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ENERGY MINIMIZERS FOR AN ASYMPTOTIC MEMS MODEL WITH HETEROGENEOUS DIELECTRIC PROPERTIES

PHILIPPE LAURENÇOT, KATERINA NIK, AND CHRISTOPH WALKER

ABSTRACT. A model for a MEMS device, consisting of a fixed bottom plate and an elastic plate, is studied. It was derived in a previous work as a reinforced limit when the thickness of the insulating layer covering the bottom plate tends to zero. This asymptotic model inherits the dielectric properties of the insulating layer. It involves the electrostatic potential in the device and the deformation of the elastic plate defining the geometry of the device. The electrostatic potential is given by an elliptic equation with mixed boundary conditions in the possibly non-Lipschitz region between the two plates. The deformation of the elastic plate is supposed to be a critical point of an energy functional which, in turn, depends on the electrostatic potential due to the force exerted by the latter on the elastic plate. The energy functional is shown to have a minimizer giving the geometry of the device. Moreover, the corresponding Euler-Lagrange equation is computed and the maximal regularity of the electrostatic potential is established.

1. INTRODUCTION

The modeling and analysis of microelectromechanical systems (MEMS) has attracted a lot of interest in recent years, see, e.g., [10, 11, 21, 22, 30, 31, 35] and the references therein. Idealized devices often consist of a rigid dielectric ground plate above which an elastic dielectric plate is suspended. Applying a voltage difference between the two plates induces a competition between attractive electrostatic Coulomb forces and restoring mechanical forces, the latter resulting from the elasticity of the upper plate. When electrostatic forces dominate mechanical forces, the two plates may come into contact, a phenomenon usually referred to as pull-in instability or touchdown. From a mathematical point of view, this phenomenon may be accounted for in different ways. In fact, in most mathematical models considered so far in the MEMS literature, the pull-in instability is revealed as a singularity in the corresponding mathematical equations which coincides with a breakdown of the model, see [10, 21, 31] and the references therein. There is a close connection between the singular character of the touchdown and the fact that the modeling does not account for the thickness of the plates. Indeed, coating the ground plate with a thin insulating layer prevents a direct contact of the plates, so that a touchdown of the elastic plate on the insulating layer does not interrupt the operation of the device [6, 17, 24, 25]. Due to the presence of this layer, the MEMS device features heterogeneous dielectric properties (with a jump of the permittivity at the interface separating the coated ground plate and the free space beneath the elastic plate) and the electrostatic potential solves a free boundary transmission problem in the non-smooth domain enclosed between the two plates [17]. The shape of the domain itself is given by a partial differential equation governing the deflection of the elastic plate from rest, which, in turn, involves the electrostatic force exerted on the latter. The mathematical treatment of such a model is rather

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complex, see [17, Section 5] and [18]. It is thus desirable to derive simpler and more tractable models. As the modeling involves two small spatial scales – the aspect ratio ε of the device and the thickness d of the insulating layer – a variety of reduced models may be obtained. For instance, the assumption of a vanishing aspect ratio of the device, when either the ratio d/ε has a positive finite limit [2, 6, 20, 24, 25] or converges to zero, see [10, 30, 31] and the references therein, leads to a model which no longer involves a free boundary. Indeed, in that case, the electrostatic potential can be computed explicitly in terms of the deflection of the elastic plate and the model reduces to a single equation for the deflection, with the drawback that some important information on the electrostatic potential may thus be lost.

For this reason an intermediate model is derived in [16] by letting only the thickness of the insulating layer d go to zero (keeping the aspect ratio of the device of order one). Assuming an appropriate scaling of the dielectric permittivity in dependence on the layer's thickness (in order to keep relevant information of the dielectric heterogeneity of the device) and using a Gamma convergence approach, the resulting energy, which is the building block of the model, is computed. The next step is the mathematical analysis of the thus derived model, in which stationary solutions correspond to critical points of the energy, while the dynamics is described by the gradient flow associated with the energy. The aim of the present work is to show the existence of a particular class of stationary solutions, which are additionally energy minimizers, and to identify the corresponding Euler-Lagrange equations.

Let us provide beforehand a more precise description of the MEMS configuration under study. We consider an idealized MEMS device composed of two rectangular two-dimensional dielectric plates: a fixed ground plate above which an elastic plate, with the same shape at rest, is suspended and clamped in only one direction while free in the other. We assume that the device is homogeneous in the free direction and that it is thus sufficient to consider only a cross-section of the device orthogonal to the free direction. The shape of the ground plate and that of the elastic plate at rest are then represented by $D := (-L, L) \subset \mathbb{R}$, the ground plate being located at $z = -H$ with $H > 0$ and covered with an infinitesimally thin dielectric layer (in consistency with the aforementioned limit). The vertical deflection of the elastic plate from its rest position at $z = 0$ is described by a function $u : \bar{D} \rightarrow [-H, \infty)$ satisfying the clamped boundary conditions

$$u(\pm L) = \partial_x u(\pm L) = 0, \quad (1.1)$$

so that its graph

$$\mathfrak{G}(u) := \{(x, u(x)) : x \in \bar{D}\}$$

represents the elastic plate and

$$\Omega(u) := \{(x, z) \in D \times \mathbb{R} : -H < z < u(x)\}$$

is the free space between the elastic plate and the ground plate. Since we do not exclude the possibility of contact between the two plates, we introduce the *coincidence set*

$$\mathcal{C}(u) := \{x \in D : u(x) = -H\}$$

and let

$$\Sigma(u) := \{(x, -H) : x \in D, u(x) > -H\} = (D \setminus \mathcal{C}(u)) \times \{-H\}$$

be the part of the ground plate which is not in contact with the elastic plate. A touchdown of the elastic plate on the ground plate corresponds to a non-empty coincidence set, in which case $\Sigma(u)$ is a strict subset of $D \times \{-H\}$. Note that the free space $\Omega(u)$ then has a different geometry with at least two connected components, which may not be Lipschitz domains due to cusps (independent of the smoothness of the function u). In Figure 1 the different situations with empty and non-empty coincidence sets are depicted.

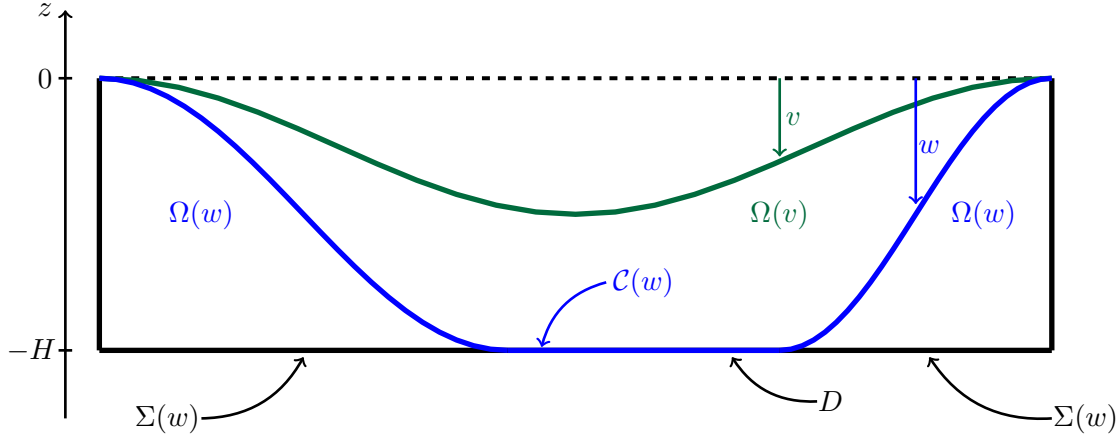


FIGURE 1. Geometry of $\Omega(u)$ for a state $u = v$ with empty coincidence set (green) and a state $u = w$ with non-empty coincidence set (blue).

As already mentioned, the building block of the model studied in this paper is the total energy $E(u)$ of the device at a state u given by

$$E(u) := E_m(u) + E_e(u)$$

and derived in [16] in the limit of an infinitesimally small insulating layer. It consists of the mechanical energy $E_m(u)$ and the electrostatic energy $E_e(u)$. The former is given by

$$E_m(u) := \frac{\beta}{2} \|\partial_x^2 u\|_{L_2(D)}^2 + \left(\frac{\tau}{2} + \frac{\alpha}{4} \|\partial_x u\|_{L_2(D)}^2 \right) \|\partial_x u\|_{L_2(D)}^2$$

with $\beta > 0$ and $\tau, \alpha \geq 0$, taking into account bending and external- and self-stretching effects of the elastic plate. The electrostatic energy is

$$E_e(u) := -\frac{1}{2} \int_{\Omega(u)} |\nabla \psi_u|^2 \, d(x, z) - \frac{1}{2} \int_D \sigma(x) |\psi_u(x, -H) - \mathfrak{h}_u(x)|^2 \, dx, \quad (1.2)$$

where ψ_u is the electrostatic potential in the device and solves the elliptic equation with mixed boundary conditions

$$\Delta \psi_u = 0 \quad \text{in } \Omega(u), \quad (1.3a)$$

$$\psi_u = h_u \quad \text{on } \partial\Omega(u) \setminus \Sigma(u), \quad (1.3b)$$

$$-\partial_z \psi_u + \sigma(\psi_u - \mathfrak{h}_u) = 0 \quad \text{on } \Sigma(u). \quad (1.3c)$$

In (1.3), the function σ represents the properties of the dielectric permittivity inherited from the insulating layer while the functions h_u and \mathfrak{h}_u determining the boundary values of ψ_u on $\partial\Omega(u)$ are of the form

$$h_u(x, z) := h(x, z, u(x)), \quad (x, z) \in \bar{D} \times [-H, \infty), \quad \mathfrak{h}_u(x) := \mathfrak{h}(x, u(x)), \quad x \in \bar{D}, \quad (1.4)$$

for some prescribed functions

$$h : \bar{D} \times [-H, \infty) \times [-H, \infty) \rightarrow \mathbb{R}, \quad \mathfrak{h} : \bar{D} \times [-H, \infty) \rightarrow \mathbb{R}.$$

The main results of this work are the existence of at least one minimizer of the total energy E and the derivation of the corresponding Euler-Lagrange equation. This requires, of course, first to study the

well-posedness of the elliptic problem (1.3) subject to its mixed boundary conditions. A first step in that direction is to guarantee that the electrostatic energy E_e is well-defined, which turns out to require some care. Indeed, it should be pointed out that $\Omega(u)$ is a non-smooth domain with corners and possibly features turning points, for instance when $\mathcal{C}(u)$ includes an interval, see Figure 1. Thus, $\Omega(u)$ might consist of several components no longer having a Lipschitz boundary, so that traces have first to be given a meaning. Once this matter is settled, the existence of a variational solution ψ_u to (1.3) readily follows from the Lax-Milgram Theorem and the electrostatic energy is then well-defined. This paves the way to the proof of the existence of minimizers of the total energy by the direct method of calculus of variations but does not yet allow us to conclude. Indeed, since E involves two contributions with opposite signs, it might be unbounded from below. We overcome this difficulty by adding a penalization term to the total energy. This additional term can be removed afterwards, thanks to an *a priori* upper bound on the minimizers which follows from the corresponding Euler-Lagrange equation. However, it turns out that the derivation of the latter requires additional regularity of the electrostatic potential ψ_u . Such a regularity is actually not obvious, as the highest expected smoothness of the boundary of $\Omega(u)$ is Lipschitz regularity (when the coincidence set $\mathcal{C}(u)$ is empty). Consequently, one needs to establish sufficient regularity for ψ_u both for states u with empty and with non-empty coincidence sets $\mathcal{C}(u)$. In particular, this will ensure a well-defined normal trace of the gradient of ψ_u on $\Sigma(u)$ as required by (1.3c) and on the part of $\mathfrak{G}(u)$ lying above $\Sigma(u)$ as required by (2.6a) below. The above mentioned difficulties are actually not the only ones that we face in the forthcoming analysis. To name but a few, the electrostatic energy $E_e(u)$ features a nonlocal and intricate dependence upon the state u and appropriate continuity properties are needed in the minimizing procedure. This requires a thorough understanding of the dependence of ψ_u on the state u , this dependence being due to the domain $\Omega(u)$ as well as the functions h_u and \mathfrak{h}_u . Also, due to the prescribed constraint $u \geq -H$, the Euler-Lagrange equation solved by minimizers is in fact a variational inequality.

2. MAIN RESULTS

Throughout this work we shall assume that

$$\sigma \in C^2(\bar{D}), \quad \sigma(x) > 0, \quad x \in \bar{D}. \quad (2.1a)$$

As for the functions h_u and \mathfrak{h}_u appearing in (1.3) we shall assume in the following that

$$h \in C^2(\bar{D} \times [-H, \infty) \times [-H, \infty)), \quad \mathfrak{h} \in C^1(\bar{D} \times [-H, \infty)), \quad (2.1b)$$

satisfy

$$\partial_z h(x, -H, w) = \sigma(x)[h(x, -H, w) - \mathfrak{h}(x, w)], \quad (x, w) \in D \times [-H, \infty). \quad (2.1c)$$

Assumption (2.1c) allows us later to rewrite (1.3) as an elliptic equation with homogeneous boundary conditions. In the following, we shall use the notation introduced in (1.4).

A simple example of boundary functions (h, \mathfrak{h}) satisfying (2.1b) and (2.1c) may be derived from [17, Example 5.5] with the scaling from [16]:

Example 2.1. *Let $V > 0$ and set*

$$h(x, z, w) := V \frac{1 + \sigma(x)(H + z)}{1 + \sigma(x)(H + w)}, \quad (x, z, w) \in \bar{D} \times [-H, \infty) \times [-H, \infty),$$

and $\mathfrak{h} \equiv 0$. Then (h, \mathfrak{h}) clearly satisfies (2.1b) and (2.1c), the former being a consequence of the regularity (2.1a) of σ . Note that $h_u(x, u(x)) = V, x \in D$, for a given state u ; that is, in this example the electrostatic potential is kept constant to the value V along the elastic plate, see (1.3b).

2.1. The Electrostatic Potential. We first turn to the existence of an electrostatic potential for a given state u . To have an appropriate functional setting for u we introduce

$$\bar{S} := \{u \in H^2(D) \cap H_0^1(D) : -H \leq u \text{ in } D\}, \quad (2.2)$$

and point out that $\mathcal{C}(u) = \emptyset$ if and only if u belongs to the interior of \bar{S} ; that is, $u \in S$, where

$$S := \{u \in H^2(D) \cap H_0^1(D) : -H < u \text{ in } D\}.$$

Note that $H^2(D)$ is embedded in $C(\bar{D})$ so that $\Omega(u)$ is well-defined for $u \in \bar{S}$. Regarding the well-posedness of (1.3) we shall prove the following result.

Theorem 2.2. *Suppose (2.1). For each $u \in \bar{S}$ there exists a unique strong solution $\psi_u \in H^2(\Omega(u))$ to (1.3). Moreover, given $\kappa > 0$ and $r \in [2, \infty)$, there are $c(\kappa) > 0$ and $c(r, \kappa) > 0$ such that*

$$\|\psi_u\|_{H^2(\Omega(u))} + \|\partial_x \psi_u(\cdot, -H)\|_{L_2(D \setminus \mathcal{C}(u))} \leq c(\kappa), \quad \|\partial_z \psi_u(\cdot, u)\|_{L_r(D \setminus \mathcal{C}(u))} \leq c(r, \kappa)$$

for each $u \in \bar{S}$ with $\|u\|_{H^2(D)} \leq \kappa$.

Theorem 2.2 is an immediate consequence of Lemma 3.1, Theorem 3.2, and (3.6) below.

2.2. Existence of Energy Minimizers. Owing to Theorem 2.2, the total energy is well-defined on the set

$$\bar{S}_0 := \{u \in H^2(D) : u(\pm L) = \partial_x u(\pm L) = 0, -H \leq u \text{ in } D\} \subset \bar{S},$$

taking into account the clamped boundary conditions (1.1). We shall now focus on the existence of energy minimizers on \bar{S}_0 . We have already observed that the total energy E is the sum of two terms E_m and E_e with different signs. Hence, the coercivity of E is not obvious. However, if $\alpha > 0$, the first order term in the mechanical energy E_m is quartic and thus dominates the negative contribution coming from the electrostatic energy E_e . This property allows us to follow the lines of [17, Section 5] to derive the coercivity of E based on the following growth assumption for h : there is a constant $K > 0$ such that

$$|\partial_x h(x, z, w)| + |\partial_z h(x, z, w)| \leq K \sqrt{\frac{1+w^2}{H+w}}, \quad |\partial_w h(x, z, w)| \leq \frac{K}{\sqrt{H+w}}, \quad (2.3a)$$

for $(x, z, w) \in \bar{D} \times [-H, \infty) \times [-H, \infty)$ and

$$|h(x, -H, w)| + |\mathfrak{h}(x, w)| \leq K, \quad (x, w) \in \bar{D} \times [-H, \infty). \quad (2.3b)$$

This approach no longer works if $\alpha = 0$ and the coercivity of E is not granted. To remedy this drawback, we shall use a regularized energy functional (see (6.1) below), which includes a penalization term ensuring its coercivity if, in addition to (2.3), we assume that

$$|h(x, w, w)| + |h(\pm L, z, w)| \leq K, \quad (x, z, w) \in \bar{D} \times [-H, \infty) \times [-H, \infty), \quad (2.4a)$$

and

$$|\partial_x h(x, w, w)| + |\partial_z h(x, w, w)| + |\partial_w h(x, w, w)| + |\partial_w \mathfrak{h}(x, w)| \leq K, \quad (x, w) \in \bar{D} \times [-H, \infty). \quad (2.4b)$$

We complete the analysis when $\alpha = 0$ by showing that minimizers of the regularized energy functional for a suitable choice of the penalization parameter give rise to a minimizer of E , establishing indirectly that E is bounded from below in that case as well. Consequently, in both cases we can prove the existence of at least one energy minimizer as stated in the next result.

Theorem 2.3. *Assume (2.1) and (2.3) and, either $\alpha > 0$, or $\alpha = 0$ and (2.4). Then the total energy E has at least one minimizer u_* in \bar{S}_0 ; that is, $u_* \in \bar{S}_0$ and*

$$E(u_*) = \min_{\bar{S}_0} E. \quad (2.5)$$

At this point, no further qualitative information on energy minimizers u_* is available, and a particularly interesting question, which is yet left unanswered by our analysis, is whether the coincidence set $\mathcal{C}(u_*)$ is empty or not. Another interesting open issue is the uniqueness of minimizers. The proof of Theorem 2.3 is given in Section 6 for $\alpha = 0$ and in Section 7 for $\alpha > 0$.

2.3. Euler-Lagrange Equation. We next aim at deriving the Euler-Lagrange equation satisfied by minimizers of the total energy E . Recalling the prescribed constraint $u \geq -H$ for $u \in \bar{S}_0$, we are dealing with an obstacle problem and the resulting equation is actually a variational inequality. For the precise statement we introduce, for a given $u \in \bar{S}$, the function $g(u) : D \rightarrow \mathbb{R}$ by setting

$$\begin{aligned} g(u)(x) := & \frac{1}{2}(1 + |\partial_x u(x)|^2) [\partial_z \psi_u - (\partial_z h)_u - (\partial_w h)_u]^2(x, u(x)) \\ & + \sigma(x) [\psi_u(x, -H) - \mathfrak{h}_u(x)] (\partial_w \mathfrak{h})_u(x) \\ & - \frac{1}{2} \left[|(\partial_x h)_u|^2 + ((\partial_z h)_u + (\partial_w h)_u)^2 \right] (x, u(x)) \end{aligned} \quad (2.6a)$$

for $x \in D \setminus \mathcal{C}(u)$ while setting

$$\begin{aligned} g(u)(x) := & \frac{1}{2} |(\partial_w h)_u|^2(x, -H) + \sigma(x) [h(x, -H, -H) - \mathfrak{h}_u(x)] (\partial_w \mathfrak{h})_u(x) \\ & - \frac{1}{2} \left[|(\partial_x h)_u|^2 + ((\partial_z h)_u + (\partial_w h)_u)^2 \right] (x, -H) \end{aligned} \quad (2.6b)$$

for $x \in \mathcal{C}(u)$. In fact, $g(u)$ represents the electrostatic force exerted on the elastic plate and is computed as the differential (in a suitable sense) of the electrostatic energy $E_e(u)$ with respect to u . We emphasize here that the regularity properties of ψ_u established in Theorem 2.2 are of utmost importance to guarantee that $g(u)$ is well-defined on $D \setminus \mathcal{C}(u)$, since it features the trace of $\partial_z \psi_u$ on $\mathfrak{G}(u)$. With this notation, we are able to identify the variational inequality solved (in a weak sense) by energy minimizers.

Theorem 2.4. *Assume (2.1). Assume that $u \in \bar{S}_0$ is a minimizer of E on \bar{S}_0 . Then $g(u) \in L_2(D)$ and u is an H^2 -weak solution to the variational inequality*

$$\beta \partial_x^4 u - (\tau + \alpha \|\partial_x u\|_{L_2(D)}^2) \partial_x^2 u + \partial \mathbb{1}_{\bar{S}_0}(u) \ni -g(u) \quad \text{in } D \quad (2.7)$$

where $\partial \mathbb{1}_{\bar{S}_0}$ denotes the subdifferential of the indicator function $\mathbb{1}_{\bar{S}_0}$ of the closed convex subset \bar{S}_0 of $H^2(D)$; that is,

$$\int_D \left\{ \beta \partial_x^2 u \partial_x^2 (w - u) + [\tau + \alpha \|\partial_x u\|_{L_2(D)}^2] \partial_x u \partial_x (w - u) \right\} dx \geq - \int_D g(u)(w - u) dx$$

for all $w \in \bar{S}_0$.

At this point, we do not know whether minimizers of E in \bar{S}_0 are the only solutions to (2.7), a question closely connected to the uniqueness issue for (2.7). It is, however, expected that the set of solutions to (2.7) exhibits a complex structure. Indeed, in the much simpler situation studied in [20], the minimizer may coexist with other steady states, depending on the boundary values of the electrostatic potential.

The proof of Theorem 2.4 is given in Section 6 for $\alpha = 0$ and in Section 7 for $\alpha > 0$. It relies on the computation of the shape derivative of the electrostatic energy $E_e(u)$, which is performed in Section 5.

Remark 2.5. *It is also possible to minimize the total energy E on the set \bar{S} (instead on \bar{S}_0). Then the corresponding minimizer in \bar{S} satisfies instead of the clamped boundary conditions (1.1) the Navier or pinned boundary conditions $u(\pm L) = \partial_x^2 u(\pm L) = 0$. With this change, the statements of Theorem 2.3 and Theorem 2.4 remain true when \bar{S}_0 is replaced everywhere by \bar{S} .*

Now, combining Theorem 2.3 and Theorem 2.4 we obtain the existence of a stationary configuration of the MEMS device given as a solution to the force balance (2.7):

Corollary 2.6. *Assume (2.1) and (2.3) and, either $\alpha > 0$, or $\alpha = 0$ and (2.4). Then there is a solution $u_* \in \bar{S}_0$ to the variational inequality (2.7).*

The subsequent sections are dedicated to the proofs of the results stated in this section.

Throughout the paper, we impose assumptions (2.1) and set

$$\sigma_{min} := \min_{\bar{D}}\{\sigma\} > 0, \quad \bar{\sigma} := \|\sigma\|_{C^2(\bar{D})} < \infty. \quad (2.8)$$

3. EXISTENCE AND H^2 -REGULARITY OF THE ELECTROSTATIC POTENTIAL ψ_u

This section is dedicated to the proof of Theorem 2.2; that is, to the existence and regularity of a unique solution ψ_u to (1.3). We first recall some basic properties of the boundary function h_v which are established in [17, Lemma 3.10] and rely on the properties (2.1b) and (2.1c) of h and \mathfrak{h} .

