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Output feedback stabilization of an underactuated cascade network of interconnected linear PDE systems using a backstepping approach

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Abstract

We detail in this article the development of a robust stabilizing output feedback control law for an underactuated cascade network of n systems of two heterodirectional linear first-order hyperbolic Partial Differential Equations interconnected through their boundaries. Only one of the subsystems is actuated. The proposed approach combines successive backstepping transformations that present a specific cascade structure. With these transformations it is possible to rewrite the original network system as a simple system for which all the in-domain coupling terms have been removed. One can then design a strictly proper stabilizing control law.The proposed control law is proved to be robust to small delays in the actuation. Finally, a boundary observer is designed, enabling stabilization by output feedback.

Key words: Hyperbolic Partial Differential Equations; difference systems; backstepping; delay-robustness.

1 Introduction

In this paper we develop a linear output feedback control law that achieves stabilization for an underactuated cascade network of n sub-systems of two heterodirectional linear first-order hyperbolic Partial Differential Equations (PDEs). More precisely, the different sub-systems are interconnected through their boundaries but only one of them (located at one extremity of the network) is actuated. The control law is designed using a backsteppin approach.The robustness (with respect to delays and uncertainties) is guaranteed by the integral structure of the proposed control law, which is consequently strictly proper.

The control of network of PDEs is an active research topic as this class of systems naturally arises in multiple industrial processes. Among them, we can cite electric power transmission across long distance using high voltage direct current transmission (HVDC) networks [2,37,38], where the electrical interconnection of the transmission lines at the network nodes leads to a coupling of the corresponding line PDEs at their boundaries. Similar kinds of networks can be found considering density-flow systems [11,13,27], open canals [21,22,26], the design of a virtual flow meter pilot [25] or the case of the Rijke tube [20] (even if, in this case an ODE is sandwiched between two PDEs systems). Finally, freeway traffic can also be modeled by hyperbolic PDEs [10]. For this class of systems, traffic control strategies are developed and implemented for the traffic management infrastructures, including ramp metering and varying speed limits. If the suppression of the stop-and-go traffic oscillations on a freeway located either upstream or downstream of the ramp has been solved in [45], the general case of a network composed of an interconnection of different roads (with different properties) and for which a limited number of actuators (ramp metering) is available remains unsolved. Thus, to envision the most general real applications, the questions of the controllability and control design for such network of PDEs have to be considered. In most of the cases presented above, only the PDE located at one extremity of the network can be actuated. These systems are consequently underactuated which makes their stabilization particularly challenging. To tackle this problem, different approaches have been proposed in the literature. PI boundary controllers have for instance been considered in [11,13] for fully actuated networks (i.e. with one control per set of heterodirectional PDEs), establishing explicit stability conditions using appropriate quadratic Lyapunov functions. In [37,38], the authors consider a flatness-based design of a feedforward control of tree-like transmission

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networks. Similar cases of interconnected problems have been considered in [42], with a velocity recirculation in a wave equation. The exact boundary controllability of nodal profile for quasilinear hyperbolic systems with interface conditions in a tree-like networks has been assessed in [22,44] using the method of characteristics. However, this last approach does not provide an explicit formulation of the corresponding feedback laws. Moreover, even if the proposed method is straightforward to implement, it may require solving a set of PDEs online, which is computationally expensive. More recently in [40], the authors have considered the output feedback control of a semilinear hyperbolic system with a structure analogous to the one considered in this paper, using the dynamics on the characteristic lines. Again however, the proposed state-feedback control law requires solving a set of PDEs online. Similarly to what is done in [39,41] it may be possible to explicitly rewrite such a control law for the case of linear systems but this has not been proved yet. Finally, in recent contributions [1], the authors have considered the delay-robust stabilization of an under-actuated scalar PDE-ODE-PDE system (the last PDE being a transport equation).

Considering linear hyperbolic systems, it has been possible, using the backstepping approach [30], to design explicit stabilizing control law for a large class of problems [6,17,16,28]. The stabilization was guaranteed by mapping the original system to a target system that converges in finite-time to its equilibrium (by cancelling all the reflection terms located at the actuated boundary). Recent contributions [4] have highlighted the necessity of a change of strategy to guarantee the existence of robustness margins. More precisely, it has been observed [19,31] that for many feedback systems, the introduction of arbitrarily small time-delays in the loop may cause instability under linear feedback. For hyperbolic systems, it has then appeared to be necessary to preserve some amount of the boundary reflection in the target system (and in so far renouncing to finite-time convergence) to preserve some robustness margins. The design of the corresponding delay-robust control laws has been achieved using the backstepping approach to rewrite the considered systems as neutral systems with distributed delays [4,8] and adjusting the analysis methods developed for time delay systems [23,24]. Recently, this approach has been successfully used [5] to tackle the problem of delay robust state feedback stabilization of an underactuated network of two interconnected subsystems of two coupled PDEs. The proposed approach cannot be straightforwardly extended for a network composed of a higher number of subsystems due to the multiple interconnections. Moreover, the control law proposed in [5] requires the value of the state all over the spatial domain, which is unrealistic. To envision real applications, a state observer has to be designed.

In this paper, we generalize the approach developed in [5] to an arbitrary number of interconnected subsystems of two linear coupled PDEs. Moreover, we provide a boundary observer that estimates the state in real time only using (collocated) boundary measurements. This allows the design of an explicit output-feedback law which is strictly proper, thus guaranteeing the existence of robustness margins. More precisely, the main contribution of this paper is to provide an explicit strictly proper **output-feedback** law that ensures the robust stabilization of an **underactuated network** of n systems of two linear hyperbolic PDEs systems. The (delay-) robustness properties of such a feedback law are guaranteed by preserving the boundary reflection terms of the system making the control law strictly proper. The proposed control law is designed using the backstepping approach. By means of successive backstepping transformations, we rewrite the original network system as a simple system for which the control design is easier, as all the in-domain couplings have been moved at the actuated boundary. To encompass the complex structure of the considered network, these backstepping transformations are not simple Volterra transformations. They have to act on all the states of the system (contrary to the case of two subsystems presented in [5]). In that sense, the idea behind such a transformation is similar to the one used in [16]. Here, the proposed transformations have a cascade structure which guarantees their invertibility. The presented result is a first step on explicit backstepping-based control of linear systems with a network structure. Multiple aspects remain to be considered, including robustness-performance trade-off. Cancelling a small amount of the boundary reflection terms or preserving dissipative terms that are present in the original system while designing the target system could improve the performance while guaranteeing the existence of robustness margins. Our approach is the following: (i) A Volterra transformation is constructed, removing some of the in-domain couplings present in the different subsystems. Due to the interconnection, some undesirable integral terms appear at the boundaries between the different subsystems. (ii) A second transformation is then used to get rid of these terms, moving them at the actuated boundary. The design of the control law becomes straightforward. (iii) The corresponding boundary observer is designed using a dual approach [3,7]. (iv) The robustness properties of the closed-loop system are studied using the fact that the control law is strictly proper.

The paper is organized as follows. In Section 2, we introduce the model equations and the notations. In Section 3, we present the stabilization result: combining successive backstepping transformations, the original system can be rewritten as a simple system for which it is possible to derive a stabilizing control law. The observer design is proposed in Section 4. Some remarks about the robustness properties of the closed-loop system are given in Section 5. Some simulation results are given in Section 6. Finally, some concluding remarks are given in Section 7

2 Problem Description

2.1 Definitions and notations

In this section we detail the notations used through this paper. We denote $L^2([0,1],\mathbb{R})$ the space of real-valued square-integrable functions defined on $[0,1]$ with the standard L^2 norm, *i.e.*, for any $f \in L^2([0,1],\mathbb{R})$: $||f||_{L^2}^2 = \int_0^1 f^2(x)\mathrm{d}x$. The set $L^\infty([0,1],\mathbb{R})$ denotes the space of bounded real-valued functions defined on $[0,1]$ with the standard L^∞ norm, *i.e.*, for any $f \in L^\infty([0,1],\mathbb{R})$, $||f||_{L^\infty} = \operatorname*{ess\,sup}_{x\in[0,1]}|f(x)|$. In the following, for any $n \in \mathbb{N}$, for any $m \in \mathbb{N}$ and for any $(u,v) \in (L^2([0,1]))^n \times (L^2([0,1]))^m$, we define the norm

$$||(u,v)||_{L^2}^2 = \sum_{i=1}^{n} ||u_i||_{L^2}^2 + \sum_{i=1}^{m} ||v_i||_{L^2}^2. \tag{1}$$

The set $\mathcal{C}^p([0,1])$ (with $p \in \mathbb{N}\cup\{\infty\}$) stands for the space of real-valued functions defined on $[0,1]$ that are p times differentiable and whose p-th derivative is continuous. The geometrical set $\mathcal{T}_b$ is defined as

$$\mathcal{T}_b = \{(x,\xi) \in [0,1]^2 \text{ s.t. } \xi \leq x\}, \tag{2}$$

while we define $\mathcal{U} = \{(x,\xi) \in [0,1]^2\}$ as the unit square,

$$\mathcal{U} = \{(x,\xi) \in [0,1]^2\}. \tag{3}$$

We denote $L^\infty(\mathcal{T}_b)$ (resp. $L^\infty(\mathcal{U})$) the space of real-valued L^∞ functions on $\mathcal{T}_b$ (resp. $\mathcal{U}$). Inspired by the notations of [24], we let $D = L^2([-\tau,0],\mathbb{R}^n)$ the Banach space of L^2 functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^n$. For a function $\phi : [-\tau,\infty) \to \mathbb{R}$, we define its partial trajectory $\phi_{[t]} \in D$ by $\phi_{[t]}(\theta) \doteq \phi(t+\theta)$, $-\tau \leq \theta \leq 0$. The associated norm is given by

$$||\phi_{[t]}||_D \doteq \left(\int_{-\tau}^{0} \phi^T(t+\theta)\phi(t+\theta)d\theta\right)^{\frac{1}{2}}. \tag{4}$$

2.2 System under consideration

We consider in this paper a system composed of n subsystems of two linear hyperbolic PDEs. This class of system may appear when considering oil production systems made of networks of pipes (whose principal line is known as the manifold) [34] or traffic network systems. Each subsystem is defined by the following set of PDEs ($i \in \{1,\cdots,n\}$)

$$\partial_t u_i(t,x) + \lambda_i \partial_x u_i(t,x) = \sigma_i^+(x) v_i(t,x), \tag{5}$$
$$\partial_t v_i(t,x) - \mu_i \partial_x v_i(t,x) = \sigma_i^-(x) u_i(t,x), \tag{6}$$

evolving in $\{(t,x) \text{ s.t. } t > 0,\ x \in [0,1]\}$, where $\lambda_i > 0$ and $\mu_i > 0$ are the constant transport velocities respectively associated to equations (5)-(6) and where the in-domain coupling terms σ_i^+ and σ_i^- belong to $\mathcal{C}^0([0,1])$. The associated initial conditions are denoted $u_i^0(\cdot) = u_i(0,\cdot)$ and $v_i^0(\cdot) = v_i(0,\cdot)$ and are defined in $L^2([0,1],\mathbb{R})$. These subsystems are connected through their boundary conditions, which satisfy

$$u_i(t,0) = q_{i,i} v_i(t,0) + q_{i,i+1} u_{i+1}(t,1), \tag{7}$$
$$v_i(t,1) = \rho_{i,i} u_i(t,1) + \rho_{i,i-1} v_{i-1}(t,0) + \delta_1^i U(t), \tag{8}$$

where the different coupling terms q_{ij} and ρ_{ij} are constant and where V is an input function (control law) that takes real values. By convention $q_{n,n+1} = 0$ and $\rho_{1,0} = 0$. The notation $\delta_1^i = 1$ stands for the Kronecker symbol (i.e. $\delta_1^i = 1$ if $i = 1$, and $\delta_1^i = 0$ if $i \neq 1$). The measured output is denoted $y(t)$. We consider in this paper the case of collocated measurements. Thus, we have

$$y(t) = u_1(t,1). \tag{9}$$

Such a system is depicted in Figure 1 (in which the network structure clearly appears).

