



HAL
open science

Numerical study of the stabilization of 1D locally coupled wave equations.

Stéphane Gerbi, Chiraz Kassem, Amina Mortada, Ali Wehbe

► **To cite this version:**

Stéphane Gerbi, Chiraz Kassem, Amina Mortada, Ali Wehbe. Numerical study of the stabilization of 1D locally coupled wave equations.. 2020. hal-02524152v1

HAL Id: hal-02524152

<https://hal.science/hal-02524152v1>

Preprint submitted on 30 Mar 2020 (v1), last revised 19 Dec 2020 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

EXACT CONTROLLABILITY AND STABILIZATION OF LOCALLY COUPLED WAVE EQUATIONS

STÉPHANE GERBI, CHIRAZ KASSEM, AMINA MORTADA, AND ALI WEHBE

ABSTRACT. In this paper, we study the exact controllability and stabilization of a system of two wave equations coupled by velocities with an internal, local control acting on only one equation. We distinguish two cases. In the first one, when the waves propagate at the same speed: using a frequency domain approach combined with multiplier technique, we prove that the system is exponentially stable when the coupling region satisfies the geometric control condition GCC. Following a result of Haraux ([11]), we establish the main indirect observability inequality. This result leads, by the HUM method, to prove that the total system is exactly controllable by means of locally distributed control. In the second case, when the waves propagate at different speed, we establish an exponential decay rate in the weak energy space. Consequently, the system is exactly controllable using a result of [11]. Finally, numerically, we provide results that insure the theoretical results of [13].

CONTENTS

1	Introduction	2
	1.1 Motivation and aims.	2
	1.2 Literature	3
	1.3 Description of the paper	3
2	Well posedness and strong stability	4
3	Exponential stability and exact controllability in the case $a = 1$	5
	3.1 Exponential stability	5
	3.2 Observability and exact controllability	12
4	Exponential stability and exact controllability in the case $a \neq 1$	14
	4.1 Exponential stability in the weak energy space	14
	4.2 Observability and exact controllability.	21
5	Numerical approximation: Validation of the theoretical results	22
	5.1 Finite difference scheme in one dimensional space	22
	5.1.1 Construction of the numerical scheme	22
	5.1.2 Practical implementation and CFL condition	23
	5.1.3 Discrete energy: definition and dissipation.	24
	5.2 Numerical experiments: validation of the theoretical results	26
	5.2.1 Same propagation speed: $a = 1$	26
	5.2.1.1. <i>No damping: conservation of the total energy</i>	26
	5.2.1.2. $\omega_b \cap \omega_{c_+} \neq \emptyset$. <i>Exponential stability</i>	26
	5.2.1.3. $\omega_b \cap \omega_{c_+} = \emptyset$. <i>Unpredicted behavior</i>	27
	5.2.2 Different propagation speed: $a > 1$	27
	5.2.2.1. $\omega_b \cap \omega_{c_+} \neq \emptyset$. <i>Polynomial stability</i>	27
	5.2.2.2. $\omega_b \cap \omega_{c_+} = \emptyset$. <i>Unpredicted behavior</i>	27
	5.2.3 Different propagation speed: $a < 1$	28
	5.2.3.1. $\omega_b \cap \omega_{c_+} \neq \emptyset$. <i>Polynomial stability</i>	28
	5.2.3.2. $\omega_b \cap \omega_{c_+} = \emptyset$: <i>Unpredicted behavior</i>	28
Acknowledgments	29
References	40

2010 *Mathematics Subject Classification.* 35L10, 35B40, 93D15, 90D20.

Key words and phrases. Coupled wave equations, internal damping, exact controllability.

1. INTRODUCTION

1.1. Motivation and aims. Let Ω be an empty connected open subset of \mathbb{R}^N having a boundary Γ of class C^2 . In [6], F. Alabau et al. considered the energy decay of a system of two wave equations coupled by velocities

$$(1.1) \quad u_{tt} - a\Delta u + \rho(x, u_t) + b(x)y_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*,$$

$$(1.2) \quad y_{tt} - \Delta y - b(x)u_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*,$$

$$(1.3) \quad u = y = 0 \quad \text{on } \Gamma \times \mathbb{R}_+^*,$$

where $a > 0$ constant, $b \in C^0(\overline{\Omega}, \mathbb{R})$ and $\rho(x, u_t)$ is a non linear damping. Using an approach based on multiplier techniques, weighted nonlinear inequalities and the optimal-weight convexity method (developed in [3]), the authors established an explicit energy decay formula in terms of the behavior of the nonlinear feedback close to the origin. Their results are obtained in the case when the following three conditions are satisfied: the waves propagate at the same speed ($a = 1$), the coupling coefficient $b(x)$ is small and positive ($0 \leq b(x) \leq b_0$, $b_0 \in (0, b^*]$ where b^* is a constant depending on Ω and on the control region) and both the coupling and the damping regions satisfying an appropriated geometric conditions named Piecewise Multipliers Geometric Conditions (introduced in [17] and denoted by PMGC in short). In their work, the case where the waves are not assumed to be propagated with equal speeds (a is not necessarily equal to 1) and/or the coupling coefficient $b(x)$ is not assumed to be positive and small has been left as an open problem even when the damping term ρ is linear with respect to the second variable. Recently, C. Kassem et al. in [13], answered this important open question by studying the stabilization of the following linear system:

$$(1.4) \quad u_{tt} - a\Delta u + c(x)u_t + b(x)y_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*,$$

$$(1.5) \quad y_{tt} - \Delta y - b(x)u_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*,$$

$$(1.6) \quad u = y = 0 \quad \text{on } \Gamma \times \mathbb{R}_+^*,$$

in the case where the waves propagate with equal or different speeds and the coupling coefficient is not assumed to be positive and small. Indeed, they distinguished two cases. The first one is when the waves propagate at the same speed (i.e. $a = 1$), but unlike the works of [6], the coupling coefficient function b is not necessarily assumed to be positive and small. In this case, under the condition that the coupling region and the damping region have non empty intersection satisfying the PMGC conditions, they established an exponential energy decay rate for weak initial data. On the contrary (i.e. $a \neq 1$) they first proved the lack of the exponential stability of the system. However, under the same geometric condition, an optimal energy decay rate of type $\frac{1}{t}$ was established for smooth initial data. The aim of this paper is to investigate the exact controllability of the following system:

$$(1.7) \quad u_{tt} - a\Delta u + b(x)y_t = c(x)v_t \quad \text{in } \Omega \times \mathbb{R}_+^*,$$

$$(1.8) \quad y_{tt} - \Delta y - b(x)u_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*,$$

$$(1.9) \quad u = y = 0 \quad \text{on } \Gamma \times \mathbb{R}_+^*,$$

with the following initial data

$$(1.10) \quad u(x, 0) = u_0, \quad y(x, 0) = y_0, \quad u_t(x, 0) = u_1 \text{ and } y_t(x, 0) = y_1, \quad x \in \Omega,$$

under appropriate geometric conditions. Here, $a > 0$ constant, $b \in C^0(\overline{\Omega}, \mathbb{R})$, $c \in C^0(\overline{\Omega}, \mathbb{R}^+)$ and v is an appropriate control. The idea is to use a result of A. Haraux in [11] for which the observability of the homogeneous system associated to (1.7)-(1.9) is equivalent to the exponential stability of system (1.4)-(1.6) in an appropriate Hilbert space. So, we provide a complete analysis for the exponential stability of system (1.4)-(1.6) in different Hilbert spaces. First, when the waves propagate at the same speed (i.e., $a = 1$), under the condition that the coupling region is included in the damping region and satisfies the so-called Geometric Control Condition (GCC in Short), we establish the exponential stability of system (1.4)-(1.6). Consequently, an observability inequality of the solution of the homogeneous system associated to (1.7)-(1.9) in the space $(H_0^1(\Omega) \times L^2(\Omega))^2$

is established. This leads, by the HUM method introduced by Lions in [15], to the exact controllability of system (1.7)-(1.9) in the space $(H^{-1}(\Omega) \times L^2(\Omega))^2$. Noting that, the geometric situations covered here are richer than those considered in [6] and [13]. Furthermore, on the contrary when the waves propagate at different speeds, (i.e., $a \neq 1$), we establish the exponential stability of system (1.4)-(1.6) in the space $H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$ provided that the damping region satisfies the PMGC condition while the coupling region includes in the damping region and satisfying the GCC conditions. Consequently, an observability inequality of the solution of the homogeneous system associated to (1.7)-(1.9) is established. This leads, by the HUM method, to the exact controllability of system (1.7)-(1.9) in the space $L^2(\Omega) \times H^{-1}(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$. Finally, we perform numerical tests in the 1-D case to insure the theoretical results obtained here and in [13]. In fact, the numerical results show a better behavior than the one expected by the theoretical results.

1.2. Literature. Since the work of J. L. Lions in [15], the observability and controllability of coupled wave equations have been studied by an intensive number of publications. In [15], J. L. Lions studied the complete and partial observability and controllability of coupled systems of either hyperbolic-hyperbolic type or hyperbolic-parabolic type. These results assume that the coupling parameter is sufficiently small. In [1] and [2], F. Alabau studied the indirect boundary observability of an abstract system of two weakly coupled second order evolution equations where the coupling coefficient is strictly positive in the whole domain. In particular, using a piecewise multiplier method, she proved that, for a sufficiently large time T , the observation of the trace of the normal derivative of the first component of the solution on a part of the boundary allows us to get back a weakened energy of the initial data. Consequently, using Hilbert Uniqueness Method, she proved that the system is exactly controllable for small coupling parameter by means of one boundary control. Noting that, the situation where the waves propagate with different speeds is not covered. Later, the indirect boundary controllability of a system of two weakly coupled one-dimensional wave equations has been studied by Z. Liu and B. Rao in [19]. Using the non harmonic analysis, they established several weak observability inequalities which depend on the ratio of the wave propagation speeds and proved the indirect exact controllability. The null controllability of the reaction diffusion System has been studied by F. Ammar-Khodja et al. in [8], by deriving an observability estimate for the linearized problem. The exact controllability of a system of weakly coupled wave equations with an internal locally control acted on only one equation has been studied by A. Wehbe and W. Youssef in [22] and [23]. They showed that, for sufficiently large time, the observation of the velocity of the first component of the solution on a neighborhood of a part of the boundary allows us to get back a weakened energy of initial data of the second component, this if the coupling parameter is sufficiently small, but non-vanishing and by the HUM method, they proved that the total system is exactly controllable. F. Alabau and M. Léautaud in [5], considered a symmetric systems of two wave-type equation, where only one of them being controlled. The two equations are coupled by zero order terms, localized in part of the domain. They obtained an internal and a boundary controllability result in any space dimension, provided that both the coupling and the control regions satisfy the Geometric Control Condition.

1.3. Description of the paper. This paper is organized as follows: In section (2), first, we show that the system (1.4)-(1.6) can be reformulated into a first order evolution equation and we deduce the well posedness property of the problem by the semigroup approach. Second, by using Theorem 2.2 of [13], we show that our problem is strongly stable without geometric conditions. In section 3, we show the exponential decay rate of system (1.4)-(1.6) when the coupling region b is a subset of the damping region c and satisfies the geometric control condition GCC. After that, we show that our system is exactly controllable by using Proposition 2 of A. Haraux in [11]. In section 4, we show the exponential decay rate of system (1.4)-(1.6) in the weak energy space provided that the damping region satisfies the PMGC condition while the coupling region is a subset of the damping region and satisfies the GCC condition. Section 5 is devoted to the numerical approximation of

the problem by a finite difference discretization and to the validation of the theoretical results stated in the previous sections.

2. WELL POSEDENESS AND STRONG STABILITY

Let us define the energy space $\mathcal{H} = \left(H_0^1(\Omega) \times L^2(\Omega) \right)^2$ equipped with the following inner product and norm, respectively

$$(U, \tilde{U})_{\mathcal{H}} = a \int_{\Omega} (\nabla u \cdot \nabla \tilde{u}) dx + \int_{\Omega} v \tilde{v} dx + \int_{\Omega} (\nabla y \cdot \nabla \tilde{y}) dx + \int_{\Omega} z \tilde{z} dx, \quad \|U\|_{\mathcal{H}} = \sqrt{(U, U)_{\mathcal{H}}},$$

for all $U = (u, v, y, z)$, $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}) \in \mathcal{H}$.

Let (u, u_t, y, y_t) be a regular solution of the system (1.4)-(1.6). Its associated energy is defined by

$$E(t) = \frac{1}{2} \int_{\Omega} (|u_t|^2 + a|\nabla u|^2 + |y_t|^2 + |\nabla y|^2) dx.$$

A straightforward computations gives

$$E'(t) = - \int_{\Omega} c(x)|u_t|^2 dx \leq 0.$$

Consequently, system (1.4)-(1.6) is dissipative in the sense that its energy is non-increasing with respect to t . Setting $U = (u, u_t, y, y_t)$, system (1.4)-(1.6) may be recast as:

$$U_t = \mathcal{A}U, \quad \text{in } (0, +\infty), \quad U(0) = (u_0, u_1, y_0, y_1),$$

where the unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ is given by:

$$(2.1) \quad D(\mathcal{A}) = \left((H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \right)^2$$

and

$$(2.2) \quad \mathcal{A}U = (v, a\Delta u - bz - cv, z, \Delta y + bv), \quad \forall U = (u, v, y, z) \in D(\mathcal{A}).$$

Note that due to the fact that $c(x) \geq 0$, the operator \mathcal{A} is dissipative in \mathcal{H} . And, by applying the Lax-Milgram Theorem, it is easy to prove that the operator \mathcal{A} is maximal in \mathcal{H} i.e. $R(I - \mathcal{A}) = \mathcal{H}$. Consequently, it generates a C_0 -semigroup of contractions $(e^{t\mathcal{A}})_{t \geq 0}$. So, system (1.4)-(1.6) is well-posed in \mathcal{H} .

We need now to study the asymptotic behavior of $E(t)$. For this aim, we suppose that there exists a non empty open $\omega_{c_+} \subset \Omega$ satisfying the following condition

$$\{x \in \Omega : c(x) > 0\} \supset \bar{\omega}_{c_+}. \quad (\text{LH1})$$

On the other hand, as $b(x)$ is not identically null and continuous, then there exists a non empty open $\omega_b \subset \Omega$ such that

$$\{x \in \Omega : b(x) \neq 0\} \supset \bar{\omega}_b. \quad (\text{LH2})$$

If $\omega = \omega_{c_+} \cap \omega_b \neq \emptyset$ and condition (LH1) holds, then system (1.4)-(1.6) is strongly stable using Theorem 2.2 in [13], i.e.

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}}(u_0, u_1, y_0, y_1)\|_{\mathcal{H}} = 0 \quad \forall (u_0, u_1, y_0, y_1) \in \mathcal{H}.$$

3. EXPONENTIAL STABILITY AND EXACT CONTROLLABILITY IN THE CASE $a = 1$

3.1. Exponential stability. This subsection is devoted to study the exponential stability of system (1.4)-(1.6) in the case when the waves propagate at the same speed, i.e., $a = 1$ under an appropriate geometric conditions.

Before we state our results, we recall the Geometric Control Conditions GCC introduced by Rauch and Taylor in [21] for manifolds without boundaries and by Bardos, Lebeau and Rauch in [9] for domains with boundaries.

Definition 1. We say that a subset ω of Ω satisfies the **GCC** if every ray of the geometrical optics starting at any point $x \in \Omega$ at $t = 0$ enters the region ω in finite time T .

We recall also the Piecewise Multipliers Geometric Condition introduced by K. Liu in [17].

Definition 2. We say that ω satisfies the Piecewise Multipliers Geometric Condition (PMGC in short) if there exist $\Omega_j \subset \Omega$ having Lipschitz boundary $\Gamma_j = \partial\Omega_j$ and $x_j \in \mathbb{R}^N$, $j = 1, \dots, J$ such that $\Omega_j \cap \Omega_i = \emptyset$ for $j \neq i$ and ω contains a neighborhood in Ω of the set $\cup_{j=1}^J \gamma_j(x_j) \cup (\Omega \setminus \cup_{j=1}^J \Omega_j)$ where $\gamma_j(x_j) = \{x \in \Gamma_j : (x - x_j) \cdot \nu_j(x) > 0\}$ and ν_j is the outward unit normal vector to Γ_j .

Remark 1. The PMGC is the generalization of the Multipliers Geometric Condition (MGC in short) introduced by Lions in [15], saying that ω contains a neighborhood in Ω of the set $\{x \in \Gamma : (x - x_0) \cdot \nu(x) > 0\}$, for some $x_0 \in \mathbb{R}^N$, where ν is the outward unit normal vector to $\Gamma = \partial\Omega$.

Now, we are in position to state our first main result by the following theorem :

Theorem 3.1. (*Exponential decay rate*) Let $a = 1$. Assume that conditions (LH1) and (LH2) hold. Assume also that $\omega_b \subset \omega_{c_+}$ satisfies the geometric control conditions GCC and that $b, c \in W^{1,\infty}(\Omega)$. Then there exist positive constants $M \geq 1$, $\theta > 0$ such that for all initial data $(u_0, u_1, y_0, y_1) \in \mathcal{H}$ the energy of the system (1.4)-(1.6) satisfies the following decay rate:

$$(3.1) \quad E(t) \leq M e^{-\theta t} E(0), \quad \forall t > 0.$$

Remark 2. The geometric situations covered by Theorem 3.1 are richer than those considered in [13] and [6]. Indeed, in the previous references, the authors consider the PMGC geometric conditions that are more restrictive than GCC. On the other hand, unlike the results in [6], we have no restriction in Theorem 3.1 on the upper bound and the sign of the coupling function coefficient b . This theorem is then a generalization in the linear case of the result of [6] where the coupling coefficient considered have to satisfy $0 \leq b(x) \leq b_0$, $b_0 \in (0, b^*]$ where b^* is a constant depending on Ω and on the control region.

In order to prove Theorem 3.1, we apply a result of Huang [12] and Prüss [20]. A C_0 - semigroup of contraction $(e^{t\mathcal{A}})_{t \geq 0}$ in a Hilbert space \mathcal{H} is exponentially stable if and only if

$$i\mathbb{R} \subseteq \rho(\mathcal{A}) \tag{H1}$$

and

$$\limsup_{\beta \in \mathbb{R}, |\beta| \rightarrow +\infty} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty \tag{H2}.$$

Since the resolvent of \mathcal{A} is compact and $0 \in \rho(\mathcal{A})$, then from the fact that our system is strongly stable, we deduce that condition (H1) is satisfied. We now prove that condition (H2) holds, using an argument of contradiction. For this aim, we suppose that there exist a real sequence β_n with $\beta_n \rightarrow +\infty$ and a sequence $U_n = (u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ such that

$$(3.2) \quad \|U_n\|_{\mathcal{H}} = 1,$$

and

$$(3.3) \quad \lim_{n \rightarrow \infty} \|(i\beta_n I - \mathcal{A})U_n\|_{\mathcal{H}} = 0.$$

Next, detailing equation (3.3), we get

$$(3.4) \quad i\beta_n u_n - v_n = f_n^1 \rightarrow 0 \quad \text{in } H_0^1(\Omega),$$

$$(3.5) \quad i\beta_n v_n - \Delta u_n + bz_n + cv_n = g_n^1 \rightarrow 0 \quad \text{in } L^2(\Omega),$$

$$(3.6) \quad i\beta_n y_n - z_n = f_n^2 \rightarrow 0 \quad \text{in } H_0^1(\Omega),$$

$$(3.7) \quad i\beta_n z_n - \Delta y_n - bv_n = g_n^2 \rightarrow 0 \quad \text{in } L^2(\Omega).$$

Eliminating v_n and z_n from the previous system, we obtain the following system

$$(3.8) \quad \beta_n^2 u_n + \Delta u_n - i\beta_n b y_n - i\beta_n c u_n = -g_n^1 - b f_n^2 - i\beta_n f_n^1 - c f_n^1,$$

$$(3.9) \quad \beta_n^2 y_n + \Delta y_n + i\beta_n b u_n = -i\beta_n f_n^2 + b f_n^1 - g_n^2.$$

On the other side, we notice that v_n and z_n are uniformly bounded in $L^2(\Omega)$. It follows, from equations (3.4) and (3.6), that

$$(3.10) \quad \int_{\Omega} |y_n|^2 dx = \frac{O(1)}{\beta_n^2} \quad \text{and} \quad \int_{\Omega} |u_n|^2 dx = \frac{O(1)}{\beta_n^2}.$$

For clarity, we divide the proof into several Lemmas.

