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ALMOST PERIODIC TYPE FUNCTIONS AND DENSITIES

MARKO KOSTIĆ

ABSTRACT. In this paper, we introduce and analyze the notions of \odot_g -almost periodicity and Stepanov \odot_g -almost periodicity for functions with values in complex Banach spaces. In order to do that, we use the recently introduced notions of lower and upper (Banach) g -densities. We also analyze uniformly recurrent functions, generalized almost automorphic functions and apply our results in the qualitative analysis of solutions of inhomogeneous abstract integro-differential inclusions. We present plenty of illustrative examples, results of independent interest, questions and unsolved problems.

1. INTRODUCTION AND PRELIMINARIES

The concept of almost periodicity was first studied by H. Bohr around 1925 and later generalized by many other mathematicians (cf. the research monographs by A. S. Besicovitch [5], H. Bohr [7], A. M. Fink [16], B. M. Levitan [33] and B. M. Levitan, V. V. Zhikov [34] for the basic introduction to the theory of almost periodic functions). Almost periodic functions and almost automorphic functions have received a great attention recently, primarily from their invaluable importance in the qualitative analysis of solutions of abstract integro-differential equations in Banach spaces (cf. also the research monographs by T. Diagana [10], G. M. N'Guérékata [21]-[22], M. Kostić [29] and S. Zaidman [40]).

In order to better explain the main ideas of this paper, we will first recall the basic facts about almost periodic functions, uniformly recurrent functions and their generalizations. Throughout this paper we assume that $(E, \|\cdot\|)$ is a complex Banach space. By $C(I : E)$, $C_b(I : E)$, $C_0(I : E)$ and $BUC(I : E)$ we denote the vector spaces consisting of all continuous functions $f : I \rightarrow E$, all bounded continuous functions $f : I \rightarrow E$, all bounded continuous functions $f : I \rightarrow E$ satisfying that $\lim_{|t| \rightarrow +\infty} \|f(t)\| = 0$ and all bounded uniformly continuous functions $f : I \rightarrow E$, respectively. As is well known, $C_b(I : E)$, $C_0(I : E)$ and $BUC(I : E)$ are Banach spaces equipped with the sup-norm, denoted by $\|\cdot\|_\infty$. Let $I = \mathbb{R}$ or $I = [0, \infty)$; unless stated otherwise, we will always assume henceforth that $f : I \rightarrow E$ is a continuous function. Given $\epsilon > 0$, we call $\tau > 0$ an ϵ -period for $f(\cdot)$ iff

$$\|f(t + \tau) - f(t)\| \leq \epsilon, \quad t \in I.$$

The set constituted of all ϵ -periods for $f(\cdot)$ is denoted by $\vartheta(f, \epsilon)$. It is said that $f(\cdot)$ is almost periodic iff for each $\epsilon > 0$ the set $\vartheta(f, \epsilon)$ is relatively dense in $[0, \infty)$, which means that there exists $l > 0$ such that any subinterval of $[0, \infty)$ of length l meets $\vartheta(f, \epsilon)$. By $AP(I : E)$ we denote the vector space consisting of all almost periodic functions from the interval I into E . Equipped with the sup-norm, $AP(I : E)$

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becomes a Banach space. The function $f : I \rightarrow E$ is said to be asymptotically almost periodic iff there exist an almost periodic function $h : I \rightarrow E$ and a function $\phi \in C_0(I : E)$ such that $f(t) = h(t) + \phi(t)$ for all $t \in I$ (the existing literature is somewhat controversial about the definition of an asymptotically almost periodic $f : \mathbb{R} \rightarrow E$; in the case that $I = \mathbb{R}$, we will use here the approach of C. Zhang from [41]). This is equivalent to saying that, for every $\epsilon > 0$, we can find numbers $l > 0$ and $M > 0$ such that every subinterval of I of length l contains, at least, one number τ such that $\|f(t + \tau) - f(t)\| \leq \epsilon$ provided $|t|, |t + \tau| \geq M$.

Within the theory of topological dynamical systems, the notion of recurrence plays an important role; for more details, the reader may consult the research monographs [9] by J. de Vries and [14] by T. Eisner et al. Following A. Haraux and P. Souplet [25], we say that the function $f(\cdot)$ is uniformly recurrent iff there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n \rightarrow +\infty} \alpha_n = +\infty$ and

$$(1.1) \quad \lim_{n \rightarrow +\infty} \sup_{t \in \mathbb{R}} \|f(t + \alpha_n) - f(t)\| = 0.$$

It is well known that any almost periodic function is uniformly recurrent, while the converse statement is not true in general. Any \odot_g -almost periodic function under our consideration is uniformly recurrent, so that the class of uniformly recurrent functions plays a leadership role in our investigation. It is worth noting that the convergence of the above limit is uniform in the variable $t \in \mathbb{R}$, so that the notion of a uniformly recurrent function should not be mistakenly identified with the notion of a recurrent function in the continuous Bebutov system [4], where the author analyzed the usual Fréchet space $C(\mathbb{R})$ and the topology of uniform convergence on compact sets (cf. also the paper [8] by L. I. Danilov and references cited therein for further information in this direction).

A function $f \in L_{loc}^p(I : E)$ is said to be Stepanov p -bounded iff

$$\|f\|_{S^p} := \sup_{t \in I} \left(\int_t^{t+1} \|f(s)\|^p ds \right)^{1/p} < \infty.$$

Equipped with the above norm, the space $L_S^p(I : E)$ consisted of all Stepanov p -bounded functions is a Banach space. A function $f \in L_S^p(I : E)$ is said to be Stepanov p -almost periodic iff the function $\hat{f} : I \rightarrow L^p([0, 1] : E)$, defined by

$$(1.2) \quad \hat{f}(t)(s) := f(t + s), \quad t \in I, \quad s \in [0, 1]$$

is almost periodic. Furthermore, we say that a function $f \in L_S^p(I : E)$ is asymptotically Stepanov p -almost periodic iff there exist a Stepanov p -almost periodic function $g \in L_S^p(I : E)$ and a function $q \in L_S^p(I : E)$ such that $f(t) = g(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^p([0, 1] : E))$. It is well known that, if $1 \leq p \leq q < \infty$ and $f(\cdot)$ is (asymptotically) Stepanov q -almost periodic, then $f(\cdot)$ is (asymptotically) Stepanov p -almost periodic. It is said that $f(\cdot)$ is (asymptotically) Stepanov almost periodic iff $f(\cdot)$ is (asymptotically) Stepanov 1-almost periodic.

After recalling these definitions, we can proceed further with the description of the main ideas and aims of this paper. Albeit the definitions of an almost periodic function and a uniformly recurrent function are quite easy and understandable, the class consisting of all almost periodic functions and the class consisting of all uniformly recurrent functions are sometimes very unpleasant and difficult to deal with. For example, already H. Bohr has marked in his pioneering papers that it

is not so satisfactory to introduce the concept of almost periodicity by requiring that for each number $\epsilon > 0$ the set $\vartheta(f, \epsilon)$ is unbounded (see e.g., [7]). A bounded uniformly continuous function $f : I \rightarrow \mathbb{R}$ satisfying this property need not be almost periodic, its Bohr-Fourier coefficients cannot be defined in general, and moreover, if two bounded uniformly continuous functions $f : I \rightarrow \mathbb{R}$ and $g : I \rightarrow \mathbb{R}$ satisfy this property, then its sum $f + g : I \rightarrow \mathbb{R}$ need not satisfy this property (see [6, part I, pp. 117-118]). Furthermore, saying that for each number $\epsilon > 0$ the set $\vartheta(f, \epsilon)$ is unbounded is equivalent to saying that $f(\cdot)$ is uniformly recurrent; hence, the sum of two bounded uniformly continuous uniformly recurrent functions is not uniformly recurrent, in general. Taking into account Proposition 2.19 below, we get that the sum of two bounded uniformly continuous \odot_g -almost periodic functions is not \odot_g -almost periodic, in general. This example can be also used for proving the fact that the pointwise product of two bounded uniformly continuous, uniformly recurrent (\odot_g -almost periodic) functions is not uniformly recurrent (\odot_g -periodic), in general.