The system (5)-(6) with the boundary conditions (7)-(8) can be rewritten in the more condensed form

$$\partial_t u(t,x) + \Lambda^+ \partial_x u(t,x) = \Sigma^+(x) v(t,x), \tag{10}$$
$$\partial_t v(t,x) - \Lambda^- \partial_x v(t,x) = \Sigma^-(x) u(t,x), \tag{11}$$

with the boundary conditions

$$\begin{pmatrix} u(t,0) \\ v(t,1) \end{pmatrix} = K \begin{pmatrix} v(t,0) \\ u(t,1) \end{pmatrix} + AU(t) \tag{12}$$

where we have chosen to denote the state $u(t,x) = (u_1(t,x),\cdots,u_n(t,x))^T$, and the state $v(t,x) = (v_1(t,x),\cdots,v_n(t,x))^T$. The matrices Λ^+ and Λ^- are the diagonal matrices such that for all $1 \leq i \leq n$, we have $\Lambda_{i,i}^+ = \lambda_i$ and $\Lambda_{i,i}^- = \mu_i$. The matrices $\Sigma^+(x)$ and $\Sigma^-(x)$ are diagonal matrices such that for all $1 \leq i \leq n$, we have $\Sigma_{i,i}^+(x) = \sigma_i^+(x)$ and $\Sigma_{i,i}^-(x) = \sigma_i^-(x)$. The matrix K is defined by

$$K = \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix}, \tag{13}$$

where K_{00} and K_{11} are diagonal matrices such that for $1 \leq i \leq n$, we have $(K_{00})_{i,i} = q_{i,i}$ and $(K_{11})_{i,i} = \rho_{i,i}$; K_{01} is an upper-diagonal matrix such that for $1 \leq i \leq n-1$, $(K_{01})_{i,i+1} = q_{i,i+1}$ and K_{10} is a lower-diagonal matrix such that for $2 \leq i \leq n$, $(K_{10})_{i,i-1} = \rho_{i,i-1}$ The vector A has all its component equal to zero except one. As we consider weak solutions, the resulting system (10)–(12) is well-posed [12, Theorem A.6, page 254].

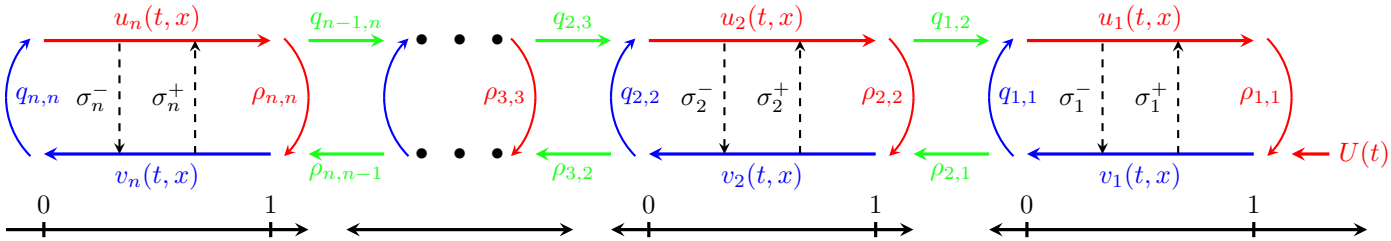

Fig. 1. Schematic representation of the interconnected system (5)-(6) with the boundary conditions (7)-(8)

Remark 1 *The positive velocities λ_i and μ_i are assumed to be constant in this paper but the presented results can be extended to the case of spatially varying coefficients whose velocities are $C^1([0,1],\mathbb{R})$-functions.*

Remark 2 *Using a simple change of variable, one can easily rewrite the system* (10)–(12) *in the framework considered in [8]. The advantage of the formulation* (10)–(12) *compared to the one used in [8] is that it emphasizes the network structure of the system (see Figure 1) and simplifies the computations.*

2.3 Problem formulation

The goal of this paper is to design an output feedback control law V such that the state (u,v) of the resulting closed-loop system (5)-(8) exponentially converges to its zero equilibrium (**stabilization problem**), *i.e.* there exist $\kappa_0 \geq 0$ and $\nu > 0$ such that for any initial condition $(u_0,v_0) \in (L^2[0,1])^n$

$$||(u,v)||_{L^2} \leq \kappa_0 \mathrm{e}^{-\nu t}||(u_0,v_0)||_{L^2},\ t \geq 0. \tag{14}$$

Moreover, we want the resulting control law to guarantee the existence of robustness margins (**robustness problem**). The main difficulty to design such a control law is due to the underactuation. Compared to classical stabilization problems [4,6,17,43], the different subsystems (u_k,v_k) can only be stabilized here through the subsystem (u_1,v_1). To design such a stabilizing control law, the proposed approach requires the following assumption

Assumption 1 *For all $1 \leq i \leq n$, the boundary couplings satisfy the following conditions*

$$\rho_{i,i} \neq 0, \quad q_{i,i} \neq 0. \tag{15}$$

For all $1 \leq k \leq n-1$, we have

$$q_{i,i+1}\rho_{i+1,i} \neq \rho_{i+1,i+1}q_{i,i}. \tag{16}$$

The conditions given in this assumption are a consequence of the choice of target systems and backstepping transformations we make in this paper. One can notice that the first condition (15) of this assumption implies (using the second condition (16)) that all the boundary coupling terms are different of zero. Even if this assumption might be conservative, the fact that the coefficients $\rho_{i+1,i}$ must be different of zero is however necessary to stabilize the different subsystems (u_k,v_k) through the subsystem (u_1,v_1). The importance of Assumption 1 will appear through the computations, while proving the existence of the backstepping kernels. Note that condition (15) can be rewritten as

$$(K_{00})_{i,i} \neq 0, \quad (K_{11})_{i,i} \neq 0, \tag{17}$$

while condition (16) can be rewritten as

$$(K_{00})_{i,i}(K_{11})_{i+1,i+1} \neq (K_{01})_{i,i+1}(K_{10})_{i+1,i}. \tag{18}$$

Regarding the robustness aspects, we consider the following assumption.

Assumption 2 *The open loop system* (10)–(12) *(i.e. $U(t) \equiv 0$) in the absence of in-domain coupling terms σ_i^+ and σ_i^- (i.e. $\sigma_i^{\cdot} \equiv 0$) is exponentially stable (in the sense given by* (14)*).*

This second (non-restrictive) assumption is required to ensure the delay-robust stabilization of (10)-(12) in the sense of [4,23]. If this assumption does not hold, then the open-loop transfer function of (10)-(12) has an infinite number of poles in the complex right half plane (RHP), which means that any linear control will lead to a zero delay margin [31, Theorem 1.2], i.e. for any control law $U(t)$ the closed-loop system (10)–(12) becomes unstable when there is an (arbitrarily small) delay $\delta > 0$ in the loop. Adjusting the approach of [8] or [5], it can easily be shown, using the transport structure of the system, that the open loop system (5)-(8) in the absence of in-domain coupling terms has equivalent stability properties to those of the system defined on $D = L^2([-\tau,0],\mathbb{R}^n)$ by

$$\phi(t) = \sum_{1\leq k\leq n^2} A_k\phi(t-\tau_k), \tag{19}$$

where the matrices A_k only depend on the boundary couplings $\rho_{i,j}$ and $q_{i,j}$ and where the delay τ_k only depend

on the velocity λ_i and μ_i (τ being the largest of these delays). More precisely, in the absence of in-domain coupling terms, the state $v(t,1)$ satisfies (19). Thus, if the delays τ_k are rationally independent, a necessary and sufficient condition for this assumption to be satisfied is given by [24, Theorem 6.1], [23]

$$\sup_{\theta_k \in [0,2\pi]^{n^2}} \text{Sp}\,(\sum_{1\le k\le n^2} A_k \exp(i\theta_k)) < 1, \qquad (20)$$

where Sp denotes the spectral radius. Note that if the delays are not rationally independent, this condition is only sufficient (see [24] for details). Condition (20) requires iterative optimization methods to be tested and is not constructive, in general. For state-feedback synthesis, some numerically tractable sufficient conditions have been proposed using Lyapunov-Krasovskii theory [14,18,33,35,36]. A simple sufficient condition has for instance been given in [15, Theorem 2.3] in the H^2 norm. Throughout the next sections, we derive, under Assumption 1 and Assumption 2, a **delay-robust** control law for the system (5)-(8).

3 State-feedback stabilization

In this section we derive a control law that guarantees the exponential stabilization of the system (5)-(6) in the sense of the L^2-norm. The methodology we use is inspired from the one proposed in the case of only two interconnected systems in [5] even if the different backstepping transformations now act on all the states. We start by using on each subsystem a Volterra transformation inspired by [17] that moves the local in-domain coupling terms to the boundaries of the different subsystems (in the form of integral terms). Due to these Volterra transformations, non-local in-domain coupling terms appear inside each subsystem. Using an invertible integral transformation, it becomes possible to simultaneously get rid of these (boundary and in-domain) coupling terms, moving them to the actuated boundary. It is then straightforward to derive a stabilizing control law.

3.1 First Volterra transformation: removing the local in-domain coupling terms

In this section, we remove the local in-domain coupling terms σ_i^+ and σ_i^- from each subsystem of (5)-(6). For each subsystem, these terms are moved to the boundary which is the closest to the actuation, i.e. the boundary located in $x=1$. Inspired by [17], this is done using Volterra transformations. These transformations are defined between $[0,x]$. More precisely, for all $i \in \{1,2,\cdots,n\}$ we consider the following Volterra transformations.

$$\begin{pmatrix}\alpha_i(t,x)\\ \beta_i(t,x)\end{pmatrix} = \begin{pmatrix}u_i(t,x)\\ v_i(t,x)\end{pmatrix} - \int_0^x \begin{pmatrix}K_i^{uu}(x,\xi) & K_i^{uv}(x,\xi)\\ K_i^{vu}(x,\xi) & K_i^{vv}(x,\xi)\end{pmatrix}\begin{pmatrix}u_i(t,\xi)\\ v_i(t,\xi)\end{pmatrix} d\xi, \qquad (21)$$

where the kernels $K_i^{\cdot\cdot}$ are L^∞-functions defined on the triangular domain $\mathcal{T}_b$. These kernels satisfy the following set of PDEs [17]

$$\lambda_i\partial_x K_i^{uu}(x,\xi) + \lambda_i\partial_\xi K_i^{uu}(x,\xi) = -\sigma_i^-(\xi)K_i^{uv}(x,\xi), \quad (22)$$
$$\lambda_i\partial_x K_i^{uv}(x,\xi) - \mu_i\partial_\xi K_i^{uv}(x,\xi) = -\sigma_i^+(\xi)K_i^{uu}(x,\xi), \quad (23)$$
$$\mu_i\partial_x K_i^{vu}(x,\xi) - \lambda_i\partial_\xi K_i^{vu}(x,\xi) = \sigma_i^-(\xi)K_i^{vv}(x,\xi), \quad (24)$$
$$\mu_i\partial_x K_i^{vv}(x,\xi) + \mu_i\partial_\xi K_i^{vv}(x,\xi) = \sigma_i^+(\xi)K_i^{vu}(x,\xi), \quad (25)$$

with the boundary conditions,

$$K_i^{vu}(x,x) = -\frac{\sigma_i^-(x)}{\lambda_i+\mu_i}, \quad K_i^{uv}(x,x) = \frac{\sigma_i^+(x)}{\lambda_i+\mu_i}, \quad (26)$$
$$K_i^{vv}(x,0) = \frac{q_{i,i}\lambda_i}{\mu_i}K_i^{vu}(x,0), \quad (27)$$
$$K_i^{uu}(x,0) = \frac{\mu_i}{q_{i,i}\lambda_i}K_i^{uv}(x,0), \quad (28)$$

Note that these equations are well defined due to Assumption 1 ($q_{i,i} \neq 0$). The well-posedness of the system (22)-(28) is assessed by the following lemma

