi}{\partial \mathbf{n}} = \mathbf{f} \cdot \mathbf{n} + \operatorname{div}_{\Gamma}(\mathbf{h} \times \mathbf{n}) \text{ on } \Gamma$$

- We set $\mathbf{F} = \mathbf{f} - \nabla \pi$. Then, \mathbf{u} is a solution of the following elliptic problem:

$$(E_T) \quad \begin{cases} -\Delta \mathbf{u} = \mathbf{F} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = g, \quad \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} & \text{on } \Gamma, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, & 1 \leq j \leq J. \end{cases}$$

- To solve the Stokes problem (S_T) , we are reduced to solve the problem (E_T) .

IV. Stokes stationary problem with normal boundary conditions

$$(S_T) \quad \begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = g, \quad \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} & \text{on } \Gamma, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, & 1 \leq j \leq J. \end{cases}$$

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- To solve the Stokes problem (S_T) , we are reduced to solve the problem (E_T) .

IV. Stokes stationary problem with normal boundary conditions

$$(S_T) \quad \begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = g, \quad \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} & \text{on } \Gamma, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, & 1 \leq j \leq J. \end{cases}$$

- We solve the pressure π :

$$\Delta \pi = \operatorname{div} \mathbf{f} \text{ in } \Omega, \quad \frac{\partial \pi}{\partial \mathbf{n}} = \mathbf{f} \cdot \mathbf{n} + \operatorname{div}_{\Gamma}(\mathbf{h} \times \mathbf{n}) \text{ on } \Gamma$$

- We set $\mathbf{F} = \mathbf{f} - \nabla \pi$. Then, \mathbf{u} is a solution of the following **elliptic** problem:

$$(E_T) \quad \begin{cases} -\Delta \mathbf{u} = \mathbf{F} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = g, \quad \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} & \text{on } \Gamma, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, & 1 \leq j \leq J. \end{cases}$$

- To solve the Stokes problem (S_T) , we are reduced to solve the problem (E_T) .

IV. Stokes stationary problem with normal boundary conditions

$$(S_T) \quad \begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = g, \quad \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} & \text{on } \Gamma, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, & 1 \leq j \leq J. \end{cases}$$

- We solve the pressure π :

$$\Delta \pi = \operatorname{div} \mathbf{f} \text{ in } \Omega, \quad \frac{\partial \pi}{\partial \mathbf{n}} = \mathbf{f} \cdot \mathbf{n} + \operatorname{div}_{\Gamma}(\mathbf{h} \times \mathbf{n}) \text{ on } \Gamma$$

- We set $\mathbf{F} = \mathbf{f} - \nabla \pi$. Then, \mathbf{u} is a solution of the following **elliptic** problem:

$$(E_T) \quad \begin{cases} -\Delta \mathbf{u} = \mathbf{F} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = g, \quad \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} & \text{on } \Gamma, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, & 1 \leq j \leq J. \end{cases}$$

- To solve the Stokes problem (S_T) , we are reduced to solve the problem (E_T) .

- Contrarily to the **Stokes** problem with **Dirichlet** boundary condition, it appears that when

$$\operatorname{div} \mathbf{f} = 0 \quad \text{in } \Omega \quad \text{and} \quad \mathbf{f} \cdot \mathbf{n} = \operatorname{div}_{\Gamma}(\mathbf{h} \times \mathbf{n}) = 0 \quad \text{on } \Gamma,$$

the pressure π can be constant.

- In the case of the **Stokes** problem with **Dirichlet**, even if

$$\operatorname{div} \mathbf{f} = 0 \quad \text{in } \Omega \quad \text{and} \quad \mathbf{f} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma,$$

we have $\Delta \mathbf{u} \cdot \mathbf{n} \neq 0$ on Γ and then π is non constant.

Study of the elliptic problem :

$$(E_T) \quad \begin{cases} -\Delta \mathbf{u} = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = g, \quad \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} & \text{on } \Gamma, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, & 1 \leq j \leq J. \end{cases}$$

Proposition (Weak and Strong solutions of (E_T))

- i) Let $\mathbf{f} \in \mathbf{L}^p(\Omega)$ with $\operatorname{div} \mathbf{f} = 0$ in Ω , $g \in W^{1-\frac{1}{p},p}(\Gamma)$ and $\mathbf{h} \in \mathbf{W}^{-\frac{1}{p},p}(\Gamma)$ satisfying the following compatibility condition:

$$\forall \mathbf{v} \in \mathbf{K}_T^{p'}(\Omega), \quad \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} + \langle \mathbf{h} \times \mathbf{n}, \mathbf{v} \rangle_{\mathbf{W}^{-\frac{1}{p},p}(\Gamma) \times \mathbf{W}^{\frac{1}{p},p'}(\Gamma)} = 0,$$

$$\mathbf{f} \cdot \mathbf{n} + \operatorname{div}_{\Gamma}(\mathbf{h} \times \mathbf{n}) = 0 \quad \text{on } \Gamma,$$

$$\int_{\Gamma} g \, d\sigma = 0.$$

Then, the problem (E_T) has a unique solution $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ satisfying the estimate:

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \left(\|\mathbf{f}\|_{\mathbf{L}^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\Gamma)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right).$$

- ii) If $g \in W^{2-1/p,p}(\Gamma)$ and $\mathbf{h} \in \mathbf{W}^{1-1/p,p}(\Gamma)$, then the solution \mathbf{u} belongs to $\mathbf{W}^{2,p}(\Omega)$.