Lemma 3.1. *Let $M > 0$.*

(a) *Given $v \in \bar{S}$ satisfying $-H \leq v(x) \leq M - H$ for $x \in D$, the function h_v belongs to $H^2(\Omega(v))$ and*

$$\begin{aligned} \|h_v\|_{H^2(\Omega(v))} &\leq C(M)(1 + \|\partial_x^2 v\|_{L_2(D)}^2), \\ \|\partial_x h_v(\cdot, -H)\|_{L_2(D)} &\leq C(M)(1 + \|\partial_x v\|_{L_2(D)}), \\ \|\partial_z h_v(\cdot, v)\|_{L_r(D)} &\leq C(M), \quad r \in [1, \infty]. \end{aligned} \quad (3.1)$$

(b) *Consider a sequence $(v_n)_{n \geq 1}$ in \bar{S} and $v \in \bar{S}$ such that*

$$-H \leq v_n(x), v(x) \leq M - H, \quad x \in D, \quad v_n \rightarrow v \text{ in } H_0^1(D). \quad (3.2)$$

Let $\Omega(M) := D \times (-H, M)$. Then

$$h_{v_n} \rightarrow h_v \quad \text{in } H^1(\Omega(M)), \quad (3.3)$$

$$h_{v_n}(\cdot, -H) \rightarrow h_v(\cdot, -H) \quad \text{in } L_2(D), \quad (3.4)$$

$$\mathfrak{h}_{v_n} \rightarrow \mathfrak{h}_v \quad \text{in } L_2(D). \quad (3.5)$$

Proof. Integrating

$$\partial_x v(x) = \partial_x v(y) + \int_y^x \partial_x^2 v(z) dz, \quad (x, y) \in [-L, L]^2,$$

with respect to $y \in [-L, L]$ and taking into account the boundary condition $v(\pm L) = 0$, we obtain

$$2L\partial_x v(x) = \int_{-L}^L \int_y^x \partial_x^2 v(z) dz dy, \quad x \in [-L, L].$$

Hence, by Hölder's inequality we get

$$\|\partial_x v\|_{L_\infty(D)} \leq \sqrt{2L} \|\partial_x^2 v\|_{L_2(D)}.$$

Using this inequality and the fact that h and its derivatives up to second order are bounded on $\bar{D} \times [-H, M] \times [-H, M]$ we derive

$$\begin{aligned} \|h_v\|_{H^2(\Omega(v))} &\leq C(M)(1 + \|\partial_x v\|_{L_2(D)} + \|\partial_x v\|_{L_\infty(D)} \|\partial_x v\|_{L_2(D)} + \|\partial_x^2 v\|_{L_2(D)}) \\ &\leq C(M)(1 + \|\partial_x^2 v\|_{L_2(D)} + \|\partial_x^2 v\|_{L_2(D)}^2), \end{aligned}$$

which yields **(a)**. As for **(b)** we first note that (3.2) and the compact embedding of $H^1(D)$ in $C(\bar{D})$ ensure that

$$v_n \rightarrow v \quad \text{in } C(\bar{D}).$$

Combining this convergence with (3.2) and the continuity properties (2.1b) of h and \mathfrak{h} readily gives (3.4) and (3.5), as well as (3.3) with the additional use of (3.2), see [17, Lemma 3.10]. \square

We shall now prove Theorem 2.2 and thus focus on (1.3), which is more conveniently considered with homogeneous boundary conditions. To this end, we introduce

$$\chi_v := \psi_v - h_v \tag{3.6}$$

for a given and fixed function $v \in \bar{S}$. Due to assumption (2.1c), problem (1.3) (with v instead of u) is then equivalent to

$$-\Delta \chi_v = \Delta h_v \quad \text{in } \Omega(v), \tag{3.7a}$$

$$\chi_v = 0 \quad \text{on } \partial\Omega(v) \setminus \Sigma(v), \tag{3.7b}$$

$$-\partial_z \chi_v + \sigma \chi_v = 0 \quad \text{on } \Sigma(v). \tag{3.7c}$$

Hence, the next result can be seen as a reformulation of Theorem 2.2 in terms of χ_v .

Theorem 3.2. *Consider a function $v \in \bar{S}$ and let $\kappa > 0$ be such that*

$$\|v\|_{H^2(D)} \leq \kappa. \tag{3.8}$$

Then there exists a unique strong solution $\chi_v \in H^2(\Omega(v))$ to (3.7) and there is $C(\kappa) > 0$ depending only on σ and κ such that

$$\|\chi_v\|_{H^2(\Omega(v))} + \|\partial_x \chi_v(\cdot, -H)\|_{L_2(D \setminus \mathcal{C}(v))} \leq C(\kappa). \tag{3.9}$$

Moreover, for any $r \in [2, \infty)$, there is $C(\kappa) > 0$ depending only on σ and κ such that

$$\|\partial_z \chi_v(\cdot, v)\|_{L_r(D \setminus \mathcal{C}(v))} \leq rC(\kappa). \tag{3.10}$$

The remainder of this section is devoted to the proof of Theorem 3.2.

3.1. Variational Solution to (3.7). We first establish the existence of a variational solution to (3.7). To this end, we introduce for $v \in \bar{S}$ the space $H_B^1(\Omega(v))$ as the closure in $H^1(\Omega(v))$ of the set

$$C_B^1(\overline{\Omega(v)}) := \left\{ \theta \in C^1(\overline{\Omega(v)}) : \theta(x, v(x)) = 0, x \in D, \theta(\pm L, z) = 0, z \in (-H, 0) \right\},$$

and shall then minimize the functional

$$\mathcal{G}(v)[\vartheta] := \frac{1}{2} \int_{\Omega(v)} |\nabla(\vartheta + h_v)|^2 d(x, z) + \frac{1}{2} \int_D \sigma(x) |\vartheta(x, -H) + h_v(x, -H) - \mathfrak{h}_v(x)|^2 dx \quad (3.11)$$

with respect to $\vartheta \in H_B^1(\Omega(v))$. Let us recall from [16, Lemma 2.2] that the trace $\vartheta(\cdot, -H) \in L_2(D)$ is well-defined for $\vartheta \in H_B^1(\Omega(v))$ (see also Lemma 3.7 below for a complete statement), while Lemma 3.1 ensures that $h_v \in H^1(\Omega(v))$ and that $h_v(\cdot, -H)$ and \mathfrak{h}_v belong to $L_2(D)$. Thus, $\mathcal{G}(v)[\vartheta]$ is well-defined for $\vartheta \in H_B^1(\Omega(v))$.

Proposition 3.3. *Let $v \in \bar{S}$. There is a unique variational solution $\chi_v \in H_B^1(\Omega(v))$ to (3.7) given as the unique minimizer of the functional $\mathcal{G}(v)$ on $H_B^1(\Omega(v))$. Moreover, χ_v is also the unique minimizer on $H_B^1(\Omega(v))$ of the functional $G_D(v)$ defined by*

$$G_D(v)[\vartheta] := \frac{1}{2} \int_{\Omega(v)} |\nabla \vartheta|^2 d(x, z) + \frac{1}{2} \int_D \sigma |\vartheta(\cdot, -H)|^2 dx - \int_{\Omega(v)} \vartheta \Delta h_v d(x, z).$$

Proof. As noted above, $\mathcal{G}(v)$ and $G_D(v)$ are both well-defined on $H_B^1(\Omega(v))$. Moreover, owing to the Poincaré inequality established in [16, Lemma 2.2], the functional $\mathcal{G}(v)$ is coercive on $H_B^1(\Omega(v))$. It thus readily follows from the Lax-Milgram Theorem that there is a unique minimizer $\chi_v \in H_B^1(\Omega(v))$ of the functional $\mathcal{G}(v)$ on $H_B^1(\Omega(v))$. Let $\vartheta \in H_B^1(\Omega(v))$. Since each connected component of $\Omega(v)$ has at most two singular points, we infer from [15, Folgerung 7.5] that we may apply Gauß' Theorem on each connected component of $\Omega(v)$ and deduce from (2.1c) that

$$\begin{aligned} \mathcal{G}(v)[\vartheta] &= \frac{1}{2} \int_{\Omega(v)} |\nabla \vartheta|^2 d(x, z) + \int_{\Omega(v)} \nabla \vartheta \cdot \nabla h_v d(x, z) + \frac{1}{2} \int_{\Omega(v)} |\nabla h_v|^2 d(x, z) \\ &\quad + \frac{1}{2} \int_D \sigma |\vartheta(\cdot, -H)|^2 dx + \int_D \sigma \vartheta(\cdot, -H) [h_v(\cdot, -H) - \mathfrak{h}_v] dx \\ &\quad + \frac{1}{2} \int_D \sigma [h_v(\cdot, -H) - \mathfrak{h}_v]^2 dx \\ &= G_D(v)[\vartheta] + \int_{\Omega(v)} \vartheta \Delta h_v d(x, z) - \int_D (\vartheta \partial_z h_v)(x, -H) dx - \int_{\Omega(v)} \vartheta \Delta h_v d(x, z) \\ &\quad + \frac{1}{2} \int_{\Omega(v)} |\nabla h_v|^2 d(x, z) + \int_D \sigma \vartheta(\cdot, -H) [h_v(\cdot, -H) - \mathfrak{h}_v] dx \\ &\quad + \frac{1}{2} \int_D \sigma [h_v(\cdot, -H) - \mathfrak{h}_v]^2 dx \\ &= G_D(v)[\vartheta] + \frac{1}{2} \int_{\Omega(v)} |\nabla h_v|^2 d(x, z) + \frac{1}{2} \int_D \sigma [h_v(\cdot, -H) - \mathfrak{h}_v]^2 dx. \end{aligned}$$

Consequently, χ_v is also the unique minimizer of the functional $G_D(v)$ on $H_B^1(\Omega(v))$. \square

For further use we state the following weak maximum principle.

Lemma 3.4. *Let $v \in \bar{S}$. Then $h_v \in C(\overline{\Omega(v)})$ and $\mathfrak{h}_v \in C(\bar{D})$ and*

$$\min \left\{ \min_{\partial\Omega(v)} h_v, \min_{\bar{D}} \mathfrak{h}_v \right\} \leq \chi_v + h_v \leq \max \left\{ \max_{\partial\Omega(v)} h_v, \max_{\bar{D}} \mathfrak{h}_v \right\}.$$

Proof. We first observe that $v \in C(\bar{D})$ which ensures, together with (2.1b), that

$$\mu_* := \min \left\{ \min_{\partial\Omega(v)} h_v, \min_{\bar{D}} \mathfrak{h}_v \right\} \quad \text{and} \quad \mu^* := \max \left\{ \max_{\partial\Omega(v)} h_v, \max_{\bar{D}} \mathfrak{h}_v \right\}$$

are well-defined and finite. Next, since χ_v is the minimizer of $\mathcal{G}(v)$ on $H_B^1(\Omega(v))$, it satisfies

$$\int_{\Omega(v)} \nabla(\chi_v + h_v) \cdot \nabla \vartheta \, d(x, z) + \int_D \sigma[(\chi_v + h_v)(\cdot, -H) - \mathfrak{h}_v] \vartheta(\cdot, -H) \, dx = 0 \quad (3.12)$$

for all $\vartheta \in H_B^1(\Omega(v))$.

Now, it follows from the definition of μ^* that $\vartheta^* := (\chi_v + h_v - \mu^*)_+$ belongs to $H_B^1(\Omega(v))$ with $\nabla \vartheta^* = \text{sign}_+(\chi_v + h_v - \mu^*) \nabla(\chi_v + h_v - \mu^*)$. Consequently, by (3.12),

$$\begin{aligned} 0 &= \int_{\Omega(v)} \nabla(\chi_v + h_v) \cdot \nabla \vartheta^* \, d(x, z) + \int_D \sigma[(\chi_v + h_v)(\cdot, -H) - \mathfrak{h}_v] \vartheta^*(\cdot, -H) \, dx \\ &= \int_{\Omega(v)} |\nabla \vartheta^*|^2 \, d(x, z) + \int_D \sigma[(\chi_v + h_v)(\cdot, -H) - \mu^* + \mu^* - \mathfrak{h}_v] \vartheta^*(\cdot, -H) \, dx \\ &\geq \int_{\Omega(v)} |\nabla \vartheta^*|^2 \, d(x, z) + \int_D \sigma[\vartheta^*(\cdot, -H)]^2 \, dx, \end{aligned}$$

where we have used the non-negativity of both $\mu^* - \mathfrak{h}_v$ and ϑ^* to derive the last inequality. We have thereby proved that $\nabla \vartheta^* = 0$ in $L_2(\Omega(v))$, which implies that $\vartheta^* = 0$ in $L_2(\Omega(v))$ thanks to the Poincaré inequality established in [16, Lemma 2.2]. In other words, $\chi_v + h_v - \mu^* \leq 0$ a.e. in $\Omega(v)$ as claimed.

Finally, a similar argument with $\vartheta_* := (\mu_* - \chi_v - h_v)_+$ leads to the inequality $\mu_* - \chi_v - h_v \leq 0$ a.e. in $\Omega(v)$ and completes the proof. \square

We now improve the regularity of χ_v as stated in Theorem 3.2 and show that χ_v belongs to $H^2(\Omega(v))$. Once this is shown, it then readily follows that χ_v is a strong solution to (3.7) (see [16, Theorem 3.5]).

As pointed out previously, for a general $v \in \bar{S}$, the set $\Omega(v)$ may consist of several connected components without Lipschitz boundaries when the coincidence set $\mathcal{C}(v)$ is non-empty. The global $H^2(\Omega(v))$ -regularity of χ_v is thus clearly not obvious. The main idea is to write the open set $D \setminus \mathcal{C}(v)$ as a countable union of disjoint open intervals $(I_j)_{j \in J}$, see [1, IX.Proposition 1.8], and to establish the H^2 -regularity for χ_v first locally on each component $\{(x, z) \in I_j \times \mathbb{R} : -H < z < v(x)\}$. This local regularity is performed in Section 3.2. The global $H^2(\Omega(v))$ -regularity is subsequently established in Section 3.3.

3.2. Local H^2 -Regularity. Let $I := (a, b)$ be an open interval in D and consider

$$v \in H^2(I) \quad \text{with} \quad v(x) > -H, \quad x \in I. \quad (3.13)$$

We define the open set $\mathcal{O}_I(v)$ by

$$\mathcal{O}_I(v) := \{(x, z) \in I \times \mathbb{R} : -H < z < v(x)\} \quad (3.14)$$

and split its boundary $\partial \mathcal{O}_I(v) = \partial \mathcal{O}_{I,D}(v) \cup \overline{\Sigma}_I$ with

$$\partial \mathcal{O}_{I,D}(v) := (\{a\} \times [-H, v(a)]) \cup (\{b\} \times [-H, v(b)]) \cup \overline{\mathfrak{G}}_I(v), \quad (3.15)$$

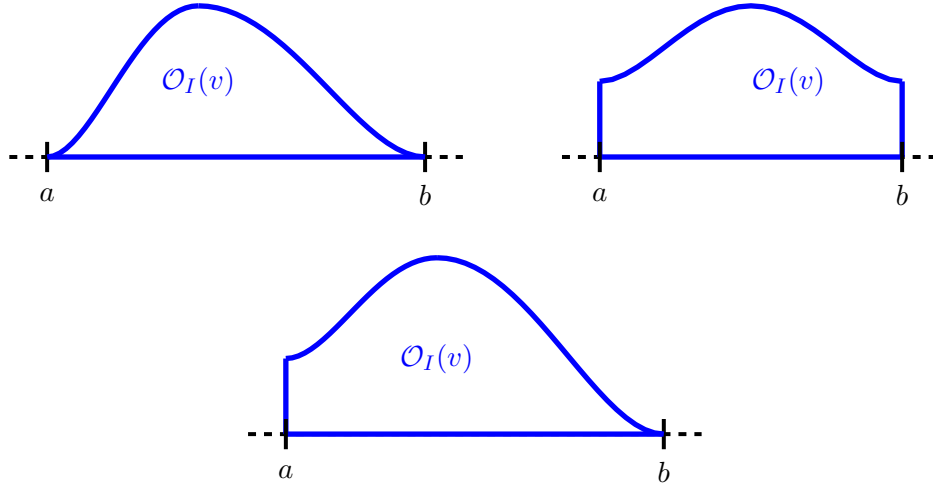


FIGURE 2. Geometry of $\mathcal{O}_I(v)$ according to the boundary values of v .

$$\overline{\Sigma}_I := [a, b] \times \{-H\}, \quad (3.16)$$

where $\Sigma_I := I \times \{-H\}$, and $\overline{\mathfrak{G}}_I(v)$ denotes the closure of the graph $\mathfrak{G}_I(v)$ of v , defined by

$$\mathfrak{G}_I(v) := \{(x, v(x)) : x \in I\}. \quad (3.17)$$

We emphasize that $\mathcal{O}_I(v)$ has no Lipschitz boundary when $v(a) + H = \partial_x v(a) = 0$ or $v(b) + H = \partial_x v(b) = 0$, as these correspond to cuspidal boundary points, see Figure 2.

Let $f \in L_2(\mathcal{O}_I(v))$ be a fixed function. The aim is to investigate the auxiliary problem

$$-\Delta \zeta_v = f \text{ in } \mathcal{O}_I(v), \quad (3.18a)$$

$$\zeta_v = 0 \text{ on } \partial \mathcal{O}_{I,D}(v), \quad (3.18b)$$

$$-\partial_z \zeta_v + \sigma \zeta_v = 0 \text{ on } \Sigma_I. \quad (3.18c)$$

We shall show the existence and uniqueness of a variational solution $\zeta_v := \zeta_{I,v} \in H^1(\mathcal{O}_I(v))$ to (3.18) and then prove its H^2 -regularity. The main difficulty encountered here is the just mentioned possible lack of Lipschitz regularity of $\mathcal{O}_I(v)$. Indeed, the trace of functions in $H^1(\mathcal{O}_I(v))$ on $\partial \mathcal{O}_I(v)$ have no meaning yet in that case, and so (3.18b) and (3.18c) are not well-defined. We shall thus first give a precise meaning to traces for functions in $H^1(\mathcal{O}_I(v))$.

Remark 3.5. Clearly, if $v \in S$, $I = D$, and $f = h_v$, then $\chi_v = \zeta_{D,v}$, so that Theorem 3.2 follows from Theorem 3.9 below in that case. Furthermore, if $I = (a, b)$ is a strict subinterval of D , $f = h_v$, and $v \in \bar{S}$ is such that $v(a) = v(b) = -H$, or $a = -L$ and $v(-L) = v(b) + H = 0$, or $b = L$ and $v(a) + H = v(L) = 0$, then $\zeta_{I,v}$ coincides – at least formally – with the restriction of χ_v to I and we shall also deduce Theorem 3.2 from Theorem 3.9. We thus do not impose that $v(a) = -H$ or $v(b) = -H$ in (3.13), so as to be able to handle simultaneously the above mentioned different cases also depicted in Figure 2.

3.2.1. Traces. As already noticed in [27], one can take advantage of the particular geometry of $\mathcal{O}_I(v)$, which lies between the graphs of two continuous functions, in order to define traces for functions in $H^1(\mathcal{O}_I(v))$ along these graphs. More precisely, one can derive the following result [16, Lemma 2.1].

Lemma 3.6 (cite[Lemma 2.1]. LNWI9] *Assume that v satisfies (3.13) and set $M_v := \|H + v\|_{L_\infty(I)}$.*

(a) *There is a linear bounded operator*

$$\Gamma_{I,v} \in \mathcal{L} \left(H^1(\mathcal{O}_I(v)), L_2(I, (H + v)dx) \right)$$

such that $\Gamma_{I,v}\vartheta = \vartheta(\cdot, v)$ for $\vartheta \in C^1(\overline{\mathcal{O}_I(v)})$ and

$$\int_I |\Gamma_{I,v}\vartheta|^2 (H + v) \, dx \leq \|\vartheta\|_{L_2(\mathcal{O}_I(v))}^2 + 2M_v \|\vartheta\|_{L_2(\mathcal{O}_I(v))} \|\partial_z \vartheta\|_{L_2(\mathcal{O}_I(v))} . \quad (3.19)$$

(b) *There is a linear bounded operator*

$$\gamma_{I,v} \in \mathcal{L} \left(H^1(\mathcal{O}_I(v)), L_2(I, (H + v)dx) \right)$$

such that $\gamma_{I,v}\vartheta = \vartheta(\cdot, -H)$ for $\vartheta \in C^1(\overline{\mathcal{O}_I(v)})$ and

$$\int_I |\gamma_{I,v}\vartheta|^2 (H + v) \, dx \leq \|\vartheta\|_{L_2(\mathcal{O}_I(v))}^2 + 2M_v \|\vartheta\|_{L_2(\mathcal{O}_I(v))} \|\partial_z \vartheta\|_{L_2(\mathcal{O}_I(v))} . \quad (3.20)$$

For simplicity, for $\vartheta \in H^1(\mathcal{O}_I(v))$, we use the notation

$$\vartheta(x, v(x)) := \Gamma_{I,v}\vartheta(x) , \quad \vartheta(x, -H) := \gamma_{I,v}\vartheta(x) , \quad x \in I .$$

We next introduce the variational setting associated with (3.18) and define the space $H_B^1(\mathcal{O}_I(v))$ as the closure in $H^1(\mathcal{O}_I(v))$ of the set

$$C_B^1(\overline{\mathcal{O}_I(v)}) := \left\{ \theta \in C^1(\overline{\mathcal{O}_I(v)}) : \theta(x, v(x)) = 0, \, x \in I, \right. \\ \left. \text{and } \theta(x, z) = 0, \, (x, z) \in \{a, b\} \times (-H, 0] \right\} .$$

Note that this is consistent with the previous definition of $H_B^1(\Omega(v))$ when $I = D$ and $v \in \bar{S}$. We have already established in [16, Lemma 2.2] a Poincaré inequality in $H_B^1(\mathcal{O}_I(v))$, as well as refined properties of the trace on $I \times \{-H\}$, which we recall now.

Lemma 3.7 ([16, Lemma 2.2]). *Assume that v satisfies (3.13) and consider $\vartheta \in H_B^1(\mathcal{O}_I(v))$. Setting $M_v := \|H + v\|_{L_\infty(I)}$, there holds*

$$\|\vartheta\|_{L_2(\mathcal{O}_I(v))} \leq 2M_v \|\partial_z \vartheta\|_{L_2(\mathcal{O}_I(v))} , \quad (3.21)$$

and the trace operator $\vartheta \mapsto \vartheta(\cdot, -H)$ maps $H_B^1(\mathcal{O}_I(v))$ to $L_2(I)$ with

$$\|\vartheta(\cdot, -H)\|_{L_2(I)}^2 \leq 2\|\vartheta\|_{L_2(\mathcal{O}_I(v))} \|\partial_z \vartheta\|_{L_2(\mathcal{O}_I(v))} . \quad (3.22)$$

3.2.2. Variational solution to (3.18). Thanks to Lemma 3.7, the trace on $I \times \{-H\}$ of a function in $H_B^1(\mathcal{O}_I(v))$ is well-defined in $L_2(I)$ and, thus, so is the functional

$$G_I(v)[\vartheta] := \frac{1}{2} \int_{\mathcal{O}_I(v)} |\nabla \vartheta|^2 \, d(x, z) + \frac{1}{2} \int_I \sigma |\vartheta(\cdot, -H)|^2 \, dx - \int_{\mathcal{O}_I(v)} f \vartheta \, d(x, z) \quad (3.23)$$

for $\vartheta \in H_B^1(\mathcal{O}_I(v))$. We now derive the existence of a unique variational solution to (3.18), or, equivalently, of a unique minimizer of $G_I(v)$ on $H_B^1(\mathcal{O}_I(v))$.

Lemma 3.8. *There is a unique variational solution $\zeta_v := \zeta_{I,v} \in H_B^1(\mathcal{O}_I(v))$ to (3.18) which satisfies*

$$\|\zeta_v\|_{H^1(\mathcal{O}_I(v))}^2 + 2\|\sqrt{\sigma}\zeta_v(\cdot, -H)\|_{L_2(I)}^2 \leq 16M_v^2(1 + 4M_v^2)\|f\|_{L_2(\mathcal{O}_I(v))}^2, \quad (3.24)$$

where $M_v := \|H + v\|_{L_\infty(I)}$.

Proof. It readily follows from (2.8), Lemma 3.7, and the Lax-Milgram Theorem that there is a unique variational solution $\zeta_v \in H_B^1(\mathcal{O}_I(v))$ to (3.18) in the sense that

$$G_I(v)[\zeta_v] \leq G_I(v)[\vartheta], \quad \vartheta \in H_B^1(\mathcal{O}_I(v)). \quad (3.25)$$

Taking $\vartheta \equiv 0$ in the previous inequality, we deduce from (3.21) and Hölder's and Young's inequalities that

$$\begin{aligned} \|\nabla\zeta_v\|_{L_2(\mathcal{O}_I(v))}^2 + \|\sqrt{\sigma}\zeta_v(\cdot, -H)\|_{L_2(I)}^2 &\leq 2\|f\|_{L_2(\mathcal{O}_I(v))}\|\zeta_v\|_{L_2(\mathcal{O}_I(v))} \\ &\leq 4M_v\|f\|_{L_2(\mathcal{O}_I(v))}\|\nabla\zeta_v\|_{L_2(\mathcal{O}_I(v))} \\ &\leq \frac{1}{2}\|\nabla\zeta_v\|_{L_2(\mathcal{O}_I(v))}^2 + 8M_v^2\|f\|_{L_2(\mathcal{O}_I(v))}^2. \end{aligned}$$

Hence,

$$\|\nabla\zeta_v\|_{L_2(\mathcal{O}_I(v))}^2 + 2\|\sqrt{\sigma}\zeta_v(\cdot, -H)\|_{L_2(I)}^2 \leq 16M_v^2\|f\|_{L_2(\mathcal{O}_I(v))}^2.$$

Combining the Poincaré inequality (3.21) and the above inequality completes the proof. \square

3.2.3. H^2 -regularity of ζ_v . We next investigate the regularity of the variational solution ζ_v to (3.18); that is, we establish a local version of Theorem 3.2.

Theorem 3.9. *Consider a function v satisfying (3.13) and let $\kappa > 0$ be such that*

$$\|v\|_{H^2(I)} \leq \kappa. \quad (3.26)$$

The variational solution $\zeta_v = \zeta_{I,v} \in H_B^1(\mathcal{O}_I(v))$ to (3.18) given by Lemma 3.8 belongs to $H^2(\mathcal{O}_I(v))$, and there is $C_1(\kappa) > 0$ depending only on σ and κ such that

$$\|\zeta_v\|_{H^2(\mathcal{O}_I(v))} + \|\partial_x\zeta_v(\cdot, -H)\|_{L_2(I)} \leq C_1(\kappa)\|f\|_{L_2(\mathcal{O}_I(v))}. \quad (3.27)$$

Moreover, there is $C_2(\kappa) > 0$ depending only on σ and κ such that, for any $r \in [2, \infty)$,

$$\|\partial_z\zeta_v(\cdot, v)\|_{L_r(I)} \leq rC_2(\kappa)\|f\|_{L_2(\mathcal{O}_I(v))}. \quad (3.28)$$

Several difficulties are encountered in the proof of Theorem 3.9, due to the low regularity of the domain $\mathcal{O}_I(v)$ which has a Lipschitz boundary if $v(a) > -H$ and $v(b) > -H$ but may have cusps otherwise, see Figure 2, and due to the mixed boundary conditions (3.18b) and (3.18c). As in [12, Section 3.3], to remedy these problems requires to construct suitable approximations of $\mathcal{O}_I(v)$ and to pay special attention in the derivation of functional inequalities and estimates on the dependence of the constants on v and I . To be more precise, we shall begin with the case where v satisfies

$$v \in W_\infty^3(I) \quad \text{and} \quad \min_{[a,b]} v > -H, \quad (3.29)$$

an assumption which is obviously stronger than (3.13). Then $\mathcal{O}_I(v)$ is a Lipschitz domain with a piecewise W_∞^3 -smooth boundary and the H^2 -regularity of ζ_v is guaranteed by [5, Theorem 2.2], see Lemma 3.10 below.