The above observation of H. Bohr has motivated us to further analyze some very specific examples of generalized almost periodic functions in more detail here (see [2] for a non-updated list of unsolved problems in the theory). First of all, we recall that B. Basit and H. Güenzler have constructed, in [3, Example 3.2], a bounded continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that its first antiderivative $t \mapsto \int_0^t f(s) ds$, $t \in \mathbb{R}$ is almost periodic, while the function $f(\cdot)$ itself is not uniformly continuous, not Stepanov almost periodic, not almost automorphic as well as

$$(1.3) \quad \sup_{t \in [-2, 0]} |f(t + \tau) - f(t)| \geq 1 \text{ for all } \tau \geq 2.$$

The construction concretely goes as follows. Define a continuous 2^{n+1} -periodic function $f_n : \mathbb{R} \rightarrow \mathbb{R}$ by $f_n(t) := \sin(2^n \pi t)$ for $t \in [2^n - 1, 2^n]$, $f_n(t) := 0$ for $t \in [-2^n, 2^n - 1)$, and extend it 2^{n+1} -periodically to the whole real axis. Then $\text{supp}(f_n) = [2^n - 1, 2^n] + 2^{n+1}\mathbb{Z}$, which simply implies that $\text{supp}(f_n)$ and $\text{supp}(f_m)$ are disjoint sets for each integers $n, m \in \mathbb{N}$ with $n \neq m$. Therefore, the function $f(x) := \sum_{n=1}^{\infty} f_n(x)$, $x \in \mathbb{R}$ is well-defined. This function satisfies all above properties, and we will provide a small contribution here by proving that the set $\vartheta(f, \epsilon)$ is empty for each number $\epsilon \in (0, 1)$:

- \triangle . Suppose that $\tau \in \vartheta(f, \epsilon)$. Due to (1.3), we have $\tau \in (0, 2)$ so that there exist two possibilities: $\tau \in (0, 1)$ or $\tau \in [1, 2)$. In the first case, there exists a sufficiently large number $n \in \mathbb{N}$ such that $(2^n + 1) - (2^n - 1 + 2^{-n-1}) > \tau$. Let $t = 2^n - 1 + 2^{-n-1}$; then $t + \tau \in (2^n, 2^n + 1)$ and therefore $f(t) = 1$ while $f(t + \tau) = 0$ so that $|f(t + \tau) - f(t)| = 1 > \epsilon$. In the second case, there exists a sufficiently large number $n \in \mathbb{N}$ such that $\tau > 2^{-n-1}$. In this case, take $t = 2^n - 2^{-n-1}$; then $t + \tau \in (2^n, 2^n + 1)$ and therefore $f(t) = -1$ while $f(t + \tau) = 0$ so that $|f(t + \tau) - f(t)| = 1 > \epsilon$.

Basically, the functions $f(\cdot)$ satisfying that there exists a number $\epsilon \in (0, 1)$ such that the set $\vartheta(f, \epsilon)$ is bounded will not occupy our attention henceforth. In connection with the above example, we would like to propose the following question:

Question 1.1. *Suppose that $f : I \rightarrow E$ is a bounded, continuous and Stepanov almost periodic. Is it true that $\vartheta(f, \epsilon) \neq \emptyset$ ($\vartheta(f, \epsilon)$ is unbounded) for all $\epsilon > 0$?*

More concretely, assume that $\alpha, \beta \in \mathbb{R}$ and $\alpha\beta^{-1}$ is a well defined irrational number. Then we know that the functions

$$m(t) := \sin\left(\frac{1}{2 + \cos \alpha t + \cos \beta t}\right), \quad t \in \mathbb{R}$$

and

$$n(t) := \cos\left(\frac{1}{2 + \cos \alpha t + \cos \beta t}\right), \quad t \in \mathbb{R}$$

are bounded, continuous and Stepanov p -almost periodic for any finite exponent $p \geq 1$ as well as that any of them is not almost periodic since it is not uniformly continuous (see e.g., [29]). Is it true that $\vartheta(m, \epsilon) \neq \emptyset$ ($\vartheta(m, \epsilon)$ is unbounded) [$\vartheta(n, \epsilon) \neq \emptyset$ ($\vartheta(n, \epsilon)$ is unbounded)] for all $\epsilon > 0$?

We continue by observing that A. Haraux and P. Souplet have proved, in [25, Theorem 1.1], that there exists a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is uniformly continuous, uniformly recurrent and unbounded. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$(1.4) \quad f(t) := \sum_{n=1}^{\infty} \frac{1}{n} \sin^2\left(\frac{t}{2^n}\right) dt, \quad t \in \mathbb{R}$$

and it is closely connected with the small divisors problem. From the argumentation given in the proof of the above-mentioned theorem, it immediately follows that the function $f(\cdot)$ given by (1.4) is neither Besicovitch almost periodic [29] nor asymptotically Stepanov almost automorphic (see Subsection 1.1 for the notion used here as well as in the formulations of Theorem 1.2-Theorem 1.5 and Theorem 1.7). The reason for that is quite simple, this function is even and enjoys the property that

$$\limsup_{t \rightarrow +\infty} \frac{1}{2t} \int_{-t}^t f(s) ds = +\infty.$$

Since the concepts of H. Weyl and A. S. Besicovitch suggest very general ways of approaching almost automorphic ([29]), it is logical to ask whether the function $f(\cdot)$ is Weyl almost automorphic. In this paper, we will prove the following result:

Theorem 1.2. *The function $f(\cdot)$, given by (1.4), is Weyl p -almost automorphic for any finite exponent $p \geq 1$ and satisfies that for each number $\tau \in \mathbb{R}$ the function $f(\cdot + \tau) - f(\cdot)$ belongs to the space $ANP(\mathbb{R} : \mathbb{C})$.*

Concerning this contribution, it is worth noting that the unbounded functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for each number $\tau \in \mathbb{R}$ the function $f(\cdot + \tau) - f(\cdot)$ belongs to the space $AP(\mathbb{R} : \mathbb{C})$ have been analyzed by A. M. Samoilenko and S. I. Trofimchuk in [36] (let us recall that the bounded functions satisfying this condition are always almost periodic due to the famous Loomis theorem). Let us also note that the function $f(\cdot)$, given by (1.4), has been employed by H. Y. Zhao and M. Fečkan in [42], for proving the fact that for each finite real numbers $M, L > 0$ the set consisting of all almost periodic functions $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $|h(t)| \leq M, t \in \mathbb{R}$ and $|h(t_1) - h(t_2)| \leq L|t_1 - t_2|, t_1, t_2 \in \mathbb{R}$ is not precompact in $C(\mathbb{R})$.

Further on, in [25, Theorem 1.2], A. Haraux and P. Souplet have proved that for each real number $c > 0$ the function $h(\cdot) = \min(c, f(\cdot))$, where $f(\cdot)$ is given by (1.4), is bounded uniformly continuous, uniformly recurrent and not asymptotically almost periodic. Since the function $h(\cdot)$ is uniformly continuous, Lemma 1.11(ii) below implies that $h(\cdot)$ is asymptotically Stepanov p -almost automorphic ($p \geq 1$)

iff $h(\cdot)$ is asymptotically almost automorphic. But, this is actually not the case because [25, Lemma 2.1] can be improved in the following manner:

Lemma 1.3. *Let $\omega : \mathbb{R} \rightarrow [0, \infty)$ be Lipschitz continuous and such that the set $\omega([0, +\infty))$ is unbounded. Define, for each finite number $c > \liminf_{t \rightarrow +\infty} \omega(t)$, the function $\omega_1 : \mathbb{R} \rightarrow [0, \infty)$ by $\omega_1(t) := \min(c, \omega(t))$, $t \in \mathbb{R}$. Then the restriction of function $\omega_1(\cdot)$ to the non-negative real axis is not asymptotically almost automorphic.*

The proof of Lemma 1.3 is almost the same as that of [25, Lemma 2.1]. The only thing worth noticing is that the existence of an almost automorphic function $\omega_1^*(\cdot)$ such that $\lim_{t \rightarrow +\infty} |\omega_1(t) - \omega_1^*(t)| = 0$ implies, as in the proof of the above-mentioned lemma, that $\omega_1^* \equiv c$; this follows by using the same arguments, almost directly from definition of almost automorphicity (we do not need the fact that the limits in the second part of proof are uniform on \mathbb{R}).