Theorem (Weak and strong solution of (\mathcal{S}_T))

i) Let \mathbf{f} , g , \mathbf{h} with:

$$\mathbf{f} \in (\mathbf{H}_0^{p'}(\operatorname{div}, \Omega))', \quad g \in W^{1-\frac{1}{p}, p}(\Gamma), \quad \mathbf{h} \in \mathbf{W}^{-\frac{1}{p}, p}(\Gamma),$$

verifying the compatibility conditions: for any \mathbf{v} in $\mathbf{K}_T^{p'}(\Omega)$:

$$\langle \mathbf{f}, \mathbf{v} \rangle_{(\mathbf{H}_0^{p'}(\operatorname{div}, \Omega))' \times \mathbf{H}_0^{p'}(\operatorname{div}, \Omega)} + \langle \mathbf{h} \times \mathbf{n}, \mathbf{v} \rangle_{\mathbf{W}^{-\frac{1}{p}, p}(\Gamma) \times \mathbf{W}^{\frac{1}{p}, p'}(\Gamma)} = 0.$$
$$\int_{\Gamma} g \, d\sigma = 0.$$

Then, the Stokes problem (\mathcal{S}_T) has a unique solution $(\mathbf{u}, \pi) \in \mathbf{W}^{1, p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ satisfying the estimate:

$$\|\mathbf{u}\|_{\mathbf{W}^{1, p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq \left(\|\mathbf{f}\|_{(\mathbf{H}_0^{p'}(\operatorname{div}, \Omega))'} + \|g\|_{W^{1-\frac{1}{p}, p}(\Gamma)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{-\frac{1}{p}, p}(\Gamma)} \right).$$

ii) If $\mathbf{f} \in \mathbf{L}^p(\Omega)$, $g \in W^{2-1/p, p}(\Gamma)$, $\mathbf{h} \in \mathbf{W}^{1-1/p, p}(\Gamma)$ then the solution (\mathbf{u}, π) of the problem (\mathcal{S}_T) given previously belongs to $\mathbf{W}^{2, p}(\Omega) \times W^{1, p}(\Omega)$.

Remark: We can consider the case of the problem (\mathcal{S}_T) with $\operatorname{div} \mathbf{u} = \chi$ in Ω , where $\chi \in L^p(\Omega)$ satisfying $\int_{\Gamma} g \, d\sigma = \langle \chi, 1 \rangle_{\Gamma}$.

Very weak solution for (S_T)

Let \mathbf{f} , χ , g , and \mathbf{h} with

$$\mathbf{f} \in (\mathbf{T}^{p'}(\Omega))', \chi \in L^p(\Omega), g \in W^{-1/p,p}(\Gamma), \mathbf{h} \in \mathbf{W}^{-1-1/p,p}(\Gamma),$$

with $\mathbf{T}^{p'}(\Omega) = \left\{ \boldsymbol{\varphi} \in \mathbf{H}_0^{p'}(\text{div}, \Omega); \text{div } \boldsymbol{\varphi} \in W_0^{1,p'}(\Omega) \right\}$ and satisfying the compatibility conditions:

$$\begin{aligned} \forall \boldsymbol{\varphi} \in \mathbf{K}_T^{p'}(\Omega), \quad \langle \mathbf{f}, \boldsymbol{\varphi} \rangle_{(\mathbf{T}^{p'}(\Omega))' \times \mathbf{T}^{p'}(\Omega)} + \langle \mathbf{h} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Gamma} &= 0. \\ \int_{\Omega} \chi \, d\mathbf{x} &= \langle g, 1 \rangle_{\Gamma}. \end{aligned}$$

Then, the Stokes problem (S_T) has exactly one solution $\mathbf{u} \in \mathbf{L}^p(\Omega)$ and $\pi \in W^{-1,p}(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C > 0$ depending only on p and Ω such that:

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{L}^p(\Omega)} + \|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} &\leq C \left(\|\mathbf{f}\|_{(\mathbf{T}^{p'}(\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma)} + \right. \\ &\quad \left. \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{-1-1/p,p}(\Gamma)} \right). \end{aligned}$$

Helmholtz Decomposition for vector fields in $L^p(\Omega)$

For any vector field $\mathbf{v} \in \mathbf{L}^p(\Omega)$, we have the first following decomposition:

$$\mathbf{v} = \mathbf{z} + \nabla \chi + \mathbf{curl} \mathbf{u},$$

- $\mathbf{z} \in \mathbf{K}_N^p(\Omega)$ is unique,
- $\chi \in W_0^{1,p}(\Omega)$ is unique,
- $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ is the unique solution, up to an additive element of the kernel $\mathbf{K}_T^p(\Omega)$, of the problem :

$$\begin{cases} -\Delta \mathbf{u} = \mathbf{curl} \mathbf{v} & \text{and} & \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0, & (\mathbf{curl} \mathbf{u} - \mathbf{v}) \times \mathbf{n} = 0 & & \text{on } \Gamma. \end{cases}$$

Helmholtz Decomposition for vector fields in $L^p(\Omega)$

For any vector field $\mathbf{v} \in \mathbf{L}^p(\Omega)$, we have the second following decomposition:

$$\mathbf{v} = \mathbf{z} + \nabla \chi + \mathbf{curl} \mathbf{u},$$

- $\mathbf{z} \in \mathbf{K}_T^p(\Omega)$ is unique,
- $\chi \in W^{1,p}(\Omega)$ is unique up to an additive constant,
- $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ is the unique solution, up to an additive element of the kernel $\mathbf{K}_N^p(\Omega)$, of the problem :

$$\begin{cases} -\Delta \mathbf{u} = \mathbf{curl} \mathbf{v} & \text{and} & \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = 0, & & & \text{on } \Gamma. \end{cases}$$

