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# Simultaneous Stabilization of Traffic Flow on Two Connected Roads

Huan Yu\*, Jean Auriol<sup>†</sup>, Miroslav Krstic\*

**Abstract**—In this paper we develop a boundary state feedback control law for a cascaded traffic flow network system: one incoming and one outgoing road connected by a junction. The macroscopic traffic dynamics on each road segment are governed by Aw-Rascle-Zhang (ARZ) model, consisting of second-order nonlinear partial differential equations (PDEs) for traffic density and velocity. Different equilibrium road conditions are considered for the two segments. For stabilization of stop-and-go traffic congestion on the two roads, we consider a ramp metering located at the connecting junction. The traffic flow rate entering from the on-ramp to the mainline junction is actuated. The objective is to simultaneously stabilize the upstream and downstream traffic to given spatially-uniform constant steady states. We design a full state feedback control law for this under-actuated network of two systems of two hetero-directional linear first-order hyperbolic PDEs interconnected through the junction boundary. Exponential Convergence to steady states in  $L^2$  sense is validated by a numerical simulation.

## I. INTRODUCTION

Freeway traffic modeling and management has been intensively investigated over the past decades. Motivations behind are to understand the formation of traffic congestion, and further to prevent or suppress instabilities of traffic flow. Macroscopic modeling of traffic dynamics is used to describe the evolution of aggregated traffic state values on road. Traffic dynamics are governed by hyperbolic PDEs. The most widely-used macroscopic traffic PDE models include the first-order Lighthill-Whitham-Richards (LWR) model and the second-order Aw-Rascle-Zhang (ARZ) model [2] [27]. The LWR model corresponds to a conservation law of the traffic density. It can predict the formation and propagation of traffic shockwaves on freeway, but fails to describe the stop-and-go phenomenon [11] [20] [25]. The oscillations of densities and velocities travel with traffic stream, cause unsafe driving conditions, increased consumptions of fuel and delay of travel time. The second-order ARZ traffic model is developed to describe this common phenomenon. It consists of a set of nonlinear hyperbolic PDEs describing the evolution of the traffic density and velocity. More recently, the macroscopic modeling of road networks based on the ARZ model has been developed in [15] [17]. We consider to use the second-order macroscopic model to describe the stop-and-go traffic congestion in this work.

Traffic control strategies are developed and implemented for the traffic management infrastructures, including ramp

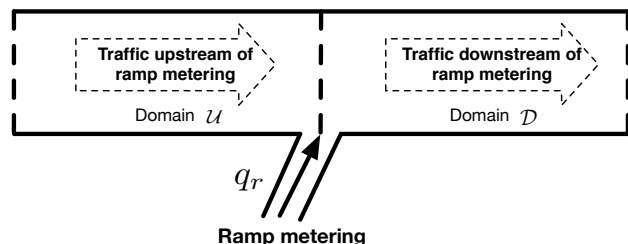


Fig. 1: Boundary feedback control of freeway traffic through a ramp metering, upstream freeway traffic and downstream traffic of the ramp are simultaneously stabilized.

metering and varying speed limits. Boundary feedback control algorithms are studied for traffic regulation on a freeway segment in [18] [19] [29] [30] [31]. In authors' previous work [29]-[30], backstepping boundary control laws for ramp metering are designed to suppress the stop-and-go traffic oscillations on freeway either upstream or downstream of the ramp. In Fig 1, the traffic flux is actuated through the traffic lights located on the ramp so that the upstream traffic in domain  $\mathcal{U}$  or the downstream traffic in domain  $\mathcal{D}$  can be controlled. However, the upstream and the downstream traffic can not be stabilized simultaneously and distinct traffic scenarios are not considered for different traffic segments. In this paper, we aim to solve two questions that remained unanswered in the previous work by reformulating the problem in a network setting and providing a more applicable control design approach. The first question is how the ramp metering control of the upstream traffic affects the downstream traffic. The second question is how we can control the downstream and the upstream traffic simultaneously. Furthermore, the control of this fundamental structure of the two connected freeway segments will be an important milestone for the control problem of the macroscopic modeling of a traffic road network.

