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RANDOM WALK ON THE SELF-AVOIDING TREE

CONG BANG HUYNH

ABSTRACT. We consider a modified version of the biased random walk on a tree constructed from the set of finite self-avoiding walks on the hexagonal lattice, and use it to construct probability measures on infinite self-avoiding walk. Under these probability measures, we prove that the infinite self-avoiding walks have the Russo-Seymour-Welsh property of the exploration curve of the supercritical Bernoulli percolation.

Keywords: Self-avoiding walk, Bernoulli percolation, random walk on tree.

1. INTRODUCTION

1.1. Background. An n -step self-avoiding walk (SAW) (or a self-avoiding walk of length n) in a regular lattice \mathbb{L} (such as the integer lattice \mathbb{Z}^2 , triangular lattice \mathbb{T} , hexagonal lattice \mathbb{T}^* , etc) is a nearest neighbor path $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_n)$ that visits no vertex more than once.

In this paper, we will focus on the case of hexagonal lattice $\mathbb{T}_+^* := \mathbb{T}^* \cap \{y \geq 0\}$ (see Figure 3). Let a be a vertex of \mathbb{T}_+^* and let b, c, d be three neighbors of a as Figure 1. In this case, we say that c (resp. d) is the right (resp. left) neighbor of a with respect to the vector \vec{ba} .

Let $\mathcal{T}_{\mathbb{T}_+^*}$ be the tree whose vertices are the finite self-avoiding walks in \mathbb{T}_+^* starting at the origin $o := (0, 0)$, where two such vertices are adjacent when one walk is a one-step extension of the other. We will call this tree the *self-avoiding tree* on \mathbb{T}_+^* . Formally, denote by Ω_n the set of self-avoiding walks of length n starting at the origin and $V := \bigcup_{n=0}^{+\infty} \Omega_n$. Two elements $x, y \in V$ are adjacent if one path is an extension by one step of the other. We then define $\mathcal{T}_{\mathbb{T}_+^*} = (V, E)$. Denote by o its root.

Remark 1. Note that each vertex (resp. an infinite branch) of $\mathcal{T}_{\mathbb{T}_+^*}$ is a finite self-avoiding path (resp. an infinite self-avoiding path). Moreover, it is easy to see that the number of vertices at generation n of $\mathcal{T}_{\mathbb{T}_+^*}$ is the number of self-avoiding walks of length n in \mathbb{T}_+^* .

For any vertex ν of $\mathcal{T}_{\mathbb{T}_+^*}$, denote by ν^{-1} its *parent* (we also say that ν is a *child* of ν^{-1}), i.e. the neighbor of ν with shortest distance from the root o . Denote by $\partial(\nu)$ the number of children of ν . In the case $\partial(\nu) \neq 0$, denote by $\nu_1, \dots, \nu_{\partial(\nu)}$ its children. We define an order on $V(\mathcal{T}_{\mathbb{T}_+^*})$ as follows: if $\nu, \mu \in V(\mathcal{T}_{\mathbb{T}_+^*})$, we say that $\nu \leq \mu$ if the simple path joining o to μ passes through ν . For each $\nu \in V(\mathcal{T}_{\mathbb{T}_+^*})$, we define the *subtree* of $\mathcal{T}_{\mathbb{T}_+^*}$ rooted at ν , denoted by $(\mathcal{T}_{\mathbb{T}_+^*})^\nu$, where $V((\mathcal{T}_{\mathbb{T}_+^*})^\nu) := \{\mu \in V(\mathcal{T}_{\mathbb{T}_+^*}) : \nu \leq \mu\}$ and $E((\mathcal{T}_{\mathbb{T}_+^*})^\nu) = E(\mathcal{T}_{\mathbb{T}_+^*})|_{V((\mathcal{T}_{\mathbb{T}_+^*})^\nu) \times V((\mathcal{T}_{\mathbb{T}_+^*})^\nu)}$. Note that ν is the root of $(\mathcal{T}_{\mathbb{T}_+^*})^\nu$ and $(\mathcal{T}_{\mathbb{T}_+^*})^o = \mathcal{T}$.

Key words and phrases. Self-avoiding walk, effective conductance, random walk on tree.

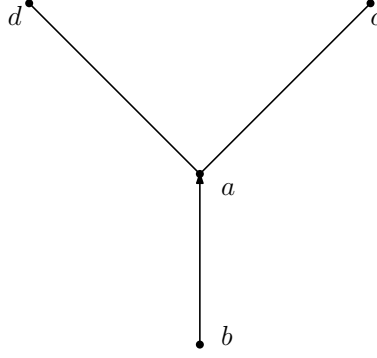


FIGURE 1. c (resp. d) is the right (resp. left) neighbor of a with respect to the vector \vec{ba} .