Lemma 3 *[17] Consider system* (22)-(28)*. There exists a unique solution $K_i^{uu}, K_i^{uv}, K_i^{vu}, K_i^{vv}$ in $L^\infty(\mathcal{T}_b)$. Moreover, there exist $L_2^{\alpha\alpha}, L_2^{\alpha\beta}, L_2^{\beta\alpha}, L_2^{\beta\beta}$ in $L^\infty(\mathcal{T}_b)$ such that*

$$u_i(t,x) = \alpha_i(t,x) - \int_0^x L_i^{\alpha\alpha}(x,\xi)\alpha_i(t,\xi)d\xi - \int_0^x L_i^{\alpha\beta}(x,\xi)\beta_i(t,\xi)d\xi, \qquad (29)$$
$$v_i(t,x) = \beta_i(t,x) - \int_0^x L_i^{\beta\alpha}(x,\xi)\alpha_i(t,\xi)d\xi - \int_0^x L_i^{\beta\beta}(x,\xi)\beta_i(t,\xi)d\xi. \qquad (30)$$

Applying the backstepping transformation (21) to the system (5)-(6) with the boundary conditions (7)-(8) yields the following target system.

$$\partial_t\alpha_i(t,x) + \lambda_i\partial_x\alpha_i(t,x) = -\lambda_i q_{i,i+1}K_i^{uu}(x,0)[\alpha_{i+1}(t,1) - \int_0^1 \left(L_{i+1}^{\alpha\alpha}(1,\xi)\alpha_{i+1}(t,\xi) + L_{i+1}^{\alpha\beta}(1,\xi)\beta_{i+1}(t,\xi)\right)d\xi], \quad (31)$$
$$\partial_t\beta_i(t,x) - \mu_i\partial_x\beta_i(t,x) = -\lambda_i q_{i,i+1}K_i^{vu}(x,0)[\alpha_{i+1}(t,1) - \int_0^1 \left(L_{i+1}^{\alpha\alpha}(1,\xi)\alpha_{i+1}(t,\xi) + L_{i+1}^{\alpha\beta}(1,\xi)\beta_{i+1}(t,\xi)\right)d\xi], \quad (32)$$

with the boundary conditions

$$\alpha_i(t,0) = q_{i,i}\beta_i(t,0) + q_{i,i+1}(\alpha_{i+1}(t,1) -$$

$$\int_0^1 \left(L_{i+1}^{\alpha\alpha}(1,\xi)\alpha_{i+1}(t,\xi) + L_{i+1}^{\alpha\beta}(1,\xi)\beta_{i+1}(t,\xi)\right) d\xi), \quad (33)$$

$$\beta_i(t,1) = \rho_{i,i}\alpha_i(t,1) + \rho_{i,i-1}\beta_{i-1}(t,0) + \delta_1^i U(t) + \int_0^1 \left(L_i^{\beta\alpha}(1,\xi) - \rho_{i,i}L_i^{\alpha\alpha}(1,\xi)\right)\alpha_i(t,\xi)d\xi + \int_0^1 \left(L_i^{\beta\beta}(1,\xi) - \rho_{i,i}L_i^{\alpha\beta}(1,\xi)\right)\beta_i(t,\xi)d\xi, \quad (34)$$

The corresponding initial conditions are denoted $\alpha^0(\cdot)$ and $\beta^0(\cdot)$. They belong to $(L^2([0,1]))^{2n}$. Due to the invertibility of the transformation (21) (as it is a Volterra transformation [30]), we immediately have the following theorem.

Theorem 4 *There exists an invertible bounded linear map $\mathcal{F} : L^2([0,1])^{2n} \to L^2([0,1])^{2n}$ such that, for every initial condition $(u^0, v^0) = (u_1^0, \cdots, u_n^0, v_1^0, \cdots, v_n^0) \in L^2([0,1])^{2n}$, if $(\alpha,\beta) \in \mathcal{C}^0([0,+\infty), L^2([0,1])^{2n})$ denotes the solution to (31)-(34) satisfying the initial data $(\alpha^0,\beta^0) = \mathcal{F}^{-1}(u^0,v^0)$, then $(u(t),v(t)) = \mathcal{F}(\alpha(t),\beta(t))$ is the solution to (5)-(8) satisfying $(u(0,\cdot),v(0,\cdot)) = (u^0,v^0)$.*

PROOF. For all $1 \le i \le n$, let us consider the integral operators $\mathcal{F}_u^i : (L^2([0,1])^n \times L^2([0,1])^n) \to L^2([0,1])$ and $\mathcal{F}_v^i : (L^2([0,1])^n \times L^2([0,1])^n) \to L^2([0,1])^2$ defined by

$$\mathcal{F}_u^i \begin{pmatrix}\alpha(x)\\ \beta(x)\end{pmatrix} = \alpha_i(x) - \int_0^x L_i^{\alpha\alpha}(x,\xi)\alpha_i(\xi)d\xi - \int_0^x L_i^{\alpha\beta}(x,\xi)\beta_i(\xi)d\xi, \quad (35)$$

$$\mathcal{F}_v^i \begin{pmatrix}\alpha(x)\\ \beta(x)\end{pmatrix} = \beta_i(x) - \int_0^x L_i^{\beta\alpha}(x,\xi)\alpha_i(\xi)d\xi - \int_0^x L_i^{\beta\beta}(x,\xi)\beta_i(\xi)d\xi. \quad (36)$$

Let us now define the operator $\mathcal{F} : (L^2([0,1])^n \times L^2([0,1])^n) \to (L^2([0,1])^n \times L^2([0,1])^n)$ as the concatenation of the n operators $\mathcal{F}_u^i$ and the n operators $\mathcal{F}_v^i$. Due to the invertibility of each sub-operator, this operator is invertible. It is straightforward [17] to show that this operator satisfies the properties given in the theorem. □

3.2 A second integral transformation

In this section we map the system (α,β) defined by (31)-(34) to a new target system for which all the undesirable integral terms present in the boundary conditions and all the non-local in-domain coupling terms have been moved at the actuated boundary. This is done using an **invertible** integral transformation, which presents a cascade structure. Although this transformation may appear similar to the so-called Fredholm transformation given in [16] it is intrinsically different since the "Fredholm part" only appear with respect to the upper off-diagonal terms. More precisely, this transformation has a cascade structure that guarantees its invertibility (which is usually not granted for an arbitrary Fredholm transformation). Moreover, this Fredholm part is combined with an Volterra part which is an important difference with [16].

3.2.1 Target system design

The objective of the transformation is to map the system (31)-(34) to the following system, that consists of transport equations coupled through their boundaries.

$$\partial_t w_i(t,x) + \lambda_i \partial_x w_i(t,x) = 0, \quad (37)$$

$$\partial_t z_i(t,x) - \mu_i \partial_x z_i(t,x) = 0, \quad (38)$$

$$w_i(t,0) = q_{ii} z_i(t,0) + q_{i,i+1} w_{i+1}(t,1) \quad (39)$$

$$z_i(t,1) = \rho_{i,i} w_i(t,1) + \rho_{i,i-1} z_{i-1}(t,0) + \delta_1^i [U(t) + \int_0^1 G^w(\xi)w(t,\xi) + G^z(\xi)z(t,\xi)d\xi] \quad (40)$$

where G^w and G^z are L^∞-matrices with one row and n columns that still have to be defined. The corresponding initial conditions are denoted $(w^0(\cdot), z^0(\cdot))$ and belong to $(L^2([0,1]))^{2n}$. This target system is pictured in Figure 2 (where we have denoted $\bar{U}(t) = U(t) + \int_0^1 G^w(\xi)w(t,\xi) + G^z(\xi)z(t,\xi)d\xi$).

The following theorem assesses the existence of an invertible transformation mapping (31)-(34) to (37)-(40).

Theorem 5 *There exists an invertible bounded linear map $\mathcal{F}_1 : L^2([0,1])^{2n} \to L^2([0,1])^{2n}$ such that, for every initial condition $(\alpha^0,\beta^0) \in L^2([0,1])^{2n}$, if $(w,z) \in \mathcal{C}^0([0,+\infty), L^2([0,1])^{2n})$ denotes the solution to (37)-(40) satisfying the initial data $(w^0,z^0) = \mathcal{F}_1^{-1}(\alpha^0,\beta^0)$, then $(\alpha(t),\beta(t)) = \mathcal{F}_1(w(t),z(t))$ is the solution of the system (31)-(34) satisfying $(\alpha(0,\cdot),\beta(0,\cdot)) = (\alpha^0,\beta^0)$.*

PROOF. Let us consider the integral operators $\mathcal{F}_1^i : L^2([0,1])^2 \to L^2([0,1])^2$ defined for all $1 \le i \le n$ by

$$\mathcal{F}_1^i \begin{pmatrix}w_i(t,x)\\ z_i(t,x)\end{pmatrix} = \begin{pmatrix}w_i(t,x)\\ z_i(t,x)\end{pmatrix} - \int_0^x \begin{pmatrix}R_i^\alpha(x,\xi) & 0\\ 0 & R_i^\beta(x,\xi)\end{pmatrix}\begin{pmatrix}w_i(t,\xi)\\ z_i(t,\xi)\end{pmatrix}d\xi + \sum_{j=i+1}^n \int_0^1 \begin{pmatrix}F_{i,j}^{\alpha\alpha}(x,\xi) & F_{i,j}^{\alpha\beta}(x,\xi)\\ F_{i,j}^{\beta\alpha}(x,\xi) & F_{i,j}^{\beta\beta}(x,\xi)\end{pmatrix}\begin{pmatrix}w_j(t,\xi)\\ z_j(t,\xi)\end{pmatrix}d\xi, \quad (41)$$

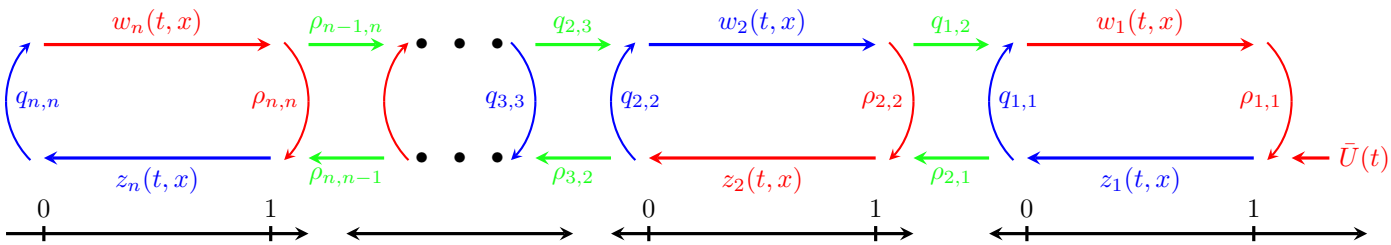

Fig. 2. Schematic representation of the interconnected system (37)-(40).

where, for $i < j \leq n$, the kernels $F^{\cdot\cdot}_{i,j}$ are L^∞-functions defined on $\mathcal{U}$, while the kernels $R^{\cdot}_i$ are L^∞-functions defined on $\mathcal{T}_b$. These kernels are defined by a set of equations given in appendix (equations (A.1)-(A.28)). Note that the well-posedness of system (A.1)-(A.28) (and consequently the existence of the kernels $F^{\cdot\cdot}_{i,j}$ and $R^{\cdot}_i$) is assessed in Theorem 12 (whose proof requires Assumption 1). In this operator, the integral terms between [0,1] are used to get rid of the integral appearing in the boundary conditions of (31)-(34) and of the non-local in-domain coupling terms. The integral terms between $[0, x]$ are used to remove the new terms that appear due to the differentation of the kernels F_{ij}. From (34) and (41), we can define the functions G^w and G^z for $\xi \in [0, 1]$ by

$$
\begin{aligned}
(G^w(\xi))_1 =& -\rho_{11} R_1^\alpha(1,\xi) + L_1^1(\xi) - \int_\xi^1 L_1^1(\nu) R_1^\alpha(\nu,\xi) d\nu,\\
(G^w(\xi))_k =& \rho_{1,1} F_{1,k}^{\alpha\alpha}(1,\xi) - F_{1,k}^{\beta\alpha}(1,\xi) + \int_0^1 (L_1^1(\nu)\\
& F_{1,k}^{\alpha\alpha}(\nu,\xi) + L_1^2(\nu) F_{1,k}^{\beta\alpha}(\nu,\xi)) d\nu,\\
(G^z(\xi))_1 =& R_1^\beta(1,\xi) + L_1^2(\xi) - \int_\xi^1 L_1^2(\nu) R_1^\beta(\nu,\xi) d\nu,\\
(G^z(\xi))_k =& \rho_{1,1} F_{1,j}^{\alpha\beta}(1,\xi) - F_{1,j}^{\beta\beta}(1,\xi) + \int_0^1 (L_1^1(\nu)\\
& F_{1,j}^{\alpha\beta}(\nu,\xi) + L_1^2(\nu) F_{1,j}^{\beta\beta}(\nu,\xi)) d\nu,
\end{aligned}
$$

where $k \geq 2$ and where we have defined for all $1 \leq i \leq n$ the functions L_i^1 and L_i^2 by

$$L_i^1(\xi) = L_i^{\beta\alpha}(1,\xi) - \rho_{i,i} L_i^{\alpha\alpha}(1,\xi), \tag{42}$$
$$L_i^2(\xi) = L_i^{\beta\beta}(1,\xi) - \rho_{i,i} L_i^{\alpha\beta}(1,\xi). \tag{43}$$