Lemma 3.2. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (3.4)-(3.7) satisfies the following estimates*

$$(3.11) \quad \int_{\Omega} c |\beta_n u_n|^2 dx = o(1) \quad \text{and} \quad \int_{\omega_{c_+}} |\beta_n u_n|^2 dx = o(1).$$

Proof. First, since U_n is uniformly bounded in \mathcal{H} , then from (3.3), we get

$$(3.12) \quad \operatorname{Re} \{ i\beta_n \| U_n \|_{\mathcal{H}}^2 - (\mathcal{A}U_n, U_n)_{\mathcal{H}} \} = \int_{\Omega} c(x) |v_n|^2 dx = o(1).$$

Under condition (LH1), it follows that

$$(3.13) \quad \int_{\omega_{c_+}} |v_n|^2 dx = o(1).$$

Then, using equations (3.12) and (3.4), we get

$$(3.14) \quad \int_{\Omega} c |\beta_n u_n|^2 dx = o(1).$$

Consequently, we have

$$\int_{\omega_{c_+}} |\beta_n u_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 3.3. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (3.4)-(3.7) satisfies the following estimates*

$$(3.15) \quad \int_{\Omega} c |\nabla u_n|^2 dx = o(1) \quad \text{and} \quad \int_{\omega_{c_+}} |\nabla u_n|^2 dx = o(1).$$

Proof. Multiplying equation (3.8) by $\bar{c}u_n$, integrating by parts and using the fact that $u_n = 0$ on Γ , we get

$$(3.16) \quad \begin{aligned} & \int_{\Omega} c |\beta_n u_n|^2 dx - \int_{\Omega} c |\nabla u_n|^2 dx - \int_{\Omega} (\nabla c \cdot \nabla u_n) \bar{u}_n dx - i \int_{\Omega} \beta_n b y_n \bar{c} u_n dx \\ & - i \int_{\Omega} \beta_n c u_n \bar{u}_n dx = \int_{\Omega} (-g_n^1 - b f_n^2 - i\beta_n f_n^1 - c f_n^1) \bar{c} u_n dx. \end{aligned}$$

Using the fact that f_n^1, f_n^2 converge to zero in $H_0^1(\Omega)$, g_n^1 converges to zero in $L^2(\Omega)$ and $\beta_n \bar{u}_n$ is uniformly bounded in $L^2(\Omega)$, we obtain

$$(3.17) \quad \int_{\Omega} (-g_n^1 - b f_n^2 - i\beta_n f_n^1 - c f_n^1) \bar{c} u_n dx = o(1).$$

Using the fact that $\nabla u_n, \beta_n y_n, \beta_n u_n$ are uniformly bounded in $L^2(\Omega)$ and $\|u_n\| = o(1)$, we get

$$(3.18) \quad - \int_{\Omega} (\nabla c \cdot \nabla u_n) \bar{u}_n dx - i \int_{\Omega} \beta_n b y_n c \bar{u}_n dx - i \int_{\Omega} \beta_n c u_n \bar{u}_n dx = o(1).$$

Inserting (3.17) and (3.18) into (3.16), we get

$$(3.19) \quad \int_{\Omega} c |\beta_n u_n|^2 dx - \int_{\Omega} c |\nabla u_n|^2 dx = o(1).$$

Finally, using estimation (3.11) in (3.19), we deduce

$$\int_{\omega_{c_+}} |\nabla u_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 3.4. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (3.4)-(3.7) satisfies the following estimate*

$$(3.20) \quad \int_{\omega_b} |\nabla y_n|^2 dx = o(1).$$

Proof. The proof contains three points.

i) First, multiplying equation (3.8) by $\frac{1}{\beta_n} \Delta \bar{y}_n$, then using Green's formula and the fact that $u_n = f_n^1 = 0$ on Γ , we obtain

$$(3.21) \quad \begin{aligned} & - \int_{\Omega} \beta_n (\nabla u_n \cdot \nabla \bar{y}_n) dx + \frac{1}{\beta_n} \int_{\Omega} \Delta u_n \Delta \bar{y}_n dx + i \int_{\Omega} (\nabla b \cdot \nabla \bar{y}_n) y_n dx \\ & + i \int_{\Omega} b |\nabla y_n|^2 dx + i \int_{\Omega} (\nabla c \cdot \nabla \bar{y}_n) u_n dx + i \int_{\Omega} c (\nabla u_n \cdot \nabla \bar{y}_n) dx \\ & = \int_{\Omega} (-g_n^1 - b f_n^2 - c f_n^1) \frac{1}{\beta_n} \Delta \bar{y}_n dx + i \int_{\Omega} (\nabla f_n^1 \cdot \nabla \bar{y}_n) dx. \end{aligned}$$

As f_n^1, f_n^2 converge to zero in $H_0^1(\Omega)$, g_n^1 converges to zero in $L^2(\Omega)$ and the fact that $\frac{1}{\beta_n} \Delta y_n, \nabla y_n$ are uniformly bounded in $L^2(\Omega)$, we have

$$(3.22) \quad \int_{\Omega} (-g_n^1 - b f_n^2 - c f_n^1) \frac{1}{\beta_n} \Delta \bar{y}_n dx + i \int_{\Omega} (\nabla f_n^1 \cdot \nabla \bar{y}_n) dx = o(1).$$

Using the fact that ∇y_n is uniformly bounded in $L^2(\Omega)$, $\|u_n\|_{L^2(\Omega)} = o(1)$, $\|y_n\|_{L^2(\Omega)} = o(1)$ and using the estimation (3.15), we get

$$(3.23) \quad i \int_{\Omega} (\nabla b \cdot \nabla \bar{y}_n) y_n dx + i \int_{\Omega} (\nabla c \cdot \nabla \bar{y}_n) u_n dx + i \int_{\Omega} c (\nabla u_n \cdot \nabla \bar{y}_n) dx = o(1).$$

Inserting now (3.22) and (3.23) into (3.21), we get

$$(3.24) \quad - \int_{\Omega} \beta_n (\nabla u_n \cdot \nabla \bar{y}_n) dx + \frac{1}{\beta_n} \int_{\Omega} \Delta u_n \Delta \bar{y}_n dx + i \int_{\Omega} b |\nabla y_n|^2 dx = o(1).$$

ii) Similarly, multiplying equation (3.9) by $\frac{1}{\beta_n} \Delta \bar{u}_n$, then using Green's formula and the fact that $y_n = f_n^2 = 0$ on Γ , we obtain

$$(3.25) \quad \begin{aligned} & - \int_{\Omega} \beta_n (\nabla y_n \cdot \nabla \bar{u}_n) dx + \frac{1}{\beta_n} \int_{\Omega} \Delta y_n \Delta \bar{u}_n dx - i \int_{\Omega} (\nabla b \cdot \nabla \bar{u}_n) u_n dx \\ & - i \int_{\Omega} b |\nabla u_n|^2 dx = \int_{\Omega} (b f_n^1 - g_n^2) \frac{1}{\beta_n} \Delta \bar{u}_n dx + i \int_{\Omega} (\nabla f_n^2 \cdot \nabla \bar{u}_n) dx. \end{aligned}$$

Using the fact that f_n^1, f_n^2 converge to zero in $H_0^1(\Omega)$, g_n^2 converges to zero in $L^2(\Omega)$ and the fact that $\frac{1}{\beta_n}\Delta u_n, \nabla u_n$ are uniformly bounded in $L^2(\Omega)$, we get

$$(3.26) \quad \int_{\Omega} (bf_n^1 - g_n^2) \frac{1}{\beta_n} \Delta \bar{u}_n dx + i \int_{\Omega} (\nabla f_n^2 \cdot \nabla \bar{u}_n) dx = o(1).$$

Also, using the fact that ∇u_n is uniformly bounded in $L^2(\Omega)$, $\|u_n\|_{L^2(\Omega)} = o(1)$, we have

$$(3.27) \quad -i \int_{\Omega} (\nabla b \cdot \nabla \bar{u}_n) u_n dx = o(1).$$

Inserting (3.26) and (3.27) into (3.25), we get

$$(3.28) \quad - \int_{\Omega} \beta_n (\nabla y_n \cdot \nabla \bar{u}_n) dx + \frac{1}{\beta_n} \int_{\Omega} \Delta y_n \Delta \bar{u}_n dx - i \int_{\Omega} b |\nabla u_n|^2 dx = o(1).$$

iii) Finally, by combining (3.24) and (3.28) and taking the imaginary part, we obtain

$$(3.29) \quad \int_{\Omega} b |\nabla y_n|^2 dx = \int_{\Omega} b |\nabla u_n|^2 dx + o(1).$$

Since $\omega_b \subset \omega_{c+}$, it follows from (3.15) and (3.29) that

$$\int_{\omega_b} |\nabla y_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 3.5. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (3.4)-(3.7) satisfies the following estimate*

$$(3.30) \quad \int_{\omega_b} |\beta_n y_n|^2 dx = o(1).$$

Proof. Multiplying equation (3.9) by $b\bar{y}_n$. Then using Green's formula and the fact that $y_n = 0$ on Γ , we obtain

$$(3.31) \quad \begin{aligned} & \int_{\Omega} b |\beta_n y_n|^2 dx - \int_{\Omega} b |\nabla y_n|^2 dx - \int_{\Omega} (\nabla b \cdot \nabla y_n) \bar{y}_n dx \\ & + i \int_{\Omega} b^2 \beta_n u_n \bar{y}_n dx = \int_{\Omega} (-i\beta_n f_n^2 + bf_n^1 - g_n^2) b \bar{y}_n dx. \end{aligned}$$

As f_n^1, f_n^2 converge to zero in $H_0^1(\Omega)$, g_n^2 converges to zero in $L^2(\Omega)$ and $\beta_n y_n$ is uniformly bounded in $L^2(\Omega)$, we get

$$(3.32) \quad \int_{\Omega} (-i\beta_n f_n^2 + bf_n^1 - g_n^2) b \bar{y}_n dx = o(1).$$

Using the fact that $\beta_n u_n$ and ∇y_n are uniformly bounded in $L^2(\Omega)$ and $\|y_n\|_{L^2(\Omega)} = o(1)$, we get

$$(3.33) \quad \int_{\Omega} (\nabla b \cdot \nabla y_n) \bar{y}_n + i \int_{\Omega} b^2 \beta_n u_n \bar{y}_n dx = o(1).$$

Inserting (3.32), (3.33) into (3.31), we obtain

$$\int_{\Omega} b |\beta_n y_n|^2 dx - \int_{\Omega} b |\nabla y_n|^2 dx = o(1).$$

Using the estimation (3.20) in the previous equation, we get

$$\int_{\Omega} b |\beta_n y_n|^2 dx = o(1).$$

This yields

$$\int_{\omega_b} |\beta_n y_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 3.6. *Let f_n be a bounded sequence in $L^2(\Omega)$. Then the solution $\phi_n \in H_0^1(\Omega) \cap H^2(\Omega)$ of the following system*

$$(3.34) \quad \begin{cases} \beta_n^2 \phi_n + \Delta \phi_n - ib\beta_n \phi_n & = f_n & \text{in } \Omega, \\ \phi_n & = 0 & \text{on } \Gamma, \end{cases}$$

verifies the following estimate

$$(3.35) \quad \int_{\Omega} (|\beta_n \phi_n|^2 + |\nabla \phi_n|^2) dx \leq C \int_{\Omega} |f_n|^2 dx,$$

where C is a constant independent of n .

Proof. Consider the following wave equation

$$(3.36) \quad \begin{cases} \phi_{tt} - \Delta \phi + b\phi_t & = 0 & \text{in } \Omega, \\ \phi & = 0 & \text{on } \Gamma. \end{cases}$$

System (3.36) is wellposed in the space $H = H_0^1(\Omega) \times L^2(\Omega)$ and since ω_b verifies GCC condition then it is exponentially stable (see [9]). Therefore, following Huang [12] and Prüss [20], we deduce that the resolvent of its corresponding operator

$$\mathcal{A}_{aux} : D(\mathcal{A}_{aux}) \longrightarrow H_0^1(\Omega) \times L^2(\Omega)$$

defined by $D(\mathcal{A}_{aux}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ and $\mathcal{A}_{aux}(\phi, \tilde{\phi}) = (\tilde{\phi}, \Delta \phi - b\tilde{\phi})$ is uniformly bounded on the imaginary axis.

On the other hand, system (3.34) can be rewritten in the form:

$$(3.37) \quad \begin{cases} i\beta_n \phi_n - \tilde{\phi}_n & = 0, \\ i\beta_n \tilde{\phi}_n - \Delta \phi_n + b\tilde{\phi}_n & = -f_n. \end{cases}$$

So,

$$(3.38) \quad (i\beta_n - \mathcal{A}_{aux}) \begin{pmatrix} \phi_n \\ \tilde{\phi}_n \end{pmatrix} = \begin{pmatrix} 0 \\ -f_n \end{pmatrix}.$$

Equivalently,

$$(3.39) \quad \begin{pmatrix} \phi_n \\ \tilde{\phi}_n \end{pmatrix} = (i\beta_n - \mathcal{A}_{aux})^{-1} \begin{pmatrix} 0 \\ -f_n \end{pmatrix}.$$

This yields

$$(3.40) \quad \begin{aligned} \|(\phi_n, \tilde{\phi}_n)\|_H^2 &\leq \|(i\beta_n - \mathcal{A}_{aux})^{-1}\|_{\mathcal{L}(H)}^2 \|(0, -f_n)\|_H^2 \\ &\leq C \int_{\Omega} |f_n|^2 dx, \end{aligned}$$

where C is a constant independent of n . Consequently, we deduce

$$\int_{\Omega} (|\beta_n \phi_n|^2 + |\nabla \phi_n|^2) dx \leq C \int_{\Omega} |f_n|^2 dx.$$

The proof is thus complete. \square

Lemma 3.7. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (3.4)-(3.7) satisfies the following estimate*

$$(3.41) \quad \int_{\Omega} |\beta_n u_n|^2 dx = o(1).$$

Proof. Taking $f_n = u_n$ in Lemma 3.6 and multiplying equation (3.8) by $\beta_n^2 \bar{\phi}_n$ where ϕ_n is a solution of (3.34). Then using Green's formula and the fact that $u_n = \phi_n = 0$ on Γ , we obtain

$$(3.42) \quad \begin{aligned} &\int_{\Omega} \beta_n^2 u_n (\beta_n^2 \bar{\phi}_n + \Delta \bar{\phi}_n) dx - i \int_{\Omega} b \beta_n y_n \beta_n^2 \bar{\phi}_n dx - i \int_{\Omega} c \beta_n u_n \beta_n^2 \bar{\phi}_n dx \\ &= \int_{\Omega} (-g_n^1 - b f_n^2 - c f_n^1) \beta_n^2 \bar{\phi}_n dx - i \int_{\Omega} \beta_n f_n^1 \beta_n^2 \bar{\phi}_n dx. \end{aligned}$$

Substituting the first equation of system (3.34) into the first term of (3.42), we get

$$(3.43) \quad \begin{aligned} & \int_{\Omega} |\beta_n u_n|^2 dx - i \int_{\Omega} \beta_n^2 \bar{\phi}_n b \beta_n u_n dx - i \int_{\Omega} b \beta_n y_n \beta_n^2 \bar{\phi}_n dx - i \int_{\Omega} c \beta_n u_n \beta_n^2 \bar{\phi}_n dx \\ &= \int_{\Omega} (-g_n^1 - b f_n^2 - c f_n^1) \beta_n^2 \bar{\phi}_n dx - i \int_{\Omega} \beta_n f_n^1 \beta_n^2 \bar{\phi}_n dx. \end{aligned}$$

As f_n^1, f_n^2 converge to zero in $H_0^1(\Omega)$ and $\beta_n^2 \bar{\phi}_n$ is uniformly bounded in $L^2(\Omega)$ due to (3.35), we get

$$(3.44) \quad \int_{\Omega} (-g_n^1 - b f_n^2 - c f_n^1) \beta_n^2 \bar{\phi}_n dx = o(1).$$

From the first equation of (3.34), we have $\beta_n^2 \bar{\phi}_n = \bar{u}_n - \Delta \bar{\phi}_n - i b \beta_n \bar{\phi}_n$. Consequently, we have

$$(3.45) \quad \begin{aligned} -i \int_{\Omega} \beta_n f_n^1 \beta_n^2 \bar{\phi}_n dx &= -i \int_{\Omega} \beta_n f_n^1 (\bar{u}_n - \Delta \bar{\phi}_n - i b \beta_n \bar{\phi}_n) dx \\ &= -i \int_{\Omega} \beta_n f_n^1 \bar{u}_n dx - i \int_{\Omega} \beta_n (\nabla \bar{\phi}_n \cdot \nabla f_n^1) dx - \int_{\Omega} b f_n^1 \beta_n^2 \bar{\phi}_n, \end{aligned}$$

which yields

$$(3.46) \quad -i \int_{\Omega} \beta_n f_n^1 \beta_n^2 \bar{\phi}_n dx = o(1),$$

because f_n^1 converges to zero in $H_0^1(\Omega)$ and $\beta_n u_n, \beta_n^2 \bar{\phi}_n, \beta_n \nabla \bar{\phi}_n$ are uniformly bounded in $L^2(\Omega)$.