We adopt the traffic network model proposed in [17]. The modeling of junction of the two connected roads conserves mass and the other traffic property as detailed later in the paper. This property is not smooth across the junction in [15]. In comparison, the solution in [17] is a weak solution of the conservative variables that guarantees the well-posedness of the closed-loop system for our control design. The mathematical model is a network of two interconnected PDE systems coupled through their boundaries. Each subsystem corresponds to a  $2 \times 2$  coupled hyperbolic system. Despite the fact that numerous theoretical results in the literature are focused on boundary control of this class of hyperbolic system based on backstepping approach [1] [6] [10] [12] [26] [28], control

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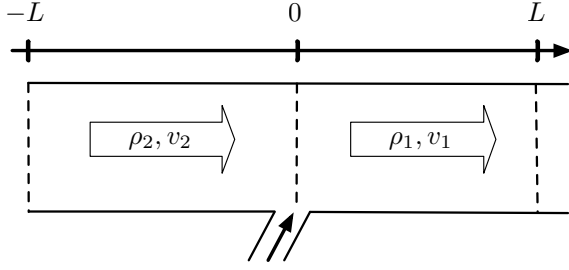


Fig. 2: Traffic flow on an incoming road and an outgoing road connected with a junction, actuation is implemented at the junction.

of the network of PDEs remains a challenging research topic. This is due to the fact that in most cases, these systems are underactuated (only the PDE located at one extremity of the network can be actuated). To tackle this problem, multiple approaches have been proposed: PI boundary controllers [7], [8], flatness based design of feedforward control laws [23], [24] and more recently backstepping-based control laws [4]. The main contribution of this paper is to provide an explicit control design that simultaneously stabilizes the traffic flow on the cascaded roads.

The paper is organized as follows. In Section II, we introduce the system under consideration. In particular, we give the PDEs describing the dynamics of the traffic density and velocity. These equations are then linearized around a given steady-state. A stabilizing state-feedback control law is obtained in Section III for this underactuated system using a backstepping approach. Some simulation results are presented in Section IV. Finally, some concluding remarks are given in Section V.

## II. PROBLEM STATEMENT

We consider a road network that contains two cascaded road segments with unidirectional traffic flow. The road conditions and properties are different for the two segments. They are assumed to have the same length  $L$  for simplicity. We can easily rescale the PDEs that describing segments with different lengths in space and the control design we propose can be directly applied. The outgoing road segment is defined on  $[0, L]$  while the incoming road segment is defined on  $[-L, 0]$ . These two segments are connected at the junction boundary  $x = 0$ . The traffic dynamics of road segments are described with PDEs and boundary conditions for the junction between the two segments, an inlet and an outlet. This allows the existence of weak solutions for the traffic network problem [17].

### A. ARZ PDE model

The evolution of traffic density  $\rho_1(x, t)$  and velocity  $v_1(x, t)$  (with  $(x, t) \in [0, L] \times [0, \infty)$ ) on the outgoing road segment and traffic density  $\rho_2(x, t)$  and velocity  $v_2(x, t)$  ( $(x, t) \in [-L, 0] \times [0, \infty)$ ) on the incoming road segment is modeled by the following ARZ model.

$$\partial_t \rho_i + \partial_x (\rho_i v_i) = 0, \quad (1)$$

$$\partial_t (\rho_i (v_i + p_i(\rho_i))) + \partial_x (\rho_i v_i (v_i + p_i(\rho_i))) = - \frac{\rho_i (v_i - V(\rho_i))}{\tau_i}, \quad (2)$$

where  $i \in \{1, 2\}$  represents either the outgoing road segment or the incoming road segment. The traffic pressure  $p_i(\rho_i)$  is defined as an increasing function of the density

$$p_i(\rho_i) = c_i \rho_i^{\gamma_i}, \quad (3)$$

where  $\gamma_i, c_i \in \mathbb{R}^+$  is defined as

$$c_i = \frac{v_m}{\rho_{m,i}^{\gamma_i}}. \quad (4)$$

The coefficient  $\gamma_i$  represents the overall drivers' aggressiveness, the positive constant  $v_m$  represents the maximum velocity and the positive constant  $\rho_{m,i}$  are the maximum densities defined as the number of vehicles per unit length. The equilibrium density-velocity relation  $V_i(\rho_i)$  is given as

$$V_i(\rho_i) = v_m \left( 1 - \left( \frac{\rho_i}{\rho_{m,i}} \right)^{\gamma_i} \right). \quad (5)$$