We will extend [25, Theorem 1.2] in the following way:

Theorem 1.4. *Let the function $f(\cdot)$ be given by (1.4), and let $c > 0$. Then the function $h(t) := \min(c, f(t))$, $t \in \mathbb{R}$ is bounded uniformly continuous, uniformly recurrent, not asymptotically (Stepanov) almost automorphic, and not (Stepanov) quasi-asymptotically almost periodic.*

Concerning this contribution, we have made a decision to further analyze the function constructed by H. Bohr on pp. 113–115 of the first part of his landmark trilogy [6]. In actual fact, the results obtained by A. M. Fink in his doctoral dissertation [18] tell us that this function is uniformly continuous (nonexpansive, in fact), uniformly recurrent and not almost periodic. The construction of this function goes as follows. Let $\tau_1 := 1$, $\tau_2 > 2$ and let the sequence $(\tau_n)_{n \in \mathbb{N}}$ of positive real numbers satisfy $\tau_n > 2 \sum_{i=1}^{n-1} i\tau_i$ for all $n \in \mathbb{N}$. Let the sequence $(f_n : \mathbb{R} \rightarrow \mathbb{R})_{n \in \mathbb{N}}$ be defined as follows. Set $f_1(x) := 1 - |x|$ for $|x| \leq 1$ and $f_1(x) := 0$, otherwise. If the functions $f_1(\cdot), \dots, f_{n-1}(\cdot)$ are already defined, set

$$f_n(x) := f_{n-1}(x) + \sum_{m=1}^{n-1} \frac{n-m}{n} \left[f_{n-1}(x - m\tau_n) + f_{n-1}(x + m\tau_n) \right], \quad x \in \mathbb{R}.$$

Then

$$|f_n(x + \tau_n) - f_n(x)| \leq \frac{1}{n}, \quad n \in \mathbb{N}, x \in \mathbb{R},$$

and the function

$$(1.5) \quad f(x) := \lim_{n \rightarrow +\infty} f_n(x), \quad x \in \mathbb{R}$$

is well defined, even and satisfies that $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}$. It is worth observing that this function also satisfies all clarified properties of function $h(\cdot)$ from Theorem 1.4:

Theorem 1.5. *The function $f : \mathbb{R} \rightarrow \mathbb{R}$, given by (1.5), is bounded uniformly continuous, uniformly recurrent, not asymptotically (Stepanov) almost automorphic, and not (Stepanov) quasi-asymptotically almost periodic.*

In Example 2.25, we will show that, for some concrete choices of sequences $(\tau_n)_{n \in \mathbb{N}}$, the function $f : \mathbb{R} \rightarrow \mathbb{R}$, given by (1.5), is Weyl p -almost automorphic for each finite exponent $p \geq 1$. Since any Stepanov p -quasi-asymptotically almost

periodic function is Weyl- p -almost periodic ($p \geq 1$) in the sense of A. S. Kovanko's approach (see [32, Proposition 2.11]), it is quite reasonable to ask the following:

Question 1.6. *Is it true that the function $f(\cdot)$, given by (1.5), is (equi-)Weyl- p -almost periodic for some (each) finite exponent $p \geq 1$?*

We would like to note that the function used by J. de Vries in [9, point 6., p. 208] can serve as a much simpler example of a bounded uniformly continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying all clarified properties of functions examined in Theorem 1.4 and Theorem 1.5: Let $(p_i)_{i \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers such that $p_i | p_{i+1}$, $i \in \mathbb{N}$ and $\lim_{i \rightarrow \infty} p_i / p_{i+1} = 0$. Define the function $f_i : [-p_i, p_i] \rightarrow [0, 1]$ by $f_i(t) := |t|/p_i$, $t \in [-p_i, p_i]$ and extend the function $f_i(\cdot)$ periodically to the whole real axis; the obtained function, denoted by the same symbol $f_i(\cdot)$, is of period $2p_i$ ($i \in \mathbb{N}$). Set

$$(1.6) \quad f(t) := \sup\{f_i(t) : i \in \mathbb{N}\}, \quad t \in \mathbb{R}.$$

We will prove the following:

Theorem 1.7. *The function $f : \mathbb{R} \rightarrow \mathbb{R}$, given by (1.6), is bounded uniformly continuous, uniformly recurrent, not asymptotically (Stepanov) almost automorphic, and not (Stepanov) quasi-asymptotically almost periodic.*

We proceed with much elementary things, by considering a general continuous function $f : I \rightarrow E$. Suppose first that there exists a number $\epsilon > 0$ such that $\vartheta(f, \epsilon) \neq \emptyset$, say $\tau \in \vartheta(f, \epsilon)$. Setting $M := \sup_{t \in I, |t| \leq \tau} \|f(t)\|$, it can be simply proved by induction that we have $\|f(t)\| \leq M + n\epsilon$ for all $t \in I$ with $|t| \in [n\tau, (n+1)\tau]$ ($n \in \mathbb{N}$). Hence, $\|f(t)\| \leq M + |t|\epsilon/\tau$ for all $t \in I$ with $|t| \in [n\tau, (n+1)\tau]$ ($n \in \mathbb{N}$), so that

$$(1.7) \quad \|f(t)\| \leq M + |t|\epsilon/\tau, \quad t \in \mathbb{R}$$

and the function $f(\cdot)$ is linearly bounded as $|t| \rightarrow +\infty$. Further on, it is clear that the assumption $\vartheta(f, \epsilon) \neq \emptyset$ for each $\epsilon > 0$ implies that $\vartheta(f, \epsilon)$ is infinite for each $\epsilon > 0$ as well as that there does not exist a finite constant M such that the interval $[0, M]$ contains the union of sets $\vartheta(f, \epsilon)$ when $\epsilon > 0$; this is a simple consequence of the fact that for each $\epsilon > 0$ we have $j\vartheta(f, \epsilon/n) \subseteq \vartheta(f, \epsilon)$ for all $j = 1, \dots, n$. Let us observe that a linear function $f : I \rightarrow \mathbb{C}$ can serve as an example of a function for which the growth order in (1.7) cannot be improved and for which the assumption $\vartheta(f, \epsilon) \neq \emptyset$ for each $\epsilon > 0$ does not imply the existence of a number $\epsilon_0 > 0$ such that the set $\vartheta(f, \epsilon_0)$ is unbounded.

To the best of our knowledge, this is the first systematic study of vector-valued uniformly recurrent functions. In this paper, we attempt to further profile the sets of ϵ -periods of uniformly recurrent functions by introducing the class of \odot_g -almost periodic functions, which is simply defined by using the notions of lower and upper (Banach) densities for the subsets of the non-negative real axis (we feel it is our duty to say that we have only partially succeeded in our mission because it is very difficult to practically control and give intrinsic characterizations of ϵ -periods). The lower and upper (Banach) m_n -densities for the subsets of \mathbb{N} , considered recently in [31], are discrete analogues of the lower and upper (Banach) g -densities considered in this paper. In the discrete setting, these densities play an important role in the field of linear chaos, for example, in definitions of frequent hypercyclicity and reiterative m_n -distributional chaos of linear continuous operators on Fréchet spaces. In the

continuous setting, these densities play an important role in the qualitative analysis of solutions to the abstract (fractional) integro-differential equations in Fréchet spaces; see e.g., the recent research monograph [30] by the author and references cited therein for a brief introduction to the theory of linear chaos. We generalize the notion of almost periodicity by analyzing several different types of (Stepanov) \odot_g -almost periodicity for functions with values in complex Banach spaces. Speaking-matter-of-factly, we analyze the lower and upper (Banach) g -densities of sets $\vartheta(f, \epsilon)$, where $\epsilon > 0$ and $g : [0, \infty) \rightarrow [1, \infty)$ is an increasing mapping satisfying the condition (1.11) below.

The organization of paper can be briefly described as follows. In Subsection 1.1, we recall the basic facts and results about generalized almost periodic functions and generalized almost automorphic functions that we will need later on. Subsection 1.2 investigates the lower and upper (Banach) g -densities for the subsets of the non-negative real line; in this subsection, we present our first significant contributions, Theorem 1.14 and Theorem 1.15, in which we transfer the main result of paper [20] by G. Grekos, V. Toma and J. Tomanová to the continuous setting and reconsider the notion and several recent results from [31].