Substituting now (3.44) and (3.46) into (3.43)

$$(3.47) \quad \int_{\Omega} |\beta_n u_n|^2 dx - i \int_{\Omega} \beta_n^2 \bar{\phi}_n b \beta_n u_n dx - i \int_{\Omega} b \beta_n y_n \beta_n^2 \bar{\phi}_n dx - i \int_{\Omega} c \beta_n u_n \beta_n^2 \bar{\phi}_n dx = o(1).$$

Finally, using estimations (3.11), (3.30) and the fact that $\beta_n^2 \bar{\phi}_n$ is uniformly bounded in $L^2(\Omega)$ into the previous equation, we obtain

$$(3.48) \quad \int_{\Omega} |\beta_n u_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 3.8. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (3.4)-(3.7) satisfies the following estimate*

$$(3.49) \quad \int_{\Omega} |\beta_n y_n|^2 dx = o(1).$$

Proof. Taking $f_n = y_n$ in Lemma 3.6. Multiplying equation (3.9) by $\beta_n^2 \bar{\phi}_n$ where ϕ_n is a solution of (3.34). Then using Green's formula and the fact that $y_n = \phi_n = 0$ on Γ , we obtain

$$(3.50) \quad \begin{aligned} & \int_{\Omega} \beta_n^2 y_n (\beta_n^2 \bar{\phi}_n + \Delta \bar{\phi}_n) dx + i \int_{\Omega} b \beta_n u_n \beta_n^2 \bar{\phi}_n dx \\ &= -i \int_{\Omega} \beta_n f_n^2 \beta_n^2 \bar{\phi}_n + \int_{\Omega} (b f_n^1 - g_n^2) \beta_n^2 \bar{\phi}_n dx. \end{aligned}$$

Then, substituting the first equation of problem (3.34) into the first term of (3.50), we get

$$(3.51) \quad \begin{aligned} & \int_{\Omega} |\beta_n y_n|^2 dx - i \int_{\Omega} b \beta_n^2 \bar{\phi}_n \beta_n y_n dx + i \int_{\Omega} b \beta_n u_n \beta_n^2 \bar{\phi}_n dx \\ &= -i \int_{\Omega} \beta_n f_n^2 \beta_n^2 \bar{\phi}_n + \int_{\Omega} (b f_n^1 - g_n^2) \beta_n^2 \bar{\phi}_n dx. \end{aligned}$$

Since $\beta_n^2 \bar{\phi}_n$ is uniformly bounded in $L^2(\Omega)$, f_n^1 converges to zero in $H_0^1(\Omega)$ and g_n^2 converges to zero in $L^2(\Omega)$, we have

$$(3.52) \quad \int_{\Omega} (-bf_n^1 - g_n^2) \beta_n^2 \bar{\phi}_n dx = o(1).$$

Moreover, using the first equation of problem (3.34) and integrating by parts yields

$$(3.53) \quad \begin{aligned} -i \int_{\Omega} \beta_n f_n^2 \beta_n^2 \bar{\phi}_n dx &= -i \int_{\Omega} (\bar{y}_n - \Delta \bar{\phi}_n - ib \beta_n \bar{\phi}_n) \beta_n f_n^2 dx \\ &= -i \int_{\Omega} f_n^2 \beta_n \bar{y}_n dx - i \int_{\Omega} \beta_n (\nabla \bar{\phi}_n \cdot \nabla f_n^2) dx - \int_{\Omega} b f_n^2 \beta_n^2 \bar{\phi}_n dx. \end{aligned}$$

Using the fact that $\beta_n y_n$, $\beta_n^2 \phi_n$ and $\beta_n \nabla \phi_n$ are uniformly bounded in $L^2(\Omega)$ and f_n^2 converges to zero in $H_0^1(\Omega)$ in (3.53), we get

$$(3.54) \quad -i \int_{\Omega} \beta_n f_n^2 \beta_n^2 \bar{\phi}_n dx = o(1).$$

Inserting (3.52), (3.54) into (3.51), we obtain

$$(3.55) \quad \int_{\Omega} |\beta_n y_n|^2 dx - i \int_{\Omega} b \beta_n^2 \bar{\phi}_n \beta_n y_n dx + i \int_{\Omega} b \beta_n u_n \beta_n^2 \bar{\phi}_n dx = o(1).$$

Finally, using (3.30), (3.41) and the fact that $\beta_n^2 \bar{\phi}_n$ is uniformly bounded in $L^2(\Omega)$, we deduce

$$\int_{\Omega} |\beta_n y_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 3.9. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (3.4)-(3.7) satisfies the following estimates*

$$(3.56) \quad \int_{\Omega} |\nabla u_n|^2 dx = o(1) \quad \text{and} \quad \int_{\Omega} |\nabla y_n|^2 dx = o(1).$$

Proof. Multiplying equation (3.8) by \bar{u}_n , applying Green's formula and using the fact that $u_n = 0$ on Γ , we get

$$(3.57) \quad \int_{\Omega} |\beta_n u_n|^2 dx - \int_{\Omega} |\nabla u_n|^2 dx - i \int_{\Omega} \beta_n b y_n \bar{u}_n dx - i \int_{\Omega} \beta_n c u_n \bar{u}_n dx = o(1).$$

Using the fact that $\beta_n u_n$, $\beta_n y_n$ are uniformly bounded in $L^2(\Omega)$, $\|u_n\| = o(1)$ and the estimation (3.41) in (3.57), we obtain

$$(3.58) \quad \int_{\Omega} |\nabla u_n|^2 dx = o(1).$$

Similarly, multiplying equation (3.9) by \bar{y}_n and applying Green's formula and using the fact that $y_n = 0$ on Γ , we get

$$(3.59) \quad \int_{\Omega} |\beta_n y_n|^2 dx - \int_{\Omega} |\nabla y_n|^2 dx + i \int_{\Omega} \beta_n b u_n \bar{y}_n dx = o(1).$$

Using the fact that $\beta_n u_n$ is uniformly bounded in $L^2(\Omega)$, $\|y_n\| = o(1)$ and (3.49) in (3.59), we obtain

$$(3.60) \quad \int_{\Omega} |\nabla y_n|^2 dx = o(1).$$

The proof is thus complete. \square

Proof of Theorem 3.1 It follows from (3.41), (3.49) and (3.56) that $\|U_n\|_{\mathcal{H}} = o(1)$ which is a contradiction with (3.2). Consequently, condition (H2) holds and the energy of system (1.4)-(1.6) decays exponentially to zero. The proof is thus complete. \square

3.2. Observability and exact controllability. First, we consider the following homogeneous system associated to (1.7)-(1.9) for $a = 1$ by:

$$(3.61) \quad \psi_{tt} - \Delta\psi + b(x)\varphi_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+,$$

$$(3.62) \quad \varphi_{tt} - \Delta\varphi - b(x)\psi_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+,$$

$$(3.63) \quad \psi = \varphi = 0 \quad \text{on } \Gamma \times \mathbb{R}_+,$$

$$(3.64) \quad \psi(\cdot, 0) = \psi_0, \psi_t(\cdot, 0) = \psi_1, \varphi(\cdot, 0) = \varphi_0, \varphi_t(\cdot, 0) = \varphi_1 \quad \text{in } \Omega.$$

Let $\Phi = (\psi, \psi_t, \varphi, \varphi_t)$ be a regular solution of system (3.61)-(3.63), its associated total energy is given by:

$$(3.65) \quad E(t) = \frac{1}{2} \int_{\Omega} (|\psi_t|^2 + |\nabla\psi|^2 + |\varphi_t|^2 + |\nabla\varphi|^2) dx.$$

A direct computation gives

$$(3.66) \quad \frac{d}{dt} E(t) = 0.$$

Thus, system (3.61)-(3.63) is conservative in the sense that its energy $E(t)$ is constant. It is also wellposed and admits a unique solution in the energy space \mathcal{H} .

Now, we establish the direct and indirect inequality given by the following theorem:

Theorem 3.10. *Let $a = 1$. Assume that conditions (LH1) and (LH2) hold. Assume also that $\omega_b \subset \omega_{c_+}$ satisfies the geometric control condition GCC and that $b, c \in W^{1,\infty}(\Omega)$. Then there exists a time T_0 such that for all $T > T_0$, there exist two constants $M_1 > 0, M_2 > 0$ such that the solution of system (3.61)-(3.63) satisfies the following observability inequalities:*

$$(3.67) \quad M_1 \|\Phi_0\|_{\mathcal{H}}^2 \leq \int_0^T \int_{\Omega} c(x) |\psi_t|^2 dx dt \leq M_2 \|\Phi_0\|_{\mathcal{H}}^2,$$

for all $\Phi_0 = (\psi_0, \psi_1, \varphi_0, \varphi_1) \in \mathcal{H}$.

Proof. The direct inequality follows from the definition of the total energy for all $T > 0$. While the proof of the inverse inequality is a direct consequence of Proposition 2 of Haraux in [11] for which the exponential stability (3.1) implies the existence of a time $T_0 > 0$ such that for all $T > T_0$ there exist two constants $M_1 > 0$ and $M_2 > 0$ such that (3.67) holds. The proof is thus complete. \square

Now, we are ready to study the exact controllability of a system (1.7)-(1.9) by using the HUM method. First, thanks to the direct inequality, the solution of the system (??) can be obtained as usual by the method of transposition (see [15] and [16]). Let $v_0 \in L^2(0, T; L^2(\omega_{c_+}))$, we choose the control

$$(3.68) \quad v(t) = -\frac{d}{dt} v_0(t) \in [H^1(0, T; L^2(\omega_{c_+}))]',$$

where the derivative $\frac{d}{dt}$ is not taken in the sense of distributions but in the sense of the duality $H^1(0, T; L^2(\omega_{c_+}))$ and its dual $[H^1(0, T; L^2(\omega_{c_+}))]'$, i.e.,

$$-\int_0^T \frac{d}{dt} v_1(t) \mu(t) dt = \int_0^T v_1(t) \frac{d}{dt} \mu(t) dt, \quad \forall \mu \in H^1(0, T; L^2(\omega_{c_+})).$$

Then we have the following result:

Theorem 3.11. *Let $T > 0$ and $a = 1$. Assume that conditions (LH1) and (LH1) hold. Assume also that $\omega_b \subset \omega_{c_+}$ satisfies the geometric control condition GCC and that $b, c \in W^{1,\infty}(\Omega)$. Given*

$$U_0 = (u_0, u_1, y_0, y_1) \in (L^2(\Omega) \times H^{-1}(\Omega))^2, \quad v = -\frac{d}{dt} v_0 \in [H^1(0, T; L^2(\omega_{c_+}))]',$$

the controlled system (1.7)-(1.9) has a unique weak solution

$$U = (u, u_t, y, y_t) \in C^0([0, T], (L^2(\Omega) \times H^{-1}(\Omega))^2).$$

Proof. Let $(\psi, \psi_t, \varphi, \varphi_t)$ be the solution of (3.61)-(3.63) associated to $\Phi_0 = (\psi_0, \psi_1, \varphi_0, \varphi_1)$. Multiplying the first equation of (1.7)-(1.9) by ψ and the second by φ and integrating by parts, we obtain

$$(3.69) \quad \begin{cases} \int_{\Omega} y_t(T)\varphi(T)dx + \int_{\Omega} u_t(T)\psi(T)dx - \int_{\Omega} y(T)\varphi_t(T)dx \\ - \int_{\Omega} u(T)\psi_t(T)dx - \int_{\Omega} bu(T)\varphi(T)dx + \int_{\Omega} by(T)\psi(T)dx = \\ \int_{\Omega} y_t(0)\varphi(0)dx + \int_{\Omega} u_t(0)\psi(0)dx - \int_{\Omega} \varphi_t(0)y(0)dx \\ - \int_{\Omega} \psi_t(0)u(0)dx - \int_{\Omega} bu(0)\varphi(0)dx + \int_{\Omega} by(0)\psi(0)dx + \int_0^T \int_{\Omega} c(x)v(t)\psi dxdt. \end{cases}$$

Note that $\mathcal{H}' = (H^{-1}(\Omega) \times L^2(\Omega))^2$. Then we have

$$(3.70) \quad \begin{cases} \langle (u_t(T, x), -u(T, x), y_t(T, x), -y(T, x)), \Phi(T) \rangle_{\mathcal{H}' \times \mathcal{H}} = \\ \langle (u_1, -u_0, y_1, -y_0), \Phi_0 \rangle_{\mathcal{H}' \times \mathcal{H}} + \int_0^T \int_{\Omega} cv(t)\psi dxdt = L(\Phi_0). \end{cases}$$

Using the direct observability inequality (3.67), we deduce that

$$(3.71) \quad \|L\|_{\mathcal{L}(\mathcal{H}, \mathbb{R})} \leq \|v_0\|_{L^2(0, T; L^2(\omega_{c_+}))} + \|U_0\|_{\mathcal{H}'}$$

Using the Riesz representation theorem, there exists an element $\mathcal{Z}(x, t) \in \mathcal{H}'$ solution of

$$(3.72) \quad L(\Phi_0) = \langle \mathcal{Z}, \Phi_0 \rangle_{\mathcal{H}' \times \mathcal{H}}, \quad \forall \Phi_0 \in \mathcal{H}.$$

Then, define the weak solution $U(x, t)$ of system (1.7)-(1.9) by $U(x, t) = \mathcal{Z}(x, t)$. The proof is thus complete. \square

Next, we consider the indirect locally internal exact controllability problem: For given $T > 0$ (sufficiently large) and initial data U_0 , does there exists a suitable control v that brings back the solution to equilibrium at time T , that is such the solution of (1.7)-(1.9) satisfies $u(T) = u_t(T) = y(T) = y_t(T) = 0$. Indeed, applying the HUM method, we obtain the following result.

Theorem 3.12. *Let $a = 1$. Assume that conditions (LH1) and (LH2) hold. Assume also that $\omega_b \subset \omega_{c_+}$ satisfies the geometric control condition GCC and that $b, c \in W^{1, \infty}(\Omega)$. For every $T > M_1$, where M_1 is given in Theorem 3.10 and for every*

$$U_0 \in (L^2(\Omega) \times H^{-1}(\Omega))^2,$$

there exists a control

$$v(t) \in [H^1(0, T; L^2(\omega_{c_+}))]',$$

such that the solution of the controlled system (1.7)-(1.9) satisfies

$$u(T) = u_t(T) = y(T) = y_t(T) = 0.$$

Proof. We will apply the HUM method. Thanks to the indirect observability inequalities (3.67), we consider the seminorm defined by

$$\|\Phi_0\|_{\mathcal{H}}^2 = \int_0^T \int_{\omega_b} |\psi_t|^2 dxdt,$$

where $\Phi = (\psi, \psi_t, \varphi, \varphi_t)$ designate the solution of the homogeneous problem (3.61)-(3.63).

Take the control $v = \frac{d}{dt}\psi_t$. Now, we solve the following time reverse problem:

$$(3.73) \quad \begin{cases} \zeta_{tt} - \Delta \zeta + b\chi_t & = c \frac{d}{dt}\psi_t & \text{in } (0, T) \times \Omega, \\ \chi_{tt} - \Delta \chi - b\zeta_t & = 0 & \text{in } (0, T) \times \Omega, \\ \chi(T) = \chi_t(T) = \zeta(T) = \zeta_t(T) & = 0. \end{cases}$$

By Theorem 3.11, the system (3.73) admits a solution

$$\Psi(x, t) = (\zeta, \zeta_t, \chi, \chi_t) \in C^0([0, T], H').$$

We define the linear operator Λ by:

$$\Lambda : \mathcal{H} = (H_0^1(\Omega) \times L^2(\Omega))^2 \rightarrow (H^{-1}(\Omega) \times L^2(\Omega))^2,$$

where

$$\Lambda \Phi_0 = (\zeta_t(0), -\zeta(0), \chi_t(0), -\chi(0)) \quad \forall \Phi_0 \in (H_0^1(\Omega) \times L^2(\Omega)).$$

In addition, we define the following linear form

$$(3.74) \quad \langle \Lambda \Phi_0, \tilde{\Phi}_0 \rangle = \int_0^T \int_{\omega_c} \psi_t \tilde{\psi}_t dx dt = (\Phi_0, \tilde{\Phi}_0)_{\mathcal{H}}, \quad \forall \tilde{\Phi}_0 \in \mathcal{H},$$

where $(\cdot, \cdot)_{\mathcal{H}}$ is the scalar product associated to the norm $\|\cdot\|_{\mathcal{H}}$.

Using Cauchy-Schwarz in (3.74), we deduce that

$$(3.75) \quad |\langle \Lambda \Phi_0, \tilde{\Phi}_0 \rangle_{\mathcal{H} \times \mathcal{H}'}| \leq \|\Phi_0\|_{\mathcal{H}} \|\tilde{\Phi}_0\|_{\mathcal{H}}, \quad \forall \Phi_0, \tilde{\Phi}_0 \in \mathcal{H}.$$

In particular, we have

$$|\langle \Lambda \Phi_0, \Phi_0 \rangle_{\mathcal{H} \times \mathcal{H}'}| = \|\Phi_0\|_{\mathcal{H}}^2 \quad \forall \Phi_0 \in \mathcal{H}.$$

Then the inverse inequality in Theorem 3.10 implies that the operator Λ is coercive and continuous over \mathcal{H} . Thanks to the Lax-Milgram theorem, we have Λ is an isomorphism from \mathcal{H} into \mathcal{H}' . In particular, for every $U_0 \in (L^2(\Omega) \times H^{-1}(\Omega))^2$, there exists a solution $\Phi_0 \in \mathcal{H}$, such that

$$\Lambda(\Phi_0) = -U_0 = (\zeta_t(0), -\zeta(0), \chi_t(0), -\chi(0)).$$

It follows from the uniqueness of the solution of problem (3.73) that

$$U = \Psi.$$

Consequently, we have

$$u(T) = u_t(T) = y(T) = y_t(T) = 0.$$

The proof is thus complete. \square

4. EXPONENTIAL STABILITY AND EXACT CONTROLLABILITY IN THE CASE $a \neq 1$

4.1. Exponential stability in the weak energy space. The aim of this subsection is to show the exponential stability of system (1.4)-(1.6) in a weak energy space in the case when the waves do not propagate with same speed, i.e., $a \neq 1$. For this, we define the weak energy space

$$D = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$$

equipped with the scalar product

$$(U, \tilde{U}) = \int_{\Omega} (a \nabla u \cdot \nabla \tilde{u} + v \tilde{v} + y \tilde{y} + (-\Delta)^{-1/2} z (-\Delta)^{-1/2} \tilde{z}) dx,$$

for all $U = (u, v, y, z) \in D$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}) \in D$.

Next, we define the unbounded linear operator $\mathcal{A}_d : D(\mathcal{A}_d) \subset D \rightarrow D$ by

$$\mathcal{A}_d U = (v, a \Delta u - bz - cv, z, \Delta y + bv),$$

$$D(\mathcal{A}_d) = ((H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)), \quad \forall U = (u, v, y, z) \in D(\mathcal{A}_d).$$

We define the partial energy associated to a solution $U = (u, u_t, y, y_t)$ of (1.4)-(1.6) by

$$e_1(t) = \frac{1}{2} (a \|\nabla u\|_{L^2(\Omega)}^2 + \|u_t\|_{L^2(\Omega)}^2).$$

We define also the weakened partial energy by

$$\tilde{e}_2(t) = \frac{1}{2} (\|y_t\|_{H^{-1}(\Omega)}^2 + \|y\|_{L^2(\Omega)}^2)$$

and the total mixed energy by

$$E_m(t) = e_1(t) + \tilde{e}_2(t).$$

In order to study the exponential decay rate, we need to assume that ω_{c_+} satisfies the geometric conditions PMGC. Then there exist $\varepsilon > 0$, subsets $\Omega_j \subset \Omega$, $j = 1, \dots, J$, with Lipschitz boundary $\Gamma_j = \partial \Omega_j$ and points $x_j \in \mathbb{R}^N$ such that $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$ and $\omega_c^+ \supset$

$\mathcal{N}_\epsilon \left(\bigcup_{j=1}^J \gamma_j(x_j) \cup (\Omega \setminus \bigcup_{j=1}^J \Omega_j) \right) \cap \Omega$ with $\mathcal{N}_\epsilon(\mathcal{O}) = \{x \in \mathbb{R}^N : d(x, \mathcal{O}) < \epsilon\}$ where $\mathcal{O} \subset \mathbb{R}^N$, $\gamma_j(x_j) = \{x \in \Gamma_j : (x - x_j) \cdot \nu_j(x) > 0\}$ where ν_j is the outward unit normal vector to Γ_j and that ω_b satisfies the GCC condition and

$$\omega_b \subset (\Omega \setminus \bigcup_{j=1}^J \Omega_j). \quad (\text{LH3})$$

Now, we are ready to establish the following main theorem of this section:

Theorem 4.1. (*Exponential decay rate*) *Let $a \neq 1$. Assume that conditions (LH1) and (LH2) hold. Assume also that ω_{c_+} satisfies the geometric conditions PMGC, ω_b satisfies GCC condition and (LH3) and $b, c \in L^\infty(\Omega)$. Then there exist positive constants $M \geq 1, \theta > 0$ such that for all initial data $(u_0, u_1, y_0, y_1) \in D$ the energy of system (1.4)-(1.6) satisfies the following decay rate:*

$$(4.1) \quad E_m(t) \leq M e^{-\theta t} E_m(0), \quad \forall t > 0.$$

In order to prove the above theorem, we apply the same strategy using Huang [12] and Prüss [20]. A C_0 - semigroup of contraction $(e^{t\mathcal{A}})_{t \geq 0}$ in a Hilbert space \mathcal{H} is exponentially stable if and only if

$$i\mathbb{R} \subseteq \rho(\mathcal{A}_d) \quad (\text{H1})$$

and

$$\limsup_{\beta \in \mathbb{R}, |\beta| \rightarrow +\infty} \|(i\beta I - \mathcal{A}_d)^{-1}\|_{\mathcal{L}(D)} < \infty \quad (\text{H2}).$$

Condition (H1) was already proved. We now prove that condition (H2) holds, using an argument of contradiction. For this aim, we suppose that there exist a real sequence β_n with $\beta_n \rightarrow +\infty$ and a sequence $U_n = (u_n, v_n, y_n, z_n) \in D(\mathcal{A}_d)$ such that

$$(4.2) \quad \|U_n\|_D = 1,$$

and

$$(4.3) \quad \lim_{n \rightarrow \infty} \|(i\beta_n I - \mathcal{A}_d)U_n\|_D = 0.$$

Next, detailing equation (4.3), we get

$$(4.4) \quad i\beta_n u_n - v_n = f_n^1 \rightarrow 0 \quad \text{in } H_0^1(\Omega),$$

$$(4.5) \quad i\beta_n v_n - a\Delta u_n + bz_n + cv_n = g_n^1 \rightarrow 0 \quad \text{in } L^2(\Omega),$$

$$(4.6) \quad i\beta_n y_n - z_n = f_n^2 \rightarrow 0 \quad \text{in } L^2(\Omega),$$

$$(4.7) \quad i\beta_n z_n - \Delta y_n - bv_n = g_n^2 \rightarrow 0 \quad \text{in } H^{-1}(\Omega).$$

Eliminating v_n and z_n from the previous system, we obtain the following system

$$(4.8) \quad \beta_n^2 u_n + a\Delta u_n - i\beta_n b y_n - i\beta_n c u_n = -g_n^1 - b f_n^2 - i\beta_n f_n^1 - c f_n^1 \quad \text{in } L^2(\Omega),$$

$$(4.9) \quad \beta_n^2 y_n + \Delta y_n + i\beta_n b u_n = -i\beta_n f_n^2 + b f_n^1 - g_n^2 \quad \text{in } H^{-1}(\Omega).$$

From (4.2), we have $\nabla u_n, v_n$ and y_n are uniformly bounded in $L^2(\Omega)$ and z_n is uniformly bounded in $H^{-1}(\Omega)$.