V. Stokes Equations with Pressure Boundary Condition

$$(S_N) \quad \begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = \mathbf{g} \times \mathbf{n}, \quad \pi = \pi_0 & \text{on } \Gamma, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \end{cases}$$

- The pressure can be found independently of the velocity as a solution of the Dirichlet problem:

$$\Delta \pi = \operatorname{div} \mathbf{f} \quad \text{in } \Omega, \quad \pi = \pi_0 \quad \text{on } \Gamma$$

- The velocity is a solution of the following system:

$$(E_N) \quad \begin{cases} -\Delta \mathbf{u} = \mathbf{F} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = \mathbf{g} \times \mathbf{n} & \text{on } \Gamma, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0, \quad \text{for all } 1 \leq i \leq I, \end{cases}$$

where $\mathbf{F} = \mathbf{f} - \nabla \pi$

Remarks:

- The condition $\operatorname{div} \mathbf{F} = 0$ in Ω is necessary to solve (E_N) .
- The condition $\operatorname{div} \mathbf{u} = 0$ in $\Omega \iff \operatorname{div} \mathbf{u} = 0$ on Γ on the one hand. On the other hand, since

$$\operatorname{div} \mathbf{u} = \operatorname{div}_\Gamma \mathbf{u}_\tau + K \mathbf{u} \cdot \mathbf{n} + \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \mathbf{n} \quad \text{sur } \Gamma,$$

where K denotes the mean curvature of Γ , the condition $\operatorname{div} \mathbf{u} = 0$ on Γ is itself equivalent, if $\mathbf{u} \times \mathbf{n} = \mathbf{0}$ on Γ , to the **Fourier-Robin** condition:

$$K \mathbf{u} \cdot \mathbf{n} + \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma.$$

That means that the problem (E_N) is equivalent to the following:

$$\begin{cases} -\Delta \mathbf{u} = \mathbf{F} & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma \\ K \mathbf{u} \cdot \mathbf{n} + \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \mathbf{n} = 0 & \text{on } \Gamma. \end{cases}$$

Using the **Inf-Sup condition**, we deduce the following theorem

Theorem (Weak and Strong solutions for (\mathcal{S}_N))

i) Let \mathbf{f} , \mathbf{g} , π_0 with

$$\mathbf{f} \in (\mathbf{H}_0^{p'}(\mathbf{curl}, \Omega))', \quad \mathbf{g} \in \mathbf{W}^{1-1/p, p}(\Gamma), \quad \pi_0 \in W^{1-1/p, p}(\Gamma),$$

satisfying the compatibility condition:

$$\forall \mathbf{v} \in \mathbf{K}_N^{p'}(\Omega), \quad \langle \mathbf{f}, \mathbf{v} \rangle_\Omega - \int_\Gamma \pi_0 \mathbf{v} \cdot \mathbf{n} \, d\sigma = 0, \quad (24)$$

with $\langle \cdot, \cdot \rangle_\Omega = \langle \cdot, \cdot \rangle_{[\mathbf{H}_0^{p'}(\mathbf{curl}, \Omega)]' \times \mathbf{H}_0^{p'}(\mathbf{curl}, \Omega)}$.

Then, the Stokes problem (\mathcal{S}_N) has a unique solution $(\mathbf{u}, \pi) \in \mathbf{W}^{1, p}(\Omega) \times W^{1, p}(\Omega)$ and satisfies the estimate:

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{W}^{1, p}(\Omega)} + \|\pi\|_{W^{1, p}(\Omega)} &\leq C \left(\|\mathbf{f}\|_{(\mathbf{H}_0^{p'}(\mathbf{curl}, \Omega))'} + \|\mathbf{g} \times \mathbf{n}\|_{\mathbf{W}^{1-1/p, p}(\Gamma)} + \right. \\ &\quad \left. + \|\pi_0\|_{W^{1-1/p, p}(\Gamma)} \right). \end{aligned}$$

ii) If $\mathbf{f} \in L^p(\Omega)$ and $\mathbf{g} \in \mathbf{W}^{2-1/p, p}(\Gamma)$, then $\mathbf{u} \in \mathbf{W}^{2, p}(\Omega)$.

Question:

What happens if the previous compatibility condition is not satisfied?

Variant of the system (S_N) :

Find $(\mathbf{u}, \pi, \mathbf{c})$ such that:

$$(S'_N) \quad \begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f} \text{ and } \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = \mathbf{g} \times \mathbf{n} & \text{on } \Gamma, \\ \pi = \pi_0 \text{ on } \Gamma_0 \text{ and } \pi = \pi_0 + c_i & \text{on } \Gamma_i, \quad 1 \leq i \leq I \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I, \end{cases}$$

where $\mathbf{c} = (c_i)_{1 \leq i \leq I}$.