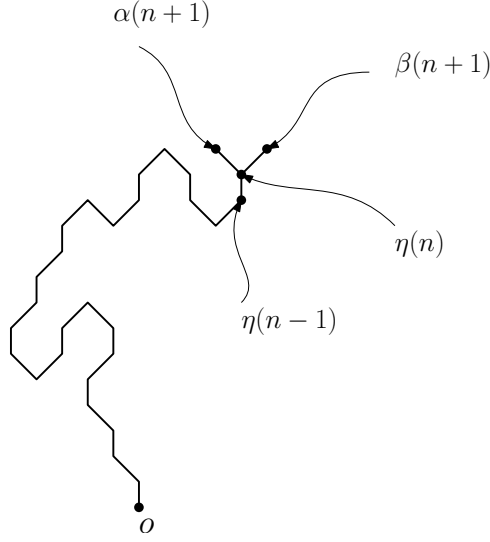


FIGURE 2. Two extensions α and β of ν

Convention: Let ν be a vertex of $\mathcal{T}_{\mathbb{T}_+^*}$ at distance n from the root (i.e $|\nu| = n$) and assume that $\partial\nu = 2$. By the construction of $\mathcal{T}_{\mathbb{T}_+^*}$, ν is a self-avoiding walk of length n starting at the origin. Let α (resp. β) be the extension by one step of ν by choosing the left (resp. right) neighbor of $\nu(n)$ (see Figure 2). We then define two children of the vertex ν in the tree $\mathcal{T}_{\mathbb{T}_+^*}$ by letting: $\nu_1 = \alpha$ and $\nu_2 = \beta$.

1.2. Random walk on the self-avoiding tree. Let $\lambda > 0$ and $\eta \in [0, 1/2]$ be such that

$$(1.1) \quad \frac{\lambda}{1 + 2\lambda} - \eta \geq 0$$

Define a Markov process $\mathbf{X} := (X_n)_{n \geq 0}$ on some probability space, taking the values in $\mathcal{T}_{\mathbb{T}_+^*}$ with the transition probability defined by the following way.

$$\mathbb{P}(X_0 = o) = 1,$$

- If o has only one child o_1 then $\mathbb{P}(X_1 = o_1) = 1$ and if o has two children o_1 and o_2 , then

$$\mathbb{P}(X_1 = o_1) = \frac{\lambda - \eta(1 + 2\lambda)}{2\lambda}; \quad \mathbb{P}(X_1 = o_2) = \frac{\lambda + \eta(1 + 2\lambda)}{2\lambda}.$$

- If $\partial(X_n) = 2$, then

$$(1.2) \quad \mathbb{P}(X_{n+1} = (X_n)_1 | X_0, \dots, X_n; X_n \neq o) = \frac{\lambda}{1 + 2\lambda} - \eta;$$

$$(1.3) \quad \mathbb{P}(X_{n+1} = (X_n)_2 | X_0, \dots, X_n; X_n \neq o) = \frac{\lambda}{1 + 2\lambda} + \eta;$$

$$(1.4) \quad \mathbb{P}(X_{n+1} = (X_n)^{-1} | X_0, \dots, X_n; X_n \neq o) = \frac{1}{1 + 2\lambda}.$$

- If $\partial(X_n) = 1$, then

$$(1.5) \quad \mathbb{P}(X_{n+1} = (X_n)_1 | X_0, \dots, X_n; X_n \neq o) = \frac{\lambda}{1 + \lambda};$$

$$(1.6) \quad \mathbb{P}(X_{n+1} = (X_n)^{-1} | X_0, \dots, X_n; X_n \neq o) = \frac{1}{1 + \lambda}.$$

(1.7)

- If $\partial(X_n) = 0$ (i.e X_n have no child), then

$$(1.8) \quad \mathbb{P}(X_{n+1} = (X_n)^{-1} | X_0, \dots, X_n) = 1.$$

The stochastic process \mathbf{X} is called *biased random walk* with parameters (λ, η) , and denoted by $RW_{\lambda, \eta}$.

Remark 2. • If $\eta = 0$, then \mathbf{X} is the biased random walk with parameter λ .

- When the bias λ converges to ∞ , we obtain a stochastic process which is called biased random walk with parameters (∞, η) , and denoted by $RW_{\infty, \eta}$.