Using now (4.2) and (4.4), we deduce that $\beta_n u_n$ is uniformly bounded in $L^2(\Omega)$. In addition, using (4.2) and (4.6), we deduce that $\beta_n y_n$ is uniformly bounded in $H^{-1}(\Omega)$. More precisely,

$$\|u_n\|_{L^2(\Omega)} = \frac{O(1)}{\beta_n} = o(1) \quad \text{and} \quad \|y_n\|_{H^{-1}(\Omega)} = \frac{O(1)}{\beta_n} = o(1).$$

Lemma 4.2. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A}_d)$ of system (4.4)-(4.7) satisfies the following estimates*

$$(4.10) \quad \int_{\Omega} c |\beta_n u_n|^2 dx = o(1) \quad \text{and} \quad \int_{\omega_{c_+}} |\beta_n u_n|^2 dx = o(1).$$

Proof. First, since U_n is uniformly bounded in D and using (4.3), we get

$$(4.11) \quad \operatorname{Re} \{ i\beta_n \| U_n \|^2 - (\mathcal{A}_d U_n, U_n) \} = \int_{\Omega} c(x) |v_n|^2 dx = o(1).$$

Next, using equations (4.11) and (4.4), we get

$$(4.12) \quad \int_{\Omega} c |\beta_n u_n|^2 dx = o(1).$$

Under condition (LH1), it follows

$$\int_{\omega_{c+}} |\beta_n u_n|^2 dx = o(1).$$

The proof is thus complete. \square

Now as ω_{c+} satisfies the PMGC condition, let the reals $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon$ and define

$$Q_i = \mathcal{N}_{\varepsilon_i} (\cup_{j=1}^J \gamma_j(x_j) \cup (\Omega \setminus \cup_{j=1}^J \Omega_j)), \quad i = 1, 2.$$

Since $\overline{\mathbb{R}^N \setminus \omega_{c+}} \cap \overline{Q_2} = \emptyset$, we can construct a function $\hat{\eta} \in C_0^\infty(\Omega)$ such that

$$\hat{\eta}(x) = 0 \quad \text{if } x \in \Omega \setminus \omega_{c+}, \quad 0 \leq \hat{\eta}(x) \leq 1, \quad \hat{\eta}(x) = 1 \quad \text{if } x \in Q_2.$$

Lemma 4.3. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A}_d)$ of system (4.4)-(4.7) satisfies the following estimates*

$$(4.13) \quad \int_{\Omega} \hat{\eta} |\nabla u_n|^2 dx = o(1) \quad \text{and} \quad \int_{Q_2 \cap \Omega} |\nabla u_n|^2 dx = o(1).$$

Proof. First, multiplying equation (4.8) by $\hat{\eta} \bar{u}_n$. Then, using Green's formula and the fact that $u_n = 0$ on Γ , we obtain

$$(4.14) \quad \begin{cases} \int_{\Omega} \hat{\eta} |\beta_n u_n|^2 dx - a \int_{\Omega} \hat{\eta} |\nabla u_n|^2 dx - a \int_{\Omega} \bar{u}_n (\nabla \hat{\eta} \cdot \nabla u_n) dx - i \beta_n \int_{\Omega} b \hat{\eta} y_n \bar{u}_n dx \\ -i \beta_n \int_{\Omega} c \hat{\eta} |u_n|^2 dx = \int_{\Omega} (-g_n^1 - b f_n^2 - i \beta_n f_n^1 - c f_n^1) \hat{\eta} \bar{u}_n dx. \end{cases}$$

As f_n^1 converges to zero in $H_0^1(\Omega)$, f_n^2, g_n^1 converge to zero in $L^2(\Omega)$ and $\beta_n u_n$ is uniformly bounded in $L^2(\Omega)$, we get

$$(4.15) \quad \int_{\Omega} (-g_n^1 - b f_n^2 - i \beta_n f_n^1 - c f_n^1) \hat{\eta} \bar{u}_n dx = o(1).$$

Using the fact that $\nabla u_n, y_n$ are uniformly bounded in $L^2(\Omega)$, $\|u_n\|_{L^2(\Omega)} = o(1)$ and estimation (4.10), we will have

$$(4.16) \quad \int_{\Omega} \hat{\eta} |\beta_n u_n|^2 dx - a \int_{\Omega} \bar{u}_n (\nabla \hat{\eta} \cdot \nabla u_n) dx - i \beta_n \int_{\Omega} b \hat{\eta} y_n \bar{u}_n dx - i \beta_n \int_{\Omega} c \hat{\eta} |u_n|^2 dx = o(1).$$

Finally, inserting (4.15) and (4.16) into (4.14), we deduce

$$\int_{\Omega} \hat{\eta} |\nabla u_n|^2 dx = o(1) \quad \text{and} \quad \int_{Q_2 \cap \Omega} |\nabla u_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 4.4. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A}_d)$ of system (4.4)-(4.7) satisfies the following estimate*

$$(4.17) \quad \int_{\omega_b} |y_n|^2 dx = o(1).$$

Proof. The proof contains two steps.

Step 1. (Boundedness of $\frac{1}{\beta_n} \nabla y_n$). Multiplying equation (4.9) by $\frac{1}{\beta_n^2} \bar{y}_n$, we obtain

$$(4.18) \quad \int_{\Omega} |y_n|^2 dx + \langle \Delta y_n, \frac{1}{\beta_n^2} \bar{y}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = -i \int_{\Omega} \frac{1}{\beta_n} f_n^2 \bar{y}_n dx + \int_{\Omega} b f_n^1 \frac{1}{\beta_n^2} \bar{y}_n dx \\ - \langle g_n^2, \frac{1}{\beta_n^2} \bar{y}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}.$$

Since f_n^1 converges to zero in $H_0^1(\Omega)$, f_n^2 converges to zero in $L^2(\Omega)$ and y_n is uniformly bounded in $L^2(\Omega)$, we get

$$(4.19) \quad -i \int_{\Omega} \frac{1}{\beta_n} f_n^2 \bar{y}_n dx + \int_{\Omega} b f_n^1 \frac{1}{\beta_n^2} \bar{y}_n dx = o(1).$$

Inserting (4.19) into (4.18), we will have after integrating by parts

$$\int_{\Omega} \left| \frac{\nabla y_n}{\beta_n} \right|^2 dx = \int_{\Omega} |y_n|^2 dx + \langle g_n^2, \frac{1}{\beta_n^2} \bar{y}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + o(1).$$

Using Cauchy-Schwarz and Young inequalities in the previous equation, we obtain that

$$\frac{1}{2} \left\| \frac{\nabla y_n}{\beta_n} \right\|_{L^2(\Omega)}^2 \leq \|y_n\|_{L^2(\Omega)}^2 + \frac{1}{2} \|g_n^2\|_{H^{-1}(\Omega)}^2 + o(1).$$

It follows, from the uniform boundedness of y_n in $L^2(\Omega)$ and g_n^2 in $H^{-1}(\Omega)$, that

$$(4.20) \quad \left\| \frac{\nabla y_n}{\beta_n} \right\|^2 = O(1).$$

Step 2. (Main asymptotic estimation). Multiplying equation (4.8) by $\hat{\eta} \frac{1}{\beta_n} \bar{y}_n$. Later, using Green's formula and the fact that $y_n = 0$ on Γ , we get

$$(4.21) \quad \begin{cases} \int_{\Omega} \hat{\eta} \beta_n u_n \bar{y}_n dx - a \int_{\Omega} \frac{1}{\beta_n} \hat{\eta} (\nabla u_n \cdot \nabla \bar{y}_n) dx - a \int_{\Omega} \frac{1}{\beta_n} (\nabla \hat{\eta} \cdot \nabla u_n) \bar{y}_n dx \\ -i \int_{\Omega} b \hat{\eta} |y_n|^2 dx - i \int_{\Omega} c u_n \hat{\eta} \bar{y}_n dx = \int_{\Omega} (-g_n^1 - b f_n^2 - i \beta_n f_n^1 - c f_n^1) \frac{\hat{\eta}}{\beta_n} \bar{y}_n dx. \end{cases}$$

Next, using the definition of $\hat{\eta}$ and equations (4.13) and (4.20), we get

$$(4.22) \quad -a \int_{\Omega} \frac{1}{\beta_n} \hat{\eta} (\nabla u_n \cdot \nabla \bar{y}_n) dx = o(1).$$

Using (4.10), (4.13) and the fact that y_n is uniformly bounded in $L^2(\Omega)$, we obtain

$$(4.23) \quad -a \int_{\Omega} \frac{1}{\beta_n} (\nabla \hat{\eta} \cdot \nabla u_n) \bar{y}_n dx - \int_{\Omega} \hat{\eta} \beta_n u_n \bar{y}_n dx - i \int_{\Omega} c u_n \hat{\eta} \bar{y}_n dx = o(1).$$

Using the fact that f_n^1 converges to zero in $H_0^1(\Omega)$, f_n^2 , g_n^1 converge to zero in $L^2(\Omega)$ and y_n is uniformly bounded in $L^2(\Omega)$, we will have

$$(4.24) \quad \int_{\Omega} (-g_n^1 - b f_n^2 - i \beta_n f_n^1 - c f_n^1) \frac{\hat{\eta}}{\beta_n} \bar{y}_n dx = o(1).$$

Finally, inserting (4.22)-(4.24) into (4.21), we get

$$\int_{\Omega} b \hat{\eta} |y_n|^2 dx = o(1).$$

It follows, from condition (LH3), that

$$\int_{\omega_b} |y_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 4.5. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A}_d)$ of system (4.4)-(4.7) satisfies the following estimate*

$$(4.25) \quad \int_{\Omega} |y_n|^2 dx = o(1).$$

Proof. Noting that ω_b satisfies the GCC condition, so we can taking $f_n = y_n$ in Lemma 3.6. Multiplying equation (4.9) by $\bar{\phi}_n$. Then, we have

$$(4.26) \quad \int_{\Omega} \beta_n^2 \bar{\phi}_n y_n dx - \langle \Delta y_n, \bar{\phi}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + i \int_{\Omega} \beta_n b u_n \bar{\phi}_n dx = -i \int_{\Omega} \beta_n f_n^2 \bar{\phi}_n dx + \int_{\Omega} b f_n^1 \bar{\phi}_n dx - \langle g_n^2, \bar{\phi}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}.$$

Using the fact that $\phi_n \in H^2(\Omega) \cap H_0^1(\Omega)$ and $y_n \in H_0^1(\Omega)$, then we have

$$(4.27) \quad - \langle \Delta y_n, \bar{\phi}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = \int_{\Omega} y_n \Delta \bar{\phi}_n dx.$$

It follows, from the first equation of (3.34) and (4.26), that

$$(4.28) \quad \int_{\Omega} |y_n|^2 dx = i \int_{\Omega} b \beta_n \bar{\phi}_n y_n dx - i \int_{\Omega} \beta_n b u_n \bar{\phi}_n dx - i \int_{\Omega} \beta_n f_n^2 \bar{\phi}_n dx + \int_{\Omega} b f_n^1 \bar{\phi}_n dx - \langle g_n^2, \bar{\phi}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}.$$

Using the fact that $\beta_n \phi_n$ is uniformly bounded in $L^2(\Omega)$, f_n^1 converges to zero in $H_0^1(\Omega)$, f_n^2 converges to zero in $L^2(\Omega)$, g_n^2 converges to zero in H^{-1} , (4.17) and $\|u_n\| = o(1)$ in equation (4.28), we obtain

$$(4.29) \quad \int_{\Omega} |y_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 4.6. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A}_d)$ of system (4.4)-(4.7) satisfies the following estimate*

$$(4.30) \quad \int_{\Omega} |\beta_n (-\Delta)^{-1/2} y_n|^2 dx = o(1).$$

Proof. Multiplying equation (4.9) by $(-\Delta)^{-1} \bar{y}_n$, then integrating by parts and using the fact that $y_n = 0$ on Γ , we get

$$(4.31) \quad \int_{\Omega} |\beta_n (-\Delta)^{-1/2} y_n|^2 dx = \int_{\Omega} |y_n|^2 dx - i \int_{\Omega} \beta_n b u_n (-\Delta)^{-1} \bar{y}_n dx - i \int_{\Omega} \beta_n (-\Delta)^{-1/2} f_n^2 (-\Delta)^{-1/2} \bar{y}_n dx + \int_{\Omega} b f_n^1 (-\Delta)^{-1} \bar{y}_n dx - \langle g_n^2, (-\Delta)^{-1} \bar{y}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}.$$

Using Cauchy-Schwarz and Poincaré inequalities, we get

$$(4.32) \quad \left| \int_{\Omega} \beta_n (-\Delta)^{-1/2} f_n^2 (-\Delta)^{-1/2} \bar{y}_n dx \right| \leq \|(-\Delta)^{-1/2} f_n^2\|_{L^2(\Omega)} \|\beta_n (-\Delta)^{-1/2} \bar{y}_n\|_{L^2(\Omega)} \leq c_0 \|f_n^2\|_{L^2(\Omega)} \|\beta_n \bar{y}_n\|_{H^{-1}(\Omega)}.$$

It follows, from the convergence to zero of f_n^2 in $L^2(\Omega)$ and the boundedness of $\beta_n y_n$ in $H^{-1}(\Omega)$,

that

$$(4.33) \quad \int_{\Omega} \beta_n (-\Delta)^{-1/2} f_n^2 (-\Delta)^{-1/2} \bar{y}_n dx = o(1).$$

Similarly, we have

$$(4.34) \quad \begin{aligned} |\langle g_n^2, (-\Delta)^{-1} \bar{y}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}| &= \int_{\Omega} (-\Delta)^{-1/2} g_n^2 (-\Delta)^{-1/2} \bar{y}_n dx \\ &\leq \|(-\Delta)^{-1/2} g_n^2\|_{L^2(\Omega)} \|(-\Delta)^{-1/2} \bar{y}_n\|_{L^2(\Omega)} \\ &\leq \|g_n^2\|_{H^{-1}(\Omega)} \|\bar{y}_n\|_{H^{-1}(\Omega)}. \end{aligned}$$

It follows, from the convergence of g_n^2 and y_n to zero in $H^{-1}(\Omega)$, that

$$(4.35) \quad \langle g_n^2, (-\Delta)^{-1} \bar{y}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = o(1).$$

Note that $(-\Delta)^{-1}$ is compact operator from L^2 to L^2 . Then $(-\Delta)^{-1} y_n$ is uniformly bounded in L^2 . Finally, using (4.10), (4.25), (4.33), (4.35) and the fact that f_n^1 converges to zero in $H_0^1(\Omega)$ into equation (4.31), we deduce

$$\int_{\Omega} |\beta_n (-\Delta)^{-1/2} y_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 4.7. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A}_d)$ of system (4.4)-(4.7) satisfies the following estimate*

$$(4.36) \quad \int_{\Omega \setminus (Q_2 \cap \Omega)} (|\nabla u_n|^2 + |\beta_n u_n|^2) dx = o(1).$$

Proof. Since $(\bar{\Omega}_j \setminus Q_2) \cap \bar{Q}_1 = \emptyset$, we define the function $\psi_j \in C_0^\infty(\mathbb{R}^N)$ by:

$$\psi_j(x) = 0 \quad \text{if } x \in Q_1, \quad 0 \leq \psi_j \leq 1, \quad \psi_j(x) = 1 \quad \text{if } x \in \bar{\Omega}_j \setminus Q_2.$$

For $m_j(x) = (x - x_j)$, we define $h_j(x) = \psi_j(x) m_j(x)$.