Next, transforming $\mathcal{O}_I(v)$ to the rectangle $\mathcal{R}_I := I \times (0, 1)$, we shall adapt the proof of [12, Lemma 4.3.1.3] to establish the identity

$$\begin{aligned} \int_{\mathcal{O}_I(v)} \partial_x^2 \zeta_v \partial_z^2 \zeta_v \, d(x, z) &= \int_{\mathcal{O}_I(v)} |\partial_x \partial_z \zeta_v|^2 \, d(x, z) + \int_I (\partial_x \zeta_v \partial_x (\sigma \zeta_v)) (\cdot, -H) \, dx \\ &\quad - \frac{1}{2} \int_I \partial_x^2 v |\partial_z \zeta_v(\cdot, v)|^2 \, dx \end{aligned} \quad (3.30)$$

in Lemma 3.11. We then shall show that the last two integrals on the right-hand side of (3.30) are controlled by the H^2 -norm of ζ_v with a sublinear dependence, a feature which will allow us to derive (3.27) when v satisfies (3.29). To this end, we shall use the embedding of the subspace

$$H_{WS}^1(\mathcal{O}_I(v)) := \left\{ P \in H^1(\mathcal{O}_I(v)) : \begin{array}{ll} P(x, -H) = 0, & x \in I, \\ P(a, z) = 0, & z \in (-H, v(a)), \end{array} \right\} \quad (3.31)$$

of $H^1(\mathcal{O}_I(v))$ in $L_r(\mathcal{O}_I(v))$ and the continuity of the trace operator from $H_{WS}^1(\mathcal{O}_I(v))$ to $L_r(\mathfrak{G}_I(v))$ for $r \in [1, \infty)$, which involves constants that do not depend on $\min_{[a,b]} \{v + H\}$, see Lemmas C.1-C.3 in Appendix C. After this preparation, we will be left with relaxing the assumption (3.29) to (3.13) and this will be achieved by an approximation argument, see Section 3.2.5.

3.2.4. H^2 -regularity of ζ_v when v satisfies (3.29). Throughout this section, we assume that v satisfies (3.29) and fix $M > 0$ such that

$$M \geq \max \{1, \|H + v\|_{L^\infty(I)}, \|\partial_x v\|_{L^\infty(I)}\}. \quad (3.32)$$

We also denote positive constants depending only on σ by C and $(C_i)_{i \geq 3}$. The dependence upon additional parameters will be indicated explicitly.

We begin with the H^2 -regularity of the variational solution ζ_v to (3.18), which follows from the analysis performed in [3–5].

Lemma 3.10. $\zeta_v \in H^2(\mathcal{O}_I(v))$.

Proof. We first recast the boundary value problem (3.18) in the framework of [5]. Owing to (3.29), the boundary of the domain $\mathcal{O}_I(v)$ includes four W_∞^3 -smooth edges $(\Gamma_i)_{1 \leq i \leq 4}$ given by

$$\begin{aligned} \Gamma_1 &:= I \times \{-H\}, & \Gamma_3 &:= \mathfrak{G}_I(v), \\ \Gamma_2 &:= \{b\} \times (-H, v(b)), & \Gamma_4 &:= \{a\} \times (-H, v(a)), \end{aligned}$$

and four vertices $(S_i)_{1 \leq i \leq 4}$

$$\begin{aligned} S_1 &:= \bar{\Gamma}_1 \cap \bar{\Gamma}_2 = (b, -H), & S_3 &:= \bar{\Gamma}_3 \cap \bar{\Gamma}_4 = (a, v(a)), \\ S_2 &:= \bar{\Gamma}_2 \cap \bar{\Gamma}_3 = (b, v(b)), & S_4 &:= \bar{\Gamma}_4 \cap \bar{\Gamma}_1 = (a, -H). \end{aligned}$$

We set

$$\begin{aligned} \mathcal{D}_\Gamma &:= \{2, 3, 4\}, & \mathcal{N}_\Gamma &:= \{1\}, \\ \mathcal{D} &:= \{2, 3\}, & \mathcal{M}_{12} &:= \{4\}, & \mathcal{M}_{21} &:= \{1\}, & \mathcal{N} &:= \emptyset, \end{aligned}$$

and note that $\mathcal{D}_\Gamma \neq \emptyset$ as required in [5].

Since $v \in W_\infty^3(I)$, the measure ω_i of the angle at S_i taken towards the interior of $\mathcal{O}_I(v)$ satisfies

$$\omega_1 = \omega_4 = \frac{\pi}{2}, \quad (\omega_2, \omega_3) \in (0, \pi)^2. \quad (3.33)$$

For $1 \leq i \leq 4$, we denote the outward unit normal vector field and the corresponding unit tangent vector field by $\boldsymbol{\nu}_i$ and $\boldsymbol{\tau}_i$, respectively. According to the geometry of $\mathcal{O}_I(v)$,

$$\begin{aligned} \boldsymbol{\nu}_1 &= (0, -1) , \quad \boldsymbol{\nu}_2 = (1, 0) , \quad \boldsymbol{\nu}_3 = \frac{(-\partial_x v, 1)}{\sqrt{1 + |\partial_x v|^2}} , \quad \boldsymbol{\nu}_4 = (-1, 0) , \\ \boldsymbol{\tau}_1 &= (1, 0) , \quad \boldsymbol{\tau}_2 = (0, 1) , \quad \boldsymbol{\tau}_3 = \frac{(-1, -\partial_x v)}{\sqrt{1 + |\partial_x v|^2}} , \quad \boldsymbol{\tau}_4 = (0, -1) . \end{aligned}$$

We also define

$$\boldsymbol{\mu}_1 := \boldsymbol{\nu}_1 , \quad \boldsymbol{\mu}_i := \boldsymbol{\tau}_i , \quad i \in \{2, 3, 4\} , \quad (3.34)$$

and note that the measure $\Psi_i \in [0, \pi]$ of the angle between $\boldsymbol{\mu}_i$ and $\boldsymbol{\tau}_i$, $1 \leq i \leq 4$, is given by

$$\Psi_1 = \frac{\pi}{2} , \quad \Psi_i = 0 , \quad i \in \{2, 3, 4\} . \quad (3.35)$$

We also set

$$\psi_1 = \phi_2 = \phi_3 = \phi_4 = 0 . \quad (3.36)$$

We finally define the boundary operator

$$\mathcal{B}_1 := -\partial_z + \sigma \text{id} \quad \text{on} \quad I \times \{-H\} .$$

Now, on the one hand, the regularity of σ implies that [5, Assumption (1.5)] is satisfied, while [5, Assumption (1.6)] obviously holds since $\mathcal{N} = \emptyset$. On the other hand, we note that $\boldsymbol{\mu}_1(S_1) = -\boldsymbol{\mu}_2(S_1)$ and $\boldsymbol{\mu}_4(S_4) = \boldsymbol{\mu}_1(S_4)$, so that [5, Assumption (2.1)] is satisfied for $i \in \{1, 4\}$ (but not for $i \in \{2, 3\}$). We then set $\varepsilon_1 = -1$ and $\varepsilon_4 = 1$. We are left with checking [5, Assumptions (2.3)-(2.4)] but this is obvious due to (3.36). We finally observe that

$$\mathcal{K} := \{(i, m) \in \{1, \dots, 4\} \times \mathbb{Z} : \lambda_{i,m} \in (-1, 0)\}$$

is empty, since

$$\begin{aligned} \lambda_{1,m} &:= \frac{\Psi_2 - \Psi_1 + m\pi}{\omega_1} = 2m - 1 \notin (-1, 0) , \\ \lambda_{2,m} &:= \frac{\Psi_3 - \Psi_2 + m\pi}{\omega_2} = \frac{m\pi}{\omega_2} \notin (-1, 0) , \\ \lambda_{3,m} &:= \frac{\Psi_4 - \Psi_3 + m\pi}{\omega_3} = \frac{m\pi}{\omega_3} \notin (-1, 0) , \\ \lambda_{4,m} &:= \frac{\Psi_1 - \Psi_4 + m\pi}{\omega_4} = 2m + 1 \notin (-1, 0) , \end{aligned}$$

for any $m \in \mathbb{Z}$. We then infer from [5, Theorem 2.2] that ζ_v has no singular part and thus belongs to $H^2(\mathcal{O}_I(v))$. \square

We now investigate the quantitative dependence of the just established H^2 -regularity of ζ_v on v and derive an H^2 -estimate, which is related to the regularity of v . To this end, we need the following identity.

Lemma 3.11.

$$\int_{\mathcal{O}_I(v)} \partial_x^2 \zeta_v \partial_z^2 \zeta_v \, d(x, z) = \int_{\mathcal{O}_I(v)} |\partial_x \partial_z \zeta_v|^2 \, d(x, z) + \int_I (\partial_x \zeta_v \partial_x (\sigma \zeta_v)) (\cdot, -H) \, dx$$

$$- \frac{1}{2} \int_I \partial_x^2 v |\partial_z \zeta_v(\cdot, v)|^2 dx .$$

The identity of Lemma 3.11 is reminiscent of [17, Lemma 3.5]. Its proof is rather technical and thus postponed to Appendix B.

The next step of the analysis is to show that the two integrals over I on the right-hand side of the identity stated in Lemma 3.11 can be controlled by the H^2 -norm of ζ_v with a mild dependence on v . To this end, we need some auxiliary functional and trace inequalities which are established in Appendix C. With this in hand, we begin with an estimate of the last integral.

Lemma 3.12. *There is $C_3(M) > 0$ such that, for any $r \in [2, \infty)$,*

$$\|\partial_z \zeta_v(\cdot, v)\|_{L_r(I)} \leq r C_3(M) \|f\|_{L_2(\mathcal{O}_I(v))}^{1/r} \left(\|\nabla \partial_z \zeta_v\|_{L_2(\mathcal{O}_I(v))} + \|f\|_{L_2(\mathcal{O}_I(v))} \right)^{(r-1)/r} . \quad (3.37)$$

In particular, there is $C_4(M) > 0$ such that

$$\left| \int_I \partial_x^2 v |\partial_z \zeta_v(\cdot, v)|^2 dx \right| \leq C_4(M) \|\partial_x^2 v\|_{L_2(I)} \left[\|f\|_{L_2(\mathcal{O}_I(v))}^{1/2} \|\nabla \partial_z \zeta_v\|_{L_2(\mathcal{O}_I(v))}^{3/2} + \|f\|_{L_2(\mathcal{O}_I(v))}^2 \right] . \quad (3.38)$$

Proof. To lighten notation, we set $\mathcal{O} := \mathcal{O}_I(v)$ and introduce $P := \partial_z \zeta_v - \sigma \zeta_v$. Since $\zeta_v \in H^2(\mathcal{O})$ by Lemma 3.10 and $\sigma \in C^2(\bar{I})$, the function P belongs to $H^1(\mathcal{O})$ and satisfies (C.2) by (3.18b) and (3.18c). In addition, we observe that $P(\cdot, v) = \partial_z \zeta_v(\cdot, v)$ by (3.18b). It then follows from Lemma C.3 that

$$\|\partial_z \zeta_v(\cdot, v)\|_{L_r(I)}^r = \|P(\cdot, v)\|_{L_r(I)}^r \leq \left(4r\sqrt{M}\right)^r \|P\|_{L_2(\mathcal{O})} \|\nabla P\|_{L_2(\mathcal{O})}^{r-1} .$$

Moreover, by (2.8) and Lemma 3.8,

$$\begin{aligned} \|P\|_{L_2(\mathcal{O})} &\leq \|\partial_z \zeta_v\|_{L_2(\mathcal{O})} + \bar{\sigma} \|\zeta_v\|_{L_2(\mathcal{O})} \leq (1 + \bar{\sigma}) \|\zeta_v\|_{H^1(\mathcal{O})} \\ &\leq 4\|H + v\|_{L_\infty(I)} \sqrt{1 + 4\|H + v\|_{L_\infty(I)}^2} (1 + \bar{\sigma}) \|f\|_{L_2(\mathcal{O})} \leq C(M) \|f\|_{L_2(\mathcal{O})} \end{aligned}$$

and

$$\begin{aligned} \|\nabla P\|_{L_2(\mathcal{O})} &\leq \|\partial_x P\|_{L_2(\mathcal{O})} + \|\partial_z P\|_{L_2(\mathcal{O})} \\ &\leq \|\partial_x \partial_z \zeta_v\|_{L_2(\mathcal{O})} + \bar{\sigma} \|\partial_x \zeta_v\|_{L_2(\mathcal{O})} + \bar{\sigma} \|\zeta_v\|_{L_2(\mathcal{O})} + \|\partial_z^2 \zeta_v\|_{L_2(\mathcal{O})} + \bar{\sigma} \|\partial_z \zeta_v\|_{L_2(\mathcal{O})} \\ &\leq \sqrt{2} \|\nabla \partial_z \zeta_v\|_{L_2(\mathcal{O})} + \bar{\sigma} \left(\sqrt{2} \|\nabla \zeta_v\|_{L_2(\mathcal{O})} + \|\zeta_v\|_{L_2(\mathcal{O})} \right) \\ &\leq \sqrt{2} \|\nabla \partial_z \zeta_v\|_{L_2(\mathcal{O})} + C(M) \|f\|_{L_2(\mathcal{O})} . \end{aligned}$$

Collecting the previous estimates, we end up with

$$\begin{aligned} \|\partial_z \zeta_v(\cdot, v)\|_{L_r(I)}^r &\leq \left(4r\sqrt{M}\right)^r C(M) \|f\|_{L_2(\mathcal{O})} \left(\sqrt{2} \|\nabla \partial_z \zeta_v\|_{L_2(\mathcal{O})} + C(M) \|f\|_{L_2(\mathcal{O})} \right)^{r-1} \\ &\leq (rC(M))^r \|f\|_{L_2(\mathcal{O})} \left(\|\nabla \partial_z \zeta_v\|_{L_2(\mathcal{O})} + \|f\|_{L_2(\mathcal{O})} \right)^{r-1} , \end{aligned}$$

from which (3.37) follows. We next deduce from (3.37) (with $r = 4$) and Hölder's inequality that

$$\begin{aligned} \left| \int_I \partial_x^2 v |\partial_z \zeta_v(\cdot, v)|^2 dx \right| &\leq \|\partial_x^2 v\|_{L_2(I)} \|\partial_z \zeta_v(\cdot, v)\|_{L_4(I)}^2 \\ &\leq 16C_3(M)^2 \|\partial_x^2 v\|_{L_2(I)} \|f\|_{L_2(\mathcal{O})}^{1/2} \left(\|\nabla \partial_z \zeta_v\|_{L_2(\mathcal{O})} + \|f\|_{L_2(\mathcal{O})} \right)^{3/2} \end{aligned}$$

$$\leq C(M) \|\partial_x^2 v\|_{L_2(I)} \|f\|_{L_2(\mathcal{O})}^{1/2} \left(\|\nabla \partial_z \zeta_v\|_{L_2(\mathcal{O})}^{3/2} + \|f\|_{L_2(\mathcal{O})}^{3/2} \right),$$

and the proof is complete. \square

We are now in a position to derive quantitative estimates in H^2 for ζ_v , which only depends on the H^2 -norm of v , even though v is assumed to be more regular.

Lemma 3.13. *There is $C_5(M) > 0$ such that*

$$\|\nabla \partial_z \zeta_v\|_{L_2(\mathcal{O}_I(v))}^2 + \|\sqrt{\sigma} \partial_x \zeta_v(\cdot, -H)\|_{L_2(I)}^2 \leq C_5(M) \left(1 + \|\partial_x^2 v\|_{L_2(I)}^4 \right) \|f\|_{L_2(\mathcal{O}_I(v))}^2, \quad (3.39a)$$

$$\|\partial_x^2 \zeta_v\|_{L_2(\mathcal{O}_I(v))}^2 \leq C_5(M) \left(1 + \|\partial_x^2 v\|_{L_2(I)}^4 \right) \|f\|_{L_2(\mathcal{O}_I(v))}^2. \quad (3.39b)$$

Proof. To lighten notation, we set $\mathcal{O} := \mathcal{O}_I(v)$. We infer from (3.18a) and Lemma 3.11 that

$$\begin{aligned} - \int_{\mathcal{O}} f \partial_z^2 \zeta_v \, d(x, z) &= \int_{\mathcal{O}} \left(\partial_x^2 \zeta_v \partial_z^2 \zeta_v + |\partial_z^2 \zeta_v|^2 \right) \, d(x, z) \\ &= \|\nabla \partial_z \zeta_v\|_{L_2(\mathcal{O})}^2 + \int_I \partial_x \zeta_v(\cdot, -H) \partial_x (\sigma \zeta_v)(\cdot, -H) \, dx \\ &\quad - \frac{1}{2} \int_I \partial_x^2 v |\partial_z \zeta_v(\cdot, v)|^2 \, dx. \end{aligned}$$

Hence, thanks to (2.8), Lemma 3.12, and Hölder's and Young's inequalities,

$$\begin{aligned} X &:= \|\nabla \partial_z \zeta_v\|_{L_2(\mathcal{O})}^2 + \|\sqrt{\sigma} \partial_x \zeta_v(\cdot, -H)\|_{L_2(I)}^2 \\ &= - \int_{\mathcal{O}} f \partial_z^2 \zeta_v \, d(x, z) - \int_I \partial_x \sigma (\zeta_v \partial_x \zeta_v)(\cdot, -H) \, dx + \frac{1}{2} \int_I \partial_x^2 v |\partial_z \zeta_v(\cdot, v)|^2 \, dx \\ &\leq \|f\|_{L_2(\mathcal{O})} \|\partial_z^2 \zeta_v\|_{L_2(\mathcal{O})} + \bar{\sigma} \|\zeta_v(\cdot, -H)\|_{L_2(I)} \|\partial_x \zeta_v(\cdot, -H)\|_{L_2(I)} \\ &\quad + \frac{C_4(M)}{2} \|\partial_x^2 v\|_{L_2(I)} \left[\|f\|_{L_2(\mathcal{O})}^{1/2} \|\nabla \partial_z \zeta_v\|_{L_2(\mathcal{O})}^{3/2} + \|f\|_{L_2(\mathcal{O})}^2 \right] \\ &\leq \frac{1}{4} \|\partial_z^2 \zeta_v\|_{L_2(\mathcal{O})}^2 + \|f\|_{L_2(\mathcal{O})}^2 + \frac{\bar{\sigma}}{\sqrt{\sigma_{\min}}} \|\zeta_v(\cdot, -H)\|_{L_2(I)} \|\sqrt{\sigma} \partial_x \zeta_v(\cdot, -H)\|_{L_2(I)} \\ &\quad + \frac{1}{4} \|\nabla \partial_z \zeta_v\|_{L_2(\mathcal{O})}^2 + C(M) \left(\|\partial_x^2 v\|_{L_2(I)}^4 + \|\partial_x^2 v\|_{L_2(I)} \right) \|f\|_{L_2(\mathcal{O})}^2 \\ &\leq \frac{1}{2} \|\nabla \partial_z \zeta_v\|_{L_2(\mathcal{O})}^2 + \frac{1}{2} \|\sqrt{\sigma} \partial_x \zeta_v(\cdot, -H)\|_{L_2(I)}^2 + \frac{\bar{\sigma}^2}{2\sigma_{\min}} \|\zeta_v(\cdot, -H)\|_{L_2(I)}^2 \\ &\quad + C(M) \left(1 + \|\partial_x^2 v\|_{L_2(I)}^4 \right) \|f\|_{L_2(\mathcal{O})}^2. \end{aligned}$$

Consequently, using once more Young's inequality,

$$X \leq \frac{\bar{\sigma}^2}{\sigma_{\min}} \|\zeta_v(\cdot, -H)\|_{L_2(I)}^2 + C(M) \left(1 + \|\partial_x^2 v\|_{L_2(I)}^4 \right) \|f\|_{L_2(\mathcal{O})}^2.$$

Now, since $\zeta_v \in H_B^1(\mathcal{O})$, it follows from (2.8), (3.32), and Lemma 3.8 that

$$2\sigma_{\min} \|\zeta_v(\cdot, -H)\|_{L_2(I)}^2 \leq 16M^2(1 + 4M^2) \|f\|_{L_2(\mathcal{O})}^2.$$

Combining the above two estimates gives (3.39a).

To complete the proof of Lemma 3.13, we simply notice that (3.18a) ensures that

$$\|\partial_x^2 \zeta_v\|_{L_2(\mathcal{O})}^2 = \|f + \partial_z^2 \zeta_v\|_{L_2(\mathcal{O})}^2 \leq 2\|\partial_z^2 \zeta_v\|_{L_2(\mathcal{O})}^2 + 2\|f\|_{L_2(\mathcal{O})}^2$$

and deduce (3.39b) from (3.39a). \square

Summarizing, we have established the following result:

Proposition 3.14. *Consider $v \in H^2(I)$ satisfying (3.29); that is,*

$$v \in W_\infty^3(I) \quad \text{and} \quad \min_{[a,b]} v > -H,$$

and fix $\kappa > 0$ such that

$$\|v\|_{H^2(I)} \leq \kappa. \quad (3.40)$$

Then the elliptic boundary value problem (3.18) has a unique strong solution $\zeta_v \in H^2(\mathcal{O}_I(v))$ which satisfies

$$\|\zeta_v\|_{H^2(\mathcal{O}_I(v))} + \|\partial_x \zeta_v(\cdot, -H)\|_{L_2(I)} \leq C_6(\kappa) \|f\|_{L_2(\mathcal{O}_I(v))}, \quad (3.41)$$

$$\|\partial_z \zeta_v(\cdot, v)\|_{L_r(I)} \leq r C_6(\kappa) \|f\|_{L_2(\mathcal{O}_I(v))}, \quad r \in [2, \infty). \quad (3.42)$$

Proof. The existence and uniqueness of a strong solution $\zeta_v \in H^2(\mathcal{O}_I(v))$ to (3.18) are consequences of Lemma 3.8 and Lemma 3.10. Next, it readily follows from (3.40) and the continuous embedding of $H^2(I)$ in $W_\infty^1(I)$ that there is $M \geq 1$ depending on κ such that

$$\|H + v\|_{L_\infty(I)} + \|\partial_x v\|_{L_\infty(I)} \leq M. \quad (3.43)$$

Due to (3.43), we deduce (3.41) from (2.8), (3.40), Lemma 3.8, and Lemma 3.13, while (3.42) follows from (3.41) and Lemma 3.12. \square

We emphasize that, though derived for functions $v \in H^2(I)$ satisfying the additional assumption (3.29), the estimates stated in Proposition 3.14 only depend on the H^2 -norm of v and, neither on its W_∞^2 -norm, nor on the value of its minimum (provided that it stays above $-H$). The outcome of Proposition 3.14 is thus likely to extend to any configuration depicted in Figure 2 under the sole assumption (3.13) and this will be shown in the next section by an approximation argument.