Theorem (Weak and Strong solutions for (\mathcal{S}'_N))

Let \mathbf{f} , \mathbf{g} and π_0 such that:

$$\mathbf{f} \in [\mathbf{H}_0^{p'}(\mathbf{curl}, \Omega)]', \quad \mathbf{g} \in \mathbf{W}^{1-1/p,p}(\Gamma), \quad \pi_0 \in W^{1-1/p,p}(\Gamma).$$

Then, the problem (\mathcal{S}'_N) has a unique solution $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$, $\pi \in W^{1,p}(\Omega)$ and constants c_1, \dots, c_I satisfying the estimate:

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \leq C(\|\mathbf{f}\|_{[\mathbf{H}_0^{p'}(\mathbf{curl}, \Omega)]'} + \|\mathbf{g}\|_{\mathbf{W}^{1-1/p,p}} + \|\pi_0\|_{W^{1-1/p,p}}),$$

and where c_1, \dots, c_I are given by

$$c_i = \langle \mathbf{f}, \nabla q_i^N \rangle_{\Omega} - \langle \pi_0, \nabla q_i^N \cdot \mathbf{n} \rangle_{\Gamma}. \quad (25)$$

In particular, if $\mathbf{f} \in \mathbf{L}^p(\Omega)$ and $\mathbf{g} \in \mathbf{W}^{2-1/p,p}(\Gamma)$, then $\mathbf{u} \in \mathbf{W}^{2,p}(\Omega)$.

Remark :

- Observe that if we suppose that the compatibility condition (24) is verified, we have that: $c_i = 0$ for all $i = 1, \dots, I$. Then, we have reduced to solve the problem (\mathcal{S}'_N) without the constant c_i and (\mathcal{S}'_N) is anything other than (\mathcal{S}_N) .

The assumption on \mathbf{f} in the previous theorem can be **weakened** by considering the space defined for all $1 < r, p < \infty$:

$$\mathbf{H}_0^{r,p}(\mathbf{curl}, \Omega) = \{\boldsymbol{\varphi} \in \mathbf{L}^r(\Omega); \mathbf{curl} \boldsymbol{\varphi} \in \mathbf{L}^p(\Omega), \boldsymbol{\varphi} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma\}.$$

which is a Banach space for the norm

$$\|\boldsymbol{\varphi}\|_{\mathbf{H}_0^{r,p}(\mathbf{curl}, \Omega)} = \|\boldsymbol{\varphi}\|_{\mathbf{L}^r(\Omega)} + \|\mathbf{curl} \boldsymbol{\varphi}\|_{\mathbf{L}^p(\Omega)}.$$

We can prove that

$$\mathcal{D}(\Omega) \text{ is dense in } \mathbf{H}_0^{r',p'}(\mathbf{curl}, \Omega)$$

and its dual space can be characterized as:

$$[\mathbf{H}_0^{r',p'}(\mathbf{curl}, \Omega)]' = \{\mathbf{F} + \mathbf{curl} \boldsymbol{\psi}; \mathbf{F} \in \mathbf{L}^r(\Omega), \boldsymbol{\psi} \in \mathbf{L}^p(\Omega)\}.$$

Theorem (Second Version for Weak solutions for (\mathcal{S}'_N))

Let \mathbf{f} , \mathbf{g} and π_0 such that

$$\mathbf{f} \in [\mathbf{H}_0^{r',p'}(\mathbf{curl}, \Omega)]', \quad \mathbf{g} \times \mathbf{n} \in \mathbf{W}^{1-1/p,p}(\Gamma), \quad \pi_0 \in W^{1-1/r,r}(\Gamma),$$

with

$$r \leq p \quad \text{and} \quad \frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}.$$

Then, the problem (\mathcal{S}'_N) has a unique solution

$$\mathbf{u} \in \mathbf{W}^{1,p}(\Omega), \quad \pi \in W^{1,r}(\Omega)$$

and constants c_1, \dots, c_I satisfying the estimate:

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,r}(\Omega)} &\leq C(\|\mathbf{f}\|_{[\mathbf{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} \\ &+ \|\mathbf{g} \times \mathbf{n}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)}), \end{aligned}$$

and c_1, \dots, c_I are given by (25), where we replace the duality brackets on Ω by

$$\langle \cdot, \cdot \rangle_{\Omega} = \langle \cdot, \cdot \rangle_{[\mathbf{H}_0^{r',p'}(\mathbf{curl}, \Omega)]' \times \mathbf{H}_0^{r',p'}(\mathbf{curl}, \Omega)}.$$

Theorem (Very weak solutions for (\mathcal{S}_N))

Let \mathbf{f} , \mathbf{g} , and π_0 with

$$\mathbf{f} \in [\mathbf{H}_0^{p'}(\mathbf{curl}, \Omega)]', \quad \mathbf{g} \in \mathbf{W}^{-1/p,p}(\Gamma), \quad \pi_0 \in W^{-1/p,p}(\Gamma),$$

and satisfying the compatibility conditions (24). Then, the Stokes problem (\mathcal{S}_N) has exactly one solution

$$\mathbf{u} \in \mathbf{L}^p(\Omega) \quad \text{and} \quad \pi \in L^p(\Omega)/\mathbb{R}.$$

Moreover, there exists a constant $C > 0$ depending only on p and Ω such that:

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{L}^p(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} &\leq C \left(\|\mathbf{f}\|_{[\mathbf{H}_0^{p'}(\mathbf{curl}, \Omega)]'} + \|\mathbf{g}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} + \right. \\ &\quad \left. + \|\pi_0\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right). \end{aligned}$$

Remark. To study the case of **Navier boundary conditions**:

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{and} \quad [\mathbf{D}(\mathbf{u})\mathbf{n}]_{\tau} = \mathbf{h},$$

it suffices to observe that

$$[2\mathbf{D}(\mathbf{v})\mathbf{n}]_{\tau} = -\mathbf{curl} \mathbf{v} \times \mathbf{n} - 2\mathbf{\Lambda} \mathbf{v} \quad \text{on } \Gamma,$$

where $\mathbf{\Lambda}$ is an operator of order 0 defined by

$$\mathbf{\Lambda} \mathbf{w} = \sum_{k=1}^2 \left(\mathbf{w}_{\tau} \cdot \frac{\partial \mathbf{n}}{\partial s_k} \right) \boldsymbol{\tau}_k,$$

and to use the results obtained with the **Navier type boundary conditions**.