Differentiating (41) with respect to space and time, using the Liebniz rule and integration by parts, one can check (using equations (A.1)-(A.28)) that for all $1 \leq i \leq n$, the operator $\mathcal{F}_1^i$ maps the system (37)-(40) into (31)-(34). We now need to prove the invertibility of the different integral operators. One can easily check that $\mathcal{F}_1^n$ is invertible as it rewrites as a Volterra operator. Then, using the cascade structure of (41), one can recursively prove the invertibility of each operator $\mathcal{F}_1^i$. Defining the operators $(\mathcal{F}_1^i)_w$ (resp. $(\mathcal{F}_1^i)_z$) as the first (resp. second) line of the operator $\mathcal{F}_1^i$ and concatenating them, we obtain the desired operator $\mathcal{F}_1$. This concludes the proof. □

Remark that, contrary to [5], the transformation (41) must act on all the PDEs states. Combining the operators $\mathcal{F}$ and $\mathcal{F}_1$, a direct consequence of this theorem is that system (37)-(40) and (5)-(8) have equivalent stability properties. Thus, we only have to find a control law $U(t)$ that ensures the exponential stability of (37)-(40). This is straightforward due to Assumption 2.

3.3 Control law and exponential stabilization

We now state the main stabilization result as follows.

Theorem 6 *System* (5)-(8) *with the following feedback control law*

$$U(t) = -\int_0^1 \begin{pmatrix} G^w(\xi) & G^z(\xi) \end{pmatrix} \mathcal{F}_1^{-1} \mathcal{F}^{-1} \begin{pmatrix} u(t,\xi) \\ v(t,\xi) \end{pmatrix} d\xi \tag{44}$$

exponentially converges to its zero equilibrium in the sense of (14).

PROOF. Using Theorem 4 and Theorem 5, the control law (44) can be rewritten as

$$U(t) = -\int_0^1 \left(G^w(\xi) w(t,\xi) + G^z(\xi) z(t,\xi)\right) d\xi. \tag{45}$$

With this control law, the boundary condition (40) rewrites

$$z_i(t,1) = \rho_{i,i} w_i(t,1) + \rho_{i,i-1} z_{i-1}(t,0).$$

Thus, due to Assumption 2, the system (37)-(40) is exponentially stable. This implies the exponential stability of (5)-(8), due to Theorem 4 and Theorem 5. □

Remark 7 *Due to the structures of the operators $\mathcal{F}$ and $\mathcal{F}_1$, one can prove using simple changes of variables under the integral (see [30] for details) that there exist $L^\infty([0,1])$ functions $G^u(\cdot)$ and $G^v(\cdot)$ such that the control law $U(t)$ defined in* (44) *rewrites*

$$U(t) = \int_0^1 G^u(\xi) u(t,\xi) + G^v(\xi) v(t,\xi) d\xi. \tag{46}$$

Remark 8 *The control law* (44) *appears as a natural control law for the stabilization of the system* (37)-(40) *(which is equivalent to* (5)-(8)*). However, one must be aware that it is a specific control design among others. When considering a real implementation, one has to deal with some robustness or performance criteria. These constraints can only be considered on a case by case basis for the moment. For instance, in a similar way to what is done in [8,9], it is possible to introduce several degrees of freedom by means of tunable parameters corresponding to the amount of reflection terms cancelled by the control law and by the observer. These degrees of freedom enable (among others) a trade-off between performance and robustness (see [9] for details). Similarly, one could consider to preserve some dissipative in-domain coupling terms that are present in the original system while designing the target system rather than removing them using the control law in order to enhance the robustness properties. In this context, the port-hamiltonian approach may be of interest [32].*

4 Output feedback stabilization

In this section we design an observer that relies on the measurements of u_1 at the right boundary, i.e. $y(t) = u_1(t,1)$. Then, using the estimates given by our observer and the control law (44), we derive an output feedback controller. The design of the observer is based on the adjoint method which has been successfully used in [3,7].

4.1 Observer design

To simplify the notations we use the condensed formulation given by equations (10)-(12). The observer equations read as follows

$$\begin{aligned}\partial_t \hat{u}(t,x) + \Lambda^+ \partial_x \hat{u}(t,x) =& - P^+(x)(u_1(t,1) - \hat{u}_1(t,1)) \\ &+ \Sigma^+(x)\hat{v}(t,x), \qquad (47)\\ \partial_t \hat{v}(t,x) - \Lambda^- \partial_x \hat{v}(t,x) =& - P^-(x)(u_1(t,1) - \hat{u}_1(t,1)) \\ &+ \Sigma^-(x)\hat{v}(t,x), \qquad (48)\end{aligned}$$

with the boundary conditions

$$\begin{pmatrix}\hat{u}(t,0)\\ \hat{v}(t,1)\end{pmatrix} = K \begin{pmatrix}\hat{v}(t,0)\\ \hat{u}(t,1)\end{pmatrix} + AU(t), \qquad (49)$$

where K is defined in (13), and where the observer gains $P^+(\cdot)$ and $P^-(\cdot)$ are $L^\infty([0,1])$-functions that have yet to be designed. The initial conditions $\hat{u}(0,\cdot)$ and $\hat{v}(0,\cdot)$ can be arbitrarily chosen in $(L^2([0,1]))^{2n}$. Defining the error estimates $\tilde{u}(t,x) = u(t,x) - \hat{u}(t,x)$ and $\tilde{v}(t,x) = v(t,x) - \hat{v}(t,x)$, we get the following error system

$$\begin{aligned}\partial_t \tilde{u}(t,x) + \Lambda^+ \partial_x \tilde{u}(t,x) =& P^+(x)(\tilde{u}_1(t,1)) \\ &+ \Sigma^+(x)\tilde{v}(t,x), \qquad (50)\\ \partial_t \tilde{v}(t,x) - \Lambda^- \partial_x \hat{v}(t,x) =& P^-(x)(\tilde{u}_1(t,1)) \\ &+ \Sigma^-(x)\tilde{u}(t,x), \qquad (51)\end{aligned}$$

with the boundary conditions

$$\begin{pmatrix}\tilde{u}(t,0)\\ \tilde{v}(t,1)\end{pmatrix} = K \begin{pmatrix}\tilde{v}(t,0)\\ \tilde{u}(t,1)\end{pmatrix}. \qquad (52)$$

The corresponding initial conditions $\tilde{u}(0,\cdot)$ and $\tilde{v}(0,\cdot)$ belong in $(L^2([0,1]))^{2n}$. Adjusting the proofs of [3, Theorem 3.2.1] or [7, Theorem 7] (which are both inspired by the adjoint lemma given in [29, p. 627]), we can show that the system (50)-(52) is the adjoint (in the sense of the weak formulation) of the system

$$\begin{aligned}\partial_t \Psi(t,x) + \Lambda^- \partial_x \Psi(t,x) =& (\Sigma^+)^T(x)\Phi(t,x), \qquad (53)\\ \partial_t \Phi(t,x) - \Lambda^+ \partial_x \Phi(t,x) =& (\Sigma^-)^T(x)\Psi(t,x), \qquad (54)\end{aligned}$$

with the boundary conditions

$$\begin{aligned}&\begin{pmatrix}\Psi(t,0)\\ \Phi(t,1)\end{pmatrix} = \tilde{K} \begin{pmatrix}\Phi(t,0)\\ \Psi(t,1)\end{pmatrix} \\ &+ \frac{A}{\lambda_1}\int_0^1 (P^+(\xi)^T\phi(t,\xi) + P^-(\xi)^T\psi(t,\xi))d\xi, \qquad (55)\end{aligned}$$

where the matrix $\tilde{K}$ is defined by

$$\begin{aligned}\tilde{K} &= \begin{pmatrix}\tilde{K}_{00} & \tilde{K}_{01}\\ \tilde{K}_{10} & \tilde{K}_{11}\end{pmatrix} \\ &= \begin{pmatrix}(\Lambda^-)^{-1}K_{00}^T\Lambda^+ & (\Lambda^-)^{-1}K_{10}^T\Lambda^-\\ (\Lambda^+)^{-1}K_{01}^T\Lambda^+ & (\Lambda^+)^{-1}K_{11}^T\Lambda^-\end{pmatrix} \qquad (56)\end{aligned}$$

where Ψ and Φ are evolving in $\{(t,x)$ s.t. $t > 0,\ x \in [0,1]\}$ with initial conditions $(\Psi(0,\cdot),\Phi(0,\cdot))$ in $(L^2([0,1]))^{2n}$. Moreover, the stability properties of (50)-(52) and (53)-(55) are equivalent. The main advantage of working with the adjoint system (53)-(55) rather than with the error system (50)-(52) is that we now have to deal with a control problem which has already been solved in the previous section. More precisely, the boundary condition (55) can be rewritten

$$\begin{pmatrix}\Psi(t,0)\\ \Phi(t,1)\end{pmatrix} = \tilde{K} \begin{pmatrix}\Phi(t,0)\\ \Psi(t,1)\end{pmatrix} + AV_1(t), \qquad (57)$$

where

$$V_1(t) = \int_0^1 (\lambda_1)^{-1}(P^+(\xi)^T\phi(t,\xi) + P^-(\xi)^T\psi(t,\xi))d\xi.$$

As system (50)-(52) and (53)-(55) have equivalent stability properties, we want to find $P^+(\xi)$ and $P^-(\xi)$ such that the solutions of (53)-(55) converge to zero. System (53)-(55) has the same structure as the system (5)-(8). To be able to find a control law $V_1(t)$ that stabilizes (53)-(55), we want to use Theorem 6. To do so, we only need to check that Assumption 1 and Assumption 2 are satisfied for (53)-(55) with this new set of couplings. To verify Assumption 1, we need to have

$$(\tilde{K}_{00})_{i,i} \neq 0, \quad (\tilde{K}_{11})_{i,i} \neq 0, \tag{58}$$

while condition (16) can be rewritten as

$$(\tilde{K}_{00})_{i,i}(\tilde{K}_{11})_{i+1,i+1} \neq (\tilde{K}_{01})_{i,i+1}(\tilde{K}_{10})_{i+1,i}. \tag{59}$$

With simple computations, we obtain

$$(\tilde{K}_{00})_{i,i} = \frac{\lambda_i}{\mu_i} q_{i,i}, (\tilde{K}_{11})_{i+1,i+1} = \frac{\mu_{i+1}}{\lambda_{i+1}} \rho_{i+1,i+1} \tag{60}$$

$$(\tilde{K}_{01})_{i,i+1} = \frac{\mu_{i+1}\rho_{i+1,i}}{\mu_i}, \ (\tilde{K}_{10})_{i+1,i} = \frac{\lambda_i q_{i,i+1}}{\lambda_{i+1}}. \tag{61}$$

Thus, using the fact that Assumption 1 holds for the original system (10)-(12), conditions (58) and (59) are immediately satisfied. Finally, using the fact that Assumption 2 is satisfied for the original system (10)-(12), the system (50)-(52) with $P^+ \equiv 0$ and $P^- \equiv 0$ is exponentially stable in the absence of in-domain coupling terms. Since (53)-(55) and (50)-(52) have equivalent stability properties, Assumption 2 holds for (53)-(55). Using Theorem 6 and Remark 7, we know that there exist $L^\infty([0,1])$ functions $G^\psi(\cdot)$ and $G^\phi(\cdot)$, such that the control law

$$V_1(t) = \int_0^1 G^\psi(\xi)\psi(t,\xi) + G^\phi(\xi)\phi(t,\xi)d\xi, \tag{62}$$

ensures the exponential stabilization of the system (53)-(54) with the boundary condition (57). Defining the observer gains as

$$P^-(x) = (G^\psi(x))^T\lambda_1, \ P^+(x) = (G^\phi(x))^T\lambda_1, \tag{63}$$

we can guarantee the exponential stability of the system (53)-(55). Since (50)-(52) and (53)-(55) have equivalent stability properties, we immediately obtain the convergence of the state $(\tilde{u}, \tilde{v})$ to zero in the sense of (14).