3.2.5. H^2 -regularity: Proof of Theorem 3.9. We now prove the H^2 -regularity of ζ_v as stated in Theorem 3.9. We thus assume that v satisfies (3.13); that is,

$$v \in H^2(I) \quad \text{such that} \quad v(x) > -H, \quad x \in I,$$

and fix $\kappa > 0$ such that $\|v\|_{H^2(I)} \leq \kappa$. Owing to the density of $C^\infty([a, b])$ in $H^2(I)$ and since v satisfies (3.13), we employ classical approximation arguments to construct a sequence $(v_n)_{n \geq 1}$ of functions in $C^\infty([a, b])$ with the following properties:

$$\lim_{n \rightarrow \infty} \|v_n - v\|_{H^2(I)} = 0, \quad \sup_{n \geq 1} \{\|v_n\|_{H^2(I)}\} \leq 1 + \kappa, \quad (3.44a)$$

$$v_n \geq v + \frac{1}{n}, \quad n \geq 1. \quad (3.44b)$$

A first consequence of (3.44a) and the continuous embedding of $H^2(I)$ in $W_\infty^1(I)$ is that

$$\begin{aligned} \|H + v_n\|_{L_\infty(I)} + \|\partial_x v_n\|_{L_\infty(I)} &\leq C(\kappa), \quad n \geq 1, \\ \lim_{n \rightarrow \infty} \|v_n - v\|_{W_\infty^1(I)} &= 0. \end{aligned} \quad (3.45)$$

According to (3.13) and (3.44b), the function v_n satisfies (3.29) for each $n \geq 1$ and, since $\mathcal{O}_I(v) \subset \mathcal{O}_I(v_n)$, we infer from Proposition 3.14 that the strong solution ζ_{v_n} to (3.18) with v_n instead of v (and f replaced by its trivial extension to $\mathcal{O}_I(v_n)$) satisfies

$$\|\zeta_{v_n}\|_{H^2(\mathcal{O}_I(v_n))} + \|\partial_x \zeta_{v_n}(\cdot, -H)\|_{L_2(I)} \leq C_7(\kappa) \|f\|_{L_2(\mathcal{O}_I(v))}, \quad (3.46)$$

$$\|\partial_z \zeta_{v_n}(\cdot, v_n)\|_{L_r(I)} \leq r C_7(\kappa) \|f\|_{L_2(\mathcal{O}_I(v))}, \quad r \in [2, \infty). \quad (3.47)$$

Using again the inclusion $\mathcal{O}_I(v) \subset \mathcal{O}_I(v_n)$, we deduce from (3.46) that $(\zeta_{v_n})_{n \geq 1}$ is bounded in $H^2(\mathcal{O}_I(v))$. Consequently, recalling that $H^1(\mathcal{O}_I(v))$ is compactly embedded in $L_2(\mathcal{O}_I(v))$ (despite the non-Lipschitz character of $\mathcal{O}_I(v)$, see [23, Theorem 11.21] or [28, I.Theorem 1.4]), there are a subsequence of $(\zeta_{v_n})_{n \geq 1}$ (not relabeled) and $\phi \in H^2(\mathcal{O}_I(v))$ such that

$$\begin{aligned} \zeta_{v_n} &\rightharpoonup \phi \quad \text{in } H^2(\mathcal{O}_I(v)), \\ \zeta_{v_n} &\longrightarrow \phi \quad \text{in } H^1(\mathcal{O}_I(v)). \end{aligned} \quad (3.48)$$

Let us first check that $\phi \in H_B^1(\mathcal{O}_I(v))$. On the one hand, since both ϕ and ζ_{v_n} belong to $H^1(\mathcal{O}_I(v))$, we infer from (3.19) that

$$\int_I |(\phi - \zeta_{v_n})(\cdot, v)|^2 (H + v) \, dx \leq C(\kappa) \|\phi - \zeta_{v_n}\|_{H^1(\mathcal{O}_I(v))}^2.$$

Hence, by (3.48),

$$\lim_{n \rightarrow \infty} \int_I |(\phi - \zeta_{v_n})(\cdot, v)|^2 (H + v) \, dx = 0.$$

On the other hand, since $\zeta_{v_n} \in H_B^1(\mathcal{O}_I(v_n))$ and $v_n \geq v$, it follows from Lemma A.1 and (3.46) that

$$\begin{aligned} \int_I |\zeta_{v_n}(\cdot, v)|^2 (H + v) \, dx &= \int_I |\zeta_{v_n}(\cdot, v) - \zeta_{v_n}(\cdot, v_n)|^2 (H + v) \, dx \\ &\leq \|(v - v_n)(H + v)\|_{L_\infty(I)} \|\partial_z \zeta_{v_n}\|_{L_2(\mathcal{O}_I(v_n))} \\ &\leq C(\kappa) \|v - v_n\|_{L_\infty(I)} \|f\|_{L_2(\mathcal{O}_I(v))}. \end{aligned}$$

Hence, by (3.45),

$$\lim_{n \rightarrow \infty} \int_I |\zeta_{v_n}(\cdot, v)|^2 (H + v) \, dx = 0.$$

Combining the previous two limits, we deduce

$$\int_I |\phi(\cdot, v)|^2 (H + v) \, dx = 0,$$

so that $\phi \in H_B^1(\mathcal{O}_I(v))$. In particular, for $n \geq 1$, due to the inclusion $\mathcal{O}_I(v) \subset \mathcal{O}_I(v_n)$, the function ϕ also belongs to $H_B^1(\mathcal{O}_I(v_n))$ and we infer from (3.22) and (3.48) that

$$\lim_{n \rightarrow \infty} \int_I |(\zeta_{v_n} - \phi)(\cdot, -H)|^2 \, dx = 0. \quad (3.49)$$

We next recall that ζ_{v_n} is the unique solution in $H_B^1(\mathcal{O}_I(v_n))$ to

$$\int_{\mathcal{O}_I(v_n)} \nabla \zeta_{v_n} \cdot \nabla \vartheta \, d(x, z) + \int_I \sigma \zeta_{v_n}(\cdot, -H) \vartheta(\cdot, -H) \, dx = \int_{\mathcal{O}_I(v_n)} f \vartheta \, dx \quad (3.50)$$

for all $\vartheta \in H_B^1(\mathcal{O}_I(v_n))$. Now, since $H_B^1(\mathcal{O}_I(v)) \subset H_B^1(\mathcal{O}_I(v_n))$, we can take $\vartheta \in H_B^1(\mathcal{O}_I(v))$ in (3.50) and use the convergences (3.48) and (3.49) to pass to the limit $n \rightarrow \infty$ and conclude that $\phi \in H_B^1(\mathcal{O}_I(v))$ satisfies the variational formulation of (3.18). Therefore, Lemma 3.8 guarantees that $\phi = \zeta_v$. We have thus shown that $\zeta_v \in H^2(\mathcal{O}_I(v))$ and it follows from (3.46) and (3.48) that

$$\|\zeta_v\|_{H^2(\mathcal{O}_I(v))} \leq \liminf_{n \rightarrow \infty} \|\zeta_{v_n}\|_{H^2(\mathcal{O}_I(v))} \leq \liminf_{n \rightarrow \infty} \|\zeta_{v_n}\|_{H^2(\mathcal{O}_I(v_n))} \leq C_7(\kappa) \|f\|_{L_2(\mathcal{O}_I(v))} . \quad (3.51)$$

A further consequence of (3.20) and (3.48) is that $(\partial_x \zeta_{v_n}(\cdot, -H))_{n \geq 1}$ converges to $\partial_x \zeta_v(\cdot, -H)$ in $L_2(I, (H+v)dx)$, which, together with the positivity of $H+v$ in I , implies that $(\partial_x \zeta_{v_n}(\cdot, -H))_{n \geq 1}$ converges to $\partial_x \zeta_v(\cdot, -H)$ in $L_2(a+\varepsilon, b-\varepsilon)$ for any $\varepsilon \in (0, (b-a)/2)$. Combining this convergence with (3.46) and using Fatou's lemma to take the limit $\varepsilon \rightarrow 0$ give

$$\|\partial_x \zeta_v(\cdot, -H)\|_{L_2(I)} \leq C_7(\kappa) \|f\|_{L_2(\mathcal{O}_I(v))} . \quad (3.52)$$

Finally, by (3.19) and (3.46),

$$\int_I |(\partial_z \zeta_{v_n} - \partial_z \zeta_v)(\cdot, v)|^2 (H+v) \, dx \leq C(\kappa) \|\partial_z \zeta_{v_n} - \partial_z \zeta_v\|_{L_2(\mathcal{O}_I(v))} .$$

Hence, by (3.48),

$$\lim_{n \rightarrow \infty} \int_I |(\partial_z \zeta_{v_n} - \partial_z \zeta_v)(\cdot, v)|^2 (H+v) \, dx = 0 . \quad (3.53)$$

Moreover, owing to Lemma A.1, (3.46), and the properties $\zeta_{v_n} \in H_B^1(\mathcal{O}_I(v_n))$ and $v_n \geq v$,

$$\begin{aligned} \int_I |\partial_z \zeta_{v_n}(\cdot, v) - \partial_z \zeta_{v_n}(\cdot, v_n)|^2 (H+v) \, dx &\leq \|(v - v_n)(H+v)\|_{L_\infty(I)} \|\partial_z^2 \zeta_{v_n}\|_{L_2(\mathcal{O}_I(v_n))}^2 \\ &\leq C(\kappa) \|v - v_n\|_{L_\infty(I)} , \end{aligned}$$

and it follows from (3.45) that

$$\lim_{n \rightarrow \infty} \int_I |\partial_z \zeta_{v_n}(\cdot, v) - \partial_z \zeta_{v_n}(\cdot, v_n)|^2 (H+v) \, dx = 0 . \quad (3.54)$$

Gathering (3.53) and (3.54) leads us to

$$\lim_{n \rightarrow \infty} \int_I |\partial_z \zeta_v(\cdot, v) - \partial_z \zeta_{v_n}(\cdot, v_n)|^2 (H+v) \, dx = 0 . \quad (3.55)$$

Since $H+v > 0$ in I , we may extract a further subsequence (not relabeled) such that $(\partial_z \zeta_{v_n}(\cdot, v_n))_{n \geq 1}$ converges a.e. in I to $\partial_z \zeta_v(\cdot, v)$. We then use Fatou's lemma to pass to the limit $n \rightarrow \infty$ in (3.47) and conclude that

$$\|\partial_z \zeta_v(\cdot, v)\|_{L_r(I)} \leq r C_7(\kappa) \|f\|_{L_2(\mathcal{O}_I(v))} , \quad r \in [2, \infty) ,$$

thereby completing the proof of Theorem 3.9.

3.3. Global H^2 -regularity of χ_v : Proof of Theorem 3.2 and Theorem 2.2. Finally, we prove Theorem 3.2 and Theorem 2.2 for which we consider an arbitrary function v in \bar{S} and $\kappa > 0$ satisfying (3.8). According to [1, IX.Proposition 1.8] we can write the open set $D \setminus \mathcal{C}(v)$ as a countable union of disjoint open intervals $(I_j)_{j \in J}$; that is,

$$D \setminus \mathcal{C}(v) = \bigcup_{j \in J} I_j.$$

Hence, $\Omega(v)$ is the disjoint union of the open domains $\mathcal{O}_{I_j}(v)$. Now recall from Proposition 3.3 that $\chi_v \in H_B^1(\Omega(v))$ is the unique minimizer on $H_B^1(\Omega(v))$ of the functional

$$G_D(v)[\vartheta] = \frac{1}{2} \int_{\Omega(v)} |\nabla \vartheta|^2 \, d(x, z) + \frac{1}{2} \int_D \sigma |\vartheta(\cdot, -H)|^2 \, dx - \int_{\Omega(v)} \vartheta \Delta h_v \, d(x, z), \quad \vartheta \in H_B^1(\Omega(v)).$$

Furthermore, since Δh_v belongs to $L_2(\Omega(v))$ by Lemma 3.1, it follows from the definition of $H_B^1(\Omega(v))$ that

$$G_D(v)[\vartheta] = \sum_{j \in J} G_{I_j}(v)[\vartheta], \quad \vartheta \in H_B^1(\Omega(v)),$$

where $G_{I_j}(v)[\vartheta]$ is defined by (3.23) with $f := \Delta h_v \mathbf{1}_{\mathcal{O}_{I_j}(v)}$. Restricting to $\vartheta \in H_B^1(\mathcal{O}_{I_j}(v))$, it thus readily follows that $\chi_v \mathbf{1}_{\mathcal{O}_{I_j}(v)}$ is a minimizer of $G_{I_j}(v)$ on $H_B^1(\mathcal{O}_{I_j}(v))$. Consequently, $\chi_v \mathbf{1}_{\mathcal{O}_{I_j}(v)} = \zeta_{I_j, v}$ by Lemma 3.8. Hence Theorem 3.9 yields

$$\|\chi_v\|_{H^2(\mathcal{O}_{I_j}(v))} + \|\partial_x \chi_v(\cdot, -H)\|_{L_2(I_j)} \leq C_1(\kappa) \|\Delta h_v\|_{L_2(\mathcal{O}_{I_j}(v))}$$

and

$$\|\partial_z \chi_v(\cdot, v)\|_{L_r(I_j)} \leq r C_2(\kappa) \|\Delta h_v\|_{L_2(\mathcal{O}_{I_j}(v))}, \quad r \in [2, \infty),$$

with constants $C_1(\kappa)$ and $C_2(\kappa)$ not depending on I_j . Therefore, summing with respect to $j \in J$, we conclude that $\chi_v \in H^2(\Omega(v))$ and satisfies (3.9) and (3.10), since $\|\Delta h_v\|_{L_2(\Omega(v))} \leq c(\kappa)$ by Lemma 3.1. Therefore, as in [16, Theorem 3.5], we may use the version of Gauß' Theorem stated in [15, Folgerung 7.5] in the variational characterization of χ_v featuring $\mathcal{G}(v)$ to deduce that $\chi_v \in H^2(\Omega(v))$ is indeed a strong solution to (3.7). This proves Theorem 3.2. Owing to (3.6) and Lemma 3.1, this also entails Theorem 2.2.

4. CONTINUITY OF χ_v WITH RESPECT TO v

In this section we derive continuity properties of χ_v and its gradient trace $\partial_z \chi_v(\cdot, v)$ with respect to $v \in \bar{S}$. The latter will also yield the continuity of the function g defined in (2.6). Throughout this section we denote positive constants depending only on σ by C . The dependence upon additional parameters will be indicated explicitly.

4.1. H^1 -Continuity: Γ -convergence of \mathcal{G} . Let us recall that, according to Proposition 3.3, χ_v is the unique minimizer on $H_B^1(\Omega(v))$ of the functional $\mathcal{G}(v)$ introduced in (3.11) as

$$\mathcal{G}(v)[\vartheta] = \frac{1}{2} \int_{\Omega(v)} |\nabla(\vartheta + h_v)|^2 \, d(x, z) + \frac{1}{2} \int_D \sigma(x) |\vartheta(x, -H) + h_v(x, -H) - \mathfrak{h}_v(x)|^2 \, dx$$

for $\vartheta \in H_B^1(\Omega(v))$. Now, in order to derive continuity properties of χ_v (and ψ_v) with respect to $v \in \bar{S}$, we first prove a Gamma convergence result for the set of functionals $\{\mathcal{G}(v), v \in \bar{S}\}$. More precisely, given

$M > 0$ we set as before $\Omega(M) := D \times (-H, M)$ and, for $v \in \bar{S}$ such that $v \leq M - H$, we extend the functional $\mathcal{G}(v)$ to $L_2(\Omega(M))$ by defining

$$\mathcal{G}(v)[\vartheta] := \infty, \quad \vartheta \in L_2(\Omega(M)) \setminus H_B^1(\Omega(v)).$$

With these notations we have:

Proposition 4.1. *Let $M > 0$ and consider a sequence $(v_n)_{n \geq 1}$ in \bar{S} and $v \in \bar{S}$ such that*

$$-H \leq v_n(x), v(x) \leq M - H, \quad x \in D, \quad v_n \rightarrow v \text{ in } H^1(D). \quad (4.1)$$

Then

$$\Gamma - \lim_{n \rightarrow \infty} \mathcal{G}(v_n) = \mathcal{G}(v) \quad \text{in } L_2(\Omega(M)).$$

Proof. The proof is very similar to that of [17, Proposition 3.11].

(i) *Asymptotic weak lower semi-continuity.* Given a sequence $(\vartheta_n)_{n \geq 1}$ in $L_2(\Omega(M))$ and $\vartheta \in L_2(\Omega(M))$ satisfying

$$\vartheta_n \rightarrow \vartheta \text{ in } L_2(\Omega(M)), \quad (4.2)$$

we shall show that

$$\mathcal{G}(v)[\vartheta] \leq \liminf_{n \rightarrow \infty} \mathcal{G}(v_n)[\vartheta_n]. \quad (4.3)$$

We may assume without of loss of generality that

$$\vartheta_n \in H_B^1(\Omega(v_n)), \quad n \geq 1, \quad \mathcal{G}_\infty := \sup_{n \geq 1} \mathcal{G}(v_n)[\vartheta_n] < \infty. \quad (4.4)$$

Let $n \geq 1$ and denote the extension by zero of ϑ_n to $\Omega(M) \setminus \Omega(v_n)$ by ϑ_n . Then $\vartheta_n \in H_B^1(\Omega(M))$ and it follows from (4.1), (4.2), (4.4), and Lemma 3.1 (b) that the sequence $(\vartheta_n)_{n \geq 1}$ is bounded in $H_B^1(\Omega(M))$. Since $\Omega(M)$ is a Lipschitz domain, the compactness of the embedding of $H^1(\Omega(M))$ in $H^{3/4}(\Omega(M))$ [12, Theorem 1.4.3.2], the continuity of the trace operator from $H^{3/4}(\Omega(M))$ to $L_2(\partial\Omega(M))$ (see, e.g., [12, Theorem 1.5.1.2], [26], or [34, Satz 8.7]) and (4.2) ensure that there is a subsequence of $(\vartheta_n)_{n \geq 1}$ (not relabeled) such that

$$\vartheta_n \rightharpoonup \vartheta \text{ in } H_B^1(\Omega(M)), \quad (4.5)$$

$$\vartheta_n \rightarrow \vartheta \text{ in } L_2(\partial\Omega(M)). \quad (4.6)$$

In particular, $\vartheta \in H^1(\Omega(v))$ and its trace $\vartheta(\cdot, v)$ is well-defined in $L_2(D, (H+v) dx)$ according to Lemma 3.6. Similarly, for each $n \geq 1$, $\vartheta \in H^1(\Omega(v_n))$ and its trace $\vartheta(\cdot, v_n)$ is well-defined in $L_2(D, (H+v_n) dx)$. Consequently, for $n \geq 1$,

$$\begin{aligned} \int_D (H+v)(H+v_n) |\vartheta(\cdot, v)|^2 dx &\leq 2 \int_D (H+v)(H+v_n) |\vartheta(\cdot, v) - \vartheta(\cdot, v_n)|^2 dx \\ &\quad + 2 \int_D (H+v)(H+v_n) |\vartheta(\cdot, v_n)|^2 dx. \end{aligned} \quad (4.7)$$

On the one hand, by Lemma A.1 and (4.1),

$$\begin{aligned} \int_D (H+v)(H+v_n) |\vartheta(\cdot, v) - \vartheta(\cdot, v_n)|^2 dx &\leq \|(H+v)(H+v_n)(v-v_n)\|_{L_\infty(D)} \|\partial_z \vartheta\|_{L_2(\Omega(M))}^2 \\ &\leq M^2 \|v-v_n\|_{L_\infty(D)} \|\partial_z \vartheta\|_{L_2(\Omega(M))}^2. \end{aligned} \quad (4.8)$$

On the other hand, since $\vartheta_n \in H_B^1(\Omega(v_n))$, we infer from (4.1) and Lemma 3.6 that

$$\begin{aligned}
& \int_D (H+v)(H+v_n)|\vartheta(\cdot, v_n)|^2 dx \\
&= \int_D (H+v)(H+v_n)|\vartheta(\cdot, v_n) - \vartheta_n(\cdot, v_n)|^2 dx \\
&\leq M \int_D (H+v_n)|\vartheta(\cdot, v_n) - \vartheta_n(\cdot, v_n)|^2 dx \\
&\leq M \left[\|\vartheta - \vartheta_n\|_{L_2(\Omega(v_n))}^2 + 2\|H+v_n\|_{L_\infty(D)}\|\vartheta - \vartheta_n\|_{L_2(\Omega(v_n))}\|\partial_z(\vartheta - \vartheta_n)\|_{L_2(\Omega(v_n))} \right] \\
&\leq M\|\vartheta - \vartheta_n\|_{L_2(\Omega(M))} \left[\sup_{m \geq 1} \|\vartheta - \vartheta_m\|_{L_2(\Omega(M))} + 2M \sup_{m \geq 1} \|\partial_z(\vartheta - \vartheta_m)\|_{L_2(\Omega(M))} \right] \\
&\leq 2M(1+M)\|\vartheta - \vartheta_n\|_{L_2(\Omega(M))} \left[\|\vartheta\|_{H^1(\Omega(M))} + \sup_{m \geq 1} \|\vartheta_m\|_{H^1(\Omega(M))} \right]. \tag{4.9}
\end{aligned}$$

Now, it readily follows from (4.1), (4.2), (4.5), (4.8), (4.9), and the continuous embedding of $H_0^1(D)$ in $C(\bar{D})$ that the right-hand side of (4.7) converges to zero as $n \rightarrow \infty$. Therefore,

$$\lim_{n \rightarrow \infty} \int_D (H+v)(H+v_n)|\vartheta(\cdot, v)|^2 dx = 0,$$

and we use Fatou's lemma to conclude that

$$\vartheta(\cdot, v) = 0 \quad \text{in} \quad L_2(D, (H+v)^2 dx).$$

Combining this result with (4.5) and (4.6) implies that

$$\vartheta \in H_B^1(\Omega(v)). \tag{4.10}$$

Now, we infer from (3.3), (4.1), (4.5), (4.10), and the continuous embedding of $H_0^1(D)$ in $C(\bar{D})$ that

$$\begin{aligned}
\int_{\Omega(v)} |\nabla(\vartheta + h_v)|^2 d(x, z) &= \int_{\Omega(M)} |\nabla(\vartheta + h_v)|^2 d(x, z) - \int_{\Omega(M) \setminus \Omega(v)} |\nabla h_v|^2 d(x, z) \\
&\leq \liminf_{n \rightarrow \infty} \int_{\Omega(M)} |\nabla(\vartheta_n + h_{v_n})|^2 d(x, z) - \lim_{n \rightarrow \infty} \int_{\Omega(M) \setminus \Omega(v_n)} |\nabla h_{v_n}|^2 d(x, z) \\
&= \liminf_{n \rightarrow \infty} \int_{\Omega(v_n)} |\nabla(\vartheta_n + h_{v_n})|^2 d(x, z).
\end{aligned}$$

Also, from (4.6) and Lemma 3.1 we deduce that

$$\lim_{n \rightarrow \infty} \int_D \sigma |(\vartheta_n + h_{v_n})(\cdot, -H) - \mathfrak{h}_{v_n}|^2 dx = \int_D \sigma |(\vartheta + h_v)(\cdot, -H) - \mathfrak{h}_v|^2 dx.$$

Gathering the outcome of the above analysis gives (4.3).

(ii) *Recovery sequence.* Consider $\vartheta \in H_B^1(\Omega(v))$ and introduce the function $\bar{\vartheta}$ defined on

$$\hat{\Omega}(M) := D \times (-2H - M, M)$$

by

$$\bar{\vartheta}(x, z) := \begin{cases} 0, & x \in D, v(x) < z < M, \\ \vartheta(x, z), & x \in D, -H < z \leq v(x), \\ \vartheta(x, -2H - z), & x \in D, -2H - v(x) < z \leq -H, \\ 0, & x \in D, -2H - M < z \leq -2H - v(x), \end{cases}$$

which is the extension of ϑ by zero in $\Omega(M) \setminus \Omega(v)$ and the reflection of the thus obtained function to $D \times (-2H - M, -H)$. Then $\bar{\vartheta} \in H_0^1(\hat{\Omega}(M))$, so that $F := -\Delta \bar{\vartheta} \in H^{-1}(\hat{\Omega}(M))$.

Let $n \geq 1$. Since

$$\hat{\Omega}(v_n) := \Omega(v_n) \cup (D \times (-2H - M, -H]) \subset \hat{\Omega}(M),$$

the distribution F can also be considered as an element of $H^{-1}(\hat{\Omega}(v_n))$ by restriction. Then there is a unique variational solution $\hat{\vartheta}_n \in H_0^1(\hat{\Omega}(v_n)) \subset H_0^1(\hat{\Omega}(M))$ to

$$-\Delta \hat{\vartheta}_n = F \quad \text{in } \hat{\Omega}(v_n), \quad \hat{\vartheta}_n = 0 \quad \text{on } \partial \hat{\Omega}(v_n).$$

Owing to (4.1) and the continuous embedding of $H_0^1(D)$ in $C(\bar{D})$,

$$d_H(\hat{\Omega}(v_n), \hat{\Omega}(v)) \leq \|v_n - v\|_{L^\infty(D)} \rightarrow 0,$$

where d_H stands for the Hausdorff distance in $\hat{\Omega}(M)$, see [14, Section 2.2.3]. Since $\overline{\hat{\Omega}(M)} \setminus \hat{\Omega}(v_n)$ has a single connected component for all $n \geq 1$, it follows from [33, Theorem 4.1] and [14, Theorem 3.2.5] that $\hat{\vartheta}_n \rightarrow \hat{\vartheta}$ in $H_0^1(\hat{\Omega}(M))$, where $\hat{\vartheta} \in H_0^1(\hat{\Omega}(M))$ is the unique variational solution to

$$-\Delta \hat{\vartheta} = F \quad \text{in } \hat{\Omega}(M), \quad \hat{\vartheta} = 0 \quad \text{on } \partial \hat{\Omega}(M).$$

Clearly, $\hat{\vartheta} = \bar{\vartheta}$ by uniqueness, so that $\hat{\vartheta}_n \rightarrow \bar{\vartheta}$ in $H_0^1(\hat{\Omega}(M))$. Setting $\vartheta_n := \hat{\vartheta}_n \mathbf{1}_{\Omega(v_n)} \in H^1(\Omega(M))$, $n \geq 1$, this convergence implies that

$$\vartheta_n \rightarrow \bar{\vartheta} \quad \text{in } H^1(\Omega(M)). \quad (4.11)$$

Since $\vartheta_n = 0$ in $\Omega(M) \setminus \Omega(v_n)$ we obtain from (3.3), (4.1), and (4.11) that

$$\begin{aligned} \int_{\Omega(v)} |\nabla(\vartheta + h_v)|^2 d(x, z) &= \int_{\Omega(v)} (|\nabla \vartheta|^2 + 2\nabla \vartheta \cdot \nabla h_v + |\nabla h_v|^2) d(x, z) \\ &= \int_{\Omega(M)} (|\nabla \vartheta|^2 + 2\nabla \vartheta \cdot \nabla h_v) d(x, z) + \int_{\Omega(v)} |\nabla h_v|^2 d(x, z) \\ &= \lim_{n \rightarrow \infty} \int_{\Omega(M)} (|\nabla \vartheta_n|^2 + 2\nabla \vartheta_n \cdot \nabla h_{v_n}) d(x, z) \\ &\quad + \lim_{n \rightarrow \infty} \int_{\Omega(v_n)} |\nabla h_{v_n}|^2 d(x, z) \\ &= \lim_{n \rightarrow \infty} \int_{\Omega(v_n)} |\nabla(\vartheta_n + h_{v_n})|^2 d(x, z). \end{aligned}$$

Moreover, the continuity of the trace from $H^1(\Omega(M))$ to $L_2(D \times \{-H\})$ and (4.11) entail that

$$\vartheta_n(\cdot, -H) \rightarrow \bar{\vartheta}(\cdot, -H) = \vartheta(\cdot, -H) \quad \text{in } L_2(D).$$

These two properties, along with (3.4) and (3.5), imply that

$$\mathcal{G}(v)[\vartheta] = \lim_{n \rightarrow \infty} \mathcal{G}(v_n)[\vartheta_n];$$

that is, $(\vartheta_n)_{n \geq 1}$ is a recovery sequence for ϑ and the claim is proved. \square

The Fundamental Theorem of Γ -convergence, see [9, Corollary 7.20], then yields the following continuous dependence of χ_v on $v \in \bar{S}$:

Corollary 4.2. *Suppose (4.1) and assume further that there is $\kappa > 0$ such that*

$$\|v\|_{H^2(D)} \leq \kappa \quad \text{and} \quad \|v_n\|_{H^2(D)} \leq \kappa, \quad n \geq 1. \quad (4.12)$$

Then

$$\lim_{n \rightarrow \infty} \mathcal{G}(v_n)[\chi_{v_n}] = \mathcal{G}(v)[\chi_v] \quad (4.13)$$

and

$$\lim_{n \rightarrow \infty} \|\chi_{v_n} - \chi_v\|_{H^1(\Omega(M))} = \lim_{n \rightarrow \infty} \|\chi_{v_n}(\cdot, -H) - \chi_v(\cdot, -H)\|_{L_r(D)} = 0, \quad r \in [1, \infty). \quad (4.14)$$

Proof. It readily follows from (4.1), (4.12), and Theorem 3.2 that

$$(\chi_{v_n})_{n \geq 1} \text{ is bounded in } H^1(\Omega(M)) \quad (4.15)$$

and thus relatively compact in $L_2(\Omega(M))$ by [12, Theorem 1.4.5.2]. According to Proposition 4.1, we deduce from the Fundamental Theorem of Γ -convergence, see [9, Corollary 7.20], that any cluster point of $(\chi_{v_n})_{n \geq 1}$ in $L_2(\Omega(M))$ is a minimizer of $\mathcal{G}(v)$ and thus coincides with χ_v by Proposition 3.3. Therefore,

$$\lim_{n \rightarrow \infty} \|\chi_{v_n} - \chi_v\|_{L_2(\Omega(M))} = 0, \quad (4.16)$$

and, using once more [9, Corollary 7.20], we obtain (4.13).