In Section 2, we analyze \odot_g -almost periodic functions, uniformly recurrent functions and their Stepanov generalizations. With the notation explained below, we say that a continuous function $f : I \rightarrow E$ is \odot_g -almost periodic iff for each $\epsilon > 0$ we have $\odot_g(\vartheta(f, \epsilon)) > 0$; see Definition 2.1, in which the symbol \odot_g denotes exactly one of the densities \underline{d}_{gc} , \overline{d}_{gc} , $\underline{Bd}_{l;gc}$, $\underline{Bd}_{u;gc}$, $\overline{Bd}_{l;gc}$ or $\overline{Bd}_{u;gc}$. In the paragraph following Definition 2.1, we collect the basic properties of \odot_g -almost periodic functions and uniformly recurrent functions. The main purpose of Proposition 2.2 is to clarify the supremum formula for uniformly recurrent functions; in Proposition 2.3, we prove that any almost periodic function $f : I \rightarrow E$ is \odot_g -almost periodic. All introduced concepts are equivalent in case $g(x) \equiv x$, and reduced then to the concept of almost periodicity (Proposition 2.4). After that, in Proposition 2.5, we prove that the almost periodicity is equivalent with the $\underline{Bd}_{l;gc}$ -almost periodicity and $\underline{Bd}_{u;gc}$ -almost periodicity for every increasing mapping $g(\cdot)$ satisfying the condition (1.11).

Definition 2.9 introduces the notions of asymptotical uniform recurrence and asymptotical \odot_g -almost periodicity, while Proposition 2.10 restates all results from Section 2 proved by then in this context. We introduce the notion of (asymptotical) Stepanov p -uniform recurrence and (asymptotical) Stepanov (p, \odot_g) -almost periodicity in Definition 2.11. The main purpose of Theorem 2.13 is to show that any asymptotically uniform recurrent, quasi-asymptotically almost periodic function is asymptotically almost periodic; the Stepanov analogue of this statement is also considered here. Proposition 2.15 shows that the uniform recurrence and asymptotical almost automorphicity (asymptotical almost periodicity) implies almost automorphicity (almost periodicity), for the usually considered classes and Stepanov classes. Further on, in Theorem 2.16 and Proposition 2.17, we prove that any uniformly continuous (asymptotically) Stepanov p -uniformly recurrent [(asymptotically) Stepanov (p, \odot_g) -almost periodic/Stepanov p -quasi-asymptotically almost periodic] function $f : I \rightarrow E$ is asymptotically uniformly recurrent [asymptotically \odot_g -almost periodic, quasi-asymptotically almost periodic].

Proposition 2.19 clarifies an interesting result which shows that for any (asymptotically) uniformly continuous, uniformly recurrent function we can find an increasing mapping $g : [0, \infty) \rightarrow [1, \infty)$ such that (1.11) holds and $f(\cdot)$ is (asymptotically)

\cdot_g -almost periodic for $\cdot_g \in \{d_{gc}, \bar{d}_{gc}\}$ (see also Remark 2.20, where we use the densities $\overline{Bd}_{l:gc}(\cdot)$ and $\overline{Bd}_{u:gc}(\cdot)$). In Example 2.23, we prove that the compactly almost automorphic function constructed by A. M. Fink in [17] is not asymptotically uniformly recurrent; the proofs of Theorem 1.2, Theorem 1.4, Theorem 1.5 and Theorem 1.7 are provided after that.

The main aim of Section 3, which is written in a concise, semi-heuristical manner, is to investigate the existence and uniqueness of uniformly recurrent and \odot_g -almost periodic type solutions of abstract integro-differential equations in Banach spaces; in this section, we pay special attention to the invariance of (asymptotical) uniform recurrence and (asymptotical) \odot_g -almost periodicity under the actions of convolution products. For simplicity, we will not consider two-parameter uniformly recurrent (\odot_g -almost periodic) functions, composition principles and applications to abstract semilinear integro-differential equations.

We use the standard notation throughout the paper. If X is also a complex Banach space, then $L(E, X)$ stands for the space of all continuous linear mappings from E into X ; $L(E) \equiv L(E, E)$. Given $s \in \mathbb{R}$ in advance, set $\lfloor s \rfloor := \sup\{l \in \mathbb{Z} : s \geq l\}$ and $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. The function $sign : \mathbb{R} \rightarrow \{-1, 0, 1\}$ is defined by $sign(t) := -1$ (0 , 1) iff $t < 0$ ($t = 0$, $t > 0$); if $c \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, then we define $cA := \{ca : a \in A\}$. If $\psi : \mathbb{R} \rightarrow \mathbb{C}$ and $f : \mathbb{R} \rightarrow E$ are measurable functions, the convolution product $\psi * f$ is defined by $\psi * f(t) := \int_{-\infty}^{\infty} \psi(t-s)f(s) ds$, $t \in \mathbb{R}$, if this integral exists in the Bochner sense. Let us recall that a function $f : (0, \infty) \rightarrow \mathbb{R}$ is called subadditive iff $f(x+y) \leq f(x) + f(y)$, $x, y > 0$. A continuous version of Fekete's lemma states that, for every measurable subadditive function $f : (0, \infty) \rightarrow \mathbb{R}$, the limit $\lim_{t \rightarrow +\infty} \frac{f(t)}{t}$ exists in $[-\infty, \infty)$ and $\lim_{t \rightarrow +\infty} \frac{f(t)}{t} = \inf_{t > 0} \frac{f(t)}{t}$ (see e.g., [27, Theorem 6.6.1]). We will use the following simple lemma:

Lemma 1.8. *There do not exist $k \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that*

$$(1.8) \quad sign\left(\cos((n+k)\pi\sqrt{2})\right) = sign\left(\cos(n\pi\sqrt{2})\right), \quad n \in \mathbb{Z}, |n| \geq n_0.$$

Proof. Since $\cos(n\pi\sqrt{2}) \neq 0$ for all $n \in \mathbb{Z}$, it is clear that (1.8) is equivalent to saying that $\cos((n+k)\pi\sqrt{2}) \cdot \cos(n\pi\sqrt{2}) > 0$, $n \in \mathbb{Z}$, $|n| \geq n_0$. If $k \in \mathbb{N}$ satisfies the above condition and $k\pi\sqrt{2} = 2k_0\pi + a$ for some numbers $k_0 \in \mathbb{Z}$ and $a \in (0, 2\pi)$, then we get from the above: $\cos(n\pi\sqrt{2} + a) \cdot \cos(n\pi\sqrt{2}) > 0$, $n \in \mathbb{Z}$, $|n| \geq n_0$. This cannot be satisfied because the set $\{e^{in\pi\sqrt{2}} : n \in \mathbb{Z}, |n| \geq n_0\}$ is dense in the unit sphere and $\cos x = \Re(e^{ix})$, $x \in \mathbb{R}$. \square

1.1. Almost periodic functions, almost automorphic functions and their generalizations. Let $f \in AP(I : E)$. Then the Bohr-Fourier coefficient $P_r(f) := \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t e^{-irs} f(s) ds$ exists for all $r \in \mathbb{R}$; furthermore, if $P_r(f) = 0$ for all $r \in \mathbb{R}$, then $f(t) = 0$ for all $t \in \mathbb{R}$, and $\sigma(f) := \{r \in \mathbb{R} : P_r(f) \neq 0\}$ is at most countable. By $AP(\Lambda : E)$, where Λ is a non-empty subset of \mathbb{R} , we denote the vector subspace of $AP(I : E)$ consisting of all functions $f \in AP(I : E)$ for which the inclusion $\sigma(f) \subseteq \Lambda$ holds good. It can be easily seen that $AP(\Lambda : E)$ is a closed subspace of $AP(I : E)$ and therefore Banach space itself.

Let us recall that $f(\cdot)$ is anti-periodic iff there exists $p > 0$ such that $f(x+p) = -f(x)$, $x \in I$. Any such function needs to be periodic, as it can be easily proved. Given $\epsilon > 0$, we call $\tau > 0$ an ϵ -antiperiod for $f(\cdot)$ iff $\|f(t+\tau) + f(t)\| \leq \epsilon$, $t \in I$. By $\vartheta_{ap}(f, \epsilon)$ we denote the set of all ϵ -antiperiods for $f(\cdot)$. It is said that $f(\cdot)$ is

almost anti-periodic iff for each $\epsilon > 0$ the set $\vartheta_{ap}(f, \epsilon)$ is relatively dense in $[0, \infty)$ (see [29] for more details). We know that any anti-periodic function needs to be almost anti-periodic as well as that any almost anti-periodic function needs to be almost periodic. Denote by $ANP_0(I : E)$ the linear span of almost anti-periodic functions from I into E . Then $ANP_0(I : E)$ is a linear subspace of $AP(I : E)$ and the linear closure of $ANP_0(I : E)$ in $AP(I : E)$, denoted by $ANP(I : E)$, satisfies

$$(1.9) \quad ANP(I : E) = AP_{\mathbb{R} \setminus \{0\}}(I : E).$$

The (Stepanov) quasi-asymptotically almost periodic functions have been recently analyzed in [32]. For our further work, it will be necessary to recall the following definition:

Definition 1.9. Suppose that $I = [0, \infty)$ or $I = \mathbb{R}$.