Based on the definitions of (3), (4) and (5), the following relation between  $V_i(\rho_i)$  and  $p_i(\rho_i)$  is satisfied on both segments,

$$V(\rho_i) + p_i(\rho_i) = v_m, \quad (6)$$

where the marginal stability is satisfied for each segment. The linear stability analysis of one-segment traffic can be found in [30]. We define the following variables

$$w_i = v_i + p_i(\rho_i), \quad (7)$$

which are interpreted as traffic "friction" or drivers' property which transport in the traffic flow with the vehicles' velocity. This property represents the heterogeneity of the traffic flow with respect to the equilibrium density-velocity relation  $V_i(\rho_i)$ . The maximum velocity  $v_m$  is assumed to be the same for the two segments while the maximum density  $\rho_{m,i}$  and coefficient  $\gamma_i$  are allowed to vary. The positive constant  $\tau_i$  are the relaxation time that represent the time scale for traffic velocity  $v_i$  adapting to the equilibrium density-velocity relation  $V_i(\rho_i)$ . Finally, we denote the traffic flow rate on each road as

$$q_i = \rho_i v_i. \quad (8)$$

The equilibrium flow and density relations, also known as the fundamental diagram, are then given by

$$Q_i(\rho_i) = \rho_i V(\rho_i) = \rho_i v_m \left( 1 - \left( \frac{\rho_i}{\rho_{m,i}} \right)^{\gamma_i} \right). \quad (9)$$

We assume that the equilibrium traffic relations are different for the two segments due to the change of road situations. The illustration is given in Fig 3. The critical density  $\rho_c$  segregates the free and congested regimes of traffic states. For the fundamental diagram in (9), the critical density is given by  $\rho_{c,i} = \frac{\rho_{m,i}}{(1+\gamma_i)^{1/\gamma_i}}$ . The traffic is in the free regime

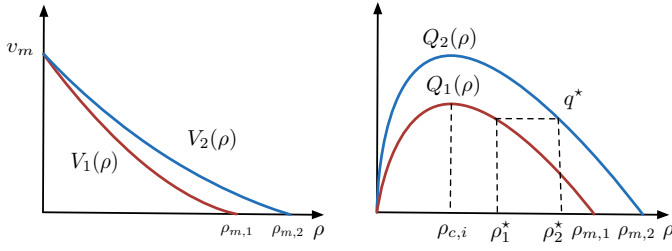


Fig. 3: The equilibrium density and velocity relation  $V_i(\rho)$  on the left, the equilibrium density and flux relation  $Q_i(\rho)$  on the right

when the density satisfies  $\rho_i < \rho_{c,i}$ . The traffic is defined as the congested traffic when the density satisfies  $\rho_i > \rho_{c,i}$ . The traffic flux reaches its maximum equilibrium value at the critical density  $Q_1(\rho_{c,i})$  which is smaller than road capacity (maximum allowable flow rate by road attributes).

In this work, we are concerned with the congested traffic and assume that both segments are in the congested regime. Fig.3. illustrates the situation that the upstream road segment for  $x \in [-L, 0]$  has more lanes than the downstream road segment for  $x \in [0, L]$ . Therefore, the maximum density  $\rho_{m,2} > \rho_{m,1}$  and the maximum speed limit  $v_m$  is assumed to be the same for the two segments. The upstream maximum traffic flux  $Q_2(\rho_c)$  is reduced in the downstream for  $Q_1(\rho_c)$ , due to change of road conditions from the segment 2 to segment 1.