VI. Oseen and Navier-Stokes Problem with Pressure Boundary Condition

We are interested to study the following problem:

Find \mathbf{u} , q and $\boldsymbol{\alpha} \in \mathbb{R}^I$ satisfying:

$$(\mathcal{NS}) \quad \begin{cases} -\Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla q = \mathbf{f} & \text{and } \operatorname{div} \mathbf{u} = \chi & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = \mathbf{g} & & \text{on } \Gamma, \\ q = q_0 \text{ on } \Gamma_0 \text{ and } q = q_0 + \alpha_i & & \text{on } \Gamma_i, i = 1, \dots, I, \\ \int_{\Gamma_i} \mathbf{u} \cdot \mathbf{n} \, d\boldsymbol{\sigma} = 0, & i = 1, \dots, I, \end{cases}$$

- Note that $\boldsymbol{\alpha}$ is a supplementary unknown Stokes which depends in fact on \mathbf{u}
- If we take $\chi = 0$ and $\mathbf{g} = \mathbf{0}$, unlike the Navier-Stokes problem with Dirichlet boundary conditions de Dirichlet, the property: $\int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{u} \, dx = 0$ does not hold.
- But, we have

$$\mathbf{u} \cdot \nabla \mathbf{u} = \operatorname{curl} \mathbf{u} \times \mathbf{u} + \frac{1}{2} \nabla |\mathbf{u}|^2$$

We rewrite then (\mathcal{NS}) under the following form:

$$(\mathcal{NS}_N) \quad \left\{ \begin{array}{ll} -\Delta \mathbf{u} + \mathbf{curl} \mathbf{u} \times \mathbf{u} + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = \chi & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = \mathbf{g} & \text{on } \Gamma, \\ \pi = \pi_0 \text{ sur } \Gamma_0 \text{ et } \pi = \pi_0 + \alpha_i & \text{on } \Gamma_i, \quad i = 1, \dots, I, \\ \int_{\Gamma_i} \mathbf{u} \cdot \mathbf{n} \, d\sigma = 0, \quad i = 1, \dots, I, \end{array} \right.$$

Remarks.

- We can search directly weak solutions $\mathbf{u} \in \mathbf{H}^1(\Omega)$ and $\pi \in L^2(\Omega)$ of the system (\mathcal{NS}_N) by using a fixed point method.
- We can then obtain solutions $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ for $p > 2$ thanks to the Stokes problem theory.
- The case $p < 2$ to study the (\mathcal{NS}_N) system is more complicated.
- For this reason, we will study the Oseen problem (\mathcal{OS}_N) .

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- The case $p < 2$ to study the (\mathcal{NS}_N) system is more complicated.
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Study of problem (\mathcal{OS}_N)

$$(\mathcal{OS}_N) \quad \begin{cases} -\Delta \mathbf{u} + \mathbf{curl} \mathbf{a} \times \mathbf{u} + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma, \\ \pi = \pi_0 + c_i & \text{on } \Gamma_i, \quad 0 = 1, \dots, I, \\ \int_{\Gamma_i} \mathbf{u} \cdot \mathbf{n} \, d\sigma = 0, \quad i = 1, \dots, I, \end{cases}$$

where we have take $\chi = 0$ and $\mathbf{g} = \mathbf{0}$. We suppose also that

$$\mathbf{curl} \mathbf{a} \in L^{3/2}(\Omega)$$

We introduce the following Hilbert space:

$$\mathbf{V}_N = \left\{ \mathbf{v} \in H^1(\Omega); \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma \right. \\ \left. \text{and } \int_{\Gamma_i} \mathbf{v} \cdot \mathbf{n} = 0, 1 \leq i \leq I \right\}$$

and recall that

$$\mathbf{v} \mapsto \left(\int_{\Omega} |\operatorname{curl} \mathbf{v}|^2 \right)^{1/2}$$

is a norm on \mathbf{V}_N equivalent to the full norm of $\mathbf{H}^1(\Omega)$.

Before establishing the result of existence of a **weak solution** for the problem (\mathcal{OS}_N) , we will see in what functional space it is reasonable to take π_0 and to find the **pressure** π appearing in (\mathcal{OS}_N) , knowing that we are first interesting to velocity fields in

$$\mathbf{u} \in \mathbf{H}^1(\Omega) \quad \text{with} \quad \mathbf{f} \in \mathbf{L}^{6/5}(\Omega).$$

With a such vector \mathbf{u} , we have

$$\mathbf{curl} \mathbf{a} \times \mathbf{u} \in \mathbf{L}^{6/5}(\Omega) \hookrightarrow \mathbf{H}^{-1}(\Omega).$$