4.2 Output feedback law

The estimates given by the proposed observer can be used in a observer-controller to derive an output feedback law guaranteeing the exponential stability of the zero equilibrium.

Theorem 9 *Consider the system composed of* (10)-(12) *and of the observer system* (47)-(49) *along with the control law*

$$\begin{aligned} U(t) &= \int_0^1 \begin{pmatrix} G^w(\xi) & G^z(\xi) \end{pmatrix} \mathcal{F}_1^{-1}\mathcal{F}^{-1} \begin{pmatrix} \hat{u}(t,\xi) \\ \hat{v}(t,\xi) \end{pmatrix} d\xi \\ &= \int_0^1 G^u(\xi)\hat{u}(t,\xi) + G^v(\xi)\hat{v}(t,\xi)d\xi, \end{aligned} \tag{64}$$

where G^u and G^v are defined in the proof of Theorem 5. Then for any initial condition $(u_0, v_0) \in (L^2([0,1]))^{2n}$, for any observer initial condition $(\hat{u}_0, \hat{v}_0) \in (L^2([0,1]))^{2n}$, its solution $(u, v, \hat{u}, \hat{v})$ exponentially converges to its zero equilibrium in the sense of (14).

PROOF. We have $\hat{u} = -\tilde{u} + u$ and $\hat{v} = -\tilde{v} + v$. Therefore, one has

$$\begin{aligned} U(t) &= \int_0^1 \begin{pmatrix} G^w(\xi) & G^z(\xi) \end{pmatrix} \mathcal{F}_1^{-1}\mathcal{F}^{-1} \begin{pmatrix} u(t,\xi) \\ v(t,\xi) \end{pmatrix} d\xi \\ &+ \int_0^1 \begin{pmatrix} G^w(\xi) & G^z(\xi) \end{pmatrix} \mathcal{F}_1^{-1}\mathcal{F}^{-1} \begin{pmatrix} \tilde{u}(t,\xi) \\ \tilde{v}(t,\xi) \end{pmatrix} d\xi. \end{aligned} \tag{65}$$

We already know that $(\tilde{u}, \tilde{v})$ exponentially converges to zero. Using the backstepping operators $\mathcal{F}$ and $\mathcal{F}_1$, one can prove (the computations are identical to the ones done above) that the closed-loop system (10)-(12) with the control law (65) can be mapped to the system (37)-(40), the only difference being the $z_1(t,1)$ boundary condition that now rewrites

$$\begin{aligned} z_1(t,1) &= \rho_{1,1}w_1(t,1) \\ &+ \int_0^1 \begin{pmatrix} G^w(\xi) & G^z(\xi) \end{pmatrix} \mathcal{F}_1^{-1}\mathcal{F}^{-1} \begin{pmatrix} \tilde{u}(t,\xi) \\ \tilde{v}(t,\xi) \end{pmatrix} d\xi. \end{aligned} \tag{66}$$

This system can be rewritten in a similar framework to the one considered in [8] by performing the change of variables $\bar{x} = -x$ for all the even subsystems. Then, following the approach given [8] which is based on the method of the characteristics and on Fubini's theorem, we can show [8, Theorem 2] that

$$\begin{aligned} z_i(t,1) &= \sum_{1\leq k\leq n^2} A_k z_l(t-\tau_k, 1) \\ &+ \delta_1^i \Big(\int_0^1 \begin{pmatrix} G^w(\xi) & G^z(\xi) \end{pmatrix} \mathcal{F}_1^{-1}\mathcal{F}^{-1} \begin{pmatrix} \tilde{u}(t,\xi) \\ \tilde{v}(t,\xi) \end{pmatrix}\Big), \end{aligned} \tag{67}$$

where the matrices A_k and the delays τ_k are identical to the ones defined in (19) and where we recall that δ_1^i stands for the Kronecker symbol. Due to Assumption 2,

the principal part of (67) generates an exponentially stable semigroup. As the integral part exponentially converges to zero, differentiating (67) and using the variation of constants formula (using [24, Theorem 7.6, page 32]), we have the convergence of $\dot{z}(t,1)$ to zero. This implies the exponential convergence of $z(t,1)$ to a constant that can only be zero since the integral part converges to zero. As stated above, following the approach of [8], it can be shown that the system (67) and (37)-(40) have equivalent stability properties (see [8, Theorem 3] for more details). This implies the exponential convergence of the state (w,z) in the sense of (14). Using the invertibility of the different backstepping transformations, we obtain the exponential convergence of the state (u,v) in the sense of (14) This concludes the proof. □

5 Some remarks on robustness

In this section we state some important remarks on the robustness properties of the output-feedback control law designed in the previous section. The robustness can be assessed with respect to uncertainties on the different coupling terms, on the transport velocities but also with respect to delays in the actuation or in the measurements. The robustness properties of the proposed output feedback control law rely on the fact that such a control law is an integral operator that does not cancel the boundary couplings (it is strictly proper). However, the complete rigorous proof is extremely technical and out of the scope of this paper. As the computations are identical to the ones done in [3,8,9] we have chosen not to give the details here. Thus, this section should be considered as an opening section in which we present some results that are out of the scope of this paper and for which only a sketch of the proof is given. We start proving the delay-robustness of the state-feedback law (44). More precisely, we can state the following theorem.

Theorem 10 *The control law U(t) defined in* (44) *delay-robustly stabilizes the system* (5)-(8)*. That is, there exists $\delta^* > 0$ such that, for all $\delta \in [0,\delta^*]$, $U(t-\delta)$ exponentially stabilizes* (5)-(8)*.*

PROOF. The proof is a consequence of Assumption 2 and of the computations done in [8]. We only recall the main ideas. Consider the system (5)-(8) with the delayed control law $U(t-\delta)$. Using the equivalence between (5)-(8) and (37)-(40) (stated by Theorem 4 and Theorem 5), the two systems have equivalent stability properties. Then, using the method of characteristics, one can easily prove (see [8] for details) that for all $1 \le i \le n$, for $t \ge \tau = \max_{k,l}\{\frac{1}{\lambda_k} + \frac{1}{\mu_l}\}$ the state $z_i(t,1)$ satisfies the following difference system

$$z_i(t,1) = \sum_{k=1}^{n^2} A_k z_l(t-\tau_k,1) + \delta_1^i (U(t-\delta) - \sum_{l=1}^{m} \int_0^{\tau} G_l(s) z_l(t-s,1) ds), \qquad (68)$$

where the matrices A_k and the delays τ_k are identical to the ones defined in (19) and where G_l are $L^\infty([0,\tau])$ function that only depend on G^w and G^z and where the control law $U(t-\delta)$ rewrites $U(t-\delta) = \sum_{l=1}^{m} \int_0^{\tau} G_l(s) z_l(t-s-\delta,1)$. As the principal part of (68) generates an exponentially stable semi-group due to Assumption 2 and as the control law only acts on the integral part of equation (68), we can directly apply [8, Theorem 5] to conclude to the existence of $\delta^* > 0$ such that, for any $\delta \in [0,\delta^*]$, the system (5)-(8) with the control law $U(t-\delta)$ is exponentially stable. □

Considering the output feedback law given in (64), we have the following theorem, whose proof can be obtained adjusting the approach of [9].

Theorem 11 *The control law U(t) defined in* (64) *delay-robustly stabilizes the system* (5)-(8) *with respect to delays in the actuation or in the measurements. That is, there exists $\delta^* > 0$ such that, for all $\delta_0 \in [0,\delta^*]$, for all $\delta_1 \in [0,\delta^*]$, the control law $U(t-\delta_0)$ based on the delayed measurements $y(t-\delta_1)$ exponentially stabilizes* (5)-(8)*. Moreover, the control law* (64) *is robust with respect to uncertainties on the transport velocities and to uncertainties on the different coupling parameters.*

The proof of the delay-robustness relies on the fact that the controller has been chosen as strictly proper (we only cancel integral terms that vanish at high frequency). We can then use the same ideas as the one used to prove Theorem 10 and the ones given in [9]. The goal is to rewrite the plant-observer system as a neutral system and to consider the corresponding associated characteristic equation. As it is out of the scope of this paper, we choose to omit it. The robustness with respect to uncertainties can be obtained adjusting the approach of [8, Theorem 5] or [9].

6 Simulation results

In this section we illustrate our results with simulations. Let us consider the unstable system (10)-(12) for which the coefficients are defined by

$$\Lambda^+ = \mathrm{diag}(1, 1.7, 1.2),\ \Lambda^- = \mathrm{diag}(1.5, 1.8, 1.8),$$
$$\Sigma^+ = \mathrm{diag}(0.3, 0, 0.3),\ \Sigma^- = \mathrm{diag}(0.2, 0, 0.2),$$
$$K_{00} = \mathrm{diag}(0.4, 0.6, 0.4), K_{11} = \mathrm{diag}(1, 0.5, 0.5),$$
$$K_{01} = \begin{pmatrix} 0 & 0.6 & 0 \\ 0 & 0 & 0.8 \\ 0 & 0 & 0 \end{pmatrix}, K_{10} = \begin{pmatrix} 0 & 0 & 0 \\ 0.4 & 0 & 0 \\ 0 & 0.7 & 0 \end{pmatrix}.$$

The parameters values are chosen such that the original open-loop system is unstable (as it can be seen in Figure 3). However, the boundary coupling terms are chosen such that Assumption 1 and Assumption 2 are fulfilled. We consider a C^1 initial condition. To illustrate Remark 7, we cancel a part of the boundary reflection by mean of a tunable parameter κ. Namely, the term $-\kappa u_1(t,1)$ is added to the control law. The algorithm we use is adapted from the one proposed in [6]. Using the method of characteristics, we write the integral equations associated to the kernel PDE-systems. These integral equations are solved using a fixed-point algorithm. These kernels are then used to compute the control law. Finally, the original system (10)-(12) is simulated using a Godunov's discretization scheme. Figure 3 pictures the L^2-norm of the state (u,v) using the control law (44) in presence of a small delay in the loop ($\delta = 0.3$ s) for different values of κ. Finally, Figure 4 depicts the control effort in the absence of delay for the different control laws.

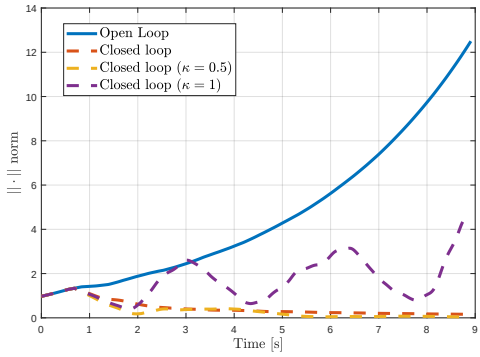

Fig. 3. Time evolution of the L^2-norm of system (10)-(12) in open-loop and in closed-loop for different values of κ in presence of a 0.3s delay.