We are left with proving (4.14). To this end, we first observe that, since $\Omega(M)$ is a Lipschitz domain, [12, Theorem 1.4.3.2, Theorem 1.4.5.2] imply that $H^1(\Omega(M))$ compactly embeds in $W_q^{3/2q}(\Omega(M))$ for $q \geq 2$. Thus, the continuity of the trace operator from $W_q^{3/2q}(\Omega(M))$ to $L_q(\partial\Omega(M))$ (see [12, Theorem 1.5.1.2] and [26]), along with (4.15) and (4.16), ensure that there is a subsequence of $(\chi_{v_n})_{n \geq 1}$ (not relabeled) such that

$$\chi_{v_n} \rightharpoonup \chi_v \quad \text{in } H_B^1(\Omega(M)), \quad (4.17)$$

$$\chi_{v_n}(\cdot, -H) \rightarrow \chi_v(\cdot, -H) \quad \text{in } L_q(D), \quad q \geq 2. \quad (4.18)$$

Notice that (4.18) yields the second assertion of (4.14). It now follows from (3.3), (3.4), (3.5), (4.13), and (4.18) that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\nabla(\chi_{v_n} + h_{v_n})\|_{L_2(\Omega(M))}^2 &= \lim_{n \rightarrow \infty} \|\nabla(\chi_{v_n} + h_{v_n})\|_{L_2(\Omega(v_n))}^2 + \lim_{n \rightarrow \infty} \|\nabla h_{v_n}\|_{L_2(\Omega(M) \setminus \Omega(v_n))}^2 \\ &= \|\nabla(\chi_v + h_v)\|_{L_2(\Omega(v))}^2 + \|\nabla h_v\|_{L_2(\Omega(M) \setminus \Omega(v))}^2 \\ &= \|\nabla(\chi_v + h_v)\|_{L_2(\Omega(M))}^2. \end{aligned}$$

This property, along with (3.3) and (4.17), guarantees that $(\nabla \chi_{v_n})_{n \geq 1}$ converges to $\nabla \chi_v$ in $L_2(\Omega(M))$ and the proof of (4.14) is complete. \square

4.2. Continuity of $\partial_z \chi_v(\cdot, v)$ with Respect to v . Finally, in order to establish the continuity of the function g defined in (2.6) we need also to investigate the continuous dependence of the gradient trace $\partial_z \chi_v(\cdot, v)$ on $v \in \bar{S}$, the main difficulty arising when $\mathcal{C}(v) \neq \emptyset$. In this regard we note:

Proposition 4.3. *Consider $v \in \bar{S}$ and a sequence $(v_n)_{n \geq 1}$ in \bar{S} such that*

$$\|v\|_{H^2(D)} + \sup_{n \geq 1} \|v_n\|_{H^2(D)} \leq \kappa \quad \text{and} \quad \lim_{n \rightarrow \infty} \|v_n - v\|_{H^1(D)} = 0. \quad (4.19)$$

Then

$$\ell(v_n) \rightarrow \ell(v) \quad \text{in } L_r(D) \quad \text{for } r \in [1, \infty), \quad (4.20)$$

where $\ell(v)$ is given by

$$\ell(v)(x) := \begin{cases} \partial_z \chi_v(x, v(x)), & x \in D \setminus \mathcal{C}(v), \\ 0, & x \in \mathcal{C}(v). \end{cases}$$

Proof. Thanks to (4.19) and the continuous embedding of $H^2(D)$ in $L_\infty(D)$, we may fix $M > H$ (only depending on κ) such that

$$-H \leq v_n(x), v(x) \leq M - H, \quad x \in \bar{D}, \quad n \geq 1. \quad (4.21)$$

Step 1. We first establish an estimate ensuring that there is no concentration of $\partial_z \chi_v(\cdot, v)$ on small subsets of $D \setminus \mathcal{C}(v)$. Indeed, since $\chi_v \in H^2(\Omega(v))$ we have $\chi_v(x, \cdot) \in H^2((-H, v(x)))$ for a.a. $x \in D \setminus \mathcal{C}(v)$, so that it follows from the boundary conditions (3.18b) and (3.18c) that

$$\begin{aligned} \partial_z \chi_v(x, v(x)) &= \partial_z \chi_v(x, -H) + \int_{-H}^{v(x)} \partial_z^2 \chi_v(x, z) \, dz \\ &= \sigma(x) \chi_v(x, -H) + \int_{-H}^{v(x)} \partial_z^2 \chi_v(x, z) \, dz \\ &= \sigma(x) \left(\chi_v(x, v(x)) - \int_{-H}^{v(x)} \partial_z \chi_v(x, z) \, dz \right) + \int_{-H}^{v(x)} \partial_z^2 \chi_v(x, z) \, dz \\ &= \int_{-H}^{v(x)} (\partial_z^2 \chi_v(x, z) - \sigma(x) \partial_z \chi_v(x, z)) \, dz \end{aligned}$$

for a.a. $x \in D \setminus \mathcal{C}(v)$. Thus, for an arbitrary measurable subset $E \subset D \setminus \mathcal{C}(v)$, we infer from Hölder's inequality that

$$\begin{aligned} &\int_E |\partial_z \chi_v(x, v(x))| \, dx \\ &\leq \int_E \int_{-H}^{v(x)} (|\partial_z^2 \chi_v(x, z)| + \sigma(x) |\partial_z \chi_v(x, z)|) \, dz \, dx \\ &\leq \left(\int_E (H + v)(x) \, dx \right)^{1/2} \left(\int_{\Omega(v)} (2|\partial_z^2 \chi_v(x, z)|^2 + 2\|\sigma\|_\infty^2 |\partial_z \chi_v(x, z)|^2) \, d(x, z) \right)^{1/2} \\ &\leq C \left(\int_E (H + v)(x) \, dx \right)^{1/2} \|\chi_v\|_{H^2(\Omega(v))}. \end{aligned} \quad (4.22a)$$

Clearly, the same proof implies that, for any $n \geq 1$ and arbitrary measurable subset $E \subset D \setminus \mathcal{C}(v_n)$,

$$\int_E |\partial_z \chi_{v_n}(x, v_n(x))| dx \leq C \left(\int_E (H + v_n)(x) dx \right)^{1/2} \|\chi_{v_n}\|_{H^2(\Omega(v_n))}. \quad (4.22b)$$

Step 2. We next handle the behavior of $\partial_z \chi_v(\cdot, v)$ where v stays away from $-H$. To this end, let $\varepsilon \in (0, H/2)$ and define

$$\Lambda(\varepsilon) := \{x \in D : v(x) > -H + 2\varepsilon\}, \quad (4.23)$$

which is a non-empty open subset of D , since $v \in C(\bar{D})$ with $v(\pm L) = 0$. We can thus write it as a countable union of disjoint open intervals $(\Lambda_j(\varepsilon))_{j \in J}$, see [1, IX.Proposition 1.8]. Also, owing to (4.19) and the continuous embedding of $H^1(D)$ in $C(\bar{D})$, there is $n_\varepsilon \geq 1$ such that

$$v(x) - \varepsilon \leq v_n(x) \leq v(x) + \varepsilon, \quad x \in \bar{D}, \quad n \geq n_\varepsilon. \quad (4.24)$$

A straightforward consequence of (4.23) and (4.24) is that

$$\{(x, z) \in \Lambda(\varepsilon) \times [-H, \infty) : -H < z < v(x) - \varepsilon\} \subset \Omega(v_n), \quad n \geq n_\varepsilon. \quad (4.25)$$

Therefore, the function $X_{n,\varepsilon}$, given by

$$X_{n,\varepsilon}(x) := \partial_z \chi_v(x, v(x) - \varepsilon) - \partial_z \chi_{v_n}(x, v(x) - \varepsilon), \quad x \in \Lambda(\varepsilon), \quad n \geq n_\varepsilon,$$

is well-defined. Let $j \in J$ and $n \geq n_\varepsilon$. Since $\partial_z \chi_v$ and $\partial_z \chi_{v_n}$ both belong to $H^1(\mathcal{O}_{\Lambda_j(\varepsilon)}(v - \varepsilon))$, the set $\mathcal{O}_{\Lambda_j(\varepsilon)}(v - \varepsilon)$ being defined in (3.14), it follows from (3.19), (4.21), and the definition of $\Lambda(\varepsilon)$ that

$$\begin{aligned} \varepsilon \int_{\Lambda_j(\varepsilon)} |X_{n,\varepsilon}(x)|^2 dx &\leq \int_{\Lambda_j(\varepsilon)} |X_{n,\varepsilon}(x)|^2 (H + v(x) - \varepsilon) dx \\ &\leq \|\partial_z(\chi_v - \chi_{v_n})\|_{L_2(\mathcal{O}_{\Lambda_j(\varepsilon)}(v - \varepsilon))}^2 \\ &\quad + 2\|H + v - \varepsilon\|_{L_\infty(\Lambda_j(\varepsilon))} \|\partial_z(\chi_v - \chi_{v_n})\|_{L_2(\mathcal{O}_{\Lambda_j(\varepsilon)}(v - \varepsilon))} \|\partial_z^2(\chi_v - \chi_{v_n})\|_{L_2(\mathcal{O}_{\Lambda_j(\varepsilon)}(v - \varepsilon))} \\ &\leq \|\partial_z(\chi_v - \chi_{v_n})\|_{L_2(\mathcal{O}_{\Lambda_j(\varepsilon)}(M))}^2 \\ &\quad + C(\kappa) \|\partial_z(\chi_v - \chi_{v_n})\|_{L_2(\mathcal{O}_{\Lambda_j(\varepsilon)}(M))} \left(\|\partial_z^2 \chi_v\|_{L_2(\mathcal{O}_{\Lambda_j(\varepsilon)}(v))} + \|\partial_z^2 \chi_{v_n}\|_{L_2(\mathcal{O}_{\Lambda_j(\varepsilon)}(v_n))} \right). \end{aligned}$$

Summing the above inequality over $j \in J$ and noticing that

$$\begin{aligned} &\sum_{j \in J} \|\partial_z(\chi_v - \chi_{v_n})\|_{L_2(\mathcal{O}_{\Lambda_j(\varepsilon)}(M))} \left(\|\partial_z^2 \chi_v\|_{L_2(\mathcal{O}_{\Lambda_j(\varepsilon)}(v))} + \|\partial_z^2 \chi_{v_n}\|_{L_2(\mathcal{O}_{\Lambda_j(\varepsilon)}(v_n))} \right) \\ &\leq \left(\sum_{j \in J} \|\partial_z(\chi_v - \chi_{v_n})\|_{L_2(\mathcal{O}_{\Lambda_j(\varepsilon)}(M))}^2 \right)^{1/2} \left(\sum_{j \in J} \left(\|\partial_z^2 \chi_v\|_{L_2(\mathcal{O}_{\Lambda_j(\varepsilon)}(v))} + \|\partial_z^2 \chi_{v_n}\|_{L_2(\mathcal{O}_{\Lambda_j(\varepsilon)}(v_n))} \right)^2 \right)^{1/2} \\ &\leq \sqrt{2} \|\partial_z(\chi_v - \chi_{v_n})\|_{L_2(\Omega(M))} \left(\sum_{j \in J} \left(\|\partial_z^2 \chi_v\|_{L_2(\mathcal{O}_{\Lambda_j(\varepsilon)}(v))}^2 + \|\partial_z^2 \chi_{v_n}\|_{L_2(\mathcal{O}_{\Lambda_j(\varepsilon)}(v_n))}^2 \right) \right)^{1/2} \\ &\leq \sqrt{2} \|\partial_z(\chi_v - \chi_{v_n})\|_{L_2(\Omega(M))} \left(\|\partial_z^2 \chi_v\|_{L_2(\Omega(v))} + \|\partial_z^2 \chi_{v_n}\|_{L_2(\Omega(v_n))} \right) \\ &\leq C(\kappa) \|\partial_z(\chi_v - \chi_{v_n})\|_{L_2(\Omega(M))} \end{aligned}$$

by the Cauchy-Schwarz inequality, (4.19), Theorem 3.2, we obtain

$$\varepsilon \int_{\Lambda(\varepsilon)} |X_{n,\varepsilon}(x)|^2 dx \leq \|\partial_z(\chi_v - \chi_{v_n})\|_{L_2(\Omega(M))}^2 + C(\kappa) \|\partial_z(\chi_v - \chi_{v_n})\|_{L_2(\Omega(M))}.$$

We now infer from (4.14) and the above inequality that

$$\lim_{n \rightarrow \infty} \int_{\Lambda(\varepsilon)} |X_{n,\varepsilon}(x)|^2 dx = 0. \quad (4.26)$$

We next set

$$Y_n(x) := \partial_z \chi_v(x, v(x)) - \partial_z \chi_{v_n}(x, v_n(x)), \quad x \in \Lambda(\varepsilon), \quad n \geq n_\varepsilon.$$

Using (4.24) and Hölder's and Young's inequalities, we obtain, for $j \in J$,

$$\begin{aligned} \|Y_n\|_{L_1(\Lambda_j(\varepsilon))} &\leq \|X_{n,\varepsilon}\|_{L_1(\Lambda_j(\varepsilon))} + \int_{\Lambda_j(\varepsilon)} \left| \int_{v-\varepsilon}^v \partial_z^2 \chi_v(\cdot, z) dz - \int_{v-\varepsilon}^{v_n} \partial_z^2 \chi_{v_n}(\cdot, z) dz \right| dx \\ &\leq \|X_{n,\varepsilon}\|_{L_1(\Lambda_j(\varepsilon))} + \int_{\Lambda_j(\varepsilon)} \int_{v-\varepsilon}^v |\partial_z^2 \chi_v(\cdot, z)| dz dx + \int_{\Lambda_j(\varepsilon)} \int_{v-\varepsilon}^{v_n} |\partial_z^2 \chi_{v_n}(\cdot, z)| dz dx \\ &\leq \|X_{n,\varepsilon}\|_{L_1(\Lambda_j(\varepsilon))} + \sqrt{\varepsilon |\Lambda_j(\varepsilon)|} \left(\int_{\Lambda_j(\varepsilon)} \int_{v-\varepsilon}^v |\partial_z^2 \chi_v(\cdot, z)|^2 dz dx \right)^{1/2} \\ &\quad + \sqrt{2\varepsilon |\Lambda_j(\varepsilon)|} \left(\int_{\Lambda_j(\varepsilon)} \int_{v-\varepsilon}^{v_n} |\partial_z^2 \chi_{v_n}(\cdot, z)|^2 dz dx \right)^{1/2} \\ &\leq \|X_{n,\varepsilon}\|_{L_1(\Lambda_j(\varepsilon))} + \frac{\sqrt{\varepsilon}}{2} |\Lambda_j(\varepsilon)| + \frac{\sqrt{\varepsilon}}{2} \int_{\Lambda_j(\varepsilon)} \int_{-H}^v |\partial_z^2 \chi_v(\cdot, z)|^2 dz dx \\ &\quad + \frac{\sqrt{\varepsilon}}{2} |\Lambda_j(\varepsilon)| + \sqrt{\varepsilon} \int_{\Lambda_j(\varepsilon)} \int_{-H}^{v_n} |\partial_z^2 \chi_{v_n}(\cdot, z)|^2 dz dx. \end{aligned}$$

Summing over $j \in J$ and using (4.19) and Theorem 3.2 give

$$\begin{aligned} \|Y_n\|_{L_1(\Lambda(\varepsilon))} &\leq \|X_{n,\varepsilon}\|_{L_1(\Lambda(\varepsilon))} + \sqrt{\varepsilon} |\Lambda(\varepsilon)| + \sqrt{\varepsilon} \|\chi_v\|_{H^2(\Omega(v))} + \sqrt{\varepsilon} \|\chi_{v_n}\|_{H^2(\Omega(v_n))} \\ &\leq \|X_{n,\varepsilon}\|_{L_1(\Lambda(\varepsilon))} + C(\kappa) \sqrt{\varepsilon}. \end{aligned}$$

Owing to (4.26), we may take the limit $n \rightarrow \infty$ in the previous inequality and obtain

$$\limsup_{n \rightarrow \infty} \|Y_n\|_{L_1(\Lambda(\varepsilon))} \leq C(\kappa) \sqrt{\varepsilon}.$$

Since $\Lambda(\varepsilon) \subset \Lambda(\delta)$ for all $\delta \in (0, \varepsilon)$, we infer from the above inequality that

$$\limsup_{n \rightarrow \infty} \|Y_n\|_{L_1(\Lambda(\varepsilon))} \leq \limsup_{n \rightarrow \infty} \|Y_n\|_{L_1(\Lambda(\delta))} \leq C(\kappa) \sqrt{\delta}$$

and we may pass to the limit $\delta \rightarrow 0$ to conclude that

$$\lim_{n \rightarrow \infty} \|Y_n\|_{L_1(\Lambda(\varepsilon))} = 0, \quad \varepsilon \in (0, H/2). \quad (4.27)$$

Step 3. Finally, we infer from (4.19), (4.21), (4.22), and Theorem 3.2 that

$$\|\ell(v_n) - \ell(v)\|_{L_1(D)}$$

$$\begin{aligned}
&\leq \int_{\Lambda(\varepsilon)} |\ell(v_n) - \ell(v)| \, dx + \int_{D \setminus \Lambda(\varepsilon)} |\ell(v_n)| \, dx + \int_{D \setminus \Lambda(\varepsilon)} |\ell(v)| \, dx \\
&= \|Y_n\|_{L_1(\Lambda(\varepsilon))} + \int_{(D \setminus \Lambda(\varepsilon)) \setminus \mathcal{C}(v_n)} |\partial_z \chi_{v_n}(\cdot, v_n)| \, dx + \int_{(D \setminus \Lambda(\varepsilon)) \setminus \mathcal{C}(v)} |\partial_z \chi_v(\cdot, v)| \, dx \\
&\leq \|Y_n\|_{L_1(\Lambda(\varepsilon))} + C \left(\int_{(D \setminus \Lambda(\varepsilon)) \setminus \mathcal{C}(v_n)} (H + v_n)(x) \, dx \right)^{1/2} \|\chi_{v_n}\|_{H^2(\Omega(v_n))} \\
&\quad + C \left(\int_{(D \setminus \Lambda(\varepsilon)) \setminus \mathcal{C}(v)} (H + v)(x) \, dx \right)^{1/2} \|\chi_v\|_{H^2(\Omega(v))} \\
&\leq \|Y_n\|_{L_1(\Lambda(\varepsilon))} + C(\kappa) \left(\int_{D \setminus \Lambda(\varepsilon)} (H + v)(x) \, dx \right)^{1/2} + C(\kappa) \left(\int_{D \setminus \Lambda(\varepsilon)} (H + v_n)(x) \, dx \right)^{1/2}.
\end{aligned}$$

Since $0 \leq H + v \leq 2\varepsilon$ and $0 \leq H + v_n \leq 3\varepsilon$ in $D \setminus \Lambda(\varepsilon)$ for $n \geq n_\varepsilon$ by (4.23) and (4.24), we further obtain

$$\|\ell(v_n) - \ell(v)\|_{L_1(D)} \leq \|Y_n\|_{L_1(\Lambda(\varepsilon))} + C(\kappa)\sqrt{\varepsilon}.$$

We now first let $n \rightarrow \infty$ with the help of (4.27) and then take the limit $\varepsilon \rightarrow 0$ to conclude that

$$\lim_{n \rightarrow \infty} \|\ell(v_n) - \ell(v)\|_{L_1(D)} = 0. \quad (4.28)$$

Finally, given $r \in [1, \infty)$, we infer from Hölder's inequality, Lemma 3.1, (3.10), and (4.19) that

$$\begin{aligned}
\|\ell(v_n) - \ell(v)\|_{L_r(D)} &\leq \|\ell(v_n) - \ell(v)\|_{L_1(D)}^{1/(2r-1)} \|\ell(v_n) - \ell(v)\|_{L_{2r}(D)}^{2(r-1)/(2r-1)} \\
&\leq \|\ell(v_n) - \ell(v)\|_{L_1(D)}^{1/(2r-1)} \left(\|\ell(v_n)\|_{L_{2r}(D)}^{2(r-1)/(2r-1)} + \|\ell(v)\|_{L_{2r}(D)}^{2(r-1)/(2r-1)} \right) \\
&\leq C(\kappa, r) \|\ell(v_n) - \ell(v)\|_{L_1(D)}^{1/(2r-1)}
\end{aligned}$$

and the assertion follows from (4.28). \square

Summarizing the outcome of this section, we have obtained continuity properties of the electrostatic energy E_e and the function g introduced in (2.6).

Theorem 4.4. *The electrostatic energy $E_e : \bar{S} \rightarrow \mathbb{R}$ is continuous for the weak topology of $H^2(D)$. The function $g : \bar{S} \rightarrow L_r(D)$ is continuous for each $r \in [1, \infty)$, the set \bar{S} being still endowed with the weak topology of $H^2(D)$.*

Proof. Let us first recall that, if $(v_n)_{n \geq 1}$ is a sequence in \bar{S} converging weakly in $H^2(D)$ to $v \in \bar{S}$, then there is $\kappa > 0$ such that (4.12) and (4.19) hold true. Consequently, we infer from Corollary 4.2 that

$$\lim_{n \rightarrow \infty} E_e(v_n) = - \lim_{n \rightarrow \infty} \mathcal{G}(v_n)[\chi_{v_n}] = -\mathcal{G}(v)[\chi_v] = E_e(v),$$

thereby establishing the stated continuity of E_e . Next, let $v \in \bar{S}$. Since $\partial_x v = 0$ a.e. in $\mathcal{C}(v)$, it follows from (2.6) and Proposition 4.3 that

$$\begin{aligned}
g(v)(x) &= \frac{1}{2} (1 + |\partial_x v(x)|^2) [\ell(v)(x) - (\partial_w h)_v(x, v(x))]^2 \\
&\quad + \sigma(x) [\chi_v(x, -H) + h_v(x, -H) - \mathfrak{h}_v(x)] (\partial_w \mathfrak{h})_v(x)
\end{aligned}$$

$$-\frac{1}{2} \left[|(\partial_x h)_v|^2 + ((\partial_z h)_v + (\partial_w h)_v)^2 \right] (x, v(x))$$

for $x \in D$. The stated continuity of g then readily follows from Proposition 4.3 and the C^1 -regularity of h and \mathfrak{h} (see also Lemma 3.1**(b)**). \square

5. SHAPE DERIVATIVE OF THE ELECTROSTATIC ENERGY

In this section we investigate differentiability properties of the electrostatic energy

$$E_e(u) = -\frac{1}{2} \int_{\Omega(u)} |\nabla \psi_u|^2 d(x, z) - \frac{1}{2} \int_D \sigma(x) |\psi_u(x, -H) - \mathfrak{h}_u(x)|^2 dx$$

with respect to $u \in \bar{S}$, where ψ_u is the strong solution to (1.3), see Theorem 2.2. Owing to the dependence of ψ_u on the domain $\Omega(u)$ this resembles the computation of a shape derivative, a topic which has received considerable attention in recent years, see [8, 14, 32] and the references therein. Note that we may write alternatively $E_e(u) = -\mathcal{G}(u)[\psi_u - h_u]$, since $\chi_u = \psi_u - h_u$ is the strong solution to (3.7) (with $v = u$) given by Theorem 3.2.

As might be expected, the switch between boundary conditions for ψ_u when $\mathcal{C}(u) \neq \emptyset$ generates additional difficulties and we begin with the differentiability of ψ_u with respect to $u \in S$.