- (i) A bounded continuous function $f : I \rightarrow E$ is said to be quasi-asymptotically almost periodic iff for each $\epsilon > 0$ there exists a finite number $L(\epsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\epsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\epsilon, \tau) > 0$ such that

$$\|f(t + \tau) - f(t)\| \leq \epsilon, \text{ provided } t \in I \text{ and } |t| \geq M(\epsilon, \tau).$$

Denote by $Q - AAP(I : E)$ the set consisting of all quasi-asymptotically almost periodic functions from I into E .

- (ii) Let $f \in L_S^p(I : E)$. Then it is said $f(\cdot)$ is Stepanov p -quasi-asymptotically almost periodic iff for each $\epsilon > 0$ there exists a finite number $L(\epsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\epsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\epsilon, \tau) > 0$ such that

$$\int_t^{t+1} \|f(s + \tau) - f(s)\|^p ds \leq \epsilon^p, \text{ provided } t \in I \text{ and } |t| \geq M(\epsilon, \tau).$$

Denote by $S^pQ - AAP(I : E)$ the set consisting of all Stepanov p -quasi-asymptotically almost periodic functions from I into E .

Let us recall that that for each number $p \in [1, \infty)$ we have that $Q - AAP(I : E) \subseteq S^pQ - AAP(I : E)$ as well as that any asymptotically Stepanov p -almost periodic function is Stepanov p -quasi-asymptotically almost periodic. Furthermore, if $1 \leq p \leq q < \infty$, then $S^qQ - AAP(I : E) \subseteq S^pQ - AAP(I : E)$ and for any function $f \in L_S^p(I : E)$, we have that $f(\cdot)$ is Stepanov p -quasi-asymptotically almost periodic iff the function $\hat{f} : I \rightarrow L^p([0, 1] : E)$, defined by (1.2), is quasi-asymptotically almost periodic. It is said that $f(\cdot)$ is Stepanov quasi-asymptotically almost periodic iff $f(\cdot)$ is Stepanov 1-quasi-asymptotically almost periodic. Any asymptotically almost periodic function $f : I \rightarrow E$ is quasi-asymptotically almost periodic.

Let us also recall the notion of an (equi-)Weyl- p -almost periodic function (cf. [29, Section 2.3] for more details).

Definition 1.10. Let $1 \leq p < \infty$ and $f \in L_{loc}^p(I : E)$.

- (i) We say that the function $f(\cdot)$ is equi-Weyl- p -almost periodic, $f \in e-W_{ap}^p(I : E)$ for short, iff for each $\epsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$

such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\sup_{x \in I} \left[\frac{1}{l} \int_x^{x+l} \|f(t+\tau) - f(t)\|^p dt \right]^{1/p} \leq \epsilon.$$

- (ii) We say that the function $f(\cdot)$ is Weyl- p -almost periodic, $f \in W_{ap}^p(I : E)$ for short, iff for each $\epsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\lim_{l \rightarrow \infty} \sup_{x \in I} \left[\frac{1}{l} \int_x^{x+l} \|f(t+\tau) - f(t)\|^p dt \right]^{1/p} \leq \epsilon.$$

Before recollecting the material about almost automorphic type functions, let us recall that $PAP_0(\mathbb{R} : E)$ stands for the space consisting of all pseudo-ergodic components, i.e., bounded continuous functions $\Phi : \mathbb{R} \rightarrow E$ such that

$$\lim_{l \rightarrow \infty} \frac{1}{2l} \int_{-l}^l \|\Phi(s)\| ds = 0.$$

For simplicity, we will mostly use the usual ergodic components henceforth.

Let $f : \mathbb{R} \rightarrow E$ be continuous. Then it is said that $f(\cdot)$ is almost automorphic iff for every real sequence (b_n) there exist a subsequence (a_n) of (b_n) and a map $g : \mathbb{R} \rightarrow E$ such that

$$(1.10) \quad \lim_{n \rightarrow \infty} f(t + a_n) = g(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} g(t - a_n) = f(t),$$

pointwise for $t \in \mathbb{R}$. If the convergence of limits appearing in (1.10) is uniform on compact subsets of \mathbb{R} , then we say that $f(\cdot)$ is compactly almost automorphic. It is worth noting that an almost automorphic function $f(\cdot)$ is compactly almost automorphic iff it is uniformly continuous as well as that an almost automorphic function is always bounded.

Let $p \in [1, \infty)$. A function $f \in L_{loc}^p(\mathbb{R} : E)$ is called Stepanov p -almost automorphic (see e.g., G. M. N'Guérékata and A. Pankov [23]) iff for every real sequence (a_n) , there exist a subsequence (a_{n_k}) and a function $g \in L_{loc}^p(\mathbb{R} : E)$ such that

$$\lim_{k \rightarrow \infty} \int_t^{t+1} \|f(a_{n_k} + s) - g(s)\|^p ds = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \int_t^{t+1} \|g(s - a_{n_k}) - f(s)\|^p ds = 0$$

for each $t \in \mathbb{R}$; a function $f \in L_{loc}^p(I : E)$ is called asymptotically Stepanov p -almost automorphic iff there exists an S^p -almost automorphic function $g(\cdot)$ and a function $q \in L_S^p(I : E)$ such that $f(t) = g(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^p([0, 1] : E))$ (see also the paper [13] by H.-S. Ding and S.-M. Wan for the case that $I = \mathbb{R}$). Any Stepanov p -almost automorphic function $f(\cdot)$ has to be Stepanov p -bounded. Furthermore, if $1 \leq p \leq q < \infty$ and a function $f(\cdot)$ is (asymptotically) Stepanov q -almost automorphic, then $f(\cdot)$ is (asymptotically) Stepanov p -almost automorphic. We say that a function $f(\cdot)$ is (asymptotically) Stepanov almost automorphic iff $f(\cdot)$ is (asymptotically) Stepanov 1-almost automorphic. Let us recall that any uniformly continuous Stepanov almost periodic (automorphic) function $f(\cdot)$ is almost periodic (automorphic); see [11, Theorem 3.3].

The following lemma can be deduced by using an elementary argumentation involving [28, Proposition 3.1], the above-mentioned theorem (cf. also [26, Lemma 1]) and a simple observation that any uniformly continuous function $q \in C_0(I : L^p([0, 1] : E))$ belongs to the space $C_0(I : E)$:

Lemma 1.11. *Let $f : I \rightarrow E$ be uniformly continuous and $p \in [1, \infty)$.*

- (i) *If $f(\cdot)$ is asymptotically Stepanov p -almost periodic, then $f(\cdot)$ is asymptotically almost periodic.*
- (ii) *If $f(\cdot)$ is asymptotically Stepanov p -almost automorphic, then $f(\cdot)$ is asymptotically almost automorphic.*

We also need the following important definition.

Definition 1.12. (S. Abbas, [1]) Let $p \in [1, \infty)$. Then we say that a function $f \in L^p_{loc}(\mathbb{R} : E)$ is Weyl p -almost automorphic iff for every real sequence (s_n) , there exist a subsequence (s_{n_k}) and a function $f^* \in L^p_{loc}(\mathbb{R} : E)$ such that

$$\lim_{k \rightarrow +\infty} \lim_{l \rightarrow +\infty} \frac{1}{2l} \int_{-l}^l \|f(t + s_{n_k} + x) - f^*(t + x)\|^p dx = 0$$

and

$$\lim_{k \rightarrow +\infty} \lim_{l \rightarrow +\infty} \frac{1}{2l} \int_{-l}^l \|f^*(t - s_{n_k} + x) - f(t + x)\|^p dx = 0$$

for each $t \in \mathbb{R}$.