Since $\Delta \mathbf{u} \in \mathbf{H}^{-1}(\Omega)$, we deduce from the first equation in (\mathcal{OS}_N) that $\nabla \pi \in \mathbf{H}^{-1}(\Omega)$. Then the pressure π belongs to $L^2(\Omega)$. Furthermore,

$$-\Delta \pi = \operatorname{div} \mathbf{f} - \operatorname{div} (\mathbf{curl} \mathbf{a} \times \mathbf{u}) \quad \text{in } \Omega,$$

so that $\Delta \pi \in \mathbf{W}^{-1,6/5}(\Omega)$ and the **trace** of π on Γ belongs to $H^{-1/2}(\Gamma)$ so that we must assume that

$$\pi_0 \in H^{-1/2}(\Gamma).$$

Theorem

Let

$$\mathbf{f} \in \mathbf{L}^{6/5}(\Omega), \quad \pi_0 \in H^{-1/2}(\Gamma) \quad \text{and} \quad \mathbf{a} \in \mathcal{D}'(\Omega) \quad \text{such that} \quad \mathbf{curl} \mathbf{a} \in \mathbf{L}^{3/2}(\Omega).$$

Then, the problem:

$$\text{Find } (\mathbf{u}, \pi, \mathbf{c}) \in \mathbf{V}_N \times L^2(\Omega) \times \mathbb{R}^{I+1} \text{ satisfying } (\mathcal{OS}_N) \text{ with } \langle \pi, 1 \rangle_\Gamma = 0 \quad (26)$$

is equivalent to the problem: Find $\mathbf{u} \in \mathbf{V}_N$ such that

$$\forall \mathbf{v} \in \mathbf{V}_N, \quad \int_{\Omega} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \mathbf{v} \, dx + \int_{\Omega} (\mathbf{curl} \mathbf{a} \times \mathbf{u}) \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx - \langle \pi_0, \mathbf{v} \cdot \mathbf{n} \rangle_\Gamma \quad (27)$$

and find constants c_0, \dots, c_I satisfying $\sum_{i=0}^I c_i \text{mes } \Gamma_i + \langle \pi_0, 1 \rangle_\Gamma = 0$ and such that for any $i = 1, \dots, I$:

$$c_i - c_0 = \int_{\Omega} \mathbf{f} \cdot \nabla q_i^N \, dx - \int_{\Omega} (\mathbf{curl} \mathbf{a} \times \mathbf{u}) \cdot \nabla q_i^N \, dx - \langle \pi_0, \nabla q_i^N \cdot \mathbf{n} \rangle_\Gamma. \quad (28)$$

Using the **Lax Milgram theorem** and some **regularity** result of the **Laplacian**, we prove the following theorem.

Theorem

Let $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$, $\mathbf{curl} \mathbf{a} \in \mathbf{L}^{3/2}(\Omega)$ and $\pi_0 \in H^{-1/2}(\Gamma)$, then the problem (56) has a unique solution

$(\mathbf{u}, \pi, \mathbf{c}) \in \mathbf{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^{I+1}$ with $\langle \pi, 1 \rangle_\Gamma = 0$ and we have the following estimates:

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \leq C(\|\mathbf{f}\|_{\mathbf{L}^{6/5}(\Omega)} + \|\pi_0\|_{H^{-1/2}(\Gamma)}), \quad (29)$$

$$\|\pi\|_{L^2(\Omega)} \leq C(1 + \|\mathbf{curl} \mathbf{a}\|_{\mathbf{L}^{3/2}(\Omega)})(\|\mathbf{f}\|_{\mathbf{L}^{6/5}(\Omega)} + \|\pi_0\|_{H^{-1/2}(\Gamma)}), \quad (30)$$

where $\mathbf{c} = (c_0, \dots, c_I)$. Moreover, if $\pi_0 \in W^{1/6,6/5}(\Gamma)$ and Ω is $\mathcal{C}^{2,1}$, then $\mathbf{u} \in \mathbf{W}^{2,6/5}(\Omega)$ and $\pi \in W^{1,6/5}(\Omega)$.

Remark

Even if the pressure π change in $\pi - c_0$, the system (OS_N) is equivalent to the following type-Oseen problem:

$$\begin{cases} -\Delta \mathbf{u} + \mathbf{curl} \mathbf{a} \times \mathbf{u} + \nabla \pi = \mathbf{f} & \text{and } \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = \mathbf{0} & & \text{on } \Gamma, \\ \pi = \pi_0 & \text{on } \Gamma_0, & \text{and } \pi = \pi_0 + \alpha_i, \quad i = 1, \dots, I, & \text{on } \Gamma_i, \\ \int_{\Gamma_i} \mathbf{u} \cdot \mathbf{n} \, d\sigma = 0, & i = 1, \dots, I, & & \end{cases}$$

where the unknowns constants satisfy for any $i = 1, \dots, I$:

$$\alpha_i = \int_{\Omega} \mathbf{f} \cdot \nabla q_i^N \, dx - \int_{\Omega} (\mathbf{curl} \mathbf{a} \times \mathbf{u}) \cdot \nabla q_i^N \, dx - \langle \pi_0, \nabla q_i^N \cdot \mathbf{n} \rangle_{\Gamma}.$$

But, it is clear that the new pressure π does not satisfy the condition $\langle \pi, 1 \rangle_{\Gamma} = 0$.

Remark

If we suppose that

$$\mathbf{f} \in [\mathbf{H}_0^{6,2}(\mathbf{curl}, \Omega)]', \quad \mathbf{curl} \mathbf{a} \in \mathbf{L}^{3/2}(\Omega) \quad \text{and} \quad \pi_0 \in H^{-1/2}(\Gamma),$$

then the problems (26) and (27)-(28) are again equivalent, with the difference that we use here the *duality brackets* between $[\mathbf{H}_0^{6,2}(\mathbf{curl}, \Omega)]'$ and $\mathbf{H}_0^{6,2}(\mathbf{curl}, \Omega)$ in place of the integral on Ω in the right hand side of (27) and the *density* of $\mathcal{D}_\sigma(\bar{\Omega}) \times \mathcal{D}(\bar{\Omega})$ in the space

$$\mathcal{M} = \{(\mathbf{u}, \pi) \in \mathbf{H}_\sigma^1(\Omega) \times L^2(\Omega); \quad -\Delta \mathbf{u} + \nabla \pi \in [\mathbf{H}_0^{6,2}(\mathbf{curl}, \Omega)]'\}.$$

It is easy now to extend the previous theorem to the case where

- $\mathbf{f} \in [\mathbf{H}_0^{6,2}(\mathbf{curl}, \Omega)]'$,
- the *divergence operator* does not vanish
- the *boundary conditions* are *nonhomogeneous*.