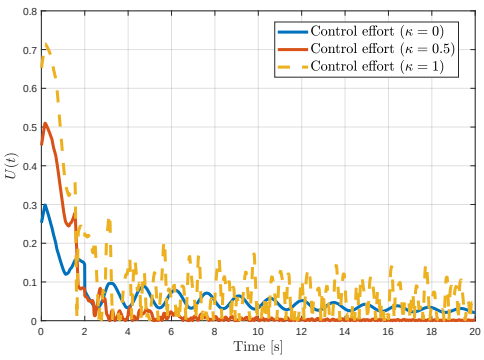

Fig. 4. Time evolution of the control effort $U(t)$ for different values of κ (no delay).

One can notice that a cancellation of a part of the reflection increases the convergence rate ($\kappa = 0.5$ compared to $\kappa = 0$). However, if κ increases, the robustness margins decrease in the same time. This explains why the system diverges for $\kappa = 1$ in presence of a delay $\delta = 0.3s$.This illustrates the trade-off performance-robustness we previously mentioned. For $\kappa = 0$, the closed-loop convergence rate is imposed by the natural boundary dissipative terms of the system. It would be interesting in future contributions to compute the maximum amount of the reflection term ρ_1 that could be cancelled to increase the convergence rate while preserving reasonnable delay-robustness properties.

7 Concluding remarks

In this paper, an explicit stabilizing output feedback control law has been designed for an underactuated cascade network of n subsystems of two coupled hyperbolic PDEs connected through their boundaries, the control law being located at one boundary of the network. The proposed method combines successive backstepping transformations that reduce the PDEs network to a simple network without in-domain coupling terms. The control design becomes then straightforward. The corresponding boundary observer (used to obtain the corresponding output feedback law) has been designed using a dual approach. Some remarks about the robustness to small delays (in the actuation and in the measurements) of the proposed output-feedback law have been assessed. These robustness properties are ensured due to the integral structure of the control law (which is consequently strictly proper). The present results are a first step towards the robust stabilization of networks of hyperbolic systems. To be able to tackle the wide diversity of physical systems (traffic networks, electricity networks...), our next contributions will focus on networks of hyperbolic systems with an arbitrary number of PDEs in each subsystem and on networks with a more complex graph structure. Finally, this raises the general question of the stabilization of underactuated hyperbolic systems.

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A Derivation of the kernel equations

In this Appendix, we give the equations satisfied by the kernels $R_i^{\cdot}$ and $F_{i,j}^{\cdot\cdot}$. They are obtained by differentiating (41) with respect to space and time, by using the Liebniz rule and integration by parts, and by plugging the obtained expressions into the system (31). We then show the well-posedness of these equations. This is necessary to prove the existence of the transformation (41). In what follows, i and j are two integers that are smaller than n. Moreover, we assume that $j > i$. By convention and to avoid useless case distinctions, we assume that $F_{i,j}^{\cdot\cdot} \equiv 0$ if $j \leq i$ or if $j \geq n+1$.

Kernel equations for R_i^α and R_i^β

The kernels R_i^α and R_i^β satisfy for all $(x,\xi) \in \mathcal{T}_b$ the following set of PDEs

$$\partial_x R_i^\beta(x,\xi) + \partial_\xi R_i^\beta(x,\xi) = 0, \tag{A.1}$$
$$\partial_x R_i^\alpha(x,\xi) + \partial_\xi R_i^\alpha(x,\xi) = 0, \tag{A.2}$$

along with the boundary conditions

$$R_i^\beta(x,0) = -\frac{\mu_{i+1}}{\mu_i}\rho_{i+1,i} F_{i,i+1}^{\beta\beta}(x,1), \tag{A.3}$$
$$q_{i,i}\lambda_i R_i^\alpha(x,0) - \mu_{i+1}\rho_{i+1,i} F_{i,i+1}^{\alpha\beta}(x,1) = 0, \tag{A.4}$$

Kernel equations for $F_{i,i+1}^{\cdot\cdot}$

Due to the presence of in-domain coupling terms in equation (32), we have to distinguish the case $j = i+1$. The kernels $F_{i,i+1}^{\cdot\cdot}(\cdot,\cdot)$ satisfy for all $(x,\xi) \in \mathcal{U}$ the following set of PDEs

$$\lambda_{i+1}\partial_\xi F_{i,i+1}^{\alpha\alpha}(x,\xi) + \lambda_i \partial_x \partial_\xi F_{i,i+1}^{\alpha\alpha}(x,\xi) = \lambda_i q_{i,i+1} K_i^{uu}(x,0)$$
$$[R_{i+1}^\alpha(1,\xi) + L_{i+1}^{\alpha\alpha}(1,\xi) - \int_0^1 L_{i+1}^{\alpha\alpha}(1,\nu) R_{i+1}^\alpha(\nu,\xi)d\nu], \tag{A.5}$$
$$-\mu_{i+1}\partial_\xi F_{i,i+1}^{\alpha\beta}(x,\xi) + \lambda_i \partial_x F_{i,i+1}^{\alpha\beta}(x,\xi) = \lambda_i q_{i,i+1} K_i^{uu}(x,0)$$
$$[L_{i+1}^{\alpha\beta}(1,\xi) - \int_0^1 L_{i+1}^{\alpha\beta}(1,\nu) R_{i+1}^\beta(\nu,\xi)d\nu], \tag{A.6}$$
$$\mu_i \partial_x F_{i,i+1}^{\beta\alpha}(x,\xi) - \lambda_{i+1}\partial_\xi F_{i,i+1}^{\beta\alpha}(x,\xi) = -\lambda_i q_{i,i+1} K_i^{vu}(x,0)$$
$$[R_{i+1}^\alpha(1,\xi) + L_{i+1}^{\alpha\alpha}(1,\xi) - \int_\xi^1 L_{i+1}^{\alpha\alpha}(1,\nu) R_{i+1}^\alpha(\nu,\xi)d\nu], \tag{A.7}$$
$$\mu_{i+1}\partial_\xi F_{i,i+1}^{\beta\beta}(x,\xi) + \mu_i \partial_x F_{i,i+1}^{\beta\beta}(x,\xi) = -\lambda_i q_{i,i+1} K_i^{vu}(x,0)$$
$$[L_{i+1}^{\alpha\beta}(1,\xi) - \int_\xi^1 L_{i+1}^{\alpha\beta}(1,\nu) R_{i+1}^\beta(\nu,\xi)d\nu], \tag{A.8}$$

with the boundary conditions

$$\mu_{i+1}\rho_{i+1,i+1} F_{i,i+1}^{\alpha\beta}(x,1) = \lambda_i q_{i,i+1} R_i^\alpha(x,0) + \lambda_{i+1} F_{i,i+1}^{\alpha\alpha}(x,1) - \lambda_i q_{i,i+1} K_i^{uu}(x,0), \tag{A.9}$$
$$F_{i,i+1}^{\alpha\alpha}(x,0) = \frac{1}{\lambda_{i+1} q_{i+1,i+1}}(\mu_{i+1} F_{i,i+1}^{\alpha\beta}(x,0) - \mu_{i+2} F_{i,i+2}^{\alpha\beta}(x,1)), \tag{A.10}$$
$$F_{i,i+1}^{\beta\beta}(x,0) = \frac{1}{\mu_{i+1}}(\lambda_{i+1} q_{i+1,i+1} F_{i,i+1}^{\beta\alpha}(x,0) + \mu_{i+2}\rho_{i+2,i+1} F_{i,i+2}^{\beta\beta}(x,1)), \tag{A.11}$$
$$F_{i,i+1}^{\beta\alpha}(x,1) = \frac{1}{\lambda_{i+1}}(\mu_{i+1}\rho_{i+1,i+1} F_{i,i+1}^{\beta\beta}(x,1) + \lambda_i q_{i,i+1} K_i^{vu}(x,0)), \tag{A.12}$$

and

$$F_{i,i+1}^{\alpha\alpha}(0,\xi) = q_{i,i} F_{i,i+1}^{\beta\alpha}(0,\xi) - q_{i,i+1} R_{i+1}^\alpha(1,\xi) - q_{i,i+1} L_{i+1}^{\alpha\alpha}(1,\xi) + \int_\xi^1 L_{i+1}^{\alpha\alpha}(1,\nu) R_{i+1}^\alpha(\nu,\xi)d\nu, \tag{A.13}$$
$$F_{i,i+1}^{\alpha\beta}(0,\xi) = q_{i,i} F_{i,i+1}^{\beta\beta}(0,\xi) - q_{i,i+1} L_{i+1}^{\alpha\beta}(1,\xi) - \int_\xi^1 L_{i+1}^{\alpha\beta}(1,\nu) R_{i+1}^\beta(\nu,\xi)d\nu, \tag{A.14}$$
$$F_{i,i+1}^{\beta\alpha}(0,\xi) = \frac{1}{\rho_{i+1,i}}(R_{i+1}^\beta(1,\xi) - L_{i+1}^1(\xi) \int_\xi^1 L_{i+1}^1(\nu) R_{i+1}^\alpha(\nu,\xi)d\nu), \tag{A.15}$$
$$F_{i,i+1}^{\beta\beta}(0,\xi) = \frac{1}{\rho_{i+1,i}}(\rho_{i+1,i+1} R_{i+1}^\alpha(1,\xi) - L_{i+1}^2(\xi)$$

$$\int_{\xi}^{1} L_{i+1}^{2}(\nu)R_{i+1}^{\beta}(\nu,\xi)d\nu), \tag{A.16}$$

where L_i^1 and L_i^2 are defined by (42)-(43).

Kernel equations for $F_{i,j}^{\cdot\cdot}$

We can now consider the case $j > i+1$. The kernels $F_{i,j}^{\beta\alpha}(\cdot,\cdot)$ and $F_{i,j}^{\beta\beta}(\cdot,\cdot)$ satisfy for all $(x,\xi) \in \mathcal{U}$ the following set of PDEs

$$\lambda_j \partial_\xi F_{i,j}^{\alpha\alpha}(t,x) + \lambda_i \partial_x F_{i,j}^{\alpha\alpha}(t,x) = \lambda_i q_{i,i+1} K_i^{uu}(x,0) [\int_0^1 (L_{i+1}^{\alpha\alpha}(1,\nu)F_{i+1,j}^{\alpha\alpha}(\nu,\xi) + L_{i+1}^{\alpha\beta}(1,\nu)F_{i+1,j}^{\beta\alpha}(\nu,\xi))d\nu - F_{i+1,j}^{\alpha\alpha}(1,\xi)], \tag{A.17}$$

$$-\mu_j \partial_\xi F_{i,j}^{\alpha\beta}(t,x) + \lambda_i \partial_x F_{i,j}^{\alpha\beta}(t,x) = \lambda_i q_{i,i+1} K_i^{uu}(x,0) [\int_0^1 (L_{i+1}^{\alpha\alpha}(1,\nu)F_{i+1,j}^{\alpha\beta}(\nu,\xi) + L_{i+1}^{\alpha\beta}(1,\nu)F_{i+1,j}^{\beta\beta}(\nu,\xi))d\nu - F_{i+1,j}^{\alpha\beta}(1,\xi)], \tag{A.18}$$

$$-\lambda_j \partial_\xi F_{i,j}^{\beta\alpha}(t,x) + \mu_i \partial_x F_{i,j}^{\beta\alpha}(t,x) = -\lambda_i q_{i,i+1} K_i^{vu}(x,0) [\int_0^1 (L_{i+1}^{\alpha\alpha}(1,\nu)F_{i+1,j}^{\alpha\alpha}(\nu,\xi) + L_{i+1}^{\beta\alpha}(1,\nu)F_{i+1,j}^{\beta\alpha}(\nu,\xi))d\nu - F_{i+1,j}^{\alpha\alpha}(1,\xi)], \tag{A.19}$$

$$\mu_j \partial_\xi F_{i,j}^{\beta\beta}(t,x) + \mu_i \partial_x F_{i,j}^{\beta\beta}(t,x) = -\lambda_i q_{i,i+1} K_i^{vu}(x,0) [\int_0^1 (L_{i+1}^{\alpha\alpha}(1,\nu)F_{i+1,j}^{\alpha\beta}(\nu,\xi) + L_{i+1}^{\alpha\beta}(1,\nu)F_{i+1,j}^{\beta\beta}(\nu,\xi))d\nu - F_{i+1,j}^{\alpha\beta}(1,\xi)], \tag{A.20}$$