Before we switch to the next subsection, we would like to observe that the Weyl p -almost automorphicity does not imply the Besicovitch p -unboundedness, in general (see e.g., [29, Definition 3.1.2] and [29, Definition 3.1.4] for the notion); a simple counterexample with $p = 1$ is given by the function $h(x) := \sqrt{|x|}$, $x \in \mathbb{R}$, which is Weyl (1-)almost automorphic with the limit function $h^* \equiv h$. This simply follows from the fact that for each numbers $t, \omega \in \mathbb{R}$ we have

$$\lim_{l \rightarrow +\infty} \frac{1}{2l} \int_{-l}^l |h(t + x + \omega) - h(t + x)| dx = 0.$$

1.2. Lower and upper (Banach) g -densities. Unless stated otherwise, in this paper we will always assume that $g : [0, \infty) \rightarrow [1, \infty)$ is an increasing mapping satisfying that there exists a finite number $L \geq 1$ such that

$$(1.11) \quad x \leq Lg(x), \quad x \geq 0,$$

which clearly implies $\liminf_{x \rightarrow +\infty} g(x)/x > 0$. If $A \subseteq [0, \infty)$ and $a, b \geq 0$, then we define $A(a, b) := \{x \in A ; x \in [a, b]\}$.

For simplicity and better exposition, in this paper we will use the Lebesgue measure $m(\cdot)$ on the non-negative real line, only. The use of Lebesgue measure is sufficiently enough for our analyses of uniformly continuous \odot_g -almost periodic functions; we feel it is our duty to say that the general case is much more complicated and almost not considered below.

Let us define (cf. [30]-[31] for more details):

- (i) The lower g -density of A , denoted in short by $\underline{d}_{gc}(A)$, as follows

$$\underline{d}_{gc}(A) := \liminf_{x \rightarrow +\infty} \frac{m(A(0, g(x)))}{x};$$

- (ii) the upper g -density of A , denoted in short by $\bar{d}_{gc}(A)$, as follows

$$\bar{d}_{gc}(A) := \limsup_{x \rightarrow +\infty} \frac{m(A(0, g(x)))}{x},$$

as well as:

(i) the lower l ; g -Banach density of A , denoted in short by $\underline{Bd}_{l;g}(A)$, as follows

$$\underline{Bd}_{l;g}(A) := \liminf_{x \rightarrow +\infty} \liminf_{y \rightarrow +\infty} \frac{m(A(y, y + g(x)))}{x};$$

(ii) the lower u ; g -Banach density of A , denoted in short by $\underline{Bd}_{u;g}(A)$, as follows

$$\underline{Bd}_{u;g}(A) := \limsup_{x \rightarrow +\infty} \liminf_{y \rightarrow +\infty} \frac{m(A(y, y + g(x)))}{x};$$

(iii) the (upper) l ; g -Banach density of A , denoted in short by $\overline{Bd}_{l;g}(A)$, as follows

$$\overline{Bd}_{l;g}(A) := \liminf_{x \rightarrow +\infty} \limsup_{y \rightarrow +\infty} \frac{m(A(y, y + g(x)))}{x};$$

(iv) the (upper) u ; g -Banach density of A , denoted in short by $\overline{Bd}_{u;g}(A)$, as follows

$$\overline{Bd}_{u;g}(A) := \limsup_{x \rightarrow +\infty} \limsup_{y \rightarrow +\infty} \frac{m(A(y, y + g(x)))}{x}.$$

Remark 1.13. It is worth noting that, for every set $A \subseteq [0, \infty)$, we have

$$\begin{aligned} & \liminf_{x \rightarrow +\infty} \limsup_{y \rightarrow +\infty} \frac{m([I \setminus A](y, y + g(x)))}{x} \\ &= \liminf_{x \rightarrow +\infty} \limsup_{y \rightarrow +\infty} \left[\frac{g(x) - m(A(y, y + g(x)))}{x} \right] \\ (1.12) \quad &= \liminf_{x \rightarrow +\infty} \left[\frac{g(x)}{x} - \liminf_{y \rightarrow +\infty} \frac{m(A(y, y + g(x)))}{x} \right]. \end{aligned}$$

Similarly,

$$\begin{aligned} & \limsup_{x \rightarrow +\infty} \limsup_{y \rightarrow +\infty} \frac{m([I \setminus A](y, y + g(x)))}{x} \\ (1.13) \quad &= \limsup_{x \rightarrow +\infty} \left[\frac{g(x)}{x} - \liminf_{y \rightarrow +\infty} \frac{m(A(y, y + g(x)))}{x} \right], \end{aligned}$$

$$(1.14) \quad \liminf_{x \rightarrow +\infty} \frac{m([I \setminus A](0, g(x)))}{x} = \liminf_{x \rightarrow +\infty} \left[\frac{g(x)}{x} - \limsup_{x \rightarrow +\infty} \frac{m(A(0, g(x)))}{x} \right]$$

and

$$(1.15) \quad \limsup_{x \rightarrow +\infty} \frac{m([I \setminus A](0, g(x)))}{x} = \limsup_{x \rightarrow +\infty} \left[\frac{g(x)}{x} - \liminf_{x \rightarrow +\infty} \frac{m(A(0, g(x)))}{x} \right].$$

Case $g(x) := (1 + |x|)^q$, $x \geq 0$ is the most important ($q \geq 1$), when we denote the corresponding densities by $\underline{d}_{qc}(A)$, $\overline{d}_{qc}(A)$, $\underline{Bd}_{l;qc}(A)$, $\underline{Bd}_{u;qc}(A)$, $\overline{Bd}_{l;qc}(A)$ and $\overline{Bd}_{u;qc}(A)$. Arguing similarly as in [31, Example 2.1(i)], for each number $q > 1$ we can simply construct a set $A \subseteq [0, \infty)$ such that $\overline{Bd}_{l;qc}(A) = 0$ and $\overline{Bd}_{u;qc}(A) = +\infty$; using the construction given in [31, Example 2.1(ii)], for each number $q > 1$ we can simply construct a set $A \subseteq [0, \infty)$ such that $\underline{d}_{qc}(A) = +\infty$ and $\overline{Bd}_{u;qc}(A) = 0$

so that the case $q > 1$ is not standard. Further on, if $q = 1$, then we get the usual concepts of lower and upper Banach densities: in this case, we have the following

Theorem 1.14. *Let $A \subseteq [0, \infty)$. Then we have*

$$(1.16) \quad \begin{aligned} \underline{Bd}_{l;1c}(A) &= \underline{Bd}_{u;1c}(A) \\ &= \sup_{x>0} \liminf_{y \rightarrow +\infty} \frac{m(A(y, y+x))}{x} = \sup_{x>0} \inf_{y \geq 0} \frac{m(A(y, y+x))}{x} := \underline{Bd}_c(A) \end{aligned}$$

and

$$(1.17) \quad \begin{aligned} \overline{Bd}_{l;1c}(A) &= \overline{Bd}_{u;1c}(A) \\ &= \inf_{x>0} \limsup_{y \rightarrow +\infty} \frac{m(A(y, y+x))}{x} = \inf_{x>0} \sup_{y \geq 0} \frac{m(A(y, y+x))}{x} := \overline{Bd}_c(A). \end{aligned}$$

Proof. Using the continuous version of Fekete's lemma, for the proof of first equality in (1.17) it suffices to show that the function

$$F(x) := \limsup_{y \rightarrow +\infty} m(A(y, y+x)), \quad x > 0$$

is subadditive, i.e., that for each fixed real numbers $x_1, x_2 > 0$ we have

$$\lim_{t \rightarrow +\infty} \sup_{t \geq y} m(A(t, t+x_1+x_2)) \leq \lim_{t \rightarrow +\infty} \sup_{t \geq y} m(A(t, t+x_1)) + \lim_{t \rightarrow +\infty} \sup_{t \geq y} m(A(t, t+x_2)).$$

This follows immediately if we prove that for each real number $y \geq 0$ we have

$$m(A(t, t+x_1+x_2)) \leq \sup_{t \geq y} m(A(t, t+x_1)) + \sup_{t \geq y} m(A(t, t+x_2)).$$

But, this is a simple consequence of the fact that for each real number $y \geq 0$ we have $t+x_1 \geq y$ and

$$m(A(t, t+x_1+x_2)) \leq m(A(t, t+x_1)) + m(A(t+x_1, t+x_1+x_2));$$

see also P. Ribenboim's paper [35]. Since

$$\limsup_{y \rightarrow +\infty} \frac{m(A(y, y+x))}{x} \leq \sup_{y \geq 0} \frac{m(A(y, y+x))}{x} \leq \liminf_{x \rightarrow +\infty} \sup_{y \geq 0} \frac{m(A(y, y+x))}{x},$$

for the proof of (1.17) it remains to be shown that

$$(1.18) \quad \liminf_{x \rightarrow +\infty} \sup_{y \geq 0} \frac{m(A(y, y+x))}{x} \leq \overline{Bd}_{u;1c}(A).$$