Multiplying equation (4.8) by $2(h_j \cdot \nabla \bar{u}_n)$ and integrating over Ω_j , using the dissipation (4.10) and the fact that ∇u_n is uniformly bounded in $L^2(\Omega)$, we obtain

$$(4.37) \quad \begin{aligned} &2\beta_n^2 \int_{\Omega_j} u_n (h_j \cdot \nabla \bar{u}_n) dx + 2a \int_{\Omega_j} \Delta u_n (h_j \cdot \nabla \bar{u}_n) dx - 2i \int_{\Omega_j} \beta_n b y_n (h_j \cdot \nabla \bar{u}_n) dx = \\ &2 \int_{\Omega_j} (-g_n^1 - b f_n^2 - c f_n^1) (h_j \cdot \nabla \bar{u}_n) dx - 2i \int_{\Omega_j} \beta_n f_n^1 (h_j \cdot \nabla \bar{u}_n) dx. \end{aligned}$$

i) **Estimation of the second member of (4.37).** First, using Green's formula and the fact that $u_n = 0$ on $(\Gamma_j \setminus \gamma_j) \cap \Gamma$ and $h_j = 0$ on γ_j , we get

$$(4.38) \quad -2i \int_{\Omega_j} \beta_n f_n^1 (h_j \cdot \nabla \bar{u}_n) dx = 2i \int_{\Omega_j} \beta_n \bar{u}_n (h_j \cdot \nabla f_n^1) dx + 2i \int_{\Omega_j} \beta_n \bar{u}_n f_n^1 (\operatorname{div} h_j) dx.$$

So, from the fact that f_n^1 converges to zero in $H_0^1(\Omega)$ and $\beta_n u_n$ is uniformly bounded in $L^2(\Omega)$, we obtain

$$(4.39) \quad -2i \int_{\Omega_j} \beta_n f_n^1 (h_j \cdot \nabla \bar{u}_n) dx = o(1).$$

Next, as f_n^1 converges to zero in $H_0^1(\Omega)$, f_n^2 , g_n^1 converge to zero in $L^2(\Omega)$ and the sequence (∇u_n) is uniformly bounded in $L^2(\Omega)$, we deduce

$$(4.40) \quad 2 \int_{\Omega_j} (-g_n^1 - b f_n^2 - c f_n^1) (h_j \cdot \nabla \bar{u}_n) dx = o(1).$$

Finally, we deduce that the second member of (4.37) is $o(1)$.

ii) **Estimation of the first integral of equation (4.37).** Using Green's formula, we get

$$(4.41) \quad \operatorname{Re} \left\{ 2 \int_{\Omega_j} \beta_n^2 u_n (h_j \cdot \nabla \bar{u}_n) dx \right\} = - \int_{\Omega_j} (\operatorname{div} h_j) |\beta_n u_n|^2 dx + \int_{\Gamma_j} (h_j \cdot \nu_j) |\beta_n u_n|^2 d\Gamma_j.$$

Since $\Psi_j = 0$ on γ_j and $u_n = 0$ on $(\Gamma_j \setminus \gamma_j) \cap \Gamma$, then we have

$$(4.42) \quad \operatorname{Re}\left\{2 \int_{\Omega_j} \beta_n^2 u_n (h_j \cdot \nabla \bar{u}_n) dx\right\} = - \int_{\Omega_j} (\operatorname{div} h_j) |\beta_n u_n|^2 dx.$$

iii) **Estimation of the second integral of equation (4.37).** Using Green's formula, we get

$$(4.43) \quad \operatorname{Re}\left\{2a \int_{\Omega_j} \Delta u_n (h_j \cdot \nabla \bar{u}_n)\right\} = -2a \operatorname{Re}\left\{\sum_{i,k=1}^N \int_{\Omega_j} \partial_i h_j^k \partial_i u_n \partial_k u_n dx\right\} + \\ a \int_{\Omega_j} (\operatorname{div} h_j) |\nabla u_n|^2 dx - a \int_{\Gamma_j} (h_j \cdot \nu_j) |\nabla u_n|^2 d\Gamma_j + 2a \operatorname{Re}\left\{\int_{\Gamma_j} \partial_{\nu_j} u_n (h_j \cdot \nabla \bar{u}_n) d\Gamma_j\right\}.$$

According to the choice of ψ_j , only the boundary terms over $(\Gamma_j \setminus \gamma_j) \cap \Gamma$ are non vanishing in (4.43). But on this part of the boundary $u_n = 0$, and consequently $\nabla u_n = (\partial_{\nu} u_n) \cdot \nu = (\partial_{\nu_j} u_n) \nu_j$. Then, we have

$$(4.44) \quad -a \int_{\Gamma_j} (h_j \cdot \nu_j) |\nabla u_n|^2 d\Gamma_j + 2a \operatorname{Re}\left\{\int_{\Gamma_j} (\partial_{\nu_j} u_n) (h_j \cdot \nabla \bar{u}_n) d\Gamma_j\right\} = \\ a \int_{(\Gamma_j \setminus \gamma_j) \cap \Gamma} (\psi_j m_j \cdot \nu_j) |\partial_{\nu_j} u_n|^2 d\Gamma_j \leq 0.$$

Inserting (4.44) into (4.43), we get

$$(4.45) \quad \operatorname{Re}\left\{2a \int_{\Omega_j} \Delta u_n (h_j \cdot \nabla \bar{u}_n)\right\} \leq -2a \operatorname{Re}\left\{\sum_{i,k=1}^N \int_{\Omega_j} \partial_i h_j^k \partial_i u_n \partial_k u_n dx\right\} + a \int_{\Omega_j} (\operatorname{div} h_j) |\nabla u_n|^2 dx.$$

iv) **The main estimation.** Inserting equations (4.39), (4.40), (4.42) and (4.45) into (4.37) and using the fact that $\psi_j = 0$ on Q_1 , we get

$$\int_{\Omega_j \setminus (Q_1 \cap \Omega_j)} \left[\operatorname{div}(\psi_j m_j) (|\beta_n u_n|^2 - a |\nabla u_n|^2) dx + 2a \sum_{i,k=1}^N \partial_i (\psi_j m_j^k) \partial_i u_n \partial_k u_n \right] dx \\ + 2i \int_{\Omega_j \setminus (Q_1 \cap \Omega_j)} \beta_n b y_n (\psi_j m_j \cdot \nabla \bar{u}_n) dx \leq o(1).$$

Thus, summing over j and using the fact that $\psi_j = 1$ on $\bar{\Omega}_j \setminus Q_2$, we get

$$(4.46) \quad N \int_{\Omega \setminus (Q_2 \cap \Omega)} |\beta_n u_n|^2 dx + (2 - N)a \int_{\Omega \setminus (Q_2 \cap \Omega)} |\nabla u_n|^2 dx \\ + 2 \operatorname{Re}\left\{i \sum_{j=1}^J \int_{\Omega_j \setminus (Q_1 \cap \Omega_j)} \beta_n b y_n (\psi_j m_j \cdot \nabla \bar{u}_n) dx\right\} \\ \leq - \sum_{j=1}^J \int_{Q_2 \cap \Omega_j} \left[\operatorname{div}(\psi_j m_j) (|\beta_n u_n|^2 - a |\nabla u_n|^2) dx + 2a \sum_{i,k=1}^N \partial_i (\psi_j m_j^k) \partial_i u_n \partial_k u_n \right] dx + o(1).$$

Using (4.10) and (4.13), we deduce

$$(4.47) \quad - \sum_{j=1}^J \int_{Q_2 \cap \Omega_j} \left[\operatorname{div}(\psi_j m_j) (|\beta_n u_n|^2 - a |\nabla u_n|^2) dx + 2a \sum_{i,k=1}^N \partial_i (\psi_j m_j^k) \partial_i u_n \partial_k u_n \right] dx = o(1).$$

Inserting (4.47) in (4.46), we obtain

$$(4.48) \quad N \int_{\Omega \setminus (Q_2 \cap \Omega)} |\beta_n u_n|^2 dx + (2 - N)a \int_{\Omega \setminus (Q_2 \cap \Omega)} |\nabla u_n|^2 dx + \\ 2 \operatorname{Re}\left\{i \sum_{j=1}^J \int_{\Omega_j \setminus (Q_1 \cap \Omega_j)} \beta_n b y_n (\psi_j m_j \cdot \nabla \bar{u}_n) dx\right\} \leq o(1).$$

Under condition (LH3) and the definition of ψ_j , we will have

$$2\operatorname{Re} \left\{ i \sum_{j=1}^J \int_{\Omega_j \setminus (Q_1 \cap \Omega_j)} \beta_n b y_n (\psi_j m_j \cdot \nabla \bar{u}_n) dx \right\} = 0.$$

Inserting the previous estimation into (4.48), we get

$$(4.49) \quad N \int_{\Omega \setminus (Q_2 \cap \Omega)} |\beta_n u_n|^2 dx + (2 - N)a \int_{\Omega \setminus (Q_2 \cap \Omega)} |\nabla u_n|^2 dx \leq o(1).$$

Multiplying (4.8) by $(1 - N)\bar{u}_n$. Then integrating on Ω , using Green's formula, the fact that y_n and $\beta_n u_n$ are bounded in $L^2(\Omega)$ and the estimation (4.10), we obtain

$$(4.50) \quad (1 - N) \int_{\Omega} |\beta_n u_n|^2 dx - (1 - N)a \int_{\Omega} |\nabla u_n|^2 dx = o(1).$$

Using (4.10) and (4.13) in (4.50), we deduce

$$(4.51) \quad (1 - N) \int_{\Omega \setminus (Q_2 \cap \Omega)} |\beta_n u_n|^2 dx - (1 - N)a \int_{\Omega \setminus (Q_2 \cap \Omega)} |\nabla u_n|^2 dx = o(1).$$

Finally, combining (4.49) and (4.51), we get the following estimate

$$\int_{\Omega \setminus (Q_2 \cap \Omega)} (a|\nabla u_n|^2 + |\beta_n u_n|^2) dx = o(1).$$

The proof is thus complete. \square

Proof of Theorem 3.10 It follows from (4.10) (4.13), (4.25), (4.30) and (4.36) that $\|U_n\| = o(1)$ which is a contradiction with (4.2). Consequently, condition (H2) holds and the energy of system (1.4)-(1.6) decays exponentially to zero in the weak energy space D . The proof is thus complete.

4.2. Observability and exact controllability. First, we consider the following homogeneous system associated to (1.4)-(1.6) for $a \neq 1$ by:

$$(4.52) \quad \psi_{tt} - a\Delta\psi + b(x)\varphi_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+,$$

$$(4.53) \quad \varphi_{tt} - \Delta\varphi - b(x)\psi_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+,$$

$$(4.54) \quad \psi = \varphi = 0 \quad \text{on } \Gamma \times \mathbb{R}_+,$$

$$(4.55) \quad \psi(\cdot, 0) = \psi_0, \psi_t(\cdot, 0) = \psi_1, \varphi(\cdot, 0) = \varphi_0, \varphi_t(\cdot, 0) = \varphi_1 \quad \text{in } \Omega.$$

Let $\Phi = (\psi, \psi_t, \varphi, \varphi_t)$ be a regular solution of system (3.61)-(3.63), its associated total energy is given by:

$$(4.56) \quad E_m(t) = \frac{1}{2} \left(a \|\nabla\psi\|_{L^2(\Omega)}^2 + \|\psi_t\|_{L^2(\Omega)}^2 + \|\varphi_t\|_{H^{-1}(\Omega)}^2 + \|\varphi\|_{L^2(\Omega)}^2 \right).$$

A direct computation gives

$$(4.57) \quad \frac{d}{dt} E_m(t) = 0.$$

Thus, system (4.52)-(4.54) is conservative in the sense that its energy $E_m(t)$ is constant. It is also wellposed and admits a unique solution in the energy space D .

Now, we establish the direct and indirect inequality given by the following theorem:

Theorem 4.8. *Let $0 < a \neq 1$. Assume that conditions (LH1) and (LH2) hold. Assume also that ω_{c_+} satisfies the PMGC, ω_b satisfies GCC condition and (LH3) and $b, c \in L^\infty(\Omega)$. Then there exists a time T_0 such that for all $T > T_0$, there exist two constants $C_1 > 0$, $C_2 > 0$ such that the solution of system (4.52)-(4.54) satisfies the following observability inequalities:*

$$(4.58) \quad C_1 \|\Phi_0\|_D^2 \leq \int_0^T \int_{\Omega} c(x) |\psi_t|^2 dx dt \leq C_2 \|\Phi_0\|_D^2,$$

for all $\Phi_0 = (\psi_0, \psi_1, \varphi_0, \varphi_1) \in D$.

Proof. The direct inequality follows from the definition of the total energy for all $T > 0$. While the proof of the inverse inequality is a direct consequence of Proposition 2 of Haraux in [11] for which the exponential stability (4.1) implies the existence of a time $T_0 > 0$ such that for all $T > T_0$ there exist two constants $C_1 > 0$ and $C_2 > 0$ such that (4.58) holds. \square

It is well known that observability of the homogeneous system (4.52)-(4.54) implies the exact controllability of the system .

5. NUMERICAL APPROXIMATION: VALIDATION OF THE THEORETICAL RESULTS

This section is devoted to the numerical approximation of the problem that we considered by a finite difference discretization and to the validation of the theoretical results stated in the previous sections. We will firstly construct in detail a discretization in the 1D case and we will define its corresponding discrete energy. Numerical experiments are performed to validate the theoretical results. In fact, the numerical results in 1D show an exponential stabilization in any case when $a = 1$ and a polynomial stabilization in any case in the case $a \neq 1$. They are better than expected.

5.1. Finite difference scheme in one dimensional space. We firstly introduce the finite difference scheme we will work on. Then we will construct the corresponding energy and finally we will perform numerical experiments. Let us firstly recall the problem we are considered.

Consider $\Omega = [0, 1]$. We are interested to study the controllability of the following coupled wave equations by velocities:

$$(5.1) \quad \begin{cases} u_{tt} - au_{xx} + b(x)y_t + c(x)u_t & = 0 & x \in (0, 1), t > 0 \\ y_{tt} - y_{xx} - b(x)u_t & = 0 & x \in (0, 1), t > 0 \\ u(0, t) = u(1, t) = y(0, t) = y(1, t) & = 0 & t > 0, \end{cases}$$

with the following initial data

$$(5.2) \quad u(x, 0) = u_0(x), \text{ and } y(x, 0) = y_0(x) \quad x \in (0, 1)$$

and

$$(5.3) \quad u_t(x, 0) = u_1(x) \text{ and } y_t(x, 0) = y_1(x), \quad x \in (0, 1)$$

where $a > 0$ constant, $b \in C^0([0, 1], \mathbb{R})$ and $c \in C^0([0, 1], \mathbb{R}^+)$. We will study the two cases $a = 1$ and $a \neq 1$.

5.1.1. Construction of the numerical scheme. Let N be a non negative integer. Consider the subdivision of $[0, 1]$ given by

$$0 = x_0 < x_1 < \dots < x_N < x_{N+1} = 1, \quad \text{i.e. } x_j = j\Delta x, \quad j = 0, \dots, N+1.$$

Set $t^{n+1} - t^n = \Delta t$ for all $n \in \mathbb{N}$. For $j = 0, \dots, N+1$, we denote $b_j = b(x_j)$, $c_j = c(x_j)$. The explicit finite-difference discretization of system (5.1) is thus, for $n \in \mathbb{N}$ and $j = 1, \dots, N$:

$$(5.4) \quad \begin{cases} \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} - a \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + b_j \frac{y_j^{n+1} - y_j^{n-1}}{2\Delta t} + c_j \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = 0 \\ \frac{y_j^{n+1} - 2y_j^n + y_j^{n-1}}{\Delta t^2} - \frac{y_{j+1}^n - 2y_j^n + y_{j-1}^n}{\Delta x^2} - b_j \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = 0. \\ u_0^n = u_{N+1}^n = 0 \\ y_0^n = y_{N+1}^n = 0 \end{cases}$$

According to the initial conditions given by equations (5.2), we have firstly: for $j = 1, \dots, N$,

$$(5.5) \quad u_j^0 = u_0(x_j)$$

$$(5.6) \quad y_j^0 = y_0(x_j).$$

We can use the second initial conditions (5.3) to find the values of u and y at time $t^1 = \Delta t$, by employing a “ghost” time-boundary (i.e. $t^{-1} = -\Delta t$) and the second-order central difference formula for $j = 1, \dots, N$:

$$(5.7) \quad u_1(x_j) = \frac{\partial u}{\partial t} \Big|_{x_j, 0} = \frac{u_j^1 - u_j^{-1}}{2\Delta t} + O(\Delta t^2).$$

Thus we have for $j = 1, \dots, N$:

$$(5.8) \quad u_j^{-1} = u_j^1 - 2\Delta t u_1(x_j).$$

We use the same discrete form of the initial conditions for y , for $j = 1, \dots, N$:

$$(5.9) \quad y_j^{-1} = y_j^1 - 2\Delta t y_1(x_j).$$

Setting $n = 0$, in the numerical scheme (5.4), the two preceding equalities permit us to compute $(u_j^1, y_j^1)_{j=0, N}$. Finally, the solution (u, y) can be computed at any time t^n .

5.1.2. *Practical implementation and CFL condition.* Let us denote $\lambda = \frac{\Delta t^2}{\Delta x^2}$. We easily remark that the discrete scheme (5.4) is composed of N linear systems of two equations which can be written under the form:

$$(5.10) \quad \text{for } j = 1, \dots, N, M_j \cdot \begin{pmatrix} u_j^{n+1} \\ y_j^{n+1} \end{pmatrix} = \begin{pmatrix} A_j \\ B_j \end{pmatrix}$$

where

$$M_j = \begin{pmatrix} 1 + \frac{c_j \Delta t}{2} & \frac{b_j \Delta t}{2} \\ -\frac{b_j \Delta t}{2} & 1 \end{pmatrix}$$

$$A_j = 2(1 - a\lambda)u_j^n + \left(\frac{c_j}{2}\Delta t - 1\right)u_j^{n-1} + a\lambda(u_{j+1}^n + u_{j-1}^n) + \frac{b_j}{2}\Delta t y_j^{n-1}$$

and

$$B_j = 2(1 - \lambda)y_j^n + \lambda(y_{j+1}^n + y_{j-1}^n) - y_j^{n-1} - \frac{b_j}{2}\Delta t u_j^{n-1}.$$

Thanks to the hypothesis $\forall x \in (0, 1), c(x) \geq 0$, for $j = 1, \dots, N$ the determinant of M_j given by

$$|M_j| = 1 + \frac{c_j \Delta t}{2} + \left(\frac{b_j \Delta t}{2}\right)^2,$$

is a strictly positive quantity.

Consequently, system (5.10) admits a unique solution given by: for $j = 1, \dots, N$,

$$(5.11) \quad u_j^{n+1} = (1 - a\lambda)\alpha_j u_j^n + \lambda\beta_j(u_{j+1}^n + u_{j-1}^n) + \gamma_j u_j^{n-1} - (1 - \lambda)\varrho_j y_j^n - \lambda\xi_j(y_{j+1}^n + y_{j-1}^n) + \kappa_j y_j^{n-1}$$

$$(5.12) \quad y_j^{n+1} = (1 - \lambda)\tilde{\alpha}_j y_j^n + \lambda\tilde{\beta}_j(y_{j+1}^n + y_{j-1}^n) + \tilde{\gamma}_j y_j^{n-1} + (1 - a\lambda)\tilde{\varrho}_j u_j^n + \lambda\tilde{\xi}_j(u_{j+1}^n + u_{j-1}^n) + \tilde{\kappa}_j u_j^{n-1}$$

where we have set:

$$\begin{aligned}
\alpha_j &= \frac{2}{1 + \frac{c_j}{2}\Delta t + \left(\frac{b_j\Delta t}{2}\right)^2} , & \beta_j &= \frac{a}{1 + \frac{c_j}{2}\Delta t + \left(\frac{b_j\Delta t}{2}\right)^2} , \\
\gamma_j &= \frac{\frac{c_j}{2}\Delta t + \left(\frac{b_j\Delta t}{2}\right)^2 - 1}{1 + \frac{c_j}{2}\Delta t + \left(\frac{b_j\Delta t}{2}\right)^2} , & \varrho_j &= \frac{b_j\Delta t}{1 + \frac{c_j}{2}\Delta t + \left(\frac{b_j\Delta t}{2}\right)^2} , \\
\xi_j &= \frac{b_j\Delta t}{2\left(1 + \frac{c_j}{2}\Delta t + \left(\frac{b_j\Delta t}{2}\right)^2\right)} , & \kappa_j &= \frac{b_j\Delta t}{1 + \frac{c_j}{2}\Delta t + \left(\frac{b_j\Delta t}{2}\right)^2} , \\
\tilde{\alpha}_j &= 2 - \frac{(b_j\Delta t)^2}{2\left(1 + \frac{c_j}{2}\Delta t + \left(\frac{b_j\Delta t}{2}\right)^2\right)} , & \tilde{\beta}_j &= 1 - \frac{(b_j\Delta t)^2}{4\left(1 + \frac{c_j}{2}\Delta t + \left(\frac{b_j\Delta t}{2}\right)^2\right)} , \\
\tilde{\gamma}_j &= \frac{(b_j\Delta t)^2}{2\left(1 + \frac{c_j}{2}\Delta t + \left(\frac{b_j\Delta t}{2}\right)^2\right)} - 1 , & \tilde{\varrho}_j &= \frac{b_j\Delta t}{1 + \frac{c_j}{2}\Delta t + \left(\frac{b_j\Delta t}{2}\right)^2} , \\
\tilde{\xi}_j &= \frac{ab_j\Delta t}{2\left(1 + \frac{c_j}{2}\Delta t + \left(\frac{b_j\Delta t}{2}\right)^2\right)} , & \tilde{\kappa}_j &= \left[\frac{\frac{c_j}{2}\Delta t + \left(\frac{b_j\Delta t}{2}\right)^2 - 1}{1 + \frac{c_j}{2}\Delta t + \left(\frac{b_j\Delta t}{2}\right)^2} - 1 \right] \frac{b_j\Delta t}{2} .
\end{aligned}$$

The implementation of the numerical discretization of the problem (5.1) consists finally of equations (5.5), (5.6), (5.11), (5.12) where (u^{-1}, y^{-1}) used for $n = 0$, are defined by (5.8), (5.9).