Theorem

Let $\mathbf{f} \in [\mathbf{H}_0^{6,2}(\mathbf{curl}, \Omega)]'$, $\mathbf{curl} \mathbf{a} \in L^{3/2}(\Omega)$, $\chi \in W^{1,6/5}(\Omega)$, $\pi_0 \in H^{-1/2}(\Gamma)$ and $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$. Then the problem

$$\begin{cases} -\Delta \mathbf{u} + \mathbf{curl} \mathbf{a} \times \mathbf{u} + \nabla \pi = \mathbf{f} & \text{and } \operatorname{div} \mathbf{u} = \chi & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = \mathbf{g} \times \mathbf{n} & & \text{on } \Gamma, \\ \pi = \pi_0 & \text{on } \Gamma_0, & \text{and } \pi = \pi_0 + \alpha_i, \quad i = 1, \dots, I & \text{on } \Gamma_i, \\ \int_{\Gamma_i} \mathbf{u} \cdot \mathbf{n} \, d\sigma = 0, & i = 1, \dots, I, & & \end{cases} \quad (31)$$

has a unique solution $(\mathbf{u}, \pi, \boldsymbol{\alpha}) \in \mathbf{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^I$ verifying the estimate:

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} &\leq C \left(\|\mathbf{f}\|_{[\mathbf{H}_0^{6,2}(\mathbf{curl}, \Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)} + (1 + \|\mathbf{curl} \mathbf{a}\|_{L^{3/2}(\Omega)}) \times \right. \\ &\quad \left. \times (\|\chi\|_{W^{1,6/5}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{1/2}(\Gamma)}) \right), \end{aligned}$$

$$\begin{aligned} \|\pi\|_{L^2(\Omega)} &\leq C (1 + \|\mathbf{curl} \mathbf{a}\|_{L^{3/2}(\Omega)}) \left(\|\mathbf{f}\|_{[\mathbf{H}_0^{6,2}(\mathbf{curl}, \Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)} + \right. \\ &\quad \left. + (1 + \|\mathbf{curl} \mathbf{a}\|_{L^{3/2}(\Omega)}) \times (\|\chi\|_{W^{1,6/5}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{1/2}(\Gamma)}) \right), \end{aligned}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_I)$. Moreover, if $\mathbf{f} \in L^{6/5}(\Omega)$, $\pi_0 \in W^{1/6,6/5}(\Gamma)$, $\mathbf{g} \in W^{7/6,6/5}(\Gamma)$ and Ω is $\mathcal{C}^{2,1}$, then $\mathbf{u} \in W^{2,6/5}(\Omega)$ and $\pi \in W^{1,6/5}(\Omega)$.

In the rest of this talk, we suppose that

$$\Omega \text{ is } C^{2,1}$$

and we are interested in the study of **strong solutions** $\mathbf{u} \in W^{2,p}(\Omega)$ for the system (\mathcal{OS}_N) when $p \geq 6/5$.

When $p < \frac{3}{2}$, because the embedding $W^{2,p}(\Omega) \hookrightarrow W^{1,p^*}(\Omega)$, the term $\mathbf{curl} \mathbf{a} \times \mathbf{u} \in L^p(\Omega)$ and we can use the **regularity results** on the Stokes problem. But this is not more the case when $p \geq \frac{3}{2}$ and that $\mathbf{curl} \mathbf{a}$ belongs only to $L^{3/2}(\Omega)$.

We give in the following theorem the good conditions to ensure the existence of strong solutions.

Theorem

Let $p \geq 6/5$,

$$\mathbf{f} \in L^p(\Omega), \quad \pi_0 \in W^{1-1/p,p}(\Gamma), \quad \mathbf{curl} \mathbf{a} \in L^s(\Omega)$$

with

$$s = \frac{3}{2} \text{ if } p < \frac{3}{2}, \quad s = p \text{ if } p > \frac{3}{2}, \quad s = \frac{3}{2} + \varepsilon \text{ if } p = \frac{3}{2}, \quad (32)$$

for $\varepsilon > 0$ arbitrary. Then the solution (\mathbf{u}, π) given by the previous theorem belongs to $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and satisfies the estimate:

$$\|\mathbf{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \leq C(1 + \|\mathbf{curl} \mathbf{a}\|_{L^s(\Omega)}) (\|\mathbf{f}\|_{L^p(\Omega)} + \|\pi_0\|_{W^{1-1/p,p}(\Gamma)}).$$

In the following theorem, we prove the existence of **Generalized Solutions with $p > 2$** .