### B. Boundary conditions

In this paper we consider the weak solution of the network (1)-(2). Regarding the boundary conditions connecting the two PDE systems, the Rankine-Hugoniot condition is satisfied at the junction. This condition implies a piecewise smooth solution and corresponds to the conservation of mass and of drivers' properties defined in (7) at the junction, i.e.

$$\rho_1 v_1(0^-, t) = \rho_2 v_2(0^+, t), \quad (10)$$

$$\rho_1 v_1 w_1(0^-, t) = \rho_2 v_2 w_2(0^+, t). \quad (11)$$

Thus, the flux and drivers' property are continuous across the boundary  $x = 0$ , that is

$$\rho_1(0, t) v_1(0, t) = \rho_2(0, t) v_2(0, t), \quad (12)$$

$$w_2(0, t) = w_1(0, t). \quad (13)$$

For open-loop system, we assume a constant incoming flow  $q^*$  entering the inlet boundary  $x = -L$  and a constant outgoing flow  $q^*$  at the outlet boundary for  $x = L$ , as also depicted in Fig.3:

$$q_2(-L, t) = q^*, \quad (14)$$

$$q_1(L, t) = q^*. \quad (15)$$

The control problem we solve consists in stabilizing the traffic flow in both segments around the given steady-states. We consider the actuator  $U_0(t)$  with a ramp metering located at the junction boundary  $x = 0$ , controlling the traffic flux entering from the ramp to the mainline road. Given the

flux continuity condition, we have the following boundary condition at the junction

$$\rho_1(0, t) v_1(0, t) = \rho_2(0, t) v_2(0, t) + U_0(t), \quad (16)$$

where the downstream segment flow consists of the incoming flow from the mainline upstream segment and the actuated traffic flow from the on-ramp.

### C. Steady states

The control objective is to stabilize the traffic flow in the two segments around the steady states. The steady states  $(\rho_1^*, v_1^*)$ ,  $(\rho_2^*, v_2^*)$  are chosen such that the boundary conditions (12) and (13) are satisfied, i.e.

$$\rho_1^* v_1^* = \rho_2^* v_2^* = q^*, \quad (17)$$

$$w_1^* = w_2^* = v_m, \quad (18)$$

where the steady state velocities satisfy the equilibrium density-velocity relations  $v_i^* = V_i(\rho_i)$ . The constant flux in (17)

$$Q_1(\rho_1^*) = Q_2(\rho_2^*), \quad (19)$$

and the definition of  $Q_i(\rho_i)$  in (9) yield the following relation for the steady state densities of the two segments

$$\frac{\rho_1^* \rho_{m,1}^{\gamma_1} - (\rho_1^*)^{\gamma_1+1}}{\rho_2^* \rho_{m,2}^{\gamma_2} - (\rho_2^*)^{\gamma_2+1}} = \frac{\rho_{m,1}^{\gamma_1}}{\rho_{m,2}^{\gamma_2}}. \quad (20)$$

According to (7), the constant driver's property in (18) implies that we have the same maximum velocity  $v_m$  for the two segments which corresponds to our initial assumption:

$$v_1^* + p_1^* = v_2^* + p_2^* = v_m. \quad (21)$$

Note that when vehicles' property follows the equilibrium relation  $v_i = V(\rho_i)$ , the above relation always holds given (6). In summary, we first choose the steady states density  $\rho_1^*$  and  $\rho_2^*$  such that the relation in (20) is satisfied. Then the steady states velocities are obtained as  $v_i^* = V(\rho_i^*)$ .

### D. Linearized model

We linearize the ARZ based traffic network equations  $(\rho_i, v_i)$  in (1), (2) with the boundary conditions (12), (13), (14), (15) around the steady states  $(\rho_i^*, v_i^*)$  defined in the previous section. In order to simplify the model for control design, the linearized model is given in the following Riemann variables defined as

$$\tilde{w}_i = \frac{\gamma_i p_i^*}{q^*} (\rho_i v_i - \rho_i^* v_i^*) + \frac{1}{r_i} (v_i - v_i^*), \quad (22)$$

$$\tilde{v}_i = v_i - v_i^*, \quad (23)$$

where the constant coefficients  $r_i$  are defined as

$$r_i = -\frac{v_i^*}{\gamma_i p_i^* - v_i^*}. \quad (24)$$

Given the controlled boundary at  $x = 0$  in (16) and steady states condition in (17),