For this, we will slightly adapt the arguments proposed in the proof of discrete version of this statement, given in [20]. Define

$$D = \{x \in [0, 1] : \forall L > 0 \exists \text{interval } I' \subseteq [0, \infty) \text{ s.t. } m(I') \geq L \text{ and } m(A \cap I')/m(I') \geq x\}.$$

Repeating literally the arguments given in [20, Subsection 2.1], we obtain that $\liminf_{x \rightarrow +\infty} \sup_{y \geq 0} \frac{m(A(y, y+x))}{x} \leq b := \sup D$. The proof of (1.17) will be completed if one shows that $b \leq \inf_{x > 0} (\limsup_{y \rightarrow +\infty} m(A(y, y+x))/x)$. Suppose the contrary. Then there are a positive real number $x_0 > 0$ and two real numbers $x_1, x_2 \in [0, 1]$ such that $x_1 < x_2 < b$ and

$$\limsup_{y \rightarrow +\infty} m(A(y, y+x_0)) < x_0 x_1.$$

By definition of $\limsup_{y \rightarrow +\infty} \cdot$, this implies that there exists a positive real number $y_0 > 0$ such that $m(A(y, y+x_0)) < x_0 x_1$ for all $y \geq y_0$. We will prove that there exists a sufficiently large number $L > 0$ such that every subinterval $I' \subseteq I$ with

$m(I') \geq L$ satisfies $m(A \cap I') < x_2 m(I')$, showing that $x_2 \notin D$ and implying the contradiction. To see this, suppose that $I' = [y, y + h]$ for some $h > 0$. Then there exists $q \in \mathbb{N}_0$ such that $qx_0 \leq h < (q + 1)x_0$ and therefore

$$\begin{aligned} m(A(y, y + h)) &\leq y_0 + m(A(y_0, y + h)) \leq y_0 + \sum_{j=0}^q m(A(y_0 + jx_0, y_0 + (j + 1)x_0)) \\ &\leq y_0 + (q + 1)x_0x_1 \leq y_0 + x_0x_1 + qx_0x_1 < y_0 + x_0x_1 + hx_1 < hx_2, \end{aligned}$$

for any $h > 0$ sufficiently large. The proof of (1.18) follows from (1.12)-(1.13) and (1.17), which also shows that for each subset $A \subseteq [0, \infty)$ we have

$$(1.19) \quad \overline{Bd}_c(I \setminus A) + \underline{Bd}_c(A) = 1.$$

□

Since the case $g(x) \equiv x$ is very special in our analysis, we will also prove the following result which is well known in the discrete case (we then write $\underline{d}_c(A) \equiv \underline{d}_{g_c}(A)$ and $\overline{d}_c(A) \equiv \overline{d}_{g_c}(A)$):

Theorem 1.15. *Let $A \subseteq [0, \infty)$. Then we have*

$$0 \leq \underline{Bd}_c(A) \leq \underline{d}_c(A) \leq \overline{d}_c(A) \leq \overline{Bd}_c(A) \leq 1.$$

Proof. The only non-trivial parts are $\underline{Bd}_c(A) \leq \underline{d}_c(A)$ and $\overline{d}_c(A) \leq \overline{Bd}_c(A)$; due to (1.19), it suffices to show that $\overline{d}_c(A) \leq \overline{Bd}_c(A)$. Suppose the contrary. Due to (1.17) and definition of $\limsup_{x \rightarrow +\infty} \cdot$, it follows that

$$\lim_{t \rightarrow +\infty} \sup_{t \geq x} \frac{m(A(0, t))}{t} > \inf_{x > 0} \sup_{y \geq 0} \frac{m(A(y, y + x))}{x}.$$

Since the mapping in the above limit is monotonically decreasing in variable t , we get the existence of positive real numbers $\delta > 0$, $x_0 > 0$ and $y_0 > 0$ such that

$$(1.20) \quad \frac{m(A(0, y))}{y} \geq \frac{m(A(z, z + x_0))}{x_0} + \delta, \quad y \geq y_0, \quad z \geq 0.$$

Due to (1.20), we get

$$m(A(0, y)) \leq \sum_{j=0}^{\lfloor y/x_0 \rfloor} m(A(jx_0, (j + 1)x_0)) \leq \left(\lfloor y/x_0 \rfloor + 1 \right) \left(\frac{m(A(0, y))}{y} - \delta \right) x_0,$$

i.e.,

$$\left(1 - \frac{x_0}{y} \left(\lfloor y/x_0 \rfloor + 1 \right) \right) \frac{m(A(0, y))}{y} \leq -\delta x_0 \left(\lfloor y/x_0 \rfloor + 1 \right) / y, \quad y \geq y_0.$$

After taking the limits as $y \rightarrow +\infty$, we obtain $0 \leq -\delta$, which is a contradiction. □

Let us finally note that, in the combinatorial and additive number theory, the sets with positive upper Banach density play a major role; see e.g., [19, Section 5.7, Section 5.8]. A great number of results about the lower and upper (Banach) densities, known for subsets of integers, cannot be so easily reformulated and re-considered for the subsets of the non-negative real axis. This is not the case with the statements of [31, Proposition 2.5-Proposition 2.7, Corollary 2.2], which can be simply reformulated for (Banach) g -densities; details can be left to the interested reader.

Before going any further, the author would like to thank Prof. A. Haraux for his permission given us to submit the first version of this paper to J. Fourier Anal. Appl. as an addendum to the paper [25], written in his collaboration with Prof. P. Souplet. After that, the paper enlarged to a great extent and we gave up this idea.

2. \odot_g -ALMOST PERIODIC FUNCTIONS, UNIFORMLY RECURRENT FUNCTIONS AND THEIR STEPANOV GENERALIZATIONS

We will always assume henceforth that $g : [0, \infty) \rightarrow [1, \infty)$ is an increasing mapping satisfying that there exists a finite number $L \geq 1$ such that (1.11) holds. Let \odot_g denote exactly one of the symbols \underline{d}_{gc} , \bar{d}_{gc} , $\underline{Bd}_{l;gc}$, $\underline{Bd}_{u;gc}$, $\bar{Bd}_{l;gc}$ or $\bar{Bd}_{u;gc}$.

We start by introducing the following notion:

Definition 2.1. Let $f : I \rightarrow E$ be continuous. Then it is said that $f(\cdot)$ is \odot_g -almost periodic iff for each $\epsilon > 0$ we have $\odot_g(\vartheta(f, \epsilon)) > 0$.

We will use hereafter the following fundamental properties of \odot_g -almost periodic functions and uniformly recurrent functions, collected as follows (for parts (iv)-(vi), see [5, pp. 3-4]; for parts (vii)-(viii), see [34, p. 3]):

- (i) Any constant function is \odot_g -almost periodic, and for any \odot_g -almost periodic (uniformly recurrent) function $f(\cdot)$ we have that the function $\|f(\cdot)\|$ is \odot_g -almost periodic (uniformly recurrent). Any \odot_g -almost periodic function is uniformly recurrent.
- (ii) Since for each $\epsilon > 0$ and $c \in \mathbb{C} \setminus \{0\}$ we have $\vartheta(cf, \epsilon) = \vartheta(f, \epsilon/|c|)$, the \odot_g -almost periodicity of function $f(\cdot)$ implies the \odot_g -almost periodicity of function $cf(\cdot)$. Similarly, the uniform recurrence of function $f(\cdot)$ implies the uniform recurrence of function $cf(\cdot)$.
- (iii) The set consisting of all \odot_g -almost periodic (uniformly recurrent) functions is translation invariant in the sense that for each $\tau \in I$ and any \odot_g -almost periodic (uniformly recurrent) function $f(\cdot)$, the function $f(\cdot + \tau)$ is also \odot_g -almost periodic (uniformly recurrent).
- (iv) If $(f_n(\cdot))$ is a sequence of \odot_g -almost periodic (uniformly recurrent) functions and $(f_n(\cdot))$ converges uniformly to a function $f : I \rightarrow E$, then the function $f(\cdot)$ is \odot_g -almost periodic (uniformly recurrent).
- (v) If $E = \mathbb{C}$, $\inf_{x \in I} |f(x)| > m > 0$ and $f(\cdot)$ is a bounded \odot_g -almost periodic (uniformly recurrent) function, then the function $1/f(\cdot)$ is likewise a bounded \odot_g -almost periodic (uniformly recurrent).
- (vi) If $f(\cdot)$ is a bounded \odot_g -almost periodic (uniformly recurrent) function and $g : [0, \infty) \rightarrow X$ is continuous, then the mapping $g(\|f(\cdot)\|)$ is bounded and \odot_g -almost periodic (uniformly recurrent).
- (vii) If $f(\cdot)$ is a bounded \odot_g -almost periodic (uniformly recurrent) function and $r > 0$, then the function $\|f(\cdot)\|^r$ is bounded and \odot_g -almost periodic (uniformly recurrent).