Theorem

Let

$$p > 2, \quad \mathbf{f} \in [\mathbf{H}_0^{r', p'}(\mathbf{curl}, \Omega)]', \quad \chi \in W^{1, r}(\Omega) \quad \text{and} \quad \mathbf{g} \in \mathbf{W}^{1-1/p, p}(\Gamma).$$

We suppose that $\pi_0 \in W^{1-1/r, r}(\Gamma)$ and $\mathbf{curl} \mathbf{a} \in L^s(\Omega)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$ and s satisfies:

$$s = \frac{3}{2} \text{ if } 2 < p < 3, \quad s = \frac{3}{2} + \varepsilon \text{ if } p = 3 \text{ and } s = r \text{ if } p > 3,$$

for some arbitrary $\varepsilon > 0$. Then the problem (31) has a unique solution $(\mathbf{u}, \pi, \boldsymbol{\alpha}) \in \mathbf{W}^{1, p}(\Omega) \times W^{1, r}(\Omega) \times \mathbb{R}^I$ satisfying the estimate

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{W}^{1, p}(\Omega)} + \|\pi\|_{W^{1, r}(\Omega)} &\leq C(1 + \|\mathbf{curl} \mathbf{a}\|_{L^s(\Omega)})^2 \left(\|\mathbf{f}\|_{[\mathbf{H}_0^{r', p'}(\mathbf{curl}, \Omega)]'} + \right. \\ &\quad \left. + \|\mathbf{g}\|_{\mathbf{W}^{1-1/p, p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r, r}(\Gamma)} + \|\chi\|_{W^{1, r}(\Omega)} \right) \end{aligned} \quad (33)$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_I)$.

- Generalized Solutions ($p < 2$):

Using a duality argument, we obtain the following result :

Theorem

We suppose that $p < 2$. Soit $\mathbf{f} \in [\mathbf{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, $\mathbf{curl} \mathbf{a} \in L^s(\Omega)$ and $\pi_0 \in W^{1-1/r,r}(\Gamma)$ with

$$r = 1 + \epsilon' \text{ if } p < \frac{3}{2}, \quad r = \frac{9 + 6\epsilon}{9 + 2\epsilon} \text{ if } p = \frac{3}{2} \text{ and } r = \frac{3p}{3 + p} \text{ if } \frac{3}{2} < p < 2, \quad (34)$$

$$s = (1 + \epsilon') \frac{3p}{4p - 3 - \epsilon'(3 - p)} \text{ if } p < \frac{3}{2}, \quad s = \frac{3}{2} + \epsilon \text{ if } p = \frac{3}{2} \text{ and } s = \frac{3}{2} \text{ if } \frac{3}{2} < p < 2, \quad (35)$$

where $\epsilon, \epsilon' > 0$ are arbitrary. Problem (\mathcal{OS}_N) has a unique solution $(\mathbf{u}, \pi, \boldsymbol{\alpha}) \in \mathbf{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ satisfying the estimate:

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C(1 + \|\mathbf{curl} \mathbf{a}\|_{L^s(\Omega)})^2 (\|\mathbf{f}\|_{[\mathbf{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Omega)}),$$

$$\|\pi\|_{W^{1,r}(\Omega)} \leq C(1 + \|\mathbf{curl} \mathbf{a}\|_{L^s(\Omega)})^3 (\|\mathbf{f}\|_{[\mathbf{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Omega)})$$

The Navier-Stokes problem (\mathcal{NS}_N)

$$(\mathcal{NS}_N) \quad \left\{ \begin{array}{ll} -\Delta \mathbf{u} + \mathbf{curl} \mathbf{u} \times \mathbf{u} + \nabla \pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = \chi & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = \mathbf{g} & \text{on } \Gamma, \\ \pi = \pi_0 \text{ on } \Gamma_i \text{ and } \pi = \pi_0 + c_i & \text{on } \Gamma_i, \\ \int_{\Gamma_i} \mathbf{u} \cdot \mathbf{n} \, d\boldsymbol{\sigma} = 0, \quad i = 1, \dots, I, & \end{array} \right.$$

In the search of a proof of the existence of generalized solution for the Navier-Stokes equations (\mathcal{NS}_N), we consider the case of small enough data.

Theorem

Let

$$\mathbf{f} \in [\mathbf{H}_0^{r',p'}(\mathbf{curl}, \Omega)]', \quad \chi \in W^{1,r}(\Omega), \quad \mathbf{g} \in \mathbf{W}^{1-1/p,p}(\Gamma), \quad \pi_0 \in W^{1-1/r,r}(\Gamma)$$

with

$$p > \frac{3}{2} \quad \text{and} \quad r = \frac{3p}{3+p}.$$

i) There exists a constant $\alpha_1 > 0$ such that, if

$$\|\mathbf{f}\|_{[\mathbf{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} + \|\chi\|_{W^{1,r}(\Omega)} + \|\mathbf{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} \leq \alpha_1,$$

then, there exists a solution

$$(\mathbf{u}, \pi, \mathbf{c}) \in \mathbf{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$$

to problem (\mathcal{NS}_N) verifying the estimate

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C(\|\mathbf{f}\|_{[\mathbf{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} + \|\chi\|_{W^{1,r}(\Omega)} + \|\mathbf{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)}),$$

with $c_i = \langle \mathbf{f}, \nabla q_i^N \rangle_{\Omega_{r',p'}} + \int_{\Gamma} (\chi - \pi_0) \nabla q_i^N \cdot \mathbf{n} - \int_{\Omega} (\mathbf{curl} \mathbf{u} \times \mathbf{u}) \cdot \nabla q_i^N$.

ii) Moreover, there exists a constant $\alpha_2 \in]0, \alpha_1]$ such that this solution is unique, if

$$\|f\|_{[\mathbf{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} + \|\chi\|_{W^{1,r}(\Omega)} + \|g\|_{W^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} \leq \alpha_2.$$

For Further Reading



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