By a standard von Neumann stability analysis (that is a discrete Fourier analysis, see for instance [7]), the numerical scheme is stable if and only if, the following Courant-Friedrichs-Lewy, CFL, condition holds:

$$\Delta t^2 \leq \Delta x^2 \text{ and } a \Delta t^2 \leq \Delta x^2$$

which is equivalent to

$$(5.13) \quad \Delta t \leq \min\left(1, \frac{1}{\sqrt{a}}\right) \Delta x .$$

The number $\min\left(1, \frac{1}{\sqrt{a}}\right)$ is called the CFL number and is denoted in the following by CFL .

5.1.3. Discrete energy: definition and dissipation. The aim of this section is to design a discrete energy that might be preserved in the case $c = 0$ and to obtain the dissipation of the discrete energy in the case $c > 0$. To this end, let us define:

- the discrete kinetic energy for u as: $E_{k,u}^n = \frac{1}{2} \sum_{j=1}^N \left(\frac{u_j^{n+1} - u_j^n}{\Delta t} \right)^2$
- the discrete potential energy for u as: $E_{p,u}^n = \frac{a}{2} \sum_{j=0}^N \left(\frac{u_{j+1}^n - u_j^n}{\Delta x} \right) \left(\frac{u_{j+1}^{n+1} - u_j^{n+1}}{\Delta x} \right)$
- the discrete kinetic energy for y as: $E_{k,y}^n = \frac{1}{2} \sum_{j=1}^N \left(\frac{y_j^{n+1} - y_j^n}{\Delta t} \right)^2$
- the discrete potential energy for u as: $E_{y,u}^n = \frac{1}{2} \sum_{j=0}^N \left(\frac{y_{j+1}^n - y_j^n}{\Delta x} \right) \left(\frac{y_{j+1}^{n+1} - y_j^{n+1}}{\Delta x} \right)$

The total discrete energy is then defined as

$$(5.14) \quad \mathcal{E}^n = E_{k,u}^n + E_{p,u}^n + E_{k,y}^n + E_{p,u}^n.$$

Let us prove now that this definition of the energy fulfills the two properties stated above. For this sake, we multiply the first equation of (5.4) by $(u_j^{n+1} - u_j^{n-1})$ and we sum over $j = 1, \dots, N$. We obtain:

$$(5.15) \quad \begin{aligned} & \sum_{j=1}^N \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} (u_j^{n+1} - u_j^{n-1}) - a \sum_{j=1}^N \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} (u_j^{n+1} - u_j^{n-1}) \\ & + \sum_{j=1}^N b_j \frac{y_j^{n+1} - y_j^{n-1}}{2\Delta t} (u_j^{n+1} - u_j^{n-1}) + \sum_{j=1}^N c_j \frac{(u_j^{n+1} - u_j^{n-1})^2}{2\Delta t} = 0. \end{aligned}$$

Estimation of the first term of (5.15) We firstly have:

$$(5.16) \quad \begin{aligned} \sum_{j=1}^N \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} (u_j^{n+1} - u_j^{n-1}) &= \sum_{j=1}^N \frac{u_j^{n+1} - u_j^n - (u_j^n - u_j^{n-1})}{\Delta t^2} (u_j^{n+1} - u_j^n + u_j^n - u_j^{n-1}) \\ &= \sum_{j=1}^N \left(\frac{u_j^{n+1} - u_j^n}{\Delta t} \right)^2 - \sum_{j=1}^N \left(\frac{u_j^n - u_j^{n-1}}{\Delta t} \right)^2 \\ &= 2(E_{k,u}^n - E_{k,u}^{n-1}). \end{aligned}$$

Estimation of the second term of (5.15). Using the same trick we have:

$$\begin{aligned} -a \sum_{j=1}^N \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} (u_j^{n+1} - u_j^{n-1}) &= -a \sum_{j=1}^N \frac{u_{j+1}^n - u_j^n - (u_j^n - u_{j-1}^n)}{\Delta x^2} (u_j^{n+1} - u_j^{n-1}) \\ &= -a \sum_{j=1}^N \frac{(u_{j+1}^n - u_j^n)(u_j^{n+1} - u_j^{n-1})}{\Delta x^2} \\ &\quad + a \sum_{j=1}^{N+1} \frac{(u_j^n - u_{j-1}^n)(u_j^{n+1} - u_j^{n-1})}{\Delta x^2}. \end{aligned}$$

So, by translation of index in the second term in the previous sum, we will have:

$$(5.17) \quad \begin{aligned} -a \sum_{j=1}^N \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} (u_j^{n+1} - u_j^{n-1}) &= -a \sum_{j=0}^N \frac{(u_{j+1}^n - u_j^n)(u_j^{n+1} - u_j^{n-1})}{\Delta x^2} \\ &\quad + a \sum_{j=0}^N \frac{(u_{j+1}^n - u_j^n)(u_{j+1}^{n+1} - u_{j+1}^{n-1})}{\Delta x^2} \\ &= a \sum_{j=0}^N \frac{(u_{j+1}^{n+1} - u_j^{n+1})(u_{j+1}^n - u_j^n)}{\Delta x^2} \\ &\quad - a \sum_{j=0}^N \frac{(u_{j+1}^{n-1} - u_j^{n-1})(u_{j+1}^n - u_j^n)}{\Delta x^2} \\ &= 2(E_{p,u}^n - E_{p,u}^{n-1}). \end{aligned}$$

Substituting (5.16) and (5.17) into (5.15), we get

$$(5.18) \quad 2 \left(E_{k,u}^n + E_{p,u}^n - E_{k,u}^{n-1} - E_{p,u}^{n-1} \right) + 2\Delta t \sum_{j=1}^N c_j \left(\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} \right)^2 + \sum_{j=1}^N b_j \frac{y_j^{n+1} - y_j^{n-1}}{2\Delta t} (u_j^{n+1} - u_j^{n-1}) = 0.$$

Similarly, by multiplying the second equation of (5.4) by $(y_j^{n+1} - y_j^{n-1})$, and using the same algebraic tricks, we will get:

$$(5.19) \quad 2 \left(E_{k,y}^n + E_{p,y}^n - E_{k,y}^{n-1} - E_{p,y}^{n-1} \right) - \sum_{j=1}^N b_j \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} (y_j^{n+1} - y_j^{n-1}) = 0.$$

Using the definition of the total discrete energy, (5.14), and the two equations (5.18), (5.19) leads to:

$$(5.20) \quad (\mathcal{E}^n - \mathcal{E}^{n-1}) + \Delta t \sum_{j=1}^N c_j \left(\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} \right)^2 = 0.$$

Consequently, the total discrete energy of system (5.4) is decreasing along time.

5.2. Numerical experiments: validation of the theoretical results. In every experiment, we have chosen:

$$u_0(x) = x(x-1), \quad u_1(x) = x(x-1), \quad y_0(x) = -x(x-1), \quad y_1(x) = -x(x-1).$$

The mesh size is chosen as $N = 100$ so that $\Delta x = 0.01$ and the time step is chosen as $\frac{\Delta t}{\Delta x} = CFL$.

In order to validate the different theoretical results, we have chosen different functions b and c synthesized in the list below:

- No coupling: $b_1(x) = 0$ or no dissipation $c_1(x) = 0$,
- Full coupling $b_2(x) = 1 \mathbb{1}_{(0,1)}(x)$ or full dissipation $c_2(x) = 1 \mathbb{1}_{(0,1)}(x)$,
- Partial coupling $b_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$ or partial dissipation $c_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$,
- Partial coupling $b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ or partial dissipation $c_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$,
- Partial coupling $b_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$ or partial dissipation $c_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$.

Combining the different choices of the coupling and damping functions in order to have or not $\omega_b \cap \omega_{c_+} \neq \emptyset$ will permit us to validate the theoretical results.

Let us notice that in the special case of the dimension 1, the geometric control condition GCC holds as soon as $\omega_{c_+} \neq \emptyset$.

5.2.1. Same propagation speed: $a = 1$. For every numerical simulation, the final time T is chosen as $T = 500$.

5.2.1.1. No damping: conservation of the total energy. Firstly, let us verify that when no damping are present, the discrete energy is conserved. We present in figure 2 the numerical experiment when $c = c_1 = 0$ and $b = b_3 = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$. Indeed, the total energy is conserved along time.

Remark 3. This numerical test where no damping is applied shows that without a damping term, the total energy is completely conserved. This fact suggests that the numerical scheme does not produce numerical dissipation. So the numerical behavior observed thereafter is only due to the considered model.

5.2.1.2. $\omega_b \cap \omega_{c_+} \neq \emptyset$. Exponential stability. Let us now verify the theoretical results when we suppose that $\omega_b \cap \omega_{c_+} \neq \emptyset$. For this sake, we present in figure 3, the total energy and the quantity $-\ln(E(t))/t$ versus time t for large time, where we have chosen $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$. This choice verifies the assumption that $\omega_b \cap \omega_{c_+} \neq \emptyset$ and in figure 3, it is shown that the energy is decreasing and an exponential decay is observed since it seems that $-\ln(E(t))/t$ tends to a constant as $t \rightarrow +\infty$. The final time profile confirms that u and y are small and the final profiles of u and y are smooth as expected (high frequency oscillations are exponentially dissipated).

5.2.1.3. $\omega_b \cap \omega_{c_+} = \emptyset$. *Unpredicted behavior.* At the numerical level, we are interested in the long time behavior of the solution (u, y) when we suppose that $\omega_b \cap \omega_{c_+} = \emptyset$. For this sake, we present in figure 4, the total energy and the quantity $-\ln(E(t))/t$ versus time t for large time, where we have chosen $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$. This choice verifies the assumption that $\omega_b \cap \omega_{c_+} = \emptyset$. In figure 4, it is shown that the energy is decreasing and an exponential decay is observed since it seems that $-\ln(E(t))/t$ tends to a constant as $t \rightarrow +\infty$. The final time profile confirms that u and y are small and again the couple of solution (u, y) is smooth. We have not considered this case in the theoretical study and this numerical result shows a similar behavior as in the case presented before.

So we decided to confirm this behavior by choosing $b = b_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$ and $c = c_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$. This choice verifies also the assumption that $\omega_b \cap \omega_{c_+} = \emptyset$. In figure 5, it is shown that the energy is decreasing and an exponential decay is observed since it seems that $-\ln(E(t))/t$ tends to a constant as $t \rightarrow +\infty$. The final time profile confirms that u and y are small and again the couple of solution (u, y) is smooth.

Remark 4. Let us notice that when the propagation speeds are the same for u and y , the final profiles of the solution u, y presented in figure 3(c), figure 4(c) and in figure 5(c) have the same form as the initial one, that is no spurious oscillations due to high frequency are present.

5.2.2. *Different propagation speed: $a > 1$.* We investigate now the long time behavior of (u, y) when the propagation speeds are different and specifically when $a > 1$. So we have chosen to take $a = 2$. We firstly investigate the case when the propagation speed for u is greater than the one of y namely $a > 1$. We have chosen $a = 2$.

5.2.2.1. $\omega_b \cap \omega_{c_+} \neq \emptyset$. *Polynomial stability.* Let us now verify the theoretical results when we suppose that $\omega_b \cap \omega_{c_+} \neq \emptyset$. For this sake, we present in figure 6, the total energy where we have chosen $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$.

When taking as final time $T = 500$, it seems that the energy does not tend to zero as shown in figure 6(a). This is the reason why we have chosen for the case when $a \neq 1$ as final time $T = 500\,000$ and figure 6(b) shows that the energy finally goes to zero.

To explore the speed of convergence to zero, we have plotted in figure 7 $-\ln(E(t))/t$, $t \cdot E(t)$ and finally $-\ln(E(t))/\ln(t)$ versus t . Figure 7(a) shows clearly that $-\ln(E(t))/t$ tends to zero and it permits to conclude that $E(t)$ tends to zero slower than an exponential. Figure 7(b) permits to conclude that $E(t)$ tends to zero faster than $1/t$. Finally figure 7(c) shows that $E(t)$ tends to zero as $1/t^\alpha$ with $\alpha \simeq 1.4$.

The final time profile presented in figure 7(d) confirms that u and y are small but it shows also that high frequencies for the unknown y are not completely controlled.

5.2.2.2. $\omega_b \cap \omega_{c_+} = \emptyset$. *Unpredicted behavior.* At the numerical level, we are interested in the long time behavior of the solution (u, y) when we suppose that $\omega_b \cap \omega_{c_+} = \emptyset$. For this sake, we present in figure 8, the total energy where we have chosen $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$.

Again, when taking as final time $T = 500$, it seems that the energy does not tend to zero as shown in figure 8(a). Taking as final time $T = 500\,000$, figure 8(b) shows that the energy goes finally to zero.

To explore the speed of convergence to zero, we have plotted in figure 9 $-\ln(E(t))/t$, $t \cdot E(t)$ and finally $-\ln(E(t))/\ln(t)$ versus t . Figure 9(a) shows clearly that $-\ln(E(t))/t$ tends to zero and it permits to conclude that $E(t)$ tends to zero slower than an exponential but figure 9(b) shows that $E(t)$ tends to zero slower than $1/t$. This fact is confirmed by figure 9(c) which shows that $E(t)$ tends to zero as $1/t^\alpha$ with $\alpha \simeq 0.9$. Eventually, taking a larger time could conclude that the convergence is like $1/t$.

Again, the final time profile presented in figure 9(d) confirms that u and y are small but it shows also that high frequencies for the unknown y are not completely controlled.

As for the case when the two propagation speeds were identical this results was not predicted by the theoretical results.

So we decided to confirm this behavior by choosing $b = b_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$ and $c = c_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$. Again, when taking as final time $T = 500$, it seems that the energy does not tends to zero as shown in figure 10(a). Taking as final time $T = 500\ 000$, figure 8(b) shows that the energy goes finally to zero.

To explore the speed of convergence to zero, we have plotted in figure 11 $-\ln(E(t))/t$, $t \cdot E(t)$ and finally $-\ln(E(t))/\ln(t)$ versus t . Figure 11(a) shows clearly that $-\ln(E(t))/t$ tends to zero and it permits to conclude that $E(t)$ tends to zero slower than an exponential and figure 11(b) permits to conclude that the convergence is faster than $1/t$. Finally figure 11(c) shows that $E(t)$ tends to zero as $1/t^\alpha$ with $\alpha \simeq 1.19$.

Remark 5. The final time profile presented in figure 7(d), figure 9(d) and figure 11(d) confirms that u and y are small but it shows also that high frequencies for the unknown y are not completely controlled.

5.2.3. *Different propagation speed: $a < 1$.* When $a \neq 1$, in order to see if the same behavior occurs no matter if a is greater or less than 1, we investigate now the long time behavior of (u, y) when the propagation speeds is less than the one of y namely $a < 1$. We have chosen $a = 0.5$.

5.2.3.1. $\omega_b \cap \omega_{c_+} \neq \emptyset$. *Polynomial stability.* Let us now verify the theoretical results when we suppose that $\omega_b \cap \omega_{c_+} \neq \emptyset$. For this sake, we present in figure 12(a), the total energy where we have chosen $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$.

When taking as final time $T = 500$, it seems that the energy does not tend to zero as shown in figure 12(a). Taking as final time $T = 500\ 000$, figure 12(b) shows that the energy goes finally to zero.

To explore the speed of convergence to zero, we have plotted in figure 13 $-\ln(E(t))/t$, $t \cdot E(t)$ and finally $-\ln(E(t))/\ln(t)$ versus t . Figure 13(a) shows clearly that $-\ln(E(t))/t$ tends to zero slower than an exponential. Figure 13(b) permits to conclude that $E(t)$ tends to zero faster than $1/t$. Finally figure 13(c) shows that $E(t)$ tends to zero as $1/t^\alpha$ with $\alpha \simeq 1.5$.

The final time profile confirms that u and y are small but it shows also that high frequencies for the unknown y are not completely controlled.

5.2.3.2. $\omega_b \cap \omega_{c_+} = \emptyset$: *Unpredicted behavior.* Again, the numerical level, we are interested in the long time behavior of the solution (u, y) when we suppose that $\omega_b \cap \omega_{c_+} = \emptyset$. For this sake, we present in figure 14(a), the total energy where we have chosen $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$.

Again, when taking as final time $T = 500$, it seems that the energy does not tend to zero as shown in figure 14(a). Taking as final time $T = 500\ 000$, figure 14(b) shows that the energy finally goes to zero.

To explore the speed of convergence to zero, we have plotted in figure 15 $-\ln(E(t))/t$, $t \cdot E(t)$ and finally $-\ln(E(t))/\ln(t)$ versus t . Figure 15(a) shows clearly that $-\ln(E(t))/t$ tends to zero slower than an exponential. But figure 15(b) shows that $E(t)$ tends to zero faster than $1/t$. Finally figure 15(c) shows that $E(t)$ tends to zero as $1/t^\alpha$ with $\alpha \simeq 1.25$.

Again, the final time profile presented in figure 15(d) confirms that u and y are small but it shows also that high frequencies for the unknown y are not completely controlled. This result was not predicted by the theoretical results.

So we decided to confirm this behavior by choosing $b = b_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$ and $c = c_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$. Again, when taking as final time $T = 500$, it seems that the energy does not tend to zero as shown in figure 16(a). Taking as final time $T = 500\ 000$, figure 16(b) shows that the energy goes finally to zero.

To explore the speed of convergence to zero, we have plotted in figure 17 $-\ln(E(t))/t$, $t \cdot E(t)$ and finally $-\ln(E(t))/\ln(t)$ versus t . Figure 17(a) shows clearly that $-\ln(E(t))/t$ tends to zero and it permits to conclude that $E(t)$ tends to zero slower than an exponential but figure 17(b) shows that $E(t)$ tends to zero faster than $1/t$. Finally figure 17(c) shows that $E(t)$ tends to zero as $1/t^\alpha$ with $\alpha \simeq 1.15$.

Again, the final time profile presented in figure 17(d) confirms that u and y are small but it shows also that high frequencies for the unknown y are not completely controlled.

Remark 6. The final time profile presented in figure 13(d) , figure 15(d) and figure 17(d) confirms that u and y are small but it shows also that high frequencies for the unknown y are not completely controlled.

Remark 7. When the propagation speeds are not equal, the solution (u, y) has the same behavior no matter if $a > 1$ or $a < 1$. The polynomial convergence is numerically better than $1/t$ but it will be probably be $1/t$ for greater time. For reason of computation time, we did not perform very long simulation to confirm.

ACKNOWLEDGMENTS

The authors would like to thank Professor Kaïs Ammari for fruitful discussions and insights regarding the results of the article.

The authors thank also the referees for very useful comments.