Lemma 5.1. *Let $u \in S$ be fixed and define, for $v \in S$, the transformation $\Theta_v : \Omega(u) \rightarrow \Omega(v)$ by*

$$\Theta_v(x, z) := \left(x, z + \frac{v(x) - u(x)}{H + u(x)}(z + H) \right), \quad (x, z) \in \Omega(u).$$

Then there exists a neighborhood U of u in S such that the mapping

$$U \rightarrow H_B^1(\Omega(u)), \quad v \mapsto \chi_v \circ \Theta_v$$

is continuously differentiable, where $\chi_v = \psi_v - h_v \in H_B^1(\Omega(v))$ solves (3.7), see Theorem 3.2, and S is endowed with the $H^2(D)$ -topology.

Proof. The proof follows the lines of [14, Theorem 5.3.2], a similar proof is given in [17, Lemma 4.1]. We thus only provide a very brief sketch here. Let $u \in S$ and $v \in S$. Setting $\xi_v := \chi_v \circ \Theta_v$ and performing a change of variables $(\bar{x}, \bar{z}) = \Theta_v(x, z)$, the weak formulation (3.12) satisfied by χ_v (as a critical point of $\mathcal{G}(v)$) can be written in the form

$$\begin{aligned} & \int_{\Omega(u)} J_v ((D\Theta_v)^{-1}(D\Theta_v^T)^{-1} \nabla \xi_v) \cdot \nabla \phi d(x, z) + \int_D \sigma(\xi_v \phi)(\cdot, -H) dx \\ &= - \int_{\Omega(u)} J_v ((D\Theta_v)^{-1}(\nabla h_v \circ \Theta_v)) \cdot \nabla \phi d(x, z) + \int_D \sigma[\mathfrak{h}_v - h_v(\cdot, -H)] \phi(\cdot, -H) dx \end{aligned} \quad (5.1)$$

for $\phi \in H_B^1(\Omega(u))$, where $J_v := |\det(D\Theta_v)|$. Therefore, (5.1) is equivalent to

$$F(v, \xi_v) = 0, \quad v \in S, \quad (5.2)$$

for some Fréchet differentiable function

$$F : S \times H_B^1(\Omega(u)) \rightarrow (H_B^1(\Omega(u)))'.$$

One then uses the Implicit Function Theorem to derive that ξ_v depends smoothly on v . \square

As a next step we establish the Fréchet differentiability of E_e on the open set S . For $u \in S$ recall that $g(u)$ is given by (2.6a) since $\mathcal{C}(u) = \emptyset$ in this case.

Proposition 5.2. *Let S be endowed with the $H^2(D)$ -topology. Then the electrostatic energy $E_e : S \rightarrow \mathbb{R}$ is continuously Fréchet differentiable with*

$$\partial_u E_e(u)[\vartheta] = \int_D g(u) \vartheta \, dx$$

for $u \in S$ and $\vartheta \in H^2(D) \cap H_0^1(D)$.

Proof. In this proof we shall use the notation from Lemma 5.1. We fix $u \in S$ and recall from Lemma 5.1 that the mapping $v \mapsto \xi_v = \chi_v \circ \Theta_v$ is continuously differentiable with respect to v in a neighborhood U of u in S and takes values in $H_B^1(\Omega(u))$. With $\psi_v = \chi_v + h_v$, $J_v = |\det(D\Theta_v)|$, and the change of variables $(\bar{x}, \bar{z}) = \Theta_v(x, z)$, we obtain that, for $v \in U$,

$$\begin{aligned} E_e(v) &= -\frac{1}{2} \int_{\Omega(v)} |\nabla \psi_v|^2 \, d(\bar{x}, \bar{z}) - \frac{1}{2} \int_D \sigma |\psi_v(\bar{x}, -H) - \mathfrak{h}_v(\bar{x})|^2 \, d\bar{x} \\ &= -\frac{1}{2} \int_{\Omega(u)} |(D\Theta_v^T)^{-1} \nabla \xi_v + \nabla h_v \circ \Theta_v|^2 J_v \, d(x, z) - \frac{1}{2} \int_D \sigma |(\xi_v + h_v)(x, -H) - \mathfrak{h}_v(x)|^2 \, dx. \end{aligned}$$

We introduce the functions

$$j(v) := (D\Theta_v^T)^{-1} \nabla \xi_v + \nabla h_v \circ \Theta_v \quad \text{in } \Omega(u),$$

$$m(v) := (\xi_v + h_v)(\cdot, -H) - \mathfrak{h}_v \quad \text{in } D.$$

Then, recalling that h and \mathfrak{h} are C^1 -functions in all their arguments by (2.1b), we conclude that the Fréchet derivative of E_e at u applied to $\vartheta \in H^2(D) \cap H_0^1(D)$ is given by

$$\begin{aligned} \partial_u E_e(u)[\vartheta] &= \partial_v E_e(v)[\vartheta]|_{v=u} = - \int_{\Omega(u)} j(u) \cdot (\partial_v j(v)[\vartheta]|_{v=u}) J_u \, d(x, z) \\ &\quad - \frac{1}{2} \int_{\Omega(u)} |j(u)|^2 (\partial_v J_v[\vartheta]|_{v=u}) \, d(x, z) - \int_D \sigma m(u) (\partial_v m(v)[\vartheta]|_{v=u}) \, dx. \end{aligned}$$

Using $J_u = 1$, $j(u) = \nabla \chi_u + \nabla h_u = \nabla \psi_u$ in $\Omega(u)$, and $m(u) = \psi_u - \mathfrak{h}_u$ in D , we see that

$$\begin{aligned} \partial_u E_e(u)[\vartheta] &= - \int_{\Omega(u)} \nabla \psi_u \cdot (\partial_v j(v)[\vartheta]|_{v=u}) \, d(x, z) - \frac{1}{2} \int_{\Omega(u)} |\nabla \psi_u|^2 (\partial_v J_v[\vartheta]|_{v=u}) \, d(x, z) \\ &\quad - \int_D \sigma [\psi_u(\cdot, -H) - \mathfrak{h}_u] (\partial_v m(v)[\vartheta]|_{v=u}) \, dx. \end{aligned}$$

Since

$$\partial_v J_v[\vartheta]|_{v=u} = \frac{\vartheta}{H+u} \quad \text{in } D$$

and

$$\partial_v m(v)[\vartheta]|_{v=u} = (\partial_v \xi_v[\vartheta]|_{v=u})(\cdot, -H) + (\partial_w h)_u(\cdot, -H) \vartheta - (\partial_w \mathfrak{h})_u \vartheta \quad \text{in } D,$$

it follows that

$$\partial_u E_e(u)[\vartheta] = - \int_{\Omega(u)} \nabla \psi_u \cdot (\partial_v j(v)[\vartheta]|_{v=u}) \, d(x, z) - \frac{1}{2} \int_{\Omega(u)} |\nabla \psi_u|^2 \frac{\vartheta}{H+u} \, d(x, z)$$

$$- \int_D \sigma [\psi_u(\cdot, -H) - \mathfrak{h}_u] [(\partial_v \xi_v[\vartheta]|_{v=u})(\cdot, -H) + (\partial_w h)_u(\cdot, -H) \vartheta - (\partial_w \mathfrak{h})_u \vartheta] dx.$$

Using that Θ_u is the identity on $\Omega(u)$, $D\Theta_u = \text{id}$, and that $\xi_u = \chi_u$, we compute from the definition of $j(v)$ that

$$\partial_v j(v)[\vartheta]|_{v=u} = -\partial_v(D\Theta_v^T)[\vartheta]|_{v=u} \nabla \chi_u + \partial_v(\nabla \xi_v)[\vartheta]|_{v=u} + \partial_v(\nabla h_v \circ \Theta_v)[\vartheta]|_{v=u}$$

in $\Omega(u)$. Now,

$$-\partial_v(D\Theta_v^T)[\vartheta]|_{v=u} \nabla \chi_u = -\partial_z \chi_u \nabla \left(\frac{(z+H)\vartheta}{H+u} \right) \quad \text{in } \Omega(u)$$

and

$$\partial_v(\nabla \xi_v)[\vartheta]|_{v=u} = \nabla(\partial_v \xi_v[\vartheta]|_{v=u}) \quad \text{in } \Omega(u).$$

Moreover,

$$\partial_v(\nabla h_v \circ \Theta_v)[\vartheta]|_{v=u} = \nabla((\partial_w h)_u \vartheta) + \frac{(z+H)\vartheta}{H+u} \nabla((\partial_z h)_u) \quad \text{in } \Omega(u).$$

The above three identities yield

$$\begin{aligned} \partial_u E_e(u)[\vartheta] &= - \int_{\Omega(u)} \nabla \psi_u \cdot \nabla(\partial_v \xi_v[\vartheta]|_{v=u} + (\partial_w h)_u \vartheta) d(x, z) \\ &\quad + \int_{\Omega(u)} \nabla \psi_u \cdot \left[\partial_z \chi_u \nabla \left(\frac{(z+H)\vartheta}{H+u} \right) - \frac{(z+H)\vartheta}{H+u} \nabla((\partial_z h)_u) \right] d(x, z) \\ &\quad - \frac{1}{2} \int_{\Omega(u)} |\nabla \psi_u|^2 \frac{\vartheta}{H+u} d(x, z) \\ &\quad - \int_D \sigma [\psi_u(\cdot, -H) - \mathfrak{h}_u] [(\partial_v \xi_v[\vartheta]|_{v=u})(\cdot, -H) + (\partial_w h)_u(\cdot, -H) \vartheta - (\partial_w \mathfrak{h})_u \vartheta] dx. \end{aligned} \tag{5.3}$$

Next we shall simplify the right-hand side of (5.3). Using Gauß' Theorem, the fact that ψ_u is a strong solution to (1.3a), $\vartheta = 0$ on ∂D , and the fact that $\partial_v \xi_v[\vartheta]|_{v=u}$ belongs to $H_B^1(\Omega(u))$, the first integral on the right-hand side of (5.3) can be rewritten in the form

$$\begin{aligned} &- \int_{\Omega(u)} \nabla \psi_u \cdot \nabla(\partial_v \xi_v[\vartheta]|_{v=u} + (\partial_w h)_u \vartheta) d(x, z) \\ &= - \int_D (\partial_w h)_u(x, u(x)) \vartheta(x) [\partial_z \psi_u - \partial_x u \partial_x \psi_u](x, u(x)) dx \\ &\quad + \int_D [(\partial_v \xi_v[\vartheta]|_{v=u})(x, -H) + (\partial_w h)_u(x, -H) \vartheta(x)] \partial_z \psi_u(x, -H) dx. \end{aligned}$$

Since, due to (1.3c),

$$\partial_z \psi_u(x, -H) = \sigma(x) [\psi_u(x, -H) - \mathfrak{h}_u(x)], \quad x \in D,$$

it follows that

$$\begin{aligned} &- \int_{\Omega(u)} \nabla \psi_u \cdot \nabla(\partial_v \xi_v[\vartheta]|_{v=u} + (\partial_w h)_u \vartheta) d(x, z) \\ &= - \int_D \vartheta(x) [(\partial_w h)_u(\partial_z \psi_u - \partial_x u \partial_x \psi_u)](x, u(x)) dx \end{aligned} \tag{5.4}$$

$$+ \int_D \sigma(x) [\psi_u(x, -H) - \mathfrak{h}_u(x)] [(\partial_v \xi_v [\vartheta]|_{v=u})(x, -H) + (\partial_w h)_u(x, -H) \vartheta(x)] dx.$$

On account of $(\partial_z h)_u = \partial_z \psi_u - \partial_z \chi_u$ in $\Omega(u)$, the second integral on the right-hand side of (5.3) can be written as

$$\begin{aligned} & \int_{\Omega(u)} \nabla \psi_u \cdot \left[\partial_z \chi_u \nabla \left(\frac{(z+H)\vartheta}{H+u} \right) - \frac{(z+H)\vartheta}{H+u} \nabla ((\partial_z h)_u) \right] d(x, z) \\ &= \int_{\Omega(u)} \nabla \psi_u \cdot \nabla \left(\partial_z \chi_u \frac{(z+H)\vartheta}{H+u} \right) d(x, z) - \int_{\Omega(u)} \nabla \psi_u \cdot \nabla (\partial_z \psi_u) \frac{(z+H)\vartheta}{H+u} d(x, z). \end{aligned} \quad (5.5)$$

Thanks again to Gauß' Theorem and using (1.3a) and the fact that

$$(x, z) \mapsto (z+H)\vartheta(x) \text{ vanishes on } \partial D \times (-H, 0) \text{ and on } D \times \{-H\}, \quad (5.6)$$

we obtain

$$\begin{aligned} & \int_{\Omega(u)} \nabla \psi_u \cdot \nabla \left(\partial_z \chi_u \frac{\vartheta(z+H)}{H+u} \right) d(x, z) \\ &= \int_D \vartheta(x) (\partial_z \chi_u [\partial_z \psi_u - \partial_x u \partial_x \psi_u]) (x, u(x)) dx \\ &= \int_D \vartheta(x) [(\partial_z \psi_u - (\partial_z h)_u) (\partial_z \psi_u - \partial_x u \partial_x \psi_u)] (x, u(x)) dx. \end{aligned}$$

We write the second integral in (5.5) in the form

$$- \int_{\Omega(u)} \nabla \psi_u \cdot \nabla (\partial_z \psi_u) \frac{(z+H)\vartheta}{H+u} d(x, z) = -\frac{1}{2} \int_{\Omega(u)} \partial_z (|\nabla \psi_u|^2) \frac{(z+H)\vartheta}{H+u} d(x, z)$$

and use integration by parts and (5.6) to get

$$\begin{aligned} & - \int_{\Omega(u)} \nabla \psi_u \cdot \nabla (\partial_z \psi_u) \frac{\vartheta(z+H)}{H+u} d(x, z) \\ &= \frac{1}{2} \int_{\Omega(u)} \frac{\vartheta}{H+u} |\nabla \psi_u|^2 d(x, z) - \frac{1}{2} \int_D \vartheta(x) |\nabla \psi_u(x, u(x))|^2 dx. \end{aligned}$$

Therefore, we deduce from (5.5) that

$$\begin{aligned} & \int_{\Omega(u)} \nabla \psi_u \cdot \left[\partial_z \chi_u \nabla \left(\frac{(z+H)\vartheta}{H+u} \right) - \frac{(z+H)\vartheta}{H+u} \nabla ((\partial_z h)_u) \right] d(x, z) \\ &= \frac{1}{2} \int_{\Omega(u)} \frac{\vartheta}{H+u} |\nabla \psi_u|^2 d(x, z) - \frac{1}{2} \int_D \vartheta(x) |\nabla \psi_u(x, u(x))|^2 dx \\ &+ \int_D \vartheta(x) [(\partial_z \psi_u - (\partial_z h)_u) (\partial_z \psi_u - \partial_x u \partial_x \psi_u)] (x, u(x)) dx. \end{aligned}$$

Combining this identity with (5.3) and (5.4) yields

$$\begin{aligned} \partial_u E_e(u)[\vartheta] &= \int_D \vartheta(x) \left[(\partial_z \psi_u - (\partial_z h)_u - (\partial_w h)_u) (\partial_z \psi_u - \partial_x u \partial_x \psi_u) \right] (x, u(x)) \, dx \\ &\quad - \frac{1}{2} \int_D \vartheta(x) |\nabla \psi_u(x, u(x))|^2 \, dx + \int_D \sigma(x) [\psi_u(x, -H) - \mathfrak{h}_u(x)] (\partial_w \mathfrak{h})_u(x) \vartheta(x) \, dx. \end{aligned} \quad (5.7)$$

Since (1.3b) entails $\psi_u(x, u(x)) = h(x, u(x), u(x))$, $x \in D$, we have

$$\partial_x \psi_u(x, u(x)) = (\partial_x h)_u(x, u(x)) - \partial_x u(x) [\partial_z \psi_u - (\partial_z h)_u - (\partial_w h)_u](x, u(x)), \quad x \in D,$$

and hence, for $x \in D$,

$$\begin{aligned} &\frac{1}{2} |\nabla \psi_u(x, u(x))|^2 - \left[(\partial_z \psi_u - (\partial_z h)_u - (\partial_w h)_u) (\partial_z \psi_u - \partial_x u \partial_x \psi_u) \right] (x, u(x)) \\ &= -\frac{1}{2} (1 + |\partial_x u(x)|^2) [\partial_z \psi_u - (\partial_z h)_u - (\partial_w h)_u]^2(x, u(x)) \\ &\quad + \frac{1}{2} [|(\partial_x h)_u|^2 + ((\partial_z h)_u + (\partial_w h)_u)^2](x, u(x)). \end{aligned}$$

Inserting this identity into (5.7) gives

$$\begin{aligned} \partial_u E_e(u)[\vartheta] &= \frac{1}{2} \int_D (1 + |\partial_x u(x)|^2) [\partial_z \psi_u - (\partial_z h)_u - (\partial_w h)_u]^2(x, u(x)) \vartheta(x) \, dx \\ &\quad - \frac{1}{2} \int_D [|(\partial_x h)_u|^2 + ((\partial_z h)_u + (\partial_w h)_u)^2](x, u(x)) \vartheta(x) \, dx \\ &\quad + \int_D \sigma(x) [\psi_u(x, -H) - \mathfrak{h}_u(x)] (\partial_w \mathfrak{h})_u(x) \vartheta(x) \, dx \\ &= \int_D g(u)(x) \vartheta(x) \, dx, \end{aligned}$$

according to (2.6a). Finally, the continuity of

$$\partial_u E_e : S \rightarrow \mathcal{L}(H^2(D) \cap H_0^1(D), \mathbb{R})$$

readily follows from Theorem 4.4. \square

We finally provide the differentiability property of E_e on the closed set \bar{S} . More precisely, we show that E_e admits a directional derivative at a point $u \in \bar{S}$ in any direction of $-u + S$, which is given by $g(u)$ defined in (2.6). Recall that $\mathcal{C}(u)$ may be non-empty in this case.

Proposition 5.3. *Let $u \in \bar{S}$ and $w \in S$. Then*

$$\lim_{s \rightarrow 0^+} \frac{1}{s} \left[E_e(u + s(w - u)) - E_e(u) \right] = \int_D g(u)(w - u) \, dx.$$

Proof. Fix $w \in S$ and note that

$$u_s := u + s(w - u) = (1 - s)u + sw \in S, \quad s \in (0, 1).$$

Since $u_s \in S$ for $s \in (0, 1)$, we obtain from Proposition 5.2 that

$$\frac{d}{ds} E_e(u_s) = \int_D g(u_s)(w - u) \, dx$$

for $s \in (0, 1)$. Therefore, letting $s \rightarrow 0$, we derive with the help of Theorem 4.4 that

$$\lim_{s \rightarrow 0^+} \frac{d}{ds} E_e(u_s) = \int_D g(u) (w - u) dx. \quad (5.8)$$

Now, Theorem 4.4 guarantees that $E_e(u_s) \rightarrow E_e(u)$ as $s \rightarrow 0$, so that

$$E_e(u_t) - E_e(u) = \int_0^t \frac{d}{ds} E_e(u_s) ds, \quad t \in (0, 1),$$

and we conclude from (5.8) that

$$\lim_{t \rightarrow 0^+} \frac{1}{t} (E_e(u_t) - E_e(u)) = \lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t \frac{d}{ds} E_e(u_s) ds = \int_D g(u) (w - u) dx,$$

as claimed. \square

6. PROOFS OF THEOREM 2.3 AND THEOREM 2.4 FOR $\alpha = 0$

In this section we deal with the case $\alpha = 0$ and recall that the total energy is then given by

$$E(u) = E_m(u) + E_e(u)$$

with mechanical energy

$$E_m(u) = \frac{\beta}{2} \|\partial_x^2 u\|_{L_2(D)}^2 + \frac{\tau}{2} \|\partial_x u\|_{L_2(D)}^2$$

and electrostatic energy

$$E_e(u) = -\frac{1}{2} \int_{\Omega(u)} |\nabla \psi_u|^2 d(x, z) - \frac{1}{2} \int_D \sigma(x) |\psi_u(x, -H) - \mathfrak{h}_u(x)|^2 dx.$$

6.1. Existence of a Minimizer of a Regularized Energy. As already noted in [17], the boundedness from below of the functional E is *a priori* unclear since $\alpha = 0$. To cope with this issue, we work with the regularized functional given by

$$\mathcal{E}_k(u) := E(u) + \frac{A}{2} \|(u - k)_+\|_{L_2(D)}^2, \quad u \in \bar{S}_0, \quad (6.1)$$

for $k \geq H$, where

$$A := 8 \left(\frac{K^4}{\beta} + 2K^2 \right),$$

and the constant K is introduced in (2.4).

Lemma 6.1. *For each $k \geq H$ the functional \mathcal{E}_k is bounded from below with*

$$\mathcal{E}_k(u) \geq \frac{\beta}{4} \|\partial_x^2 u\|_{L_2(D)}^2 + \frac{A}{4} \|(u - k)_+\|_{L_2(D)}^2 - c(k)$$

for some constant $c(k) > 0$.

Proof. By (2.3), (2.8), and Proposition 3.3,

$$\begin{aligned}
-E_e(u) &= \mathcal{G}(u)[\psi_u - h_u] \leq \mathcal{G}(u)[0] \\
&= \frac{1}{2} \int_{\Omega(u)} |\nabla h_u|^2 \, d(x, z) + \frac{1}{2} \int_D \sigma(x) [h_u(x, -H) - \mathfrak{h}_u(x)]^2 \, dx \\
&\leq \int_{\Omega(u)} [(\partial_x h)_u^2 + |\partial_x u|^2 (\partial_w h)_u^2 + (\partial_z h)_u^2] \, dx + \bar{\sigma} \int_D \left\{ [h_u(x, -H)]^2 + [\mathfrak{h}_u(x)]^2 \right\} \, dx \\
&\leq K^2 \int_{\Omega(u)} \left(2 \frac{1 + u(x)^2}{H + u(x)} + \frac{|\partial_x u(x)|^2}{H + u(x)} \right) \, d(x, z) + 2\bar{\sigma} K^2 |D| \\
&\leq K^2 \left(2|D| + 2\|u\|_{L_2(D)}^2 + \|\partial_x u\|_{L_2(D)}^2 \right) + 2\bar{\sigma} K^2 |D| \\
&= 2(1 + \bar{\sigma})|D|K^2 + 2K^2\|u\|_{L_2(D)}^2 + K^2\|\partial_x u\|_{L_2(D)}^2.
\end{aligned}$$

Now, since $u \in \bar{S}$,

$$\int_D |\partial_x u|^2 \, dx = - \int_D u \partial_x^2 u \, dx \leq \|u\|_{L_2(D)} \|\partial_x^2 u\|_{L_2(D)},$$

and we further obtain with the help of Young's inequality that

$$\begin{aligned}
-E_e(u) &\leq 2(1 + \bar{\sigma})|D|K^2 + 2K^2\|u\|_{L_2(D)}^2 + K^2\|u\|_{L_2(D)} \|\partial_x^2 u\|_{L_2(D)} \\
&\leq 2(1 + \bar{\sigma})|D|K^2 + \left(\frac{K^4}{\beta} + 2K^2 \right) \|u\|_{L_2(D)}^2 + \frac{\beta}{4} \|\partial_x^2 u\|_{L_2(D)}^2.
\end{aligned}$$

Using this estimate in the definition of $\mathcal{E}_k(u)$ along with

$$\begin{aligned}
\|u\|_{L_2(D)}^2 &= \int_D u^2 \mathbf{1}_{(k, \infty)}(u) \, dx + \int_D u^2 \mathbf{1}_{[-H, k]}(u) \, dx \\
&\leq 2\|(u - k)_+\|_{L_2(D)}^2 + 2k^2 \int_D \mathbf{1}_{(k, \infty)}(u) \, dx + k^2 \int_D \mathbf{1}_{[-H, k]}(u) \, dx \\
&\leq 2\|(u - k)_+\|_{L_2(D)}^2 + 2k^2 |D|,
\end{aligned}$$

we derive

$$\begin{aligned}
\mathcal{E}_k(u) &\geq \frac{\beta}{4} \|\partial_x^2 u\|_{L_2(D)}^2 - 2(1 + \bar{\sigma})|D|K^2 - \left(\frac{K^4}{\beta} + 2K^2 \right) \|u\|_{L_2(D)}^2 + \frac{A}{2} \|(u - k)_+\|_{L_2(D)}^2 \\
&\geq \frac{\beta}{4} \|\partial_x^2 u\|_{L_2(D)}^2 - c(k) + \left[\frac{A}{2} - 2 \left(\frac{K^4}{\beta} + 2K^2 \right) \right] \|(u - k)_+\|_{L_2(D)}^2 \\
&\geq \frac{\beta}{4} \|\partial_x^2 u\|_{L_2(D)}^2 + \frac{A}{4} \|(u - k)_+\|_{L_2(D)}^2 - c(k),
\end{aligned}$$

thereby completing the proof. \square

Due to the weak lower semicontinuity of E_m in $H^2(D)$ and the continuity of E_e with respect to the weak topology of $H^2(D)$ (see Theorem 4.4), Lemma 6.1 allows us to apply the direct method of the calculus of variations to derive the existence of a minimizer of \mathcal{E}_k in \bar{S}_0 .

Corollary 6.2. *For each $k \geq H$, the functional \mathcal{E}_k has at least one minimizer $u_k \in \bar{S}_0$; that is,*

$$\mathcal{E}_k(u_k) = \min_{\bar{S}_0} \mathcal{E}_k. \quad (6.2)$$

6.2. Derivation of the Euler-Lagrange Equation for the Regularized Energy. We shall next identify the Euler-Lagrange equation satisfied by a minimizer of the regularized energy \mathcal{E}_k on \bar{S}_0 .