$$\tilde{q}_1(0, t) = \tilde{q}_2(0, t) + U_0(t). \quad (25)$$

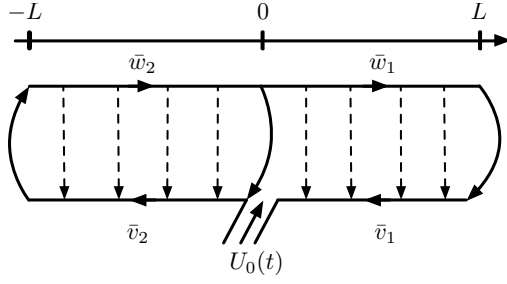


Fig. 4: Control diagram for the closed-loop system

Then we obtain the linearized model with boundary conditions

$$\partial_t \tilde{w}_i + v_i^* \partial_x \tilde{w}_i = -\frac{1}{\tau_i} \tilde{w}_i, \quad (26)$$

$$\partial_t \tilde{v}_i - (\gamma_i p_i^* - v_i^*) \partial_x \tilde{v}_i = -\frac{1}{\tau_i} \tilde{w}_i, \quad (27)$$

$$\tilde{v}_1(L, t) = r_1 \tilde{w}_1(L, t), \quad (28)$$

$$\tilde{w}_1(0, t) = \tilde{w}_2(0, t), \quad (29)$$

$$\tilde{w}_2(-L, t) = \frac{1}{r_2} \tilde{v}_2(-L, t), \quad (30)$$

$$\begin{aligned} \tilde{v}_2(0, t) = & \delta \frac{r_2}{r_1} \tilde{v}_1(0, t) + r_2(1 - \delta) \tilde{w}_2(0, t) \\ & + \frac{v_2^*(1 - r_2)}{q^*} U_0(t). \end{aligned} \quad (31)$$

Detailed calculations regarding the linearization can be obtained following [30].

For the congested regime of traffic flow,  $\rho_i^* > \frac{\rho_{m,i}}{(1+\gamma_i)^{1/\gamma_i}}$  is satisfied so that the characteristic speed  $\gamma_i p_i^* - v_i^* > 0$ . The velocity variations  $\tilde{v}_i(x, t)$  transport upstream which means the action of velocity acceleration or deceleration is repeated from the leading vehicle to the following vehicle. The following inequality is satisfied for the characteristic speeds ratio defined in (24),

$$-1 < r_i < 0. \quad (32)$$

The more congested of the traffic, the smaller of the ratio constant  $r_i$ . The control diagram for the closed-loop system (26)-(31) is given in Fig. 4. Using a spatial transformation, we get rid of the diagonal terms  $-\frac{1}{\tau_i} \tilde{w}_i$  that appear in the two equations describing the evolution of the state  $\tilde{w}_i$ . More precisely, we define for all  $x \in [-L, 0]$  and all  $t > 0$  the state  $\tilde{w}_2$  as

$$\tilde{w}_2(x, t) = \exp\left(\frac{x}{\tau_2 v_2^*}\right) \tilde{w}_2(x, t). \quad (33)$$

Similarly, the state  $\tilde{w}_1$  is defined for all  $x \in [0, L]$  and all  $t > 0$  by

$$\tilde{w}_1(x, t) = \exp\left(\frac{x}{\tau_1 v_1^*}\right) \tilde{w}_1(x, t). \quad (34)$$

One can easily check that with such a change of variables, the system (26)-(31) rewrites (for  $i \in \{1, 2\}$ ) as

$$\partial_t \tilde{w}_i + v_i^* \partial_x \tilde{w}_i = 0, \quad (35)$$

$$\partial_t \tilde{v}_i - (\gamma_i p_i^* - v_i^*) \partial_x \tilde{v}_i = c_i(x) \tilde{w}_i, \quad (36)$$

$$\tilde{v}_1(L, t) = r_1 \exp\left(-\frac{L}{\tau_1 v_1^*}\right) \tilde{w}_1(L, t), \quad (37)$$