Let $\lambda \in [0, +\infty]$ and $\eta \in [0, 1/2]$ such that $\frac{\lambda}{1+2\lambda} - \eta \geq 0$ and consider the biased random walk $RW_{\lambda, \eta}$ on $\mathcal{T}_{\mathbb{T}_+^*}$. For (λ, η) such that the biased random walk $RW_{\lambda, \eta}$ on $\mathcal{T}_{\mathbb{T}_+^*}$ is transient, then almost surely, the random walk does not visit $(\mathcal{T}_{\mathbb{T}_+^*})_k^1$ anymore after a sufficiently large time. We can then define the *limit walk*, as denoted by $\omega_{\lambda, \eta}^\infty$ in the following way:

$$\omega_{\lambda, \eta}^\infty(i) = x_i \iff \left\{ \begin{array}{l} x_i \in (\mathcal{T}_{\mathbb{T}_+^*})_i \\ \exists n_0, \forall n > n_0 : X_n \in (\mathcal{T}_{\mathbb{T}_+^*})^{x_i} \end{array} \right\}.$$

Denote by $\mathbb{P}_{\lambda, \eta}$ the law of $\omega_{\lambda, \eta}^\infty$.

1.3. Main results. Note that $\mathbb{P}_{\lambda, \eta}$ is a probability measure on the set of infinite self-avoiding paths starting at the origin (denoted by SAW_∞) in the lattice \mathbb{T}^* . By the same argument used in ([2], Section 6.2) and Remark 2, we can see that $\omega_{\infty, \eta}^\infty$ (i.e $\lambda = \infty$) can be interpreted as the exploration curve $\gamma^{1/2+\eta}$ of Bernoulli percolation on the hexagonal lattice. This is very useful because every feature of the curve $\gamma^{1/2+\eta}$ is also one for $\omega_{\infty, 0}^\infty$ and can therefore be restated in terms of the biased walk on the self-avoiding tree. One of these properties is that

¹Denote by $(\mathcal{T}_{\mathbb{T}_+^*})_k$ the set of vertices of $\mathcal{T}_{\mathbb{T}_+^*}$ at distance k from the root.

$\gamma^{1/2+\eta}$ reaches the interval $[n, 2n] \times \{0\}^2$ with a probability larger than a constant which do not depend on n . This property is called *RSW-property* (see [7] and [6] for the case $\eta = 0$; see [1] for the case $\eta > 0$).

In this paper, we prove that if $\eta > 0$ and λ is large enough, then the limit walk $\omega_{\lambda,\eta}^\infty$ has RSW-property:

Theorem 3. *For all $\eta \in]0, 1/2[$ and for all $\lambda > \max\left(4/\eta; \frac{\eta}{1-2\eta}\right)$: $\exists \varepsilon > 0, c > 0, \forall n \geq 1$, we have:*

$$\mathbb{P}_{\lambda,\eta}(\omega_{\lambda,\eta}^\infty \cap ([n, 2n] \times \{0\}) \neq \emptyset) \geq (1 - cn^{-\varepsilon})^3.$$

Recall that if \mathcal{T} is a tree, we denote by $\tilde{\mathcal{T}}$ the subtree obtained from \mathcal{T} by recursively erasing all its leaves; in terms of our dynamical self-avoiding walk model, this corresponds to preventing the path from entering traps. The reader can easily check that the limit walk is the same on these trees without leaves as in the original ones, it is sufficient to prove Theorem 3 in the case of $\tilde{\mathcal{T}}_{\mathbb{T}_+^*}$.

2. EXPLORATION CURVE OF BERNOULLI PERCOLATION ON HEXAGONAL LATTICE

First, we review some definitions of percolation theory. Percolation was introduced by Broadbent and Hammersley in 1957 [3]. For $p \in [0, 1]$, we consider the triangular lattice \mathbb{T} , a site of \mathbb{T} is open with probability p or closed with probability $1 - p$, independently of the others. This can also be seen as a random colouring (in black or white) of the faces of hexagonal lattice \mathbb{T}^* dual of \mathbb{T} .

We define the exploration curve as follows (see [2], section 6.2 for more detail). Let Ω be a simply connected subgraph of the triangular lattice and A, B be two points on its boundary. We can then divide the hexagonal cells of $\partial\Omega$ into two arcs, going from A to B in two directions (clockwise and counter-clockwise). These arcs will be denoted by \mathbb{B} and \mathbb{W} such that $A, \mathbb{B}, B, \mathbb{W}$ is in the clockwise direction. Assume that all of the hexagons in \mathbb{B} are colored in black and that all of the hexagons in \mathbb{W} are colored in white. The color of the hexagonal faces in Ω is chosen at random (black with probability p and white with probability $1 - p$), independently of the others. We define the *exploration curve* γ^p starting at A and ending at B which separates the black component containing \mathbb{B} from the white component containing \mathbb{W} . Then the exploration curve γ^p is a self-avoiding walk using the vertices and edges of hexagonal lattice \mathbb{T}^* .