Proposition 6.3. *Let $k \geq H$ and let $u \in \bar{S}_0$ be a minimizer of \mathcal{E}_k on \bar{S}_0 . Then u is an H^2 -weak solution to the variational inequality*

$$\beta \partial_x^4 u - \tau \partial_x^2 u + A(u - k)_+ + \partial \mathbb{I}_{\bar{S}_0}(u) \ni -g(u) \text{ in } D, \quad (6.3a)$$

where $\partial \mathbb{I}_{\bar{S}_0}$ is the subdifferential of the indicator function $\mathbb{I}_{\bar{S}_0}$ of the closed convex subset \bar{S}_0 of $H^2(D)$; that is,

$$\int_D \{ \beta \partial_x^2 u \partial_x^2 (w - u) + \tau \partial_x u \partial_x (w - u) + A(u - k)_+(w - u) \} dx \geq - \int_D g(u)(w - u) dx \quad (6.3b)$$

for all $w \in \bar{S}_0$.

Proof. Let $k \geq H$ be fixed. Consider a minimizer $u \in \bar{S}_0$ of \mathcal{E}_k on \bar{S}_0 and fix $w \in S_0 := \bar{S}_0 \cap S$. Owing to the convexity of \bar{S}_0 , the function $u + s(w - u) = (1 - s)u + sw$ belongs to S_0 for all $s \in (0, 1]$ and the minimizing property of u guarantees that

$$0 \leq \liminf_{s \rightarrow 0^+} \frac{1}{s} (\mathcal{E}_k(u + s(w - u)) - \mathcal{E}_k(u)).$$

Since $u \in \bar{S}_0 \subset \bar{S}$ and $w \in S_0 \subset S$, Proposition 5.3 implies that

$$0 \leq \int_D \{ \beta \partial_x^2 u \partial_x^2 (w - u) + \tau \partial_x u \partial_x (w - u) + A(u - k)_+(w - u) \} dx + \int_D g(u)(w - u) dx$$

for all $w \in S_0$. Since S_0 is dense in \bar{S}_0 and $(u, g(u))$ belongs to $H^2(D) \times L_2(D)$, this inequality also holds for any $w \in \bar{S}_0$. \square

Proposition 6.4. *There is $\kappa_0 \geq H$ depending only on K such that, if $u \in \bar{S}_0$ is any solution to the variational inequality (6.3) with $k \geq H$, then $\|u\|_{L^\infty(D)} \leq \kappa_0$.*

Proof. Owing to the continuous embedding of $H_0^1(D)$ in $C(\bar{D})$, the function u belongs to $C(\bar{D})$ with $u(\pm L) = 0$. Consequently, the set $\{x \in D : u(x) > -H\}$ is a non-empty open subset of D and we can write it as a countable union of disjoint open intervals $(I_j)_{j \in J}$, see [1, IX.Proposition 1.8]. Using once more the property $u(\pm L) = 0 > -H$, we may assume without loss of generality that $I_0 = (-L, a_0)$ and $I_1 = (b_0, L)$ for some $-L < a_0 < b_0 < L$, and $\bar{I}_j \subset (-L, L)$ for $j \in J$ with $j \geq 2$.

Step 1: Thanks to (2.3b) and (2.4a), we infer from Lemma 3.4 that $|\psi_u| \leq K$ in $\Omega(u)$. Combining this bound with (2.3), (2.4), (2.6), and (2.8) readily gives

$$g(u)(x) \geq -2\bar{\sigma}K^2 - K^2 =: -G_0, \quad x \in D. \quad (6.4)$$

Step 2: Consider first $j \in J$ with $j \geq 2$ and let $\theta \in \mathcal{D}(I_j)$. Since $u > -H$ in the support of θ , the function $u \pm \delta\theta$ belongs to S_0 for $\delta > 0$ small enough. We thus infer from (6.3b) that

$$\pm \delta \int_{I_j} \{ \beta \partial_x^2 u \partial_x^2 \theta + \tau \partial_x u \partial_x \theta + A(u - k)_+ \theta \} dx \geq \mp \delta \int_{I_j} g(u) \theta dx,$$

hence

$$\int_{I_j} \{ \beta \partial_x^2 u \partial_x^2 \theta + \tau \partial_x u \partial_x \theta + A(u - k)_+ \theta \} dx = - \int_{I_j} g(u) \theta dx.$$

Consequently, using the function S_{I_j} defined in Proposition D.1, we realize that $u - S_{I_j} \in H^2(I_j)$ is a weak solution to the boundary value problem

$$\beta \partial_x^4 w - \tau \partial_x^2 w = -G_0 - g(u) - A(u - k)_+ \quad \text{in } I_j, \quad (6.5a)$$

$$w = \partial_x w = 0 \quad \text{in } \partial I_j, \quad (6.5b)$$

the boundary conditions (6.5b) being a consequence of the definition of I_j , $j \geq 2$, the $H^2(D)$ -regularity of u , and the constraint $u \geq -H$. Taking into account that $g(u) + A(u - k)_+ \in L_2(I_j)$ by Theorem 4.4, classical elliptic regularity theory implies that $u - S_{I_j} \in H^4(I_j)$ is a strong solution to (6.5). Since the right hand side of (6.5a) is non-positive due to (6.4), it now follows from a version of Boggio's comparison principle [7, 13, 19, 29] that $u - S_{I_j} < 0$ in I_j , so that $u(x) \leq \kappa_0$ for $x \in \bar{I}_j$ and $j \geq 2$ by Proposition D.1.

Step 3: We next handle the case $j = 0$ in which $I_0 = (-L, a_0)$. We first argue as in the previous step to conclude that

$$\int_{I_0} \{ \beta \partial_x^2 u \partial_x^2 \theta + \tau \partial_x u \partial_x \theta + A(u - k)_+ \theta \} dx = - \int_{I_0} g(u) \theta dx \quad (6.6)$$

for all $\theta \in \mathcal{D}(I_0)$ and that $u(-L) = \partial_x u(-L) = u(a_0) + H = \partial_x u(a_0) = 0$. Consequently, we infer from (6.6) and Proposition D.1 that $u - S_{I_0} \in H^2(I_0)$ is a weak solution to the boundary value problem

$$\beta \partial_x^4 w - \tau \partial_x^2 w = -G_0 - g(u) - A(u - k)_+ \quad \text{in } I_0,$$

$$w = \partial_x w = 0 \quad \text{on } \partial I_0.$$

We then argue as in **Step 2** to establish that $u - S_{I_0} < 0$ in $I_0 = (-L, a_0)$. Hence, $u \leq \kappa_0$ in $[-L, a_0]$ by Proposition D.1.

Step 4: For the case I_1 we proceed as in **Step 3** using Proposition D.1 to deduce that $u \leq \kappa_0$ in $[b_0, L]$. This completes the proof. \square

6.3. Proof of Theorem 2.3 for $\alpha = 0$. Let $k \geq H$ and consider a minimizer $u_k \in \bar{S}_0$ of the functional \mathcal{E}_k on \bar{S}_0 as provided by Corollary 6.2. Then, $-H \leq u_k \leq \kappa_0$ in D according to Proposition 6.4. Therefore, if $k \geq \kappa_0$, then

$$E(u_k) = \mathcal{E}_{\kappa_0}(u_k) = \mathcal{E}_k(u_k) \leq \mathcal{E}_k(v) = E(v) + \frac{A}{2} \|(v - k)_+\|_{L_2(D)}^2, \quad v \in \bar{S}_0. \quad (6.7)$$

Now, it follows from Lemma 6.1 and the fact that $0 \in \bar{S}_0$ that, for $k \geq \kappa_0$,

$$\frac{\beta}{4} \|\partial_x^2 u_k\|_{L_2(D)}^2 \leq \mathcal{E}_{\kappa_0}(u_k) + c(\kappa_0) \leq \mathcal{E}_k(0) + c(\kappa_0) = E(0) + c(\kappa_0).$$

Therefore, $(u_k)_{k \geq \kappa_0}$ is bounded in $H^2(D)$ and there is a subsequence of $(u_k)_{k \geq \kappa_0}$ (not relabeled) which converges weakly in $H^2(D)$ and strongly in $H^1(D)$ towards some $u_* \in \bar{S}_0$. Due to the weak lower semicontinuity of E_m in $H^2(D)$ and the continuity of E_e with respect to the weak topology of $H^2(D)$ (see Theorem 4.4), we readily infer from (6.7) that

$$E(u_*) \leq E(v), \quad v \in \bar{S}_0,$$

after taking into account that

$$\lim_{k \rightarrow \infty} \|(v - k)_+\|_{L_2(D)} = 0, \quad v \in L_2(D).$$

Consequently, $u_* \in \bar{S}_0$ is a minimizer of E on \bar{S}_0 . This proves Theorem 2.3.

6.4. Proof of Theorem 2.4 for $\alpha = 0$. Let $u \in \bar{S}_0$ be any minimizer of E on \bar{S}_0 . Proceeding as in the proof of Proposition 6.3, this implies that $u \in \bar{S}_0$ is an H^2 -weak solution to the variational inequality

$$\beta \partial_x^4 u - \tau \partial_x^2 u + \partial \mathbb{I}_{\bar{S}_0}(u) \ni -g(u) \quad \text{in } D,$$

which completes the proof of Theorem 2.4.

7. PROOFS OF THEOREM 2.3 AND THEOREM 2.4 FOR $\alpha > 0$

Consider now $\alpha > 0$. In that case, the total energy is given by

$$E(u) = E_m(u) + E_e(u)$$

with mechanical energy

$$E_m(u) = \frac{\beta}{2} \|\partial_x^2 u\|_{L_2(D)}^2 + \left(\frac{\tau}{2} + \frac{\alpha}{4} \|\partial_x u\|_{L_2(D)}^2 \right) \|\partial_x u\|_{L_2(D)}^2$$

and electrostatic energy

$$E_e(u) = -\frac{1}{2} \int_{\Omega(u)} |\nabla \psi_u|^2 d(x, z) - \frac{1}{2} \int_D \sigma(x) |\psi_u(x, -H) - \mathfrak{h}_u(x)|^2 dx.$$

Observe that, since $\alpha > 0$, the mechanical energy E_m features a super-quadratic term in $\|\partial_x u\|_{L_2(D)}$ which has the following far-reaching consequence.

Lemma 7.1. *The functional E is bounded from below with*

$$E(u) \geq \frac{\beta}{4} \|\partial_x^2 u\|_{L_2(D)}^2 - c$$

for some constant $c > 0$.

Proof. As in the proof of Lemma 6.1, we deduce from (2.3), (2.8), and Proposition 3.3 that

$$-E_e(u) \leq 2(1 + \bar{\sigma})|D|K^2 + \left(\frac{K^4}{\beta} + 2K^2 \right) \|u\|_{L_2(D)}^2 + \frac{\beta}{4} \|\partial_x^2 u\|_{L_2(D)}^2.$$

Therefore, since $\|u\|_{L_2(D)} \leq 2|D| \|\partial_x u\|_{L_2(D)}$ by Poincaré's inequality, it follows from Young's inequality that

$$\begin{aligned} E(u) &\geq \frac{\beta}{4} \|\partial_x^2 u\|_{L_2(D)}^2 + \frac{\alpha}{4} \|\partial_x u\|_{L_2(D)}^4 - 2(1 + \bar{\sigma})|D|K^2 - \left(\frac{K^4}{\beta} + 2K^2 \right) \|u\|_{L_2(D)}^2 \\ &\geq \frac{\beta}{4} \|\partial_x^2 u\|_{L_2(D)}^2 + \frac{\alpha}{4} \|\partial_x u\|_{L_2(D)}^4 - 2(1 + \bar{\sigma})|D|K^2 - 4|D|^2 \left(\frac{K^4}{\beta} + 2K^2 \right) \|\partial_x u\|_{L_2(D)}^2 \\ &\geq \frac{\beta}{4} \|\partial_x^2 u\|_{L_2(D)}^2 + \frac{\alpha}{8} \|\partial_x u\|_{L_2(D)}^4 - c, \end{aligned}$$

and the proof is complete. \square

Once Lemma 7.1 is established, the existence of a minimizer of E on \bar{S}_0 follows from the weak lower semicontinuity of E_m in $H^2(D)$ and the continuity of E_e with respect to the weak topology of $H^2(D)$ (see Corollary 4.2) by the direct method of the calculus of variations, hence Theorem 2.3 for $\alpha > 0$ (see also [17, Theorem 5.1]). As for the proof of Theorem 2.4 for $\alpha > 0$, it is the same as that for $\alpha = 0$, see Section 6.4.

APPENDIX A. A TECHNICAL LEMMA

Lemma A.1. *Let I and J be two bounded intervals in \mathbb{R} , and let U be a bounded open subset of $I \times J$. Consider $\vartheta \in H^1(U)$ and functions $v \in C(\bar{I})$, $w \in C(\bar{I})$, and $\rho \in C(\bar{I})$, $\rho \geq 0$, such that*

- (a) $x \mapsto \vartheta(x, v(x))$ and $x \mapsto \vartheta(x, w(x))$ are well-defined and belong to $L_2(I, \rho dx)$;
- (b) $\{(x, z) \in I \times J : \min\{v(x), w(x)\} < z < \max\{v(x), w(x)\}\} \subset \bar{U}$.

Then

$$\int_I |\vartheta(\cdot, v) - \vartheta(\cdot, w)|^2 \rho dx \leq \|(v - w)\rho\|_{L^\infty(I)} \|\partial_z \vartheta\|_{L_2(U)}^2.$$

Proof. Owing to (b) we have, for a.a. $x \in I$,

$$|\vartheta(x, v(x)) - \vartheta(x, w(x))|^2 = \left(\int_{w(x)}^{v(x)} \partial_z \vartheta(x, z) dz \right)^2.$$

Integrating with respect to $x \in I$ after multiplication by $\rho(x)$ and using Hölder's inequality give

$$\begin{aligned} \int_I |\vartheta(x, v(x)) - \vartheta(x, w(x))|^2 \rho(x) dx &\leq \int_I |v(x) - w(x)| \left| \int_{w(x)}^{v(x)} |\partial_z \vartheta(x, z)|^2 dz \right| \rho(x) dx \\ &\leq \|\rho(v - w)\|_{L^\infty(I)} \int_U |\partial_z \vartheta(x, z)|^2 d(x, z) \end{aligned}$$

and the proof is complete. \square

APPENDIX B. PROOF OF LEMMA 3.11

The proof of Lemma 3.11 relies on the following result, which can be seen as an extension of [12, Lemma 4.3.1.3] to include Robin boundary conditions.

Lemma B.1. *Let $I := (a, b)$ and set $\mathcal{R}_I = I \times (0, 1)$. Consider $\varphi \in H^2(\mathcal{R}_I)$ and $\mu \in C^2(\bar{I})$ such that*

$$\varphi(a, \eta) = \varphi(b, \eta) = 0, \quad \eta \in (0, 1), \quad (\text{B.1a})$$

$$\varphi(x, 1) = -\partial_\eta \varphi(x, 0) + \mu(x)\varphi(x, 0) = 0, \quad x \in I. \quad (\text{B.1b})$$

Then

$$\int_{\mathcal{R}_I} \partial_x^2 \varphi \partial_\eta^2 \varphi d(x, \eta) = \int_{\mathcal{R}_I} |\partial_x \partial_\eta \varphi|^2 d(x, \eta) + \int_I (\partial_x \varphi \partial_x (\mu \varphi))(\cdot, 0) dx.$$

Proof. We put $\xi(x, \eta) := e^{-\eta\mu(x)}\varphi(x, \eta)$ and $\rho(x, \eta) := e^{\eta\mu(x)}$ for $(x, \eta) \in \mathcal{R}_I$. Owing to the regularity of φ and μ , the function ξ belongs to $H^2(\mathcal{R}_I)$ and, for $(x, \eta) \in \mathcal{R}_I$,

$$\partial_x \xi(x, \eta) = e^{-\eta\mu(x)} [\partial_x \varphi(x, \eta) - \eta \partial_x \mu(x) \varphi(x, \eta)],$$

$$\partial_\eta \xi(x, \eta) = e^{-\eta\mu(x)} [\partial_\eta \varphi(x, \eta) - \mu(x) \varphi(x, \eta)].$$

Consequently, the functions F and G , defined for $(x, \eta) \in \mathcal{R}_I$ by

$$F(x, \eta) := \rho(x, \eta) \partial_x \xi(x, \eta) = \partial_x \varphi(x, \eta) - \eta \partial_x \mu(x) \varphi(x, \eta),$$

$$G(x, \eta) := \rho(x, \eta) \partial_\eta \xi(x, \eta) = \partial_\eta \varphi(x, \eta) - \mu(x) \varphi(x, \eta),$$

satisfy

$$\begin{aligned} G(a, \eta) = G(b, \eta) = 0, & \quad \eta \in (0, 1), \\ F(x, 1) = G(x, 0) = 0, & \quad x \in I, \end{aligned}$$

since, by (B.1),

$$\begin{aligned} \partial_\eta \varphi(a, \eta) = \partial_\eta \varphi(b, \eta) = 0, & \quad \eta \in (0, 1), \\ \partial_x \varphi(x, 1) = 0, & \quad x \in I. \end{aligned} \tag{B.2}$$

We then infer from [12, Lemma 4.3.1.3] that

$$\begin{aligned} \int_{\mathcal{R}_I} \partial_x(\rho \partial_x \xi) \partial_\eta(\rho \partial_\eta \xi) \, d(x, \eta) &= \int_{\mathcal{R}_I} \partial_x F \partial_\eta G \, d(x, \eta) = \int_{\mathcal{R}_I} \partial_\eta F \partial_x G \, d(x, \eta) \\ &= \int_{\mathcal{R}_I} \partial_\eta(\rho \partial_x \xi) \partial_x(\rho \partial_\eta \xi) \, d(x, \eta); \end{aligned}$$

that is,

$$0 = \int_{\mathcal{R}_I} [\partial_x^2 \varphi \partial_\eta^2 \varphi - |\partial_x \partial_\eta \varphi|^2] \, d(x, \eta) + \sum_{j=1}^3 I_j, \tag{B.3}$$

where

$$\begin{aligned} I_1 &:= \int_{\mathcal{R}_I} [-\partial_\eta(\mu \varphi) \partial_x^2 \varphi + \partial_x(\mu \varphi) \partial_x \partial_\eta \varphi] \, d(x, \eta), \\ I_2 &:= \int_{\mathcal{R}_I} [-\partial_x(\eta \varphi \partial_x \mu) \partial_\eta^2 \varphi + \partial_\eta(\eta \varphi \partial_x \mu) \partial_x \partial_\eta \varphi] \, d(x, \eta), \\ I_3 &:= \int_{\mathcal{R}_I} [\partial_x(\eta \varphi \partial_x \mu) \partial_\eta(\tau \varphi) - \partial_\eta(\eta \varphi \partial_x \mu) \partial_x(\tau \varphi)] \, d(x, \eta). \end{aligned}$$

First, integrating by parts and using the boundary values (B.1) of φ give

$$I_3 = \int_0^1 [\eta \varphi \partial_x \mu \partial_\eta(\mu \varphi)]_{x=a}^{x=b} \, d\eta - \int_I [\eta \varphi \partial_x \mu \partial_x(\mu \varphi)]_{\eta=0}^{\eta=1} \, dx = 0$$

and

$$\begin{aligned} I_2 &= - \int_I [\partial_x(\eta \varphi \partial_x \mu) \partial_\eta \varphi]_{\eta=0}^{\eta=1} \, dx + \int_0^1 [\partial_\eta(\eta \varphi \partial_x \mu) \partial_\eta \varphi]_{x=a}^{x=b} \, d\eta \\ &= - \int_I \partial_x \mu \partial_x \varphi(\cdot, 1) \partial_\eta \varphi(\cdot, 1) \, dx + \int_0^1 \partial_\eta(\eta \varphi \partial_x \mu)(b, \cdot) \partial_\eta \varphi(b, \cdot) \, d\eta \\ &\quad - \int_0^1 \partial_\eta(\eta \varphi \partial_x \mu)(a, \cdot) \partial_\eta \varphi(a, \cdot) \, d\eta. \end{aligned}$$

Owing to (B.2) we conclude that $I_2 = 0$. Finally, we deduce from (B.1) and (B.2) after integrating by parts that

$$I_1 = - \int_0^1 [\partial_\eta(\mu \varphi) \partial_x \varphi]_{x=a}^{x=b} \, d\eta + \int_I [\partial_x(\mu \varphi) \partial_x \varphi]_{\eta=0}^{\eta=1} \, dx$$

$$\begin{aligned}
&= - \int_0^1 \mu(b) \partial_x \varphi(b, \cdot) \partial_\eta \varphi(b, \cdot) \, d\eta + \int_0^1 \mu(a) \partial_x \varphi(a, \cdot) \partial_\eta \varphi(a, \cdot) \, d\eta \\
&\quad + \int_I \partial_x(\mu\varphi)(\cdot, 1) \partial_x \varphi(\cdot, 1) \, dx - \int_I \partial_x(\mu\varphi)(\cdot, 0) \partial_x \varphi(\cdot, 0) \, dx \\
&= - \int_I \partial_x(\mu\varphi)(\cdot, 0) \partial_x \varphi(\cdot, 0) \, dx .
\end{aligned}$$

Collecting (B.3) and the formulas for I_j , $1 \leq j \leq 3$, completes the proof. \square

Proof of Lemma 3.11. For $(x, \eta) \in \mathcal{R}_I$, we define

$$\Phi(x, \eta) := \zeta_v(x, -H + \eta(H + v(x))), \quad (\text{B.4})$$

or, equivalently,

$$\zeta_v(x, z) = \Phi\left(x, \frac{H + z}{H + v(x)}\right), \quad (x, z) \in \mathcal{O}_I(v).$$

Since $\zeta_v \in H^2(\mathcal{O}_I(v))$ by Lemma 3.10 and $v \in H^2(I)$, the function Φ belongs to $H^2(\mathcal{R}_I)$ and we infer from (3.18b) and (3.18c) that

$$\begin{aligned}
\Phi(a, \eta) &= \Phi(b, \eta) = 0, & \eta &\in (0, 1), \\
\Phi(x, 1) &= -\partial_\eta \Phi(x, 0) + \sigma(x)(H + v)(x)\Phi(x, 0) = 0, & x &\in I.
\end{aligned} \quad (\text{B.5})$$

Next,

$$J := \int_{\mathcal{O}_I(v)} \partial_x^2 \zeta_v \partial_z^2 \zeta_v \, d(x, z) = \sum_{i=1}^3 J_i, \quad (\text{B.6})$$

where

$$\begin{aligned}
J_1 &:= \int_{\mathcal{R}_I} \partial_x^2 \Phi \partial_\eta^2 \Phi \frac{d(x, \eta)}{H + v}, \\
J_2 &:= \int_{\mathcal{R}_I} \left[-2\eta \frac{\partial_x v}{H + v} \partial_x \partial_\eta \Phi + \eta^2 \left(\frac{\partial_x v}{H + v} \right)^2 \partial_\eta^2 \Phi \right] \partial_\eta^2 \Phi \frac{d(x, \eta)}{H + v}, \\
J_3 &:= \int_{\mathcal{R}_I} \eta \left[2 \left(\frac{\partial_x v}{H + v} \right)^2 - \frac{\partial_x^2 v}{H + v} \right] \partial_\eta \Phi \partial_\eta^2 \Phi \frac{d(x, \eta)}{H + v}.
\end{aligned}$$

Since

$$\partial_x^2 \left(\frac{\Phi}{\sqrt{H + v}} \right) = \frac{\partial_x^2 \Phi}{\sqrt{H + v}} - \frac{\partial_x v}{(H + v)^{3/2}} \partial_x \Phi - \frac{1}{2} \partial_x \left(\frac{\partial_x v}{(H + v)^{3/2}} \right) \Phi,$$

we further obtain

$$J_1 := \sum_{i=1}^3 J_{1,i},$$

where

$$J_{1,1} := \int_{\mathcal{R}_I} \partial_x^2 \left(\frac{\Phi}{\sqrt{H + v}} \right) \partial_\eta^2 \left(\frac{\Phi}{\sqrt{H + v}} \right) \, d(x, \eta),$$

$$\begin{aligned}
J_{1,2} &:= \int_{\mathcal{R}_I} \frac{\partial_x v}{(H+v)^{3/2}} \partial_x \Phi \partial_\eta^2 \left(\frac{\Phi}{\sqrt{H+v}} \right) d(x, \eta), \\
J_{1,3} &:= \frac{1}{2} \int_{\mathcal{R}_I} \partial_x \left(\frac{\partial_x v}{(H+v)^{3/2}} \right) \Phi \partial_\eta^2 \left(\frac{\Phi}{\sqrt{H+v}} \right) d(x, \eta).
\end{aligned}$$

We first infer from (B.5) and Lemma B.1 (with $\varphi = \Phi/\sqrt{H+v}$ and $\mu = \sigma(H+v)$) that

$$\begin{aligned}
J_{1,1} &= \int_{\mathcal{R}_I} \left| \partial_x \partial_\eta \left(\frac{\Phi}{\sqrt{H+v}} \right) \right|^2 d(x, \eta) \\
&\quad + \int_I \partial_x \left(\frac{\Phi}{\sqrt{H+v}} \right) (\cdot, 0) \partial_x \left(\sigma \sqrt{H+v} \Phi \right) (\cdot, 0) dx \\
&= \int_{\mathcal{R}_I} \frac{|\partial_x \partial_\eta \Phi|^2}{H+v} d(x, \eta) - \int_{\mathcal{R}_I} \frac{\partial_x v}{(H+v)^2} \partial_\eta \Phi \partial_x \partial_\eta \Phi d(x, \eta) \\
&\quad + \frac{1}{4} \int_{\mathcal{R}_I} \frac{(\partial_x v)^2}{(H+v)^3} |\partial_\eta \Phi|^2 d(x, \eta) \\
&\quad + \int_I \partial_x \left(\frac{\Phi}{\sqrt{H+v}} \right) (\cdot, 0) \partial_x \left(\sigma \sqrt{H+v} \Phi \right) (\cdot, 0) dx.
\end{aligned}$$