$$\tilde{w}_1(0, t) = \tilde{w}_2(0, t), \quad (38)$$

$$\tilde{w}_2(-L, t) = \exp\left(\frac{-L}{\tau_2 v_2^*}\right) \frac{1}{r_2} \tilde{v}_2(-L, t), \quad (39)$$

$$\begin{aligned} \tilde{v}_2(0, t) = & \delta \frac{r_2}{r_1} \tilde{v}_1(0, t) + r_2(1 - \delta) \tilde{w}_2(0, t) \\ & + \frac{v_2^*(1 - r_2)}{q^*} U_0(t), \end{aligned} \quad (40)$$

where the spatially varying coefficients  $c_i(x)$  are defined as  $c_i(x) = -\frac{1}{\tau_i} \exp\left(-\frac{x}{\tau_i v_i^*}\right)$ , and the constant coefficient  $\delta$  is  $\delta = \frac{\gamma_1 p_1^*}{\gamma_2 p_2^*}$ , which represents the ratio related to the traffic pressure on the two segments. The corresponding initial conditions are denoted as  $(\tilde{v}_0)_i = \tilde{v}_i(\cdot, 0)$  and  $(\tilde{w}_0)_i = \tilde{w}_i(\cdot, 0)$ . The objective is to design the control law  $U_0$  to stabilize the system (35)-(40) in the sense of the  $L^2$ -norm. Such an interconnected system has already been considered in [4] in the case of an actuator located at one of the extremity of the network. It has been proved in [21] that a system can be delay-robustly stabilized only if its open-loop transfer function has a finite number of zeros on the complex right half plane. For the considered class of linear hyperbolic system, it is proved in [5] that such a condition is equivalent to requiring (35)-(40) with zero in-domain couplings (i.e.  $c_1 \equiv c_2 \equiv 0$ ) to be exponentially stable in open-loop. Necessary and sufficient stability conditions to guarantee such an open-loop stability can be obtained by writing the corresponding characteristic equations. However for the case of system (35)-(40), a simpler condition has been given in [9] in the form of the following Assumption.

*Assumption 1:* The boundary couplings of the system (35)-(40) are such that

$$\delta < \frac{1 + \exp\left(\frac{L}{\tau_2 v_2^*}\right)}{1 + \exp\left(\frac{-L}{\tau_1 v_1^*}\right)}. \quad (41)$$

Indeed, if  $c_i \equiv 0$ , it is straightforward to express  $\tilde{w}_2(t, 0)$  as a solution of a neutral system using the method of characteristics. The stability of the neutral system is only guaranteed if Assumption 1 holds. Due to the transport structure of (35)-(40), the convergence to zero of  $\tilde{w}_2(t, 0)$  implies the stabilization of the system. Then, the system (35)-(40) with zero in-domain couplings is exponentially stable in open-loop and can be robustly stabilized. For example, both the upstream and downstream segments are congested and the downstream traffic with denser traffic such that  $\delta < 1$ . It is easily observed that the assumption holds.

### III. STATE FEEDBACK CONTROL DESIGN

In this section we design a full-state feedback law that guarantees the stabilization of the system (26)-(31). Our approach is based on the backstepping methodology. Using two Volterra transformations we map the original underactuated system to a target system for which the in-domain coupling

terms  $c_1$  and  $c_2$  are moved to the actuated boundary in the form of integral couplings. We can then use the control input  $U_0(t)$  to eliminate these terms, leading the target system to be exponentially stable. As such a control law does not modify the boundary couplings, robustness margins are preserved under Assumption 1 (see [3], [5] for details).

We consider the following backstepping transformations

$$\alpha_i(x, t) = \bar{w}_i(x, t), \quad (42)$$

$$\begin{aligned} \beta_1(x, t) = & \bar{v}_1(x, t) - \int_x^L K_1^{vw}(x, \xi) \bar{w}_1(\xi, t) d\xi \\ & - \int_x^L K_1^{vv}(x, \xi) \bar{v}_1(\xi, t) d\xi, \end{aligned} \quad (43)$$

$$\begin{aligned} \beta_2(x, t) = & \bar{v}_2(x, t) - \int_{-L}^x K_2^{vw}(x, \xi) \bar{w}_2(\xi, t) d\xi \\ & - \int_{-L}^x K_2^{vv}(x, \xi) \bar{v}_2(\xi, t) d\xi, \end{aligned} \quad (44)$$