We can define this interface γ^p in an equivalent, dynamical way, informally described as follows. At each step, γ^p looks at its three neighbors on the hexagonal lattice, one of which is occupied by the previous step of γ^p . For the next step, γ^p randomly chooses one of these neighbors that has not yet occupied by γ^p . If there is just one neighbor that has not yet been occupied, then we choose this neighbor and if there are two neighbors, then we choose the right neighbor with probability p and the left neighbor with probability $1 - p$.

²Denote by $\partial^+(\mathbb{T}_+^*)$ the set of hexagons on the right side of the origin. The subset of $\partial^+(\mathbb{T}_+^*)$ which contains the hexagons at the distance k from the origin ($n \leq k \leq 2n$) is denoted by $[n, 2n] \times \{0\}$. See Figure 3.

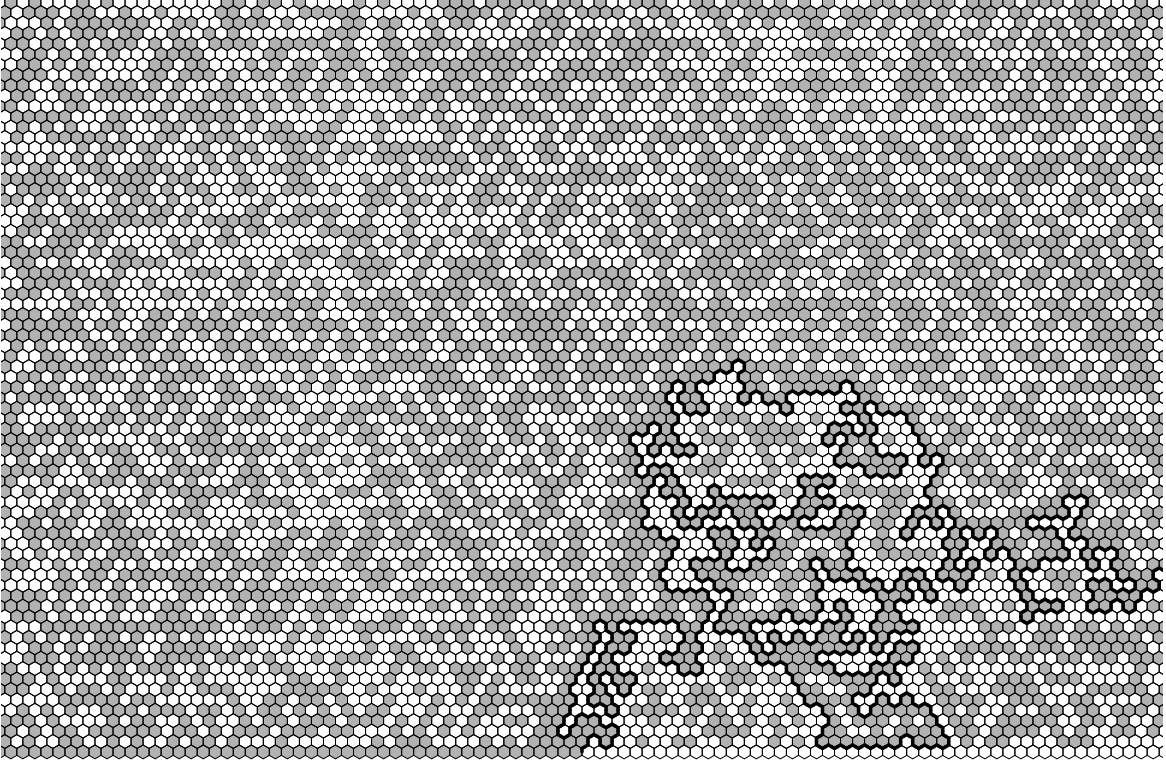


FIGURE 3. The hexagons on the right side of origin (i.e $\partial^+(\mathbb{T}_+^*)$) are colored in white and the hexagons on the left side of origin (i.e $\partial^-(\mathbb{T}_+^*)$) are colored in black.

We know that there exists $p_c = 1/2$ such that for $p < p_c$ there is almost surely no infinite cluster, while for $p > p_c$ there is almost surely an infinite cluster ([8], Theorem 4.9).

Now, we take $\Omega = \mathbb{T}_+^*$. The hexagons on the boundary of Ω ($\partial\Omega$) and on the right of origin (denoted by $\partial^+\Omega$) are colored in white and the hexagons on $\partial\Omega$ and on the left of origin ($\partial^-\Omega$) are colored in black. In this case, the exploration curve is an (random) infinite self-avoiding walk. See Figure 3.