along with the following boundary conditions

$$F_{i,j}^{\alpha\beta}(x,1) = \frac{1}{\mu_j \rho_{jj}}(\lambda_j F_{i,j}^{\alpha\alpha}(x,1) - q_{j-1,j}\lambda_{j-1}F_{i,j-1}^{\alpha\alpha}(x,0)), \tag{A.21}$$

$$F_{i,j}^{\alpha\alpha}(x,0) = \frac{1}{\lambda_j q_{j,j}}(\mu_j F_{i,j}^{\alpha\beta}(x,0) - \mu_{j+1}F_{i,j+1}^{\alpha\beta}(x,1)), \tag{A.22}$$

$$F_{i,j}^{\beta\beta}(x,0) = \frac{1}{\mu_j}(\lambda_j q_{j,j} F_{i,j}^{\beta\alpha}(x,0) + \mu_{j+1}\rho_{j+1,j}F_{i,j+1}^{\beta\beta}(x,1)), \tag{A.23}$$

$$F_{i,j}^{\beta\alpha}(x,1) = \frac{1}{\lambda_j}(\mu_j \rho_{j,j} F_{i,j}^{\beta\beta}(x,1) + \lambda_{j-1}q_{j-1,j}F_{i,j-1}^{\beta\alpha}(x,0)). \tag{A.24}$$

and

$$F_{i,j}^{\alpha\alpha}(0,\xi) = q_{i,i}F_{i,j}^{\beta\alpha}(0,\xi) + q_{i,i+1}F_{i+1,j}^{\alpha\alpha}(1,\xi) - \int_{\xi}^{1} L_{i+1}^{\alpha\alpha}(1,\nu)F_{i+1,j}^{\alpha,\alpha}(\nu,\xi) + L_{i+1}^{\alpha\beta}(1,\nu)F_{i+1,j}^{\beta,\alpha}(\nu,\xi)d\nu, \tag{A.25}$$

$$F_{i,j}^{\alpha\beta}(0,\xi) = q_{i,i}F_{i,j}^{\beta\beta}(0,\xi) + q_{i,i+1}F_{i+1,j}^{\alpha\beta}(1,\xi) - \int_{\xi}^{1} L_{i+1}^{\alpha\alpha}(1,\nu)F_{i+1,j}^{\alpha,\beta}(\nu,\xi) + L_{i+1}^{\alpha\beta}(1,\nu)F_{i+1,j}^{\beta,\beta}(\nu,\xi)d\nu, \tag{A.26}$$

$$F_{i,j}^{\beta\alpha}(0,\xi) = \frac{1}{\rho_{i+1,i}}(F_{i+1,j}^{\beta\alpha}(1,\xi) - \rho_{i+1,i+1}F_{i+1,j}^{\alpha\alpha}(1,\xi) + \int_{\xi}^{1} L_{i+1}^{1}(\nu)F_{i+1,j}^{\alpha,\alpha}(\nu,\xi) + L_{i+1}^{2}(\nu)F_{i+1,j}^{\beta,\alpha}(\nu,\xi)d\nu), \tag{A.27}$$

$$F_{i,j}^{\beta\beta}(0,\xi) = \frac{1}{\rho_{i+1,i}}(F_{i+1,j}^{\beta\beta}(1,\xi) - \rho_{i+1,i+1}F_{i+1,j}^{\alpha\beta}(1,\xi) + \int_{\xi}^{1} L_{i+1}^{1}(\nu)F_{i+1,j}^{\alpha,\beta}(\nu,\xi) + L_{i+1}^{2}(\nu)F_{i+1,j}^{\beta,\beta}(\nu,\xi)d\nu). \tag{A.28}$$

where L_i^1 and L_i^2 are defined by (42)-(43).

All the kernel PDEs are obtained through the differentiation of the transformation (41). This is also the case for the boundary conditions in $\xi = 0$ or $\xi = 1$. Regarding, the boundary conditions in $x = 0$ (equations (A.13)-(A.16) and (A.25)-(A.28)), they are required to guarantee that the boundary conditions (33)-(34) and (39)-(40) hold. More precisely, there are integral terms in (33)-(34) that do not appear in (39)-(40). Thus, the kernel boundary conditions in $(0,\xi)$ are used to remove these integral terms. These conditions are necessary to map the system (31)-(34) into the system (37)-(40). We have the following theorem.

Theorem 12 *Consider system* (A.1)-(A.28)*. There exists a unique solution R_i^α, R_i^β, $F_{i,\cdot}^{\beta\alpha}$, $F_{i,\cdot}^{\beta\beta}$, $F_{i,\cdot}^{\alpha\alpha}$ and $F_{i,\cdot}^{\alpha\beta}$ in $(L^\infty(\mathcal{T}_b))^2 \times (L^\infty(\mathcal{U}))^4$.*

To give some insights about the proof of this theorem, let us consider the following preliminary remarks.

(1) Equations (A.1)-(A.28) do not appear linear at first sight. However, one can easily notice that for a fixed value of i, the potential non-linear terms are either previously computed kernels ($L_{\cdot\cdot}^{\cdot\cdot}$ for instance or functions that depend on $F_{i+k,\cdot}^{\cdot\cdot}$ (with $k > 0$)). In other words, the kernel equations present a cascade structure which is the consequence of the cascade structure of the transformation (41). This suggests a recursive proof of the well-posedness.

(2) Some boundary conditions ((A.4) and (A.9) for instance) are not independent (in the sense as they do not only depend on previously computed kernels or on functions that depend on $F_{i+k,\cdot}^{\cdot\cdot}$ (with $k > 0$)) but are coupled with each other. We will show that due to Assumption 1, we can easily decouple them.

Using these remarks, we can now prove Theorem 12.

PROOF. As explained above, equations (A.1)-(A.28) have a cascade structure as they depend on $F^{\cdot\cdot}_{i+k,\cdot}$ (with $k > 0$). This justifies a recursive approach. Let us consider the following property defined for all $1 \leq s \leq n$:
$P(s)$: "$\forall\ n+1-s \leq i \leq n \quad \forall\ i < j \leq n$ the problem (A.1)-(A.28) has a unique solution $R^{\alpha}_i(\cdot,\cdot), R^{\beta}_i(\cdot,\cdot)$, $F^{\beta\alpha}_{i,j}(\cdot,\cdot), F^{\beta\beta}_{i,j}(\cdot,\cdot), F^{\alpha\alpha}_{i,j}(\cdot,\cdot)$ and $F^{\alpha\beta}_{i,j}(\cdot,\cdot)$ in $(L^{\infty}(\mathcal{T}_b))^2 \times (L^{\infty}(\mathcal{U}))^4$". Due to the definition of this property, we will successively prove the well-posedness of each line of the kernel matrices starting from the last one.

<u>Initialization</u> Let us prove $P(1)$. For $s = 1$, we have $i = n$. As for any $i < j$ The kernels $F^{\cdot\cdot}_{ij}$ are equal to zero, the system (A.1)-(A.28) simply rewrites

$$\partial_x R^{\beta}_i(x,\xi) + \partial_\xi R^{\beta}_i(x,\xi) = \partial_x R^{\alpha}_i(x,\xi) + \partial_\xi R^{\alpha}_i(x,\xi) = 0$$

along with the boundary conditions

$$R^{\beta}_i(x,0) = R^{\alpha}_i(x,0) = 0.$$

This system admits 0 as a unique solution. This proves $P(1)$.

<u>Induction:</u> Let us assume that the property $P(q-1)$ $(1 < q \leq n)$ is true. We consequently have $\forall\ n+1-(q-1) \leq i \leq n, \quad \forall\ i < j \leq n$ the existence and the boundedness of the kernels $R^{\alpha}_i(\cdot,\cdot), R^{\beta}_i(\cdot,\cdot), F^{\beta\alpha}_{i,j}(\cdot,\cdot)$, $F^{\beta\beta}_{i,j}(\cdot,\cdot)$, $F^{\alpha\alpha}_{i,j}(\cdot,\cdot)$ and $F^{\alpha\beta}_{i,j}(\cdot,\cdot)$. Let us denote $i = n+1-q$. We need to prove that $\forall\ j \in \mathbb{N}$ such that $i < j \leq n$ the kernels $R^{\alpha}_{n+1-q}(\cdot,\cdot), R^{\beta}_{n+1-q}(\cdot,\cdot), F^{\beta\alpha}_{n+1-q,j}(\cdot,\cdot)$, $F^{\beta\beta}_{n+1-q,j}(\cdot,\cdot)$, $F^{\alpha\alpha}_{n+1-q,j}(\cdot,\cdot)$ and $F^{\alpha\beta}_{n+1-q,j}(\cdot,\cdot)$ exist. Using the fact that $P(q)$ holds, the PDEs (A.1)-(A.2) and (A.17)-(A.20) satisfied by the different kernels (A.1)-(A.28) can be rewritten

$$\partial_x R^{\alpha}_i(x,\xi) + \partial_\xi R^{\alpha}_i(x,\xi) = 0, \tag{A.29}$$
$$\partial_x R^{\beta}_i(x,\xi) + \partial_\xi R^{\beta}_i(x,\xi) = 0, \tag{A.30}$$
$$\lambda_j \partial_\xi F^{\alpha\alpha}_{i,j}(t,x) + \lambda_i \partial_x F^{\alpha\alpha}_{i,j}(t,x) = f^{\alpha\alpha}_{i,j}(x,\xi), \tag{A.31}$$
$$\mu_j \partial_\xi F^{\alpha\beta}_{i,j}(t,x) - \lambda_i \partial_x F^{\alpha\beta}_{i,j}(t,x) = f^{\alpha\beta}_{i,j}(x,\xi), \tag{A.32}$$
$$\lambda_j \partial_\xi F^{\beta\alpha}_{i,j}(t,x) - \mu_i \partial_x F^{\beta\alpha}_{i,j}(t,x) = f^{\beta\alpha}_{i,j}(x,\xi), \tag{A.33}$$
$$\mu_j \partial_\xi F^{\beta\beta}_{i,j}(t,x) + \mu_i \partial_x F^{\beta\beta}_{i,j}(t,x) = f^{\beta\beta}_{i,j}(x,\xi), \tag{A.34}$$

where the functions $f^{\cdot\cdot}_{i,j}$ are L^{∞}-functions that only depend on the previously computed kernels. Let us now consider the boundary conditions (A.25)-(A.28). Injecting the boundary conditions (A.26)-(A.28) into (A.26)-(A.25), we obtain

$$F^{\beta\alpha}_{i,j}(0,\xi) = g^{\beta\alpha}_{i,j}(\xi), \quad F^{\alpha\beta}_{i,j}(0,\xi) = g^{\alpha\beta}_{i,j}(\xi), \tag{A.35}$$
$$F^{\alpha\alpha}_{i,j}(0,\xi) = g^{\alpha\alpha}_{i,j}(\xi), \quad F^{\beta\beta}_{i,j}(0,\xi) = g^{\beta\beta}_{i,j}(\xi), \tag{A.36}$$

where the functions $g^{\cdot\cdot}_{i,j}$ are L^{∞}-functions that only depend on the previously computed kernels. The other boundary conditions are given by

$$\begin{aligned} F^{\alpha\alpha}_{i,j}(x,0) =& \frac{1}{\lambda_j q_{j,j}}(\mu_j F^{\alpha\beta}_{i,j}(x,0) \\ &- \mu_{j+1} F^{\alpha\beta}_{i,j+1}(x,1)), \end{aligned} \tag{A.37}$$
$$\begin{aligned} F^{\alpha\beta}_{i,j}(x,1) =& \frac{1}{\mu_j \rho_{jj}}(\lambda_j F^{\alpha\alpha}_{i,j}(x,1) \\ &- q_{j-1,j}\lambda_{j-1} F^{\alpha\alpha}_{i,j-1}(x,0)), \end{aligned} \tag{A.38}$$
$$\begin{aligned} F^{\beta\alpha}_{i,j}(x,1) =& \frac{1}{\lambda_j}(\mu_j \rho_{j,j} F^{\beta\beta}_{i,j}(x,1) + \\ &h_{i,j}(x) + \lambda_{j-1} q_{j-1,j} F^{\beta\alpha}_{i,j-1}(x,0)), \end{aligned} \tag{A.39}$$
$$\begin{aligned} F^{\beta\beta}_{i,j}(x,0) =& \frac{1}{\mu_j}(\lambda_j q_{j,j} F^{\beta\alpha}_{i,j}(x,0) \\ &+ \mu_{j+1}\rho_{j+1,j} F^{\beta\beta}_{i,j+1}(x,1)), \end{aligned} \tag{A.40}$$

where the functions $h_{i,j}$ is a L^{∞}-functions that only depend on the previously computed kernels. Note that equation (A.38) only holds if $j > i+1$. If $j = i+1$, we have

$$\begin{aligned} \mu_{i+1}\rho_{i+1,i+1} F^{\alpha\beta}_{i,i+1}(x,1) =& \lambda_i q_{i,i+1} R^{\alpha}_i(x,0) \\ &+ \lambda_{i+1} F^{\alpha\alpha}_{i,i+1}(x,1) - \lambda_i q_{i,i+1} K^{uu}_i(x,0). \end{aligned} \tag{A.41}$$