where the kernels  $K_1^{vw}$  and  $K_1^{vv}$  are  $L^\infty$  functions defined on the set  $\mathcal{T}_1 = \{(x, \xi) \in [0, L]^2, \xi \geq x\}$ , while the kernels  $K_2^{vw}$  and  $K_2^{vv}$  are  $L^\infty$  functions defined on the set  $\mathcal{T}_2 = \{(x, \xi) \in [-L, 0]^2, \xi \leq x\}$ . On their corresponding domains of definition, they satisfy the following set of PDEs:

$$(\gamma_i p_i^* - v_i^*) \partial_x K_i^{vw} - v_i^* \partial_\xi K_i^{vw} = c_i(\xi) K_i^{vv}, \quad (45)$$

$$\partial_x K_i^{vv}(x, \xi) + \partial_\xi K_i^{vv}(x, \xi) = 0, \quad (46)$$

along with the boundary conditions

$$K_1^{vw}(x, x) = \frac{c_1(x)}{\gamma_1 p_1^*}, \quad K_2^{vw}(x, x) = -\frac{c_2(x)}{\gamma_2 p_2^*}, \quad (47)$$

$$K_1^{vv}(x, L) = -\exp\left(\frac{L}{\tau_1 v_1^*}\right) K_1^{vw}(x, L), \quad (48)$$

$$K_2^{vv}(x, -L) = -\exp\left(\frac{-L}{\tau_2 v_2^*}\right) K_2^{vw}(x, -L). \quad (49)$$

The well-posedness of this kernel PDE-system is guaranteed by the following lemma.

*Lemma 1:* [10] Consider system (45)-(49). There exists a unique solution  $K_1^{vw}, K_1^{vv}$  in  $L^\infty(\mathcal{T}_1)$  and  $K_2^{vw}, K_2^{vv}$  in  $L^\infty(\mathcal{T}_2)$ . The transformation (43)-(44) maps the original system (35)-(40) to the following decoupled target system

$$\partial_t \alpha_i + v_i^* \partial_x \alpha_i = 0, \quad (50)$$

$$\partial_t \beta_i - (\gamma_i p_i^* - v_i^*) \partial_x \beta_i = 0, \quad (51)$$

$$\beta_1(L, t) = r_1 \exp\left(-\frac{L}{\tau_1 v_1^*}\right) \alpha_1(L, t), \quad (52)$$

$$\alpha_1(0, t) = \alpha_2(0, t), \quad (53)$$

$$\alpha_2(-L, t) = \exp\left(\frac{-L}{\tau_2 v_2^*}\right) \frac{1}{r_2} \beta_2(-L, t), \quad (54)$$

$$\beta_2(0, t) = \delta \frac{r_2}{r_1} \beta_1(0, t) + r_2 (1 - \delta) \alpha_2(0, t). \quad (55)$$

The controlled boundary (55) is obtained by defining the control input  $U_0(t)$  as

$$U_0(t) = \frac{q^*}{v_2^*(r_2 - 1)} \left( \int_{-L}^0 K_2^{vw}(0, \xi) \bar{w}_2(\xi, t) + K_2^{vv}(0, \xi) \bar{v}_2(\xi, t) d\xi \right.$$

$$\left. - \delta \frac{r_2}{r_1} \int_0^L K_1^{vw}(0, \xi) \bar{w}_1(\xi, t) + K_1^{vv}(0, \xi) \bar{v}_1(\xi, t) d\xi \right). \quad (56)$$

We have the following theorem.

*Theorem 1:* Consider the PDE system (35)-(40) with the feedback law  $U_0$  defined in (56). Then for any  $L^2$  initial condition  $(\bar{w}_i(0, \cdot), \bar{v}_i(0, \cdot))$  the system (35)-(40) exponentially converges to 0.