Now, we recall some results of Bernoulli percolation on the hexagonal lattice. We denote $A[2n, n]$ being the event that exists a path formed of black faces which is contained in the rectangle $[0, 2n] \times [0, n]$ from $\{0\} \times [0, n]$ to $\{2n\} \times [0, n]$.

Lemma 4 ([1], Lemma 4.2). *For any $p > \frac{1}{2}$, there exists $\varepsilon = \varepsilon(p) > 0$ and $c = c(p) > 0$ such that for all $n \geq 1$,*

$$\mathbb{P}_p(A[2n, n]) \geq 1 - cn^{-\varepsilon}.$$

A simple consequence of Lemma 4 and FKG inequality [4] is the following result:

Lemma 5. *Let $p > \frac{1}{2}$, there exists $\varepsilon = \varepsilon(p) > 0$ and $c = c(p) > 0$ such that for any $n \geq 1$:*

$$(2.1) \quad \mathbb{P}_p(\gamma^p \cap ([n, 2n] \times \{0\}) \neq \emptyset) \geq (1 - cn^{-\varepsilon})^3.$$

3. PROOF OF THEOREM 3

3.1. The law of first steps of the limit walk. We consider the biased random walk $RW_{\lambda,\eta}$ on $\tilde{\mathcal{T}}_{\mathbb{T}_+^*}$. Recall that $\omega_{\lambda,\eta}^\infty$ is the associated limit walk and $\mathbb{P}_{\lambda,\eta}$ denotes its law.

Let $k \in \mathbb{N}^*$ and y_1, y_2, \dots, y_k be k elements of $V(\tilde{\mathcal{T}}_{\mathbb{T}_+^*})$ such that the path $(o, y_1, y_2, \dots, y_k)$ in $\tilde{\mathcal{T}}_{\mathbb{T}_+^*}$ is simple. For each λ such that $RW_{\lambda,\eta}$ on $\tilde{\mathcal{T}}_{\mathbb{T}_+^*}$ is transient, the law of first k steps of $\omega_{\lambda,\eta}^\infty$ is defined by:

$$(3.1) \quad \varphi^{\lambda,\eta,k}(y_1, y_2, \dots, y_k) = \mathbb{P}_{\lambda,\eta}(\omega_{\lambda,\eta}^\infty(0) = o, \omega_{\lambda,\eta}^\infty(1) = y_1, \omega_{\lambda,\eta}^\infty(2) = y_2, \dots, \omega_{\lambda,\eta}^\infty(k) = y_k).$$

Notation. Let ν be a vertex of the tree $\mathcal{T}_{\mathbb{T}_+^*}$ and let μ be a vertex of $(\mathcal{T}_{\mathbb{T}_+^*})^\nu$. Denote by $\tilde{\mathcal{C}}(\lambda, \eta, (\mathcal{T}_{\mathbb{T}_+^*})^\nu, \mu)$ for the probability of the event that the random walk $RW_{\lambda,\eta}$ on $(\mathcal{T}_{\mathbb{T}_+^*})^\nu$, started at the root (i.e $X_0 = \nu$), visits μ at its first step (i.e $X_1 = \mu$) and never returns to the root. Finally, denote by $\tilde{\mathcal{C}}(\lambda, \eta, (\mathcal{T}_{\mathbb{T}_+^*})^\nu)$ for the probability of the event that the random walk $RW_{\lambda,\eta}$ on $(\mathcal{T}_{\mathbb{T}_+^*})^\nu$, started at the root (i.e $X_0 = \nu$) and never returns to the root.

Lemma 6 ([2], Lemma 64). *Let $k \in \mathbb{N}^*$ and y_1, y_2, \dots, y_k be k elements of $V(\tilde{\mathcal{T}}_{\mathbb{T}_+^*})$ such that $(o, y_1, y_2, \dots, y_k)$ is a simple path starting at o of $\tilde{\mathcal{T}}_{\mathbb{T}_+^*}$. We then have*

$$\varphi^{\lambda,\eta,k}(y_1, y_2, \dots, y_k) = \frac{\tilde{\mathcal{C}}(\lambda, \eta, \mathcal{T}_{\mathbb{T}_+^*}, y_1)}{\tilde{\mathcal{C}}(\lambda, \eta, \mathcal{T}_{\mathbb{T}_+^*})} \times \frac{\tilde{\mathcal{C}}(\lambda, \eta, (\mathcal{T}_{\mathbb{T}_+^*})^{y_1}, y_2)}{\tilde{\mathcal{C}}(\lambda, \eta, (\mathcal{T}_{\mathbb{T}_+^*})^{y_1})} \times \dots \times \frac{\tilde{\mathcal{C}}(\lambda, \eta, (\mathcal{T}_{\mathbb{T}_+^*})^{y_{k-1}}, y_k)}{\tilde{\mathcal{C}}(\lambda, \eta, (\mathcal{T}_{\mathbb{T}_+^*})^{y_{k-1}})}.$$