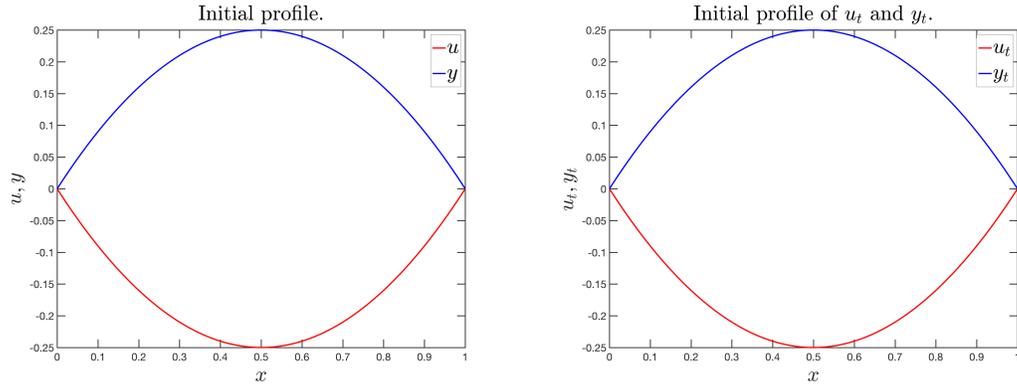
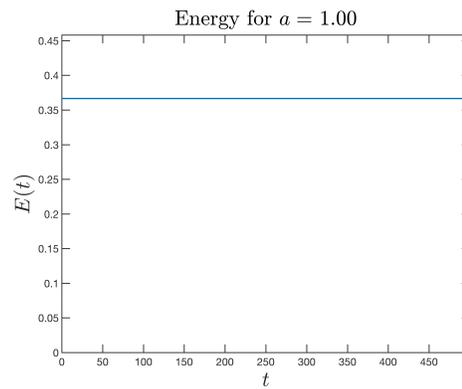
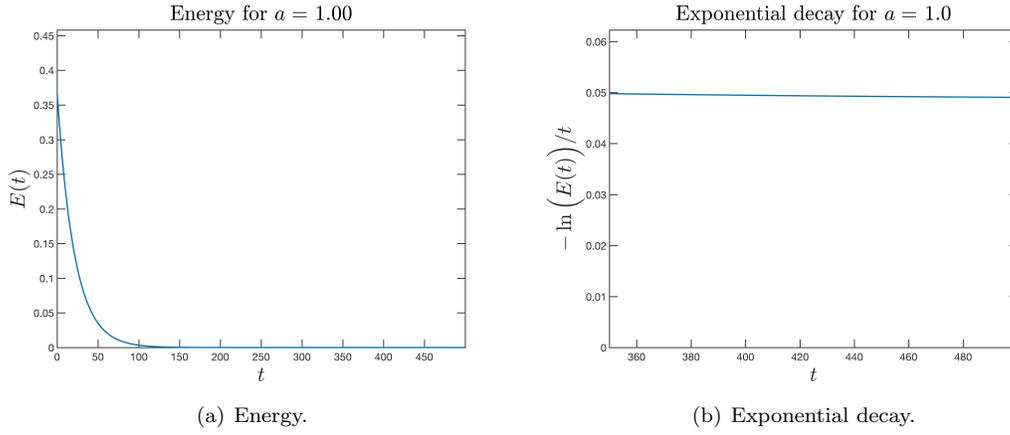


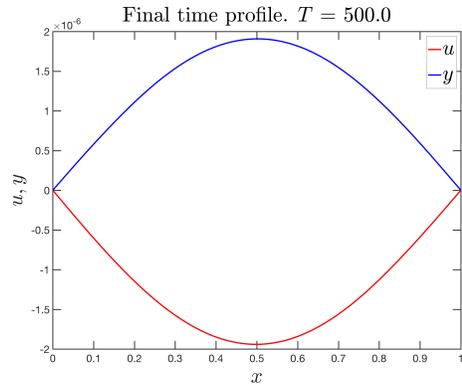
FIGURE 1. Initial profiles

FIGURE 2. No damping : $c = c_1 = 0$, partial coupling $b = b_3 = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$



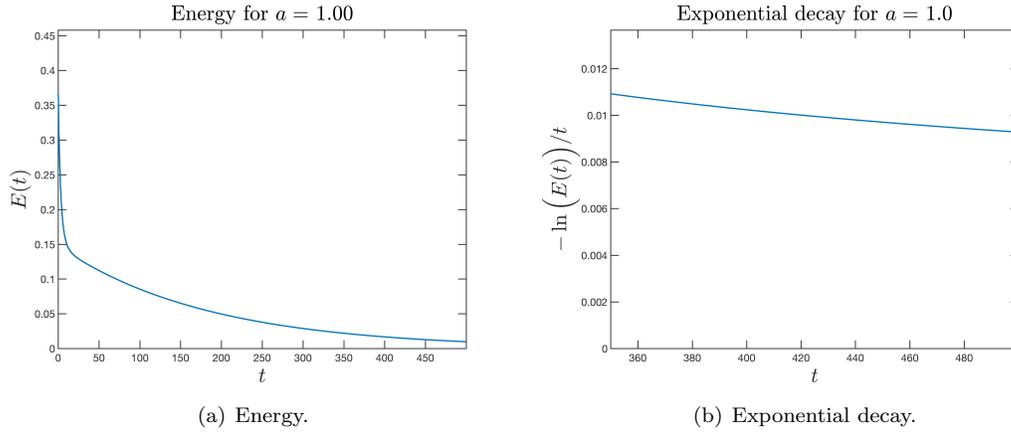
(a) Energy.

(b) Exponential decay.



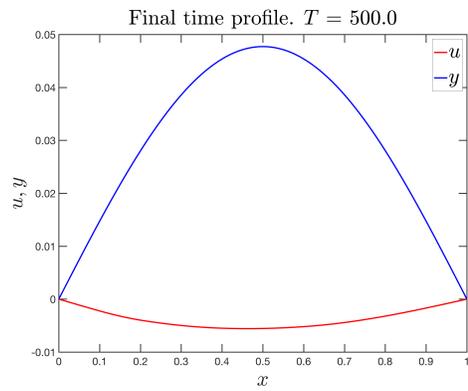
(c) Final time profile.

FIGURE 3. Long time behavior when $\omega_b \cap \omega_{c_+} \neq \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$.



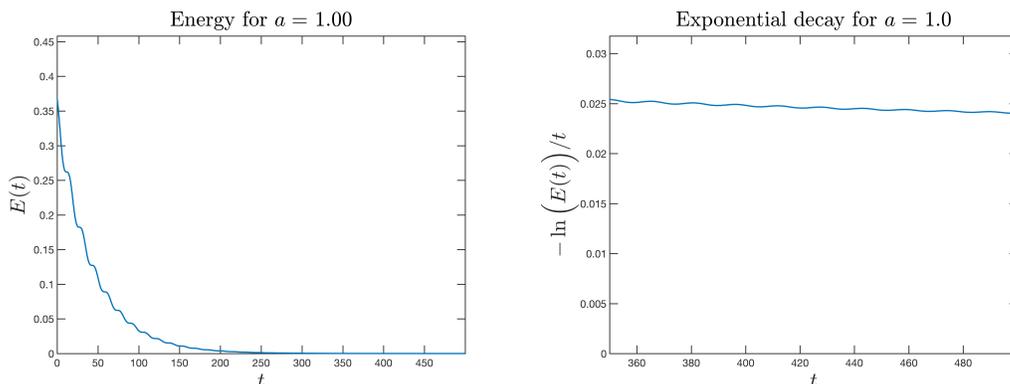
(a) Energy.

(b) Exponential decay.



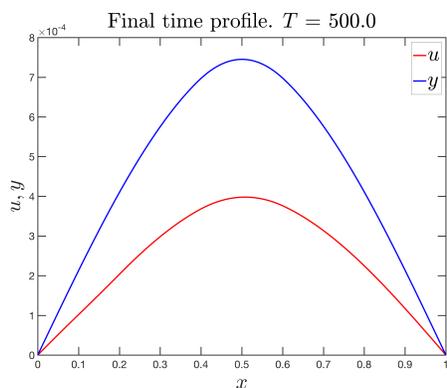
(c) Final time profile.

FIGURE 4. Long time behavior when $\omega_b \cap \omega_{c_+} = \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$.



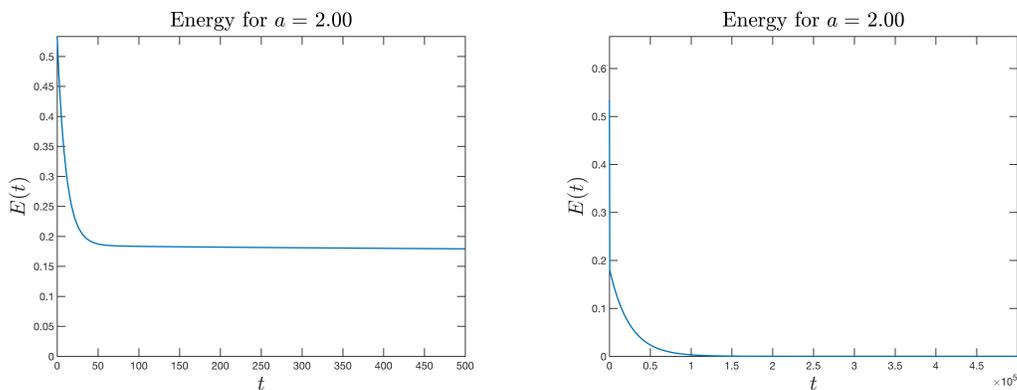
(a) Energy.

(b) Exponential decay.



(c) Final time profile.

FIGURE 5. Long time behavior when $\omega_b \cap \omega_{c_+} = \emptyset$. $b = b_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$ and $c = c_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$.



(a) Final time $T = 500$.

(b) Final time $T = 500\ 000$.

FIGURE 6. Energy when $\omega_b \cap \omega_{c_+} \neq \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$.

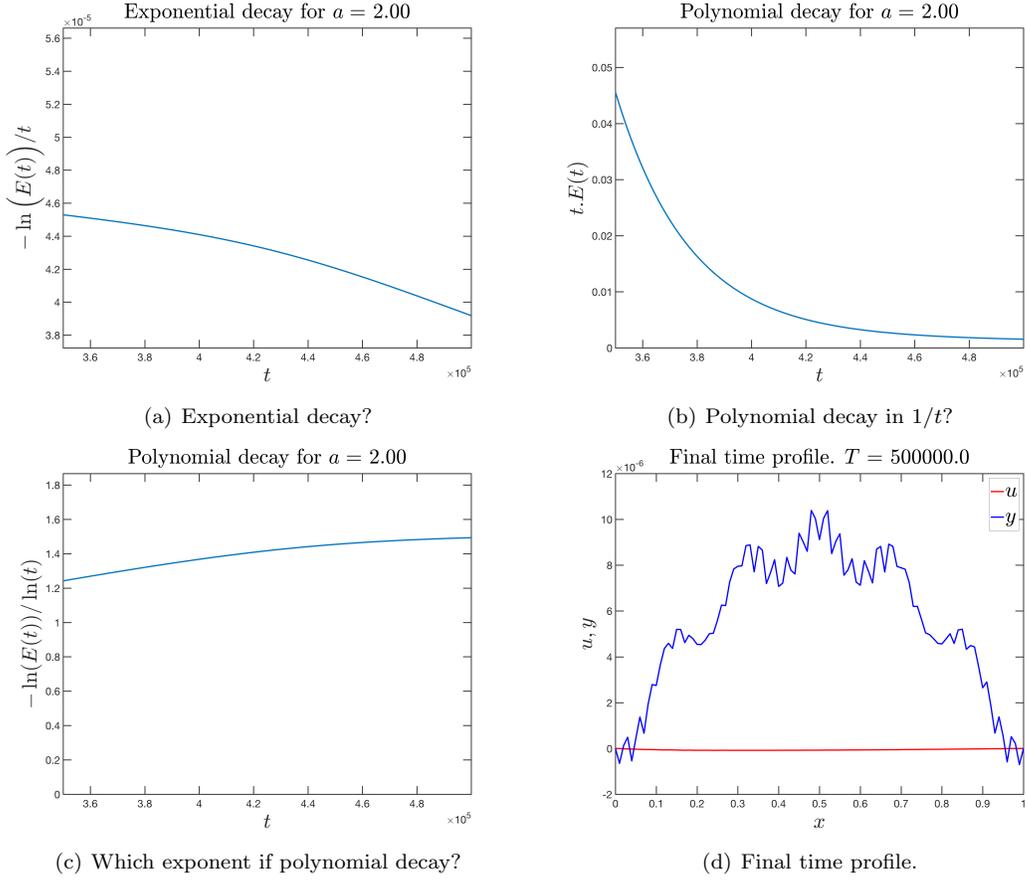


FIGURE 7. Long time behavior when $\omega_b \cap \omega_{c_+} \neq \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$.

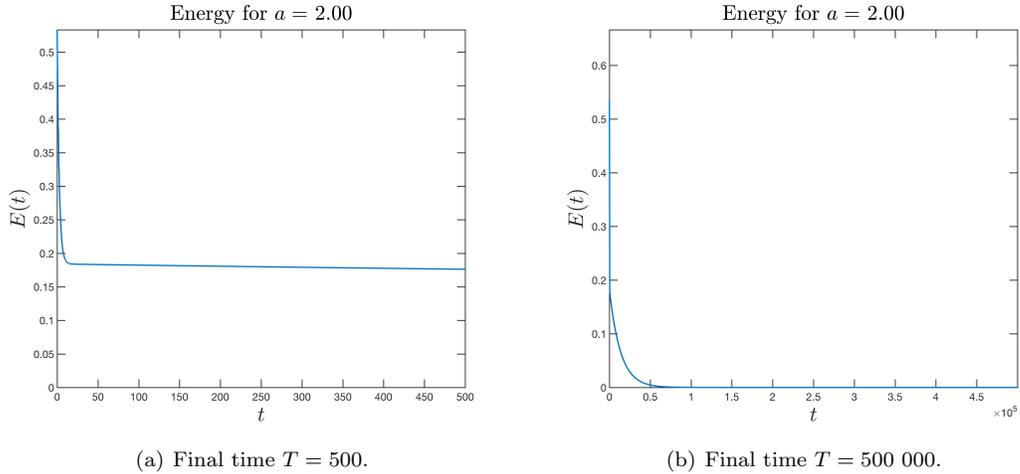


FIGURE 8. Energy when $\omega_b \cap \omega_{c_+} = \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$.

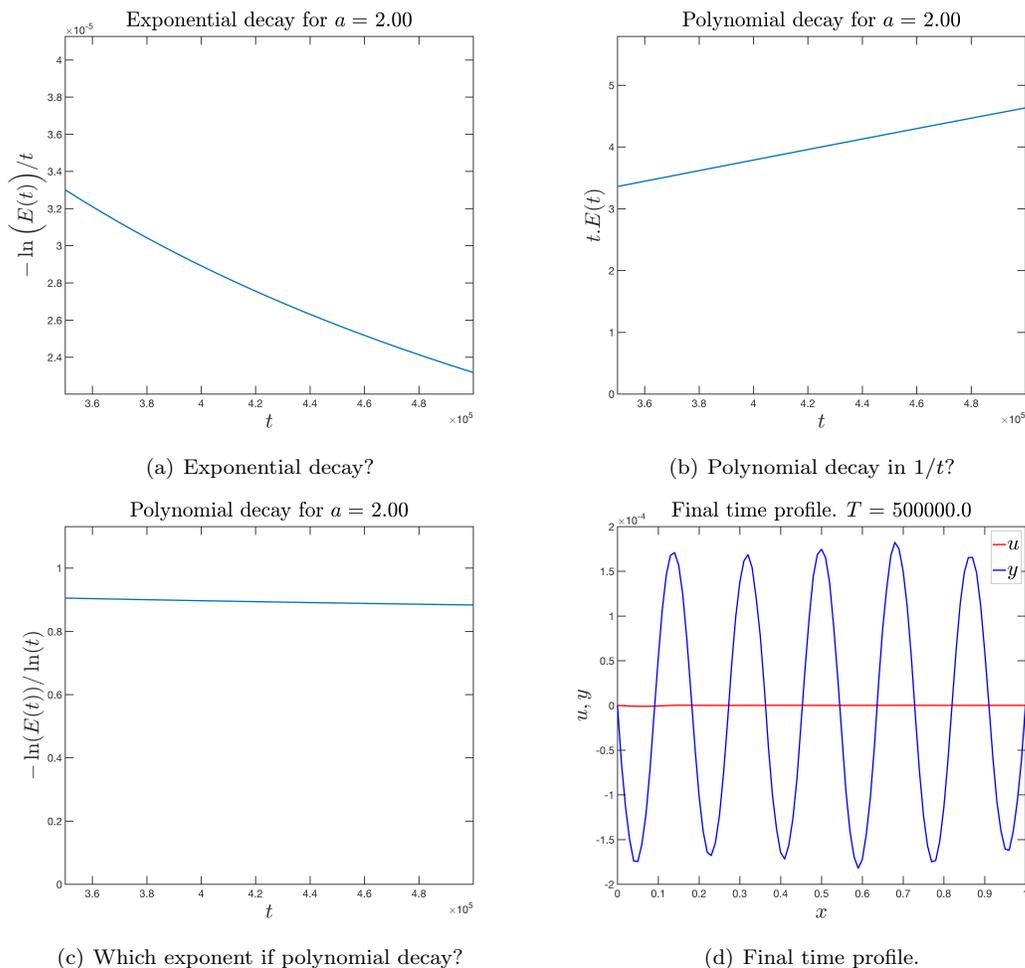


FIGURE 9. Long time behavior when $\omega_b \cap \omega_{c_+} = \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$.

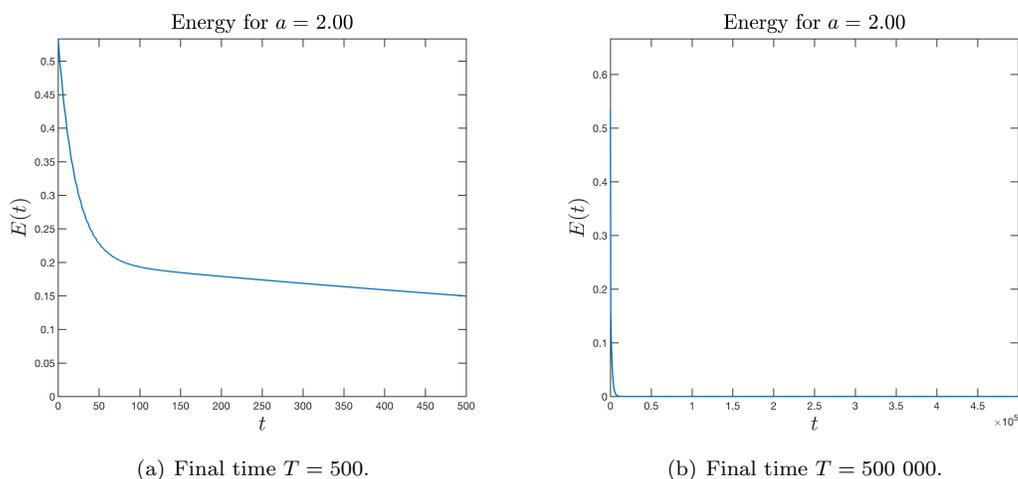


FIGURE 10. Energy when $\omega_b \cap \omega_{c_+} \neq \emptyset$. $b = b_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$.

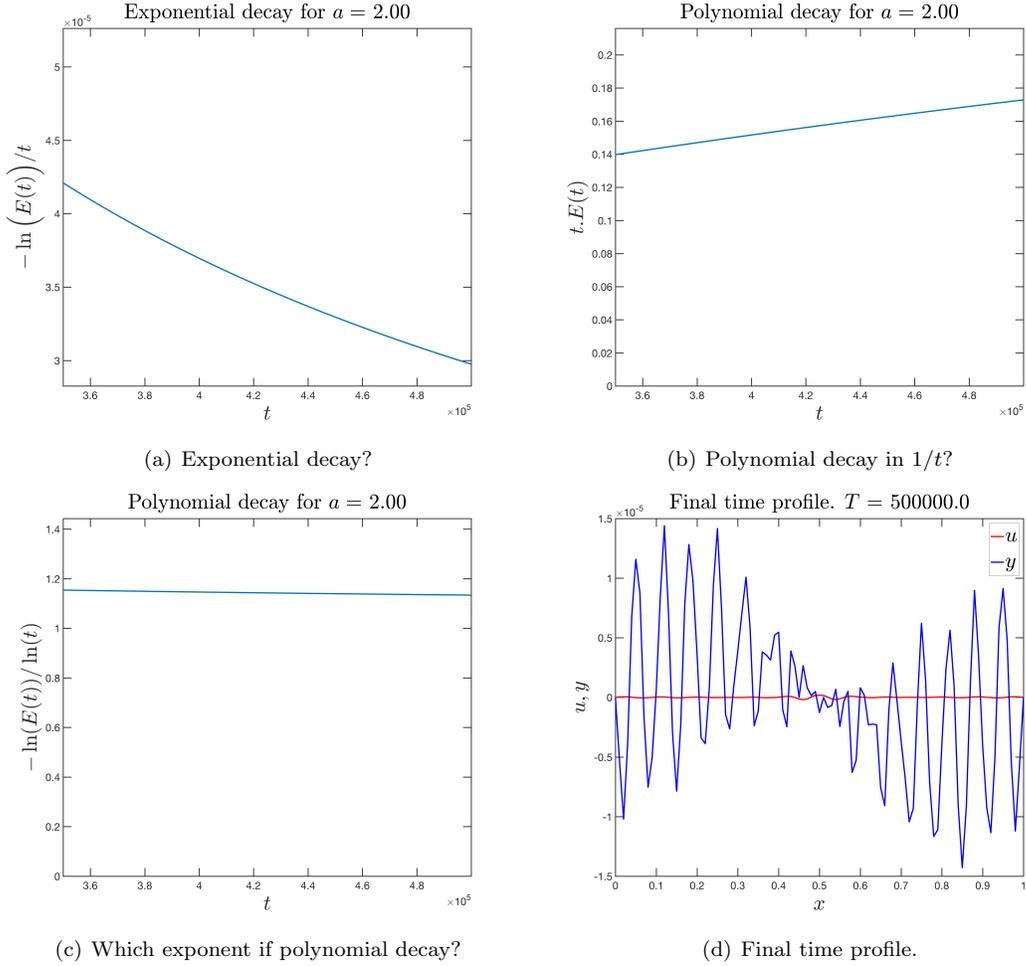


FIGURE 11. Long time behavior when $\omega_b \cap \omega_{c_+} = \emptyset$. $b = b_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$ and $c = c_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$.