*Proof:* Using the method of characteristics (see [5] for details), it is possible to express the state  $\beta_2(L, t)$  as the solution of the difference equation

$$\begin{aligned} \beta_2(0, t) = & \exp\left(\frac{-L}{\tau_2 v_2^*}\right) (1 - \delta) \beta_2(0, t - \kappa_2) \\ & + \delta \exp\left(\frac{-L}{\tau_2 v_2^*}\right) \exp\left(-\frac{L}{\tau_1 v_1^*}\right) \beta_2(0, t - \kappa_1 - \kappa_2), \end{aligned}$$

where  $\kappa_i = \frac{1}{v_i^*} + \frac{1}{\gamma_i p_i^* - v_i^*}$ . This difference system is exponentially stable due to Assumption 1. Then, it implies that  $\beta_2(0, t)$  converges to zero. Using the transport structure of (50)-(55), we have the convergence of  $(\alpha_i, \beta_i)$  to zero. Due to the invertibility of the Volterra transformations (43)-(44), the systems (35)-(40) and (50)-(55) have equivalent stability properties. This implies the exponential stability of (35)-(40). ■

#### IV. SIMULATION RESULTS

The length of each freeway segment is chosen to be  $L = 2$  km so the total length of the two connected segments are 4 km. The maximum speed limit is  $v_m = 40$  m/s. We consider six lanes for the downstream freeway segment 1. The maximum density of the road is obtained as  $\rho_{m,1} = 6/7.5$  vehicles/m = 800 vehicles/km based on the assumption that the average vehicle length is 5 m plus the minimum safety distance of 50% vehicle length. The upstream segment has less functional lanes thus its maximum density is chosen as  $\rho_{m,2} = 700$  vehicles/km. We take  $\gamma_i = 0.5$ . The steady states  $(\rho_1^*, v_1^*)$  and  $(\rho_2^*, v_2^*)$  are chosen respectively as (600 vehicles/km, 19.4 m/s) and (488.6 vehicles/km, 23.8 m/s), both of which are in the congested regime and satisfy (17),(18). The equilibrium steady state of the downstream road is more congested with higher density and lower velocity. The relaxation time of each segments are  $\tau_1 = 120$  s and  $\tau_2 = 90$  s. The sinusoid initial conditions are highlighted by blue lines. The closed-loop simulation show that the exponential convergence to the steady states is achieved simultaneously for the upstream and downstream segments in Fig 5. The ramp metering control input at junction is highlighted with red line. The stop-and-go traffic in the two cascaded segments are suppressed.

#### V. CONCLUDING REMARKS

We design a stabilizing state feedback boundary control that guarantees the simultaneous stabilization of the stop-and-go on two connected roads around given steady states. The flow rate actuation is realized with a ramp metering located at the middle junction. Our approach is based on

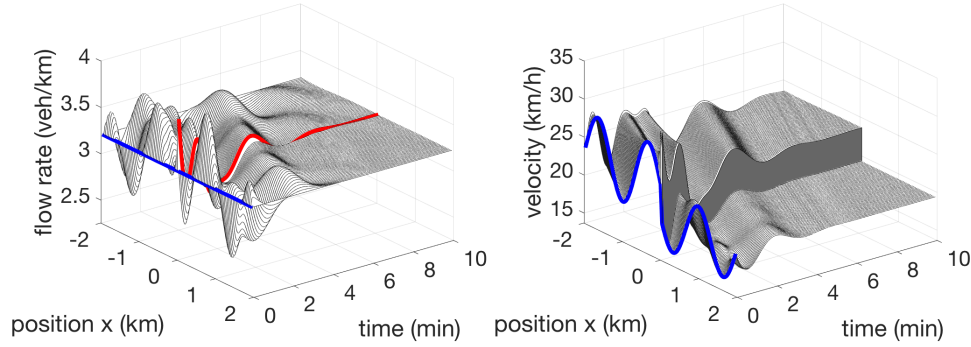


Fig. 5: Closed-loop flow rate and velocity evolution of upstream and downstream of the ramp metering in the middle.

the backstepping methodology. This is a first step towards the stabilization of traffic networks. We will consider in future work the design of an observer (in view of output-feedback stabilization) for this class of system. It is also of authors' interest to validate the model with traffic field data and to further develop control strategy to address lane-changing behaviors that arise in multi-lane congested traffic.

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