Fix $\eta \in [0, 1/2]$ and $\lambda > 0$ such that $RW_{\lambda,\eta}$ on $\mathcal{T}_{\mathbb{T}_+^*}$ is transient. For each finite path ω of $\tilde{\mathcal{T}}_{\mathbb{T}_+^*}$ starting at o , such that $\omega_{|\omega|}$ has two children, we define:

$$(3.2) \quad \alpha_\omega := \mathbb{P}\left(\omega_{\lambda,\eta}^\infty(|\omega| + 1) = (\omega_{|\omega|})_2 \mid (\omega_{\lambda,\eta}^\infty)_{|[0, |\omega|]} = \omega\right).$$

By using Lemma 6, we obtain:

$$(3.3) \quad \alpha_\omega = \frac{\tilde{\mathcal{C}}(\lambda, \eta, (\tilde{\mathcal{T}}_{\mathbb{T}_+^*})^{\omega_{(|\omega|)}}, (\omega_{|\omega|})_2)}{\tilde{\mathcal{C}}(\lambda, \eta, (\tilde{\mathcal{T}}_{\mathbb{T}_+^*})^{\omega_{(|\omega|)}})}.$$

Denote by \mathcal{A} the set of finite paths ω of $\tilde{\mathcal{T}}_{\mathbb{T}_+^*}$ such that α_ω is well defined.

Lemma 7. *For all $\eta \in]0, 1/2[$ and for all $\lambda > \max\left(4/\eta; \frac{\eta}{1-2\eta}\right)$, we have:*

$$(3.4) \quad \min_{\omega \in \mathcal{A}} \alpha_\omega \geq 1/2 + \eta/2.$$

Proof. Fix $\eta \in]0, 1/2[$ and $\lambda > 4/\eta$. Let $\omega \in \mathcal{A}$ and consider $(X_n)_{n \geq 0}$ be the random walk $RW_{\lambda,\eta}$ on $(\tilde{\mathcal{T}}_{\mathbb{T}_+^*})^{\omega_{(|\omega|)}}$ started at its root $\omega_{(|\omega|)}$. We divide $(\tilde{\mathcal{T}}_{\mathbb{T}_+^*})^{\omega_{(|\omega|)}}$ into two sub-trees \mathcal{T}_1 and \mathcal{T}_2 presented in Figure 4. We then have:

$$\begin{aligned}
(3.5) \quad \tilde{\mathcal{C}}(\lambda, \eta, (\tilde{\mathcal{T}}_{\mathbb{T}_+^*})^{\omega(|\omega|)}, (\omega_{|\omega|})_2) &\geq \mathbb{P}(X_0 = \omega_{|\omega|}; X_1 = (\omega_{|\omega|})_2) \text{ and } \forall n \geq 1 : X_n \neq \omega_{|\omega|} \\
&\geq \frac{\lambda + \eta(1 + 2\lambda)}{2\lambda} \tilde{\mathcal{C}}(\lambda, \eta, \mathcal{T}_1) \\
&\geq \left(\frac{\lambda}{1 + 2\lambda} + \eta \right) \tilde{\mathcal{C}}(\lambda, \eta, \mathcal{T}_1).
\end{aligned}$$

Let \mathbb{N} be the regular tree of degree 1 and denote by $\mathcal{C}(\lambda, \mathbb{N})$ the effective conductance of biased random walk on \mathbb{N} started at the root with parameter λ . By Rayleigh's monotonicity principle (see [5], page 35), we have:

$$(3.6) \quad \tilde{\mathcal{C}}(\lambda, \eta, \mathcal{T}_1) \geq \mathcal{C}(\lambda, \mathbb{N}).$$

On the other hand, we have:

$$(3.7) \quad \mathcal{C}(\lambda, \mathbb{N}) = \frac{\lambda - 1}{\lambda}.$$

Hence by (3.5), (3.6) and (3.7), we obtain:

$$(3.8) \quad \tilde{\mathcal{C}}(\lambda, \eta, (\tilde{\mathcal{T}}_{\mathbb{T}_+^*})^{\omega(|\omega|)}, (\omega_{|\omega|})_2) \geq \frac{\lambda - 1}{\lambda} \left(\frac{\lambda}{1 + 2\lambda} + \eta \right)$$