Next, we integrate by parts and use the boundary values (B.5) of Φ to obtain

$$\begin{aligned}
J_{1,2} &= \int_{\mathcal{R}_I} \frac{\partial_x v}{(H+v)^2} \partial_x \Phi \partial_\eta^2 \Phi d(x, \eta) \\
&= \int_I \frac{\partial_x v}{(H+v)^2} \left[\partial_x \Phi \partial_\eta \Phi \right]_{\eta=0}^{\eta=1} dx - \int_{\mathcal{R}_I} \frac{\partial_x v}{(H+v)^2} \partial_\eta \Phi \partial_x \partial_\eta \Phi d(x, \eta) \\
&= - \int_I \frac{\sigma \partial_x v}{(H+v)} \Phi(\cdot, 0) \partial_x \Phi(\cdot, 0) dx - \int_{\mathcal{R}_I} \frac{\partial_x v}{(H+v)^2} \partial_\eta \Phi \partial_x \partial_\eta \Phi d(x, \eta)
\end{aligned}$$

and

$$\begin{aligned}
J_{1,3} &= \int_{\mathcal{R}_I} \left(\frac{\partial_x^2 v}{2(H+v)^2} - \frac{3}{4} \frac{(\partial_x v)^2}{(H+v)^3} \right) \Phi \partial_\eta^2 \Phi d(x, \eta) \\
&= \int_I \left(\frac{\partial_x^2 v}{2(H+v)^2} - \frac{3}{4} \frac{(\partial_x v)^2}{(H+v)^3} \right) \left[\Phi \partial_\eta \Phi \right]_{\eta=0}^{\eta=1} dx \\
&\quad - \int_{\mathcal{R}_I} \left(\frac{\partial_x^2 v}{2(H+v)^2} - \frac{3}{4} \frac{(\partial_x v)^2}{(H+v)^3} \right) |\partial_\eta \Phi|^2 d(x, \eta) \\
&= - \int_I \sigma \left(\frac{\partial_x^2 v}{2(H+v)} - \frac{3}{4} \frac{(\partial_x v)^2}{(H+v)^2} \right) |\Phi(\cdot, 0)|^2 dx \\
&\quad - \int_{\mathcal{R}_I} \left(\frac{\partial_x^2 v}{2(H+v)^2} - \frac{3}{4} \frac{(\partial_x v)^2}{(H+v)^3} \right) |\partial_\eta \Phi|^2 d(x, \eta).
\end{aligned}$$

Next,

$$J_3 = \int_{\mathcal{R}_I} 2\eta \frac{(\partial_x v)^2}{(H+v)^3} \partial_\eta \Phi \partial_\eta^2 \Phi d(x, \eta) + J_{3,2}$$

with

$$J_{3,2} := - \int_{\mathcal{R}_I} \frac{\eta \partial_x^2 v}{2(H+v)^2} \partial_\eta (|\partial_\eta \Phi|^2) \, d(x, \eta) .$$

Integrating by parts and using (B.5) give

$$\begin{aligned} J_{3,2} &= - \int_I \frac{\partial_x^2 v}{2(H+v)^2} \left[\eta |\partial_\eta \Phi|^2 \right]_{\eta=0}^{\eta=1} \, dx + \int_{\mathcal{R}_I} \frac{\partial_x^2 v}{2(H+v)^2} |\partial_\eta \Phi|^2 \, d(x, \eta) \\ &= - \int_I \frac{\partial_x^2 v}{2(H+v)^2} |\partial_\eta \Phi(\cdot, 1)|^2 \, dx + \int_{\mathcal{R}_I} \frac{\partial_x^2 v}{2(H+v)^2} |\partial_\eta \Phi|^2 \, d(x, \eta) . \end{aligned}$$

Consequently,

$$\begin{aligned} J &= \int_{\mathcal{R}_I} (H+v) \left(\frac{\partial_x \partial_\eta \Phi}{H+v} - \frac{\eta \partial_x v}{(H+v)^2} \partial_\eta^2 \Phi - \frac{\partial_x v}{(H+v)^2} \partial_\eta \Phi \right)^2 \, d(x, \eta) \\ &\quad + J_4 - \frac{1}{2} \int_I \frac{\partial_x^2 v}{(H+v)^2} |\partial_\eta \Phi(\cdot, 1)|^2 \, dx , \end{aligned} \tag{B.7}$$

where

$$\begin{aligned} J_4 &:= \int_I \left(\partial_x \Phi \partial_x (\sigma \Phi) - \frac{\sigma \partial_x v}{2(H+v)} \Phi \partial_x \Phi - \frac{\partial_x v}{2(H+v)} \Phi \partial_x (\sigma \Phi) \right) (\cdot, 0) \, dx \\ &\quad + \frac{1}{2} \int_I \sigma \left(\frac{(\partial_x v)^2}{(H+v)^2} - \frac{\partial_x^2 v}{H+v} \right) |\Phi(\cdot, 0)|^2 \, dx . \end{aligned}$$

Now, since $H^2(\mathcal{R}_I)$ is continuously embedded in $C(\overline{\mathcal{R}_I})$ by [12, Theorem 1.4.5.2], we infer from (B.5) that

$$\Phi(a, 0) = \Phi(b, 0) = 0 .$$

Using this property along with an integration by parts, we obtain

$$\begin{aligned} &- \int_I \left(\frac{\sigma \partial_x v}{2(H+v)} \Phi \partial_x \Phi + \frac{\partial_x v}{2(H+v)} \Phi \partial_x (\sigma \Phi) \right) (\cdot, 0) \, dx \\ &= - \frac{1}{2} \int_I \frac{\partial_x v}{H+v} \partial_x (\sigma \Phi^2) \, dx \\ &= - \frac{1}{2} \left[\frac{\partial_x v}{H+v} \sigma |\Phi(\cdot, 0)|^2 \right]_{x=a}^{x=b} + \frac{1}{2} \int_I \left(\frac{\partial_x^2 v}{H+v} - \frac{(\partial_x v)^2}{(H+v)^2} \right) \sigma |\Phi(\cdot, 0)|^2 \, dx \\ &= \frac{1}{2} \int_I \left(\frac{\partial_x^2 v}{H+v} - \frac{(\partial_x v)^2}{(H+v)^2} \right) \sigma |\Phi(\cdot, 0)|^2 \, dx , \end{aligned}$$

so that J_4 reduces to

$$J_4 = \int_I (\partial_x \Phi \partial_x (\sigma \Phi)) (\cdot, 0) \, dx . \tag{B.8}$$

We then infer from (B.4), (B.7), and (B.8) that

$$J = \int_{\mathcal{O}_I(v)} |\partial_x \partial_z \zeta_v|^2 \, d(x, z) + \int_I \partial_x \zeta_v(\cdot, -H) \partial_x (\sigma \zeta_v)(\cdot, -H) \, dx - \frac{1}{2} \int_I \partial_x^2 v |\partial_z \zeta_v(\cdot, v)|^2 \, dx .$$

Combining (B.6) and the above identity completes the proof. \square

APPENDIX C. SOME FUNCTIONAL INEQUALITIES

Let $I = (a, b) \subset D$ be an open interval and consider $v \in W_\infty^3(I)$ such that $\min_{[a,b]} v > -H$. Let $M > 0$ be such that

$$M \geq \max \{1, \|H + v\|_{L_\infty(I)}, \|\partial_x v\|_{L_\infty(I)}\} . \quad (\text{C.1})$$

We derive in this section functional inequalities for functions in the subspace $H_{WS}^1(\mathcal{O}_I(v))$ of $H^1(\mathcal{O}_I(v))$ introduced in (3.31). Recall that $P \in H_{WS}^1(\mathcal{O}_I(v))$ if and only if $P \in H^1(\mathcal{O}_I(v))$ satisfies

$$P(x, -H) = 0 , \quad x \in I , \quad (\text{C.2a})$$

$$P(a, z) = 0 , \quad z \in (-H, v(a)) , \quad (\text{C.2b})$$

We begin with Poincaré and Sobolev inequalities and pay special attention to the dependence of the constants on v .

Lemma C.1. *Let $P \in H_{WS}^1(\mathcal{O}_I(v))$. Then*

$$\|P\|_{L_2(\mathcal{O}_I(v))}^2 \leq 2M \|\nabla P\|_{L_1(\mathcal{O}_I(v))} \|\partial_z P\|_{L_1(\mathcal{O}_I(v))} ,$$

where M is given by (C.1).

Proof. For $(x, \eta) \in \mathcal{R}_I = I \times (0, 1)$, we define

$$Q(x, \eta) := P(x, -H + \eta(H + v(x))) , \quad (\text{C.3})$$

and observe that the regularity of v and P implies that $Q \in H^1(\mathcal{R}_I)$. In addition, we deduce from (C.2) that

$$Q(x, 0) = 0 , \quad x \in I , \quad (\text{C.4a})$$

$$Q(a, \eta) = 0 , \quad \eta \in (0, 1) . \quad (\text{C.4b})$$

On the one hand, it follows from (C.4b) that, for a.a. $(x, \eta) \in \mathcal{R}_I$,

$$\begin{aligned} |(H + v)(x)Q(x, \eta)| &= \left| \int_a^x [(H + v(x_*))\partial_x Q(x_*, \eta) + \partial_x v(x_*)Q(x_*, \eta)] dx_* \right| \\ &\leq \mathcal{J}_1(\eta) := \int_I |(H + v(x_*))\partial_x Q(x_*, \eta) + \partial_x v(x_*)Q(x_*, \eta)| dx_* . \end{aligned} \quad (\text{C.5})$$

On the other hand, by (C.4a), we obtain, for a.a. $(x, \eta) \in \mathcal{R}_I$,

$$|Q(x, \eta)| = \left| \int_0^\eta \partial_\eta Q(x, \eta_*) d\eta_* \right| \leq \mathcal{J}_2(x) := \int_0^1 |\partial_\eta Q(x, \eta_*)| d\eta_* . \quad (\text{C.6})$$

We then infer from (C.3), (C.5), and (C.6) that

$$\begin{aligned} \|P\|_{L_2(\mathcal{O}_I(v))}^2 &= \int_{\mathcal{R}_I} (H + v(x))|Q(x, \eta)|^2 d(x, \eta) \\ &\leq \int_{\mathcal{R}_I} \mathcal{J}_1(\eta)\mathcal{J}_2(x) d(x, \eta) = \left(\int_I \mathcal{J}_2(x) dx \right) \left(\int_0^1 \mathcal{J}_1(\eta) d\eta \right) . \end{aligned} \quad (\text{C.7})$$

Now,

$$\int_I \mathcal{J}_2(x) dx = \int_{\mathcal{R}_I} |\partial_\eta Q(x, \eta)| d(x, \eta) = \int_{\mathcal{O}_I(v)} |\partial_z P(x, z)| d(x, z) \quad (\text{C.8})$$

and

$$\begin{aligned} \int_0^1 \mathcal{J}_1(\eta) \, d\eta &= \int_{\mathcal{R}_I} |(H + v(x))\partial_x Q(x, \eta) + \partial_x v(x)Q(x, \eta)| \, d(x, \eta) \\ &= \int_{\mathcal{O}_I(v)} \left| \partial_x P(x, z) + \frac{H + z}{H + v(x)} \partial_x v(x) \partial_z P(x, z) + \frac{\partial_x v(x)}{(H + v)(x)} P(x, z) \right| \, d(x, z). \end{aligned} \quad (\text{C.9})$$

It further follows from (C.2a) that, for a.a. $(x, z) \in \mathcal{O}_I(v)$,

$$|P(x, z)| = \left| \int_{-H}^z \partial_z P(x, z_*) \, dz_* \right| \leq \int_{-H}^{v(x)} |\partial_z P(x, z_*)| \, dz_*.$$

Hence,

$$\int_{\mathcal{O}_I(v)} \left| \frac{\partial_x v(x)}{(H + v)(x)} P(x, z) \right| \, d(x, z) \leq \int_{\mathcal{O}_I(v)} |\partial_x v(x)| |\partial_z P(x, z_*)| \, d(x, z_*). \quad (\text{C.10})$$

Since $0 \leq H + z \leq H + v(x)$ for $(x, z) \in \mathcal{O}_I(v)$, we deduce from (C.1), (C.9), and (C.10) that

$$\int_0^1 \mathcal{J}_1(\eta) \, d\eta \leq \int_{\mathcal{O}_I(v)} (|\partial_x P(x, z)| + 2|\partial_x v(x)| |\partial_z P(x, z)|) \, d(x, z) \leq 2M \|\nabla P\|_{L_1(\mathcal{O}_I(v))}. \quad (\text{C.11})$$

Collecting (C.7), (C.8), and (C.11) completes the proof. \square

Since $\mathcal{O}_I(v)$ is a two-dimensional domain, a classical consequence of Lemma C.1 is the continuous embedding of $H_{WS}^1(\mathcal{O}_I(v))$ in $L_r(\mathcal{O}_I(v))$ for $r \in [1, \infty)$. We stress here once more that our main concern is the precise dependence of the embedding constant on v .

Lemma C.2. *Let $P \in H_{WS}^1(\mathcal{O}_I(v))$ and $r \in [2, \infty)$. Then*

$$\|P\|_{L_r(\mathcal{O}_I(v))}^r \leq \left(2r\sqrt{M}\right)^{r-2} \|P\|_{L_2(\mathcal{O}_I(v))}^2 \|\nabla P\|_{L_2(\mathcal{O}_I(v))}^{(r-2)/2} \|\partial_z P\|_{L_2(\mathcal{O}_I(v))}^{(r-2)/2},$$

where M is given by (C.1).

Proof. Step 1. Assume first that $r \geq 4$. For $n \geq 1$, we define the truncation \mathcal{T}_n by $\mathcal{T}_n(s) := s$ for $s \in [-n, n]$ and $\mathcal{T}_n(s) := n \operatorname{sign}(s)$ for $s \in (-\infty, -n) \cup (n, \infty)$. Since \mathcal{T}_n is a Lipschitz continuous function on \mathbb{R} with $|\mathcal{T}'_n| \leq 1$ and vanishes at zero, the function $\mathcal{T}_n(P)^{r/2}$ also belongs to $H_{WS}^1(\mathcal{O}_I(v))$. We then infer from Lemma C.1, the bound $|\mathcal{T}'_n| \leq 1$, and Hölder's inequality that

$$\begin{aligned} \|\mathcal{T}_n(P)\|_{L_r(\mathcal{O}_I(v))}^r &\leq \frac{Mr^2}{2} \left\| \mathcal{T}_n(P)^{(r-2)/2} \nabla P \right\|_{L_1(\mathcal{O}_I(v))} \left\| \mathcal{T}_n(P)^{(r-2)/2} \partial_z P \right\|_{L_1(\mathcal{O}_I(v))} \\ &\leq Mr^2 \|\mathcal{T}_n(P)\|_{L_{r-2}(\mathcal{O}_I(v))}^{r-2} \|\nabla P\|_{L_2(\mathcal{O}_I(v))} \|\partial_z P\|_{L_2(\mathcal{O}_I(v))}. \end{aligned}$$

Using again Hölder's inequality, as well as the property $|\mathcal{T}_n(s)| \leq |s|$ for $s \in \mathbb{R}$, gives

$$\begin{aligned} \|\mathcal{T}_n(P)\|_{L_{r-2}(\mathcal{O}_I(v))}^{r-2} &\leq \|\mathcal{T}_n(P)\|_{L_r(\mathcal{O}_I(v))}^{r(r-4)/(r-2)} \|\mathcal{T}_n(P)\|_{L_2(\mathcal{O}_I(v))}^{4/(r-2)} \\ &\leq \|\mathcal{T}_n(P)\|_{L_r(\mathcal{O}_I(v))}^{r(r-4)/(r-2)} \|P\|_{L_2(\mathcal{O}_I(v))}^{4/(r-2)}, \end{aligned}$$

since $r \geq 4$. Combining the above two inequalities leads us to

$$\|\mathcal{T}_n(P)\|_{L_r(\mathcal{O}_I(v))}^r \leq (Mr^2)^{(r-2)/2} \|P\|_{L_2(\mathcal{O}_I(v))}^2 \|\nabla P\|_{L_2(\mathcal{O}_I(v))}^{(r-2)/2} \|\partial_z P\|_{L_2(\mathcal{O}_I(v))}^{(r-2)/2}.$$

Since the right-hand side of the above inequality does not depend on $n \geq 1$, we may take the limit $n \rightarrow \infty$ and deduce from Fatou's lemma that $P \in L_r(\mathcal{O}_I(v))$ and satisfies the stated bound for $r \geq 4$.

Step 2. Consider now $r \in [2, 4]$. By Hölder's inequality and Lemma C.2 for $r = 4$,

$$\begin{aligned} \|P\|_{L_r(\mathcal{O}_I(v))}^r &\leq \|P\|_{L_4(\mathcal{O}_I(v))}^{2(r-2)} \|P\|_{L_2(\mathcal{O}_I(v))}^{4-r} \\ &\leq (16M)^{(r-2)/2} \|P\|_{L_2(\mathcal{O}_I(v))}^2 \|\nabla P\|_{L_2(\mathcal{O}_I(v))}^{(r-2)/2} \|\partial_z P\|_{L_2(\mathcal{O}_I(v))}^{(r-2)/2}, \end{aligned}$$

and we complete the proof by noticing that $4 \leq 2r$. \square

In the same vein, we derive an estimate for the trace of $P \in H_{WS}^1(\mathcal{O}_I(v))$ on the graph $\mathfrak{G}_I(v)$ of v , the trace being here well-defined since the assumption $\min_{[a,b]} v > -H$ guarantees that $\mathcal{O}_I(v)$ is a Lipschitz domain.

Lemma C.3. *Let $P \in H_{WS}^1(\mathcal{O}_I(v))$ and $r \in [2, \infty)$. Then*

$$\|P(\cdot, v)\|_{L_r(I)}^r \leq \left(4r\sqrt{M}\right)^r \|P\|_{L_2(\mathcal{O}_I(v))} \|\nabla P\|_{L_2(\mathcal{O}_I(v))}^{(r-2)/2} \|\partial_z P\|_{L_2(\mathcal{O}_I(v))}^{r/2},$$

where M is given by (C.1).

Proof. By (C.2a) we have, for a.a. $x \in I$,

$$|P(x, v(x))|^r \leq r \int_{-H}^{v(x)} |P(x, z)|^{r-1} |\partial_z P(x, z)| \, dz.$$

Integrating over I and using Hölder's inequality lead us to

$$\|P(\cdot, v)\|_{L_r(I)}^r \leq r \|P\|_{L_{2(r-1)}(\mathcal{O}_I(v))}^{r-1} \|\partial_z P\|_{L_2(\mathcal{O}_I(v))}.$$

Since $2(r-1) \geq 2$ as $r \geq 2$, we deduce from Lemma C.2 and the above inequality that

$$\|P(\cdot, v)\|_{L_r(I)}^r \leq r \left(4(r-1)\sqrt{M}\right)^{r-2} \|P\|_{L_2(\mathcal{O}_I(v))} \|\nabla P\|_{L_2(\mathcal{O}_I(v))}^{(r-2)/2} \|\partial_z P\|_{L_2(\mathcal{O}_I(v))}^{r/2},$$

from which Lemma C.3 follows, after using that $r \left(4(r-1)\sqrt{M}\right)^{r-2} \leq \left(4r\sqrt{M}\right)^r$. \square

APPENDIX D. A UNIFORM BOUND FOR AN AUXILIARY STATIONARY PROBLEM

Proposition D.1. *Consider $G_0 > 0$, $\beta > 0$, and $\tau \geq 0$. Let $I = (a, b) \subset (-L, L)$ be an open interval and let S_I be the unique solution to the boundary value problem*

$$\beta S_I'''' - \tau S_I'' = G_0, \quad x \in I, \quad (\text{D.1})$$

supplemented with inhomogeneous Dirichlet boundary conditions

$$S_I(a) + H = S_I'(a) = S_I(b) + H = S_I'(b) = 0 \quad \text{if} \quad -L < a < b < L, \quad (\text{D.2})$$

$$S_I(-L) = S_I'(-L) = S_I(b) + H = S_I'(b) = 0 \quad \text{if} \quad -L = a < b < L, \quad (\text{D.3})$$

$$S_I(a) + H = S_I'(a) = S_I(L) = S_I'(L) = 0 \quad \text{if} \quad -L < a < b = L, \quad (\text{D.4})$$

or clamped boundary conditions

$$S_I(-L) = S_I'(-L) = S_I(L) = S_I'(L) = 0 \quad \text{if} \quad -L = a < b = L. \quad (\text{D.5})$$

There is $\kappa_0 > 0$ depending only on G_0 , β , L , H , and τ such that

$$|S_I(x)| \leq \kappa_0, \quad x \in [a, b], \quad -L \leq a < b \leq L.$$

Proof. Case 1: $-L < a < b < L$. We set $P(y) := S_I(a + (b - a)y) + H$ for $y \in [0, 1]$ and deduce from (D.1) and (D.2) that P solves the boundary-value problem

$$\begin{aligned} \beta P'''' - \tau(b - a)^2 P'' &= (b - a)^4 G_0, \quad y \in (0, 1), \\ P(0) = P'(0) = P(1) = P'(1) &= 0. \end{aligned} \tag{D.6}$$

We first infer from (D.6), the positivity of G_0 , and a version of Boggio's comparison principle [7, 13, 19, 29] that $P > 0$ in $(0, 1)$. We next multiply (D.6) by P and integrate over $(0, 1)$. After integrating by parts and using the boundary conditions, we obtain

$$\beta \|P''\|_{L_2(0,1)}^2 + \tau(b - a)^2 \|P'\|_{L_2(0,1)}^2 = (b - a)^4 G_0 \int_0^1 P(y) \, dy.$$

Since

$$|P(y)| = \left| \int_0^y (y - y_*) P''(y_*) \, dy_* \right| \leq \|P''\|_{L_2(0,1)}, \quad y \in (0, 1),$$

by (D.6), we infer from these observations that

$$\beta \|P\|_{L_\infty(0,1)}^2 \leq \beta \|P''\|_{L_2(0,1)}^2 \leq (b - a)^4 G_0 \|P\|_{L_\infty(0,1)} \leq 16L^4 G_0 \|P\|_{L_\infty(0,1)}.$$

Consequently, $0 \leq P \leq 16L^4 G_0 / \beta$ in $[0, 1]$, hence $-H \leq S_I \leq 16L^4 G_0 / \beta - H$ in $[a, b]$.

Case 2: $-L = a < b < L$. Let $Q \in \mathbb{R}_4[X]$ be such that $Q(0) = Q'(0) = Q(1) + H = Q'(1) = 0$; that is, $Q(y) = y^2(y^2 + 2(H - 1)y + 1 - 3H)$. We set $P(y) := S_I(-L + (b + L)y) - Q(y)$ for $y \in [0, 1]$ and deduce from (D.1) and (D.3) that P solves the boundary value problem

$$\begin{aligned} \beta P'''' - \tau(b + L)^2 P'' &= (b + L)^4 G_0 - \beta Q'''' + \tau(b + L)^2 Q'', \quad y \in (0, 1), \\ P(0) = P'(0) = P(1) = P'(1) &= 0. \end{aligned} \tag{D.7}$$

Arguing as in Case 1, we are led to

$$\begin{aligned} \beta \|P\|_{L_\infty(0,1)}^2 &\leq \beta \|P''\|_{L_2(0,1)}^2 + \tau(b + L)^2 \|P'\|_{L_2(0,1)}^2 \\ &\leq [(b + L)^4 G_0 + 24\beta + 14\tau(H + 1)(b + L)^2] \|P\|_{L_\infty(0,1)} \\ &\leq [16L^4 G_0 + 24\beta + 56\tau(H + 1)L^2] \|P\|_{L_\infty(0,1)}, \end{aligned}$$

since $Q'''' = 24$ and

$$-14(H + 1) \leq -12y - 6H \leq Q''(y) = 12y^2 + 12(H - 1)y + 2(1 - 3H) \leq 14 + 12H \leq 14(H + 1).$$

Consequently,

$$\|S_I\|_{L_\infty(I)} \leq \|P\|_{L_\infty(0,1)} + \|Q\|_{L_\infty(0,1)} \leq \frac{16L^4 G_0 + 24\beta + 56\tau(H + 1)L^2}{\beta} + \|Q\|_{L_\infty(0,1)}.$$

Case 3: $-L < a < b = L$. We set $P(y) := S_I(a + y(L - a)) - Q(1 - y)$ for $y \in [0, 1]$ and proceed as in the previous case to derive the same bound for $\|S_I\|_{L_\infty(I)}$.

Case 4: $-L = a < b = L$. We set $P(y) := S_I(-L + 2Ly)$ for $y \in [0, 1]$ and deduce from (D.1) and (D.5) that P solves the boundary value problem

$$\begin{aligned}\beta P'''' - 4\tau L^2 P'' &= 16L^4 G_0, & y \in (0, 1), \\ P(0) = P'(0) = P(1) = P'(1) &= 0.\end{aligned}$$

We then argue as in Case 1 to conclude that $0 \leq S_I \leq 16L^4 G_0 / \beta$ in $[-L, L]$. \square

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