Finally we recall the boundary conditions for the kernels R^{α} and R^{β}

$$R^{\beta}_i(x,0) = -\frac{\mu_{i+1}}{\mu_i}\rho_{i+1,i} F^{\beta\beta}_{i,i+1}(x,1), \tag{A.42}$$
$$q_{i,i}\lambda_i R^{\alpha}_i(x,0) - \mu_{i+1}\rho_{i+1,i} F^{\alpha\beta}_{i,i+1}(x,1) = 0. \tag{A.43}$$

The terms $F^{\beta\beta}_{i,i+1}(x,1)$ and $R^{\alpha}_i(x,0)$ appear both in (A.43) and (A.45). However, these two equations can be rewritten as

$$A_0 \begin{pmatrix} R^{\alpha}_i(x,0) \\ F^{\alpha\beta}_{i,i+1}(x,1) \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda_{i+1} F^{\alpha\alpha}_{i,i+1}(x,1) - \lambda_i q_{i,i+1} K^{uu}_i(x,0) \end{pmatrix}$$

where

$$A_0 = \begin{pmatrix} \lambda_i q_{i,i} & -\mu_{i+1}\rho_{i+1,i} \\ -\lambda_i q_{i,i+1} & \mu_{i+1}\rho_{i+1,i+1} \end{pmatrix} \tag{A.44}$$

The matrix A_0 is invertible due to Assumption 1 (equation 16). Thus, we can rewrite the boundary condition (A.38) for $F^{\alpha\beta}_{i,j}$ (for $j > i$) as

$$F^{\alpha\beta}_{i,j}(x,1) = \sum_{k=i+1}^{n} h_{i,j,k} F^{\alpha\alpha}_{i,j}(x,1) + h^2_{i,j}(x), \tag{A.45}$$

where $h_{i,j,k}$ and $h^2_{i,j}$ are known L^∞ functions. Note that if the $F^{\cdot\cdot}$-equations admit a solution, it is possible to prove the existence of the kernels $R^{\cdot}_i$ using the method of characteristics. To prove the existence of the kernels $F^{\cdot\cdot}_{i,\cdot}$ kernels we adjust the proof of [5]. Let us denote $M = \max_{1\le k\le n}(\lambda_k, \mu_k)$ and $m = \min_{1\le k\le n}(\lambda_k, \mu_k)$. Let us consider the sequence x_p defined for all positive integer p by

$$x_0 = 0, \quad x_{p+1} = \min(1, x_p + \frac{m}{M}).$$

This sequence is strictly increasing and converge after a **finite** number of iterations to 1. We denote N the first integer for which $x_N = 1$. Let us consider the property $S(p)$ defined for all $0 \le p \le N$ by $S(p)$: "For all $i < j \le n$, the PDEs (A.32)-(A.33) along with the boundary conditions (A.35), (A.39) and (A.45) admit a unique L^∞-solution on the space $\mathcal{R}^i_p = \{(x,\xi) \in \mathcal{U},\ \text{s.t. } x \le x_p - \frac{m}{M}(\xi - 1)\}$. The PDEs (A.31)-(A.34) along with the boundary conditions (A.36), (A.37) and (A.40) admit a unique L^∞-solution on the space $\mathcal{S}^i_p = \{(x,\xi) \in \mathcal{U},\ \text{s.t. } x \le \frac{m}{M}\xi + x_p\}$."

<u>Initialization:</u> Let us consider the domains $\mathcal{R}^i_0$ and $\mathcal{S}^i_0$ as represented in Figure A.1. Let us consider the characteristic lines associated to (A.31)-(A.34) (and pictured in Figure A.1). Using the method of characteristics with the boundary condition (A.31), we immediately have for all $i < j \le n$ and all $(x,\xi) \in \mathcal{R}^i_0$

$$F^{\alpha\beta}_{i,j}(x,\xi) = g^{\alpha\beta}_{i,j}(\xi + \frac{\mu_j}{\lambda_i}x) - \int_0^{\frac{x}{\lambda_i}} f^{\alpha\beta}_{i,j}(x - \lambda_i s, \xi + \mu_j s)ds,$$
$$F^{\beta\alpha}_{i,j}(x,\xi) = g^{\beta\alpha}_{i,j}(\xi + \frac{\lambda_j}{\mu_i}x) - \int_0^{\frac{x}{\mu_i}} f^{\beta\alpha}_{i,j}(x - \mu_i s, \xi + \lambda_j s)ds.$$

Similarly, we have for $(x,\xi) \in \mathcal{S}^i_0$

$$F^{\alpha\alpha}_{i,j}(x,\xi) = g^{\alpha\alpha}_{i,j}(\xi - \frac{\lambda_j}{\lambda_i}x) + \int_0^{\frac{x}{\lambda_i}} f^{\alpha\alpha}_{i,j}(x - \lambda_i s, \xi - \lambda_j s)ds,$$
$$F^{\beta\beta}_{i,j}(x,\xi) = g^{\beta\beta}_{i,j}(\xi - \frac{\mu_j}{\mu_i}x) + \int_0^{\frac{x}{\mu_i}} f^{\beta\beta}_{i,j}(x - \mu_i s, \xi - \mu_j s)ds.$$

For any $(x,\xi) \in \mathcal{R}^i_0$, we have $\xi + \frac{\lambda_j}{\mu_i}x \le 1$ and $\xi + \frac{\mu_j}{\lambda_i}x \le 1$. For any $(x,\xi) \in \mathcal{S}_0$, we have $\xi - \frac{\mu_j}{\mu_i}x \ge 0$ and $\xi - \frac{\lambda_j}{\lambda_i}x \ge 0$. Since the functions $g^{\cdot\cdot}_{i,j}$ and $f^{\cdot\cdot}_{ij}$ are L^∞ functions, we can immediately conclude that $S(0)$ holds.

<u>Induction:</u> Les us assume that the property $S(p)$ $(0 \le p \le N-1)$ is true. Let us consider $(x,\xi) \in \mathcal{R}^i_{p+1}$. Let us consider $j \in \mathbb{N}$ such that $i < j \le n$. If $(x,\xi) \in \mathcal{R}^i_0$, then we already know that $F^{\beta\alpha}_{i,j}(x,\xi)$ is well-defined. If $(x,\xi) \notin \mathcal{R}^i_0$, then using the method of characteristics,

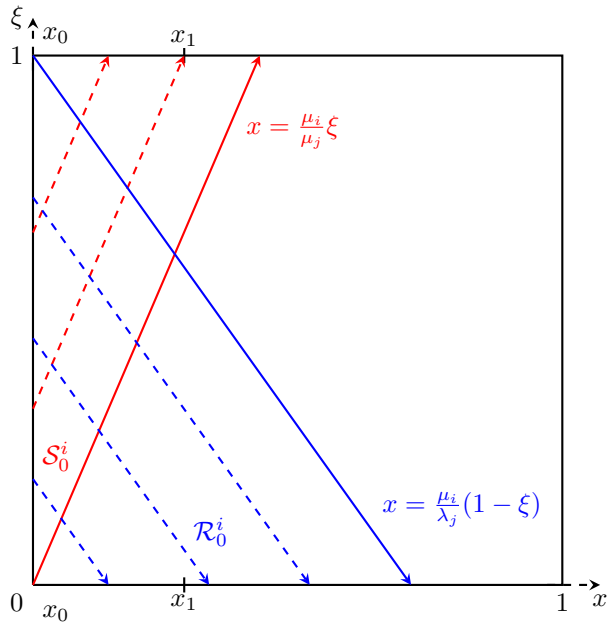

Fig. A.1. Representation of the domains $\mathcal{R}^i_0$ and $\mathcal{S}^i_0$ and of the characteristic lines associated to equations (A.31)-(A.34)

we immediately obtain if $x \le \frac{\mu_i}{\lambda_j}(1-\xi)$

$$F^{\beta\alpha}_{i,j}(x,\xi) = g^{\beta\alpha}_{i,j}(\xi + \frac{\lambda_j}{\mu_i}x) - \int_0^{\frac{x}{\mu_i}} f^{\beta\alpha}_{i,j}(x - \mu_i s, \xi + \lambda_j s)ds.$$

If $x > \frac{\mu_i}{\lambda_j}(1-\xi)$, then we have $F^{\beta\alpha}_{i,j}(x,\xi) = F^{\beta\alpha}_{i,j}(x - \frac{\mu_i}{\lambda_j}(1-\xi), 1) - \int_0^{\frac{1-\xi}{\lambda_j}} f^{\beta\alpha}_{i,j}(x - \mu_i s, \xi + \lambda_j s)ds$. Using (A.39), this gives

$$F^{\beta\alpha}_{i,j}(x,\xi) = \frac{1}{\lambda_j}(\mu_j \rho_{j,j} F^{\beta\beta}_{i,j}(x - \frac{\mu_i}{\lambda_j}(1-\xi), 1)$$
$$+ \lambda_{j-1} q_{j-1,j} F^{\beta\alpha}_{i,j-1}(x - \frac{\mu_i}{\lambda_j}(1-\xi), 0))$$
$$+ h_{i,j}(x - \frac{\mu_i}{\lambda_j}(1-\xi)) - \int_0^{\frac{1-\xi}{\lambda_j}} f^{\beta\alpha}_{i,j}(x - \mu_i s, \xi + \lambda_j s)ds.$$

Since $(x,\xi) \in \mathcal{R}^i_{p+1}$, we have $x - \frac{m}{M}(1-\xi) \le x_{p+1}$. Using the definition of x_{p+1}, we obtain $x - \frac{m}{M}(1-\xi) - \frac{m}{M} \le x_p$. As $\frac{m}{M} \le \frac{\mu_i}{\lambda_j}$, we immediately obtain

$$x - \frac{\mu_i}{\lambda_j}(1-\xi) - \frac{m}{M} \le x_p.$$

Thus, the point $(x - \frac{\mu_i}{\lambda_j}(1-\xi), 1) \in \mathcal{S}^i_p$ and this immediately implies that $F^{\beta\alpha}_{i,j}$ is a $L^\infty(\mathcal{R}^i_{p+1})$ function since $S(p)$ is true. Similar computations can be done to prove that $F^{\alpha\beta}_{i,j}$ is a $L^\infty(\mathcal{R}^i_{p+1})$ function and that $F^{\alpha\alpha}_{i,j}$ and

$F_{i,j}^{\beta\beta}$ are $L^\infty(\mathcal{S}_{p+1}^i)$ functions. This concludes the recursive proof of the property $S(k)$. It is then possible to conclude to the existence of the kernels $F_{i,j}^{\alpha\alpha}$, $F_{i,j}^{\alpha\beta}$ $F_{i,j}^{\beta\alpha}$ and $F_{i,j}^{\beta\beta}$ on the the unit square $\mathcal{U}$. It immediately implies the existence of the kernel R_i^β on the domain $\mathcal{T}_b$. Thus, we have proved $P(q)$. This concludes the proof. $\square$