Since $\lambda > \max\left(4/\eta; \frac{\eta}{1-2\eta}\right)$, by an simple computation we obtain:

$$(3.9) \quad \frac{\lambda - 1}{\lambda} \left(\frac{\lambda}{1 + 2\lambda} + \eta \right) \geq 1/2 + \eta/2.$$

By (3.8) and (3.9), we obtain:

$$(3.10) \quad \tilde{\mathcal{C}}(\lambda, \eta, (\tilde{\mathcal{T}}_{\mathbb{T}_+^*})^{\omega(|\omega|)}, (\omega_{|\omega|})_2) \geq 1/2 + \eta/2.$$

It is clear that $\tilde{\mathcal{C}}(\lambda, \eta, (\tilde{\mathcal{T}}_{\mathbb{T}_+^*})^{\omega(|\omega|)}) \leq 1$, hence we obtain:

$$\alpha_\omega \geq \tilde{\mathcal{C}}(\lambda, \eta, (\tilde{\mathcal{T}}_{\mathbb{T}_+^*})^{\omega(|\omega|)}, (\omega_{|\omega|})_2) \geq 1/2 + \eta/2,$$

this completes the proof of lemma. \square

3.2. Proof of Theorem 3. Consider the supercritical Bernoulli percolation on \mathbb{T}_+^* with parameter $1/2 + \eta/2$. Given a configuration of percolation, we construct a random path $\gamma_{\lambda, \eta}^\infty$ starting at $o = (0, 0)$ by the following way. At step n , $\gamma_{\lambda, \eta}^\infty$ looks at its three neighbors on the hexagonal lattice, one of which is occupied by the previous step of γ^p . For the next step, $\gamma_{\lambda, \eta}^\infty$ randomly chooses one of these neighbors that has not yet occupied by γ^p . If there is just one neighbor that has not yet been occupied, then we choose this neighbor. If there are two neighbors, then we choose the right neighbor and the left neighbor by the following rule. Let h be the hexagon which contains these neighbors and let γ be such that $(\gamma_{\lambda, \eta}^\infty)_{|[0, n]} = \gamma$:

- If h is black, we choose the right neighbor;

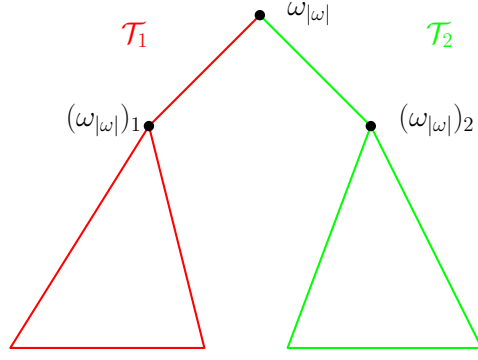


FIGURE 4.

- If h is white, we have two possibilities:
 - (1) we choose the right neighbor with probability $\frac{\alpha_\gamma - (1/2 + \eta/2)}{1 - (1/2 + \eta/2)} \geq 0$ (by Lemma 7);
 - (2) we choose the left neighbor with probability $\frac{1 - \alpha_\gamma}{1 - (1/2 + \eta/2)}$.

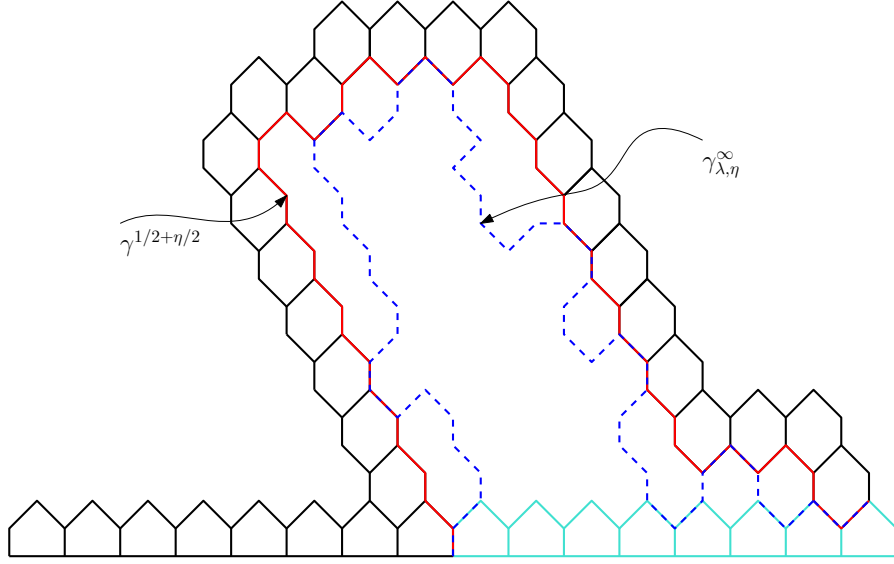


FIGURE 5. The exploration curve $\gamma^{1/2+\eta/2}$ is the red path and $\gamma_{\lambda,\eta}^\infty$ is the blue path. At each step, if $\gamma_{\lambda,\eta}^\infty$ visits a black hexagon: it always chooses the right neighbor.