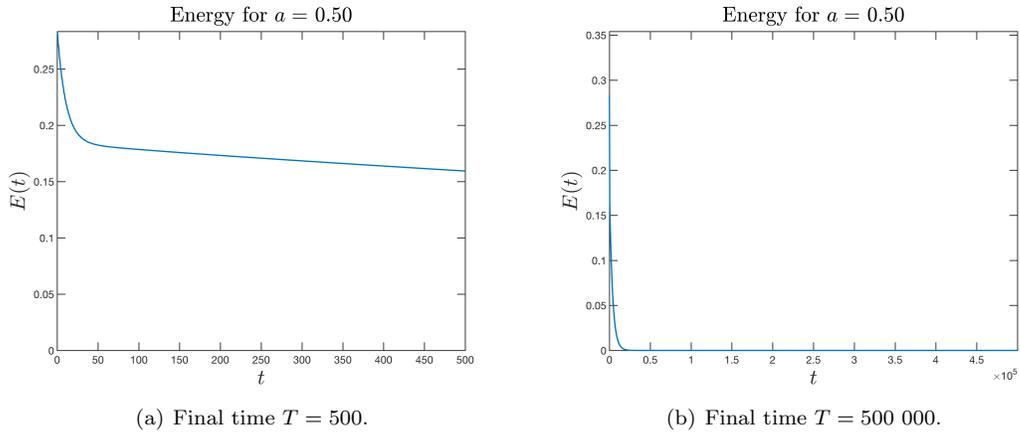


FIGURE 12. Energy when $\omega_b \cap \omega_{c_+} \neq \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$.

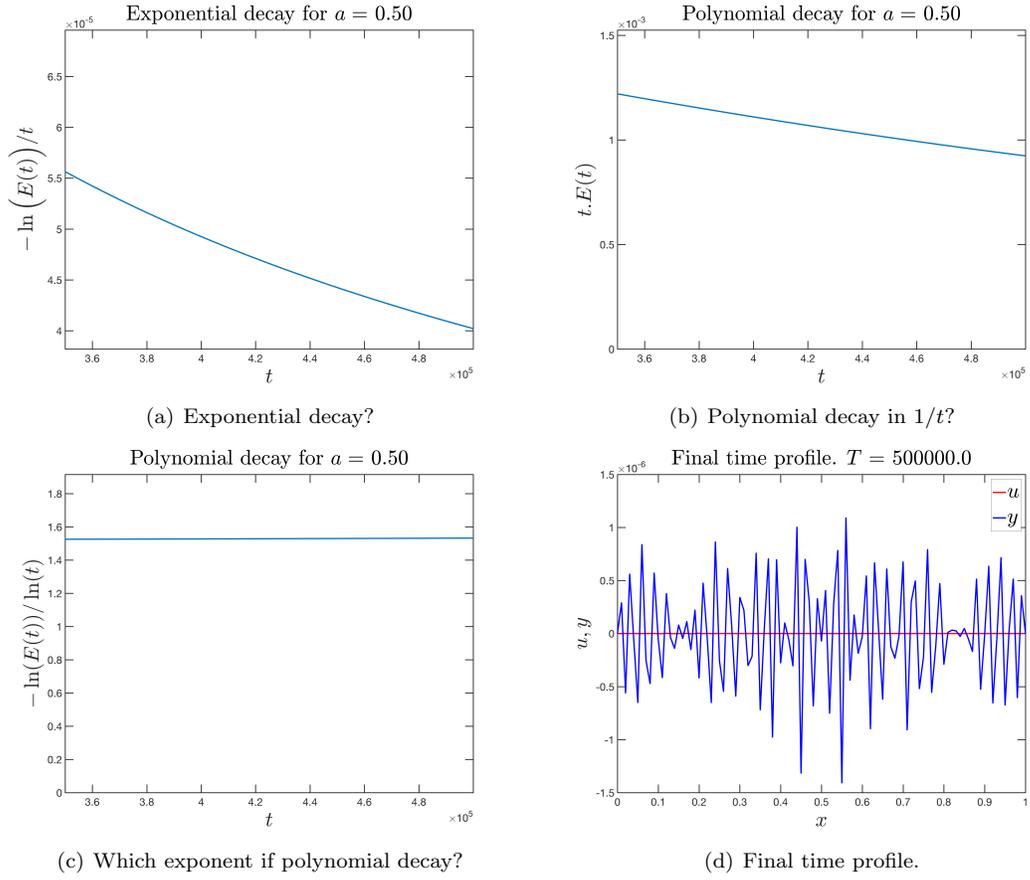


FIGURE 13. Long time behavior when $\omega_b \cap \omega_{c_+} \neq \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$.

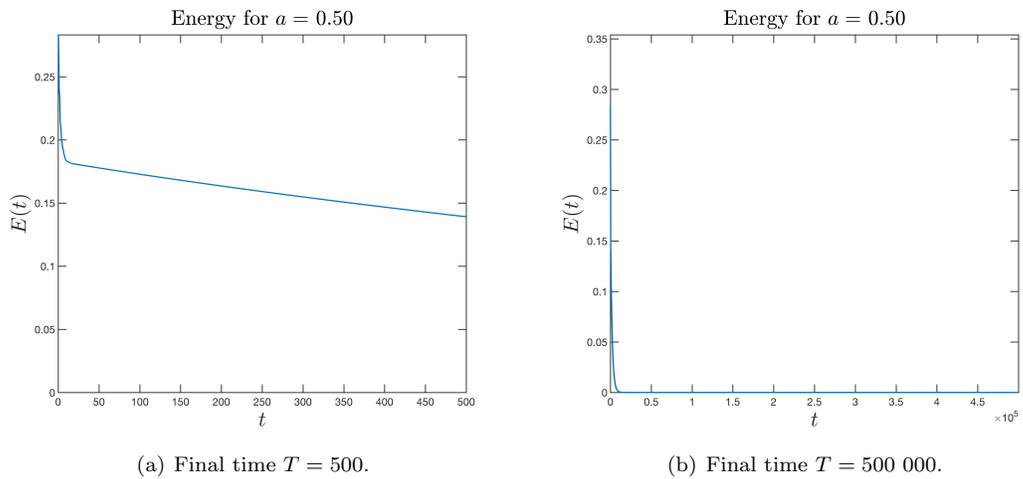
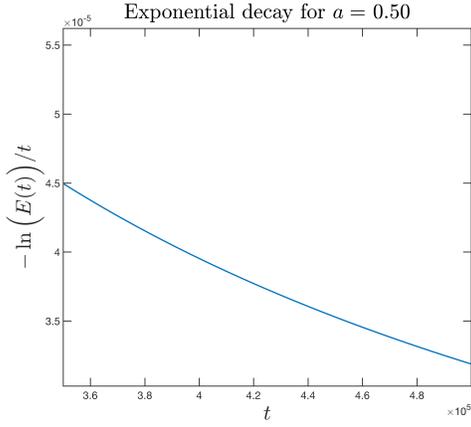
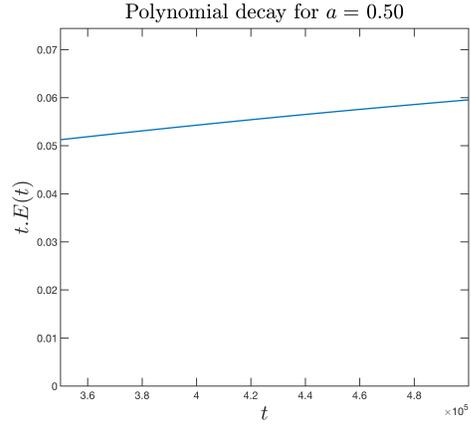
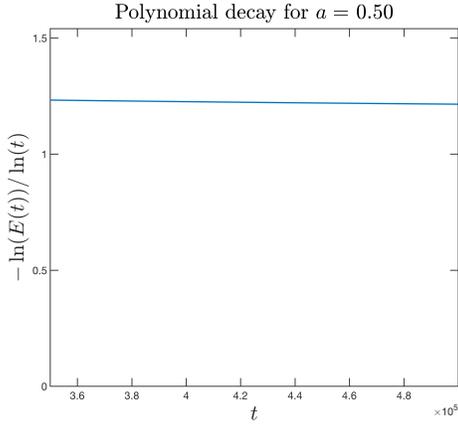


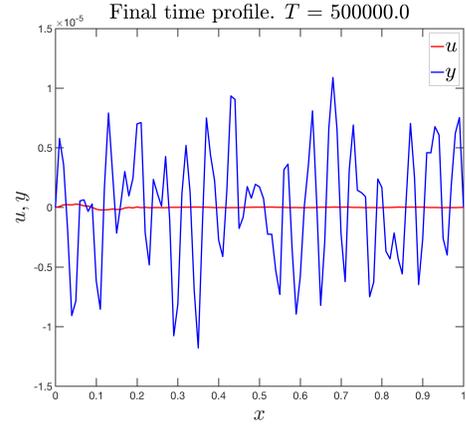
FIGURE 14. Energy when $\omega_b \cap \omega_{c_+} = \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$.



(a) Exponential decay?

(b) Polynomial decay in $1/t$?

(c) Which exponent if polynomial decay?



(d) Final time profile.

FIGURE 15. Long time behavior when $\omega_b \cap \omega_{c_+} = \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$.

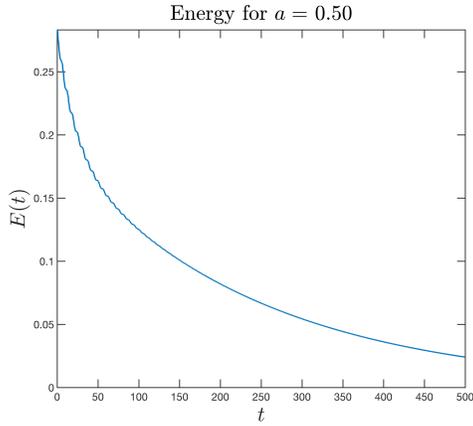
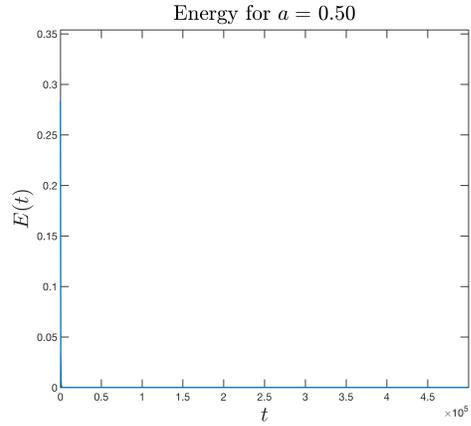
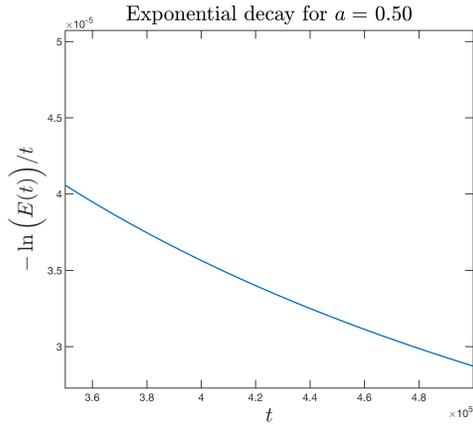
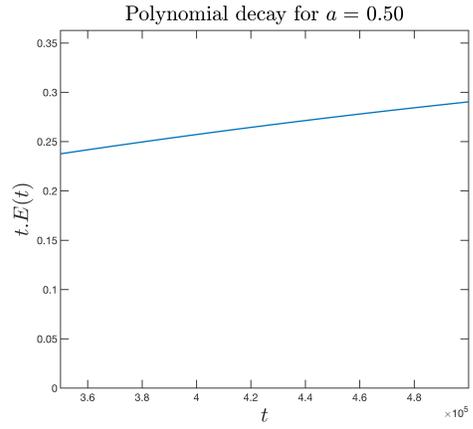
(a) Final time $T = 500$ (b) Final time $T = 500\,000$.

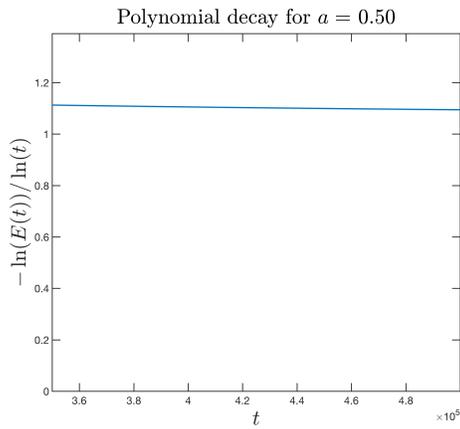
FIGURE 16. Energy when $\omega_b \cap \omega_{c_+} = \emptyset$. $b = b_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$.



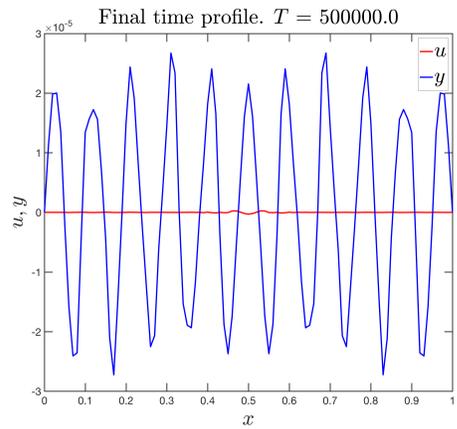
(a) Exponential decay?



(b) Polynomial decay in $1/t$?



(c) Which exponent if polynomial decay?



(d) Final time profile?

FIGURE 17. Long time behavior when $\omega_b \cap \omega_{c_+} = \emptyset$. $b = b_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$ and $c = c_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$.

REFERENCES

- [1] F. Alabau, Observabilité frontière indirecte de systèmes faiblement couplés, *C. R. Acad. Sci. Paris Sér. I Math.*, **333** (2001), 645–650.
- [2] F. Alabau-Boussouira, A two-level energy method for indirect boundary observability and controllability of weakly coupled hyperbolic systems, *SIAM Journal on Control and Optimization*, **42** (2003), 871–906.
- [3] F. Alabau-Boussouira, Convexity and weighted integral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems, *Applied Mathematics and Optimization*, **51** (2005), 61–105.
- [4] F. Alabau-Boussouira and M. Léautaud, Indirect stabilization of locally coupled wave-type systems, *ESAIM Control Optim. Calc. Var.*, **18** (2012), 548–582.
- [5] F. Alabau-Boussouira and M. Léautaud. Indirect controllability of locally coupled wave-type systems and applications. *Journal de Mathématiques Pures et Appliquées*, **99** (5) : 544–576, 2013.
- [6] F. Alabau-Boussouira, Z. Wang and L. Yu, A one-step optimal energy decay formula for indirectly nonlinearly damped hyperbolic systems coupled by velocities, *ESAIM Control Optim. Calc. Var.*, **23** (2017), 721–749.
- [7] W. F. Ames. *Numerical methods for partial differential equations*. Computer Science and Scientific Computing. Academic Press, Inc., Boston, MA, third edition, 1992.
- [8] F. Ammar-Khodja, A. Benabdallah and C. Dupaix, Null-controllability of some reaction–diffusion systems with one control force, *Journal of Mathematical Analysis and Applications*, **320** (2006), 928–943.
- [9] C. Bardos, G. Lebeau and J. Rauch, Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary, *SIAM J. Control Optim.*, **30** (1992), 1024–1065.
- [10] F. Conrad and B. Rao. Decay of solutions of the wave equation in a star-shaped domain with nonlinear boundary feedback. *Asymptotic Anal.*, **7**, (1993), 159–177.
- [11] A. Haraux, Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps, *Portugaliae mathematica*, **46** (1989), 245–258.
- [12] F. L. Huang, Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, *Ann. Differential Equations*, **1** (1985), 43–56.
- [13] C. Kassem, A. Mortada, L. Toufayli and A. Wehbe, Stabilisation locale indirecte d’un système N-d de deux équations d’ondes couplées sous conditions géométriques, *C. R. Acad. Sci. Paris Sér. I Math.*, To appear.
- [14] V. Komornik and P. Loreti, *Fourier Series in Control Theory*, Springer Monographs in Mathematics. Springer, 2005.
- [15] J. L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués: Perturbations*, Recherches en Mathématiques Appliquées. Masson, 1988.
- [16] J. L. Lions, Exact controllability, stabilization and perturbations for distributed systems, *SIAM Review*, **30** (1988), 1–68.
- [17] K. Liu, Locally distributed control and damping for the conservative systems, *SIAM J. Control Optim.*, **35** (1997), 1574–1590.
- [18] Z. Liu and B. Rao, Frequency domain approach for the polynomial stability of a system of partially damped wave equations, *Journal of Mathematical Analysis and Applications*, **335** (2007), 860–881.
- [19] Z. Liu and B. Rao, A spectral approach to the indirect boundary control of a system of weakly coupled wave equations, *Discrete Contin. Dyn. Syst.*, **23** (2009), 399–414.
- [20] J. Prüss, On the spectrum of C_0 -semigroups, *Trans. Amer. Math. Soc.*, **284** (1984), 847–857.
- [21] J. Rauch, M. Taylor and R. Phillips, Exponential decay of solutions to hyperbolic equations in bounded domains, *Indiana University Mathematics Journal*, **24** (1974), 79–86.
- [22] A. Wehbe et W. Youssef. Observabilité et contrôlabilité exacte indirecte interne par un contrôle localement distribué de systèmes d’équations couplées, *C.R. Acad. Sci. Paris, Ser. I*, **348** (2010) 1169–1173.
- [23] A. Wehbe and W. Youssef. Indirect locally internal observability of weakly coupled wave equations, *Differential Equations and Applications-DEA*, Vol. 3, No. 3, (2011), 449–462.

LABORATOIRE DE MATHÉMATIQUES UMR 5127 CNRS & UNIVERSITÉ DE SAVOIE MONT BLANC, CAMPUS SCIENTIFIQUE, 73376 LE BOURGET DU LAC CEDEX, FRANCE
 Email address: stephane.gerbi@univ-smb.fr

UNIVERSITÉ LIBANAISE, EDST, EQUIPE EDP-AN, HADATH, BEIRUT, LEBANON
 Email address: shiraz.kassem@hotmail.com

UNIVERSITÉ LIBANAISE, EDST, EQUIPE EDP-AN, HADATH, BEIRUT, LEBANON
 Email address: amina_mortada2010@hotmail.com

UNIVERSITÉ LIBANAISE, FACULTÉ DES SCIENCES 1, EDST, EQUIPE EDP-AN, HADATH, BEIRUT, LEBANON
 Email address: ali.wehbe@ul.edu.lb