Lemma 8. $(\gamma_{\lambda,\eta}^\infty)$ has the same law as $(\omega_{\lambda,\eta}^\infty)$.

Proof. First, by the construction of $\gamma_{\lambda,\eta}^\infty$ and the definition of limit walk, we have:

$$\mathbb{P}(\gamma_{\lambda,\eta}^\infty(0) = o) = \mathbb{P}(\omega_{\lambda,\eta}^\infty(0) = o) = 1.$$

Let $n > 0$ and denote by \mathcal{A} the set of self-avoiding walk of length n starting at o which can extend to infinity (i.e the set of vertices of $\tilde{\mathcal{T}}_{\mathbb{T}_+^*}$ at distance n from the root). Assume that for all $\gamma \in \mathcal{A}$, we have:

$$(3.11) \quad \mathbb{P}((\gamma_{\lambda,\eta}^\infty)_{|[0,n]} = \gamma) = \mathbb{P}((\omega_{\lambda,\eta}^\infty)_{|[0,n]} = \gamma).$$

Let γ be an element of \mathcal{A} . We have two possibilities:

- If there is only one way to extend γ to a self-avoiding walk γ_1 of length $n+1$ (i.e the vertex γ of $\tilde{\mathcal{T}}_{\mathbb{T}_+^*}$ has only one child γ_1), we then have:

$$\mathbb{P}(\gamma_{\lambda,\eta}^\infty(n+1) = \gamma_1 \mid (\gamma_{\lambda,\eta}^\infty)_{|[0,n]} = \gamma) = \mathbb{P}(\omega_{\lambda,\eta}^\infty(n+1) = \gamma_1 \mid (\omega_{\lambda,\eta}^\infty)_{|[0,n]} = \gamma) = 1.$$

- If γ has two children γ_1 and γ_2 , by the construction of $\gamma_{\lambda,\eta}^\infty$, we have:

$$(3.12) \quad \begin{aligned} \mathbb{P}(\gamma_{\lambda,\eta}^\infty(n+1) = \gamma_2 \mid (\gamma_{\lambda,\eta}^\infty)_{|[0,n]} = \gamma) &= 1/2 + \eta/2 + (1 - (1/2 + \eta/2)) \frac{\alpha_\gamma - (1/2 + \eta/2)}{1 - (1/2 + \eta/2)} \\ &= \alpha_\gamma. \end{aligned}$$

Hence by (3.3) and (3.12) we obtain:

$$\mathbb{P}(\gamma_{\lambda,\eta}^\infty(n+1) = \gamma_2 \mid (\gamma_{\lambda,\eta}^\infty)_{|[0,n]} = \gamma) = \mathbb{P}(\omega_{\lambda,\eta}^\infty(n+1) = \gamma_2 \mid (\omega_{\lambda,\eta}^\infty)_{|[0,n]} = \gamma) = \alpha_\gamma,$$

and

$$\mathbb{P}(\gamma_{\lambda,\eta}^\infty(n+1) = \gamma_1 \mid (\gamma_{\lambda,\eta}^\infty)_{|[0,n]} = \gamma) = \mathbb{P}(\omega_{\lambda,\eta}^\infty(n+1) = \gamma_1 \mid (\omega_{\lambda,\eta}^\infty)_{|[0,n]} = \gamma) = 1 - \alpha_\gamma.$$

□

Lemma 9. *We have the following inequality:*

$$\mathbb{P}(\gamma_{\lambda,\eta}^\infty \cap ([n, 2n] \times \{0\}) \neq \emptyset) \geq \mathbb{P}(\gamma^{1/2+\eta/2} \cap ([n, 2n] \times \{0\}) \neq \emptyset).$$

Proof. This is intuitively clear: informally, by the construction of $\gamma_{\lambda,\eta}^\infty$, the path $\gamma_{\lambda,\eta}^\infty$ always stays on the right of $\gamma^{1/2+\eta/2}$ (see Figure 5). A formal proof is easy but tedious to write, and is therefore omitted here. □

Theorem 3 is a straightforward consequence of Lemma 5, Lemma 8 and Lemma 9.

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