Furthermore, it can be simply shown that:

- (viii) If $f : \mathbb{R} \rightarrow E$ is a bounded \odot_g -almost periodic (uniformly recurrent) function and $\psi \in L^1(\mathbb{R})$, then the function $(\psi * f)(\cdot)$ is bounded, uniformly continuous and \odot_g -almost periodic (uniformly recurrent).
- (ix) If $f : [0, \infty) \rightarrow E$ is uniformly recurrent and belongs to the space $C_0([0, \infty) : E)$, then $f \equiv 0$.

- (x) If $f : \mathbb{R} \rightarrow E$ is \odot_g -almost periodic (uniformly recurrent), then the function $\check{f} : \mathbb{R} \rightarrow E$, defined by $\check{f}(\cdot) := f(-\cdot)$, is \odot_g -almost periodic (uniformly recurrent). If, additionally, $f|_{[0, \infty)}(\cdot) \in C_0([0, \infty) : E)$ or $\check{f}|_{[0, \infty)}(\cdot) \in C_0([0, \infty) : E)$, then $f \equiv 0$.
- (xi) If $a \in I$ and the function $f(\cdot)$ is \odot_g -almost periodic (uniformly recurrent), then the function $f(\cdot + a) - f(\cdot)$ is \odot_g -almost periodic (uniformly recurrent).

For the sake of completeness, we will include short proofs of the following two propositions (the first proposition improves the corresponding result for almost periodic functions; for almost automorphic functions, see [29, Lemma 3.9.9]):

Proposition 2.2. (*Supremum formula*) *Suppose that $f : I \rightarrow E$ is uniformly recurrent. Then we have*

$$\sup_{t \in I} \|f(t)\| = \sup_{t \geq a} \|f(t)\| \in [0, \infty], \quad a \in I.$$

Proof. Let $a \in I$, $t \in I$ and $\epsilon > 0$ be fixed. It suffices to show that

$$\|f(t)\| \leq \epsilon + \sup_{s \geq a} \|f(s)\|.$$

In order to do that, take any strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n \rightarrow +\infty} \alpha_n = +\infty$ and (1.1) holds. Let $n \in \mathbb{N}$ be such that $t + \alpha_n \geq a$. Then $\|f(t + \alpha_n) - f(t)\| \leq \epsilon$ and therefore

$$\|f(t)\| \leq \epsilon + \|f(t + \alpha_n)\| \leq \epsilon + \sup_{s \geq a} \|f(s)\|,$$

as claimed. \square

Proposition 2.3. *Any almost periodic function $f : I \rightarrow E$ is \odot_g -almost periodic.*

Proof. Let us recall that any almost periodic function is uniformly continuous. Using this fact, it can be easily seen that for each $\epsilon > 0$ there exist two finite constants $\delta > 0$ and $l > 0$ such that any segment $[y, y + g(x)]$ for $x \geq L(1 + l)$ and $y \geq 0$ contains the segment $[y, y + x/L]$ (cf. (1.11)) and therefore at least $\lfloor x/Ll \rfloor \geq 1$ disjoint intervals of length δ whose elements are ϵ -periods for $f(\cdot)$; see also [5, Corollary, p. 2]. This clearly implies $\odot_g(\vartheta(f, \epsilon)) > \delta/Ll > 0$. \square

Now we will prove the following

Proposition 2.4. *Let $f : I \rightarrow E$ be continuous and $g(x) \equiv x$. Then $f(\cdot)$ is almost periodic iff $f(\cdot)$ is \odot_g -almost periodic.*

Proof. Having in mind Proposition 2.3 and Theorem 1.15, it suffices to show that any \underline{Bd}_c -almost periodic function $f : I \rightarrow E$ is almost periodic. Towards this end, it suffices to show that any set $A \subseteq [0, \infty)$ satisfying $\underline{Bd}_c(A) > 0$ is relatively dense. Otherwise, for every real number $L > 0$, we have that there exists an interval I_L of length L which does not contain any ϵ -period of $f(\cdot)$. Thus, an unbounded set $\bigcup_{n \in \mathbb{N}} I_{2^n}$ does not contain any ϵ -period of $f(\cdot)$, which immediately implies that $\underline{Bd}_c(A) = 0$ by definition. \square

Concerning the notions of $\underline{Bd}_{l;g_c}$ -almost periodicity and $\underline{Bd}_{u;g_c}$ -almost periodicity, the things are pretty clear. In the following proposition, whose discrete analogue has been considered in [31, Proposition 2.4], we will prove that these notions are equivalent with the almost periodicity:

Proposition 2.5. *Let $f : I \rightarrow E$ be continuous and let $g : [0, \infty) \rightarrow [1, \infty)$ be an increasing mapping satisfying that there exists a finite number $L \geq 1$ such that (1.11) holds. Then $f(\cdot)$ is almost periodic iff $f(\cdot)$ is $\underline{Bd}_{l;gc}$ -almost periodic iff $f(\cdot)$ is $\underline{Bd}_{u;gc}$ -almost periodic.*

Proof. Due to Proposition 2.3 and the fact that any $\underline{Bd}_{l;gc}$ -almost periodic function is $\underline{Bd}_{u;gc}$ -almost periodic, it suffices to show that any $\underline{Bd}_{u;gc}$ -almost periodic function is almost periodic. Suppose the contrary and fix a number $x > 0$. Then there exists a number $\epsilon > 0$ such that, for every $n \in \mathbb{N}$, there exists an interval $I_n = [y_n, y_n + 2n + 2g(x)] \subseteq [0, \infty)$ of length $2n + 2g(x)$ such that the set $\vartheta(f, \epsilon)$ does not meet I_n . Then, for every $n \in \mathbb{N}$, the interval $I'_n = [y_n + n + g(x), y_n + 2n + 2g(x)]$ does not meet $\vartheta(f, \epsilon)$ and has the length $n + g(x) \geq g(x)$. This implies $m([\vartheta(f, \epsilon)](y_n + n + g(x), y_n + 2n + 2g(x))) = 0$. Hence, $\liminf_{y \rightarrow +\infty} m([\vartheta(f, \epsilon)](y, y + x)) = 0$, which contradicts condition $\underline{Bd}_{u;gc}(\vartheta(f, \epsilon)) > 0$. \square

Remark 2.6. Let $f : I \rightarrow E$ be continuous and let $c \in I \setminus \{0\}$. Define the function $f_c : I \rightarrow E$ by $f_c(t) := f(ct)$, $t \in I$. Then we have $|c|\vartheta(f, \epsilon) \subseteq \vartheta(f_c, \epsilon)$ for all $\epsilon > 0$, which simply implies that for any uniformly recurrent function $f(\cdot)$ we have that the function $f_c(\cdot)$ is uniformly recurrent. Due to Proposition 2.5 and the corresponding statement for almost periodic functions, the same holds for \odot_g -almost periodicity with $\odot_g \in \{\underline{Bd}_{l;gc}, \underline{Bd}_{u;gc}\}$. If \odot_g is one of the densities \underline{d}_{gc} , \bar{d}_{gc} , $\overline{Bd}_{l;gc}$ or $\overline{Bd}_{u;gc}$, then directly from their definitions and the definition of \odot_g -almost periodicity we may conclude, keeping in mind the fact that for any Lebesgue measurable subset $A \subseteq [0, \infty)$ the set cA is also Lebesgue measurable with $m(cA) = cm(A)$, that the \odot_g -almost periodicity of function $f(\cdot)$ implies the \odot_g -almost periodicity of function $f_c(\cdot)$ for any $c \in I \setminus \{0\}$ with $|c| \leq 1$. Assume now that \odot_g is one of the above four densities and $|c| > 1$. In this case, it is almost inevitable to impose some additional conditions on the function $g(\cdot)$ under which the \odot_g -almost periodicity of function $f(\cdot)$ implies the \odot_g -almost periodicity of function $f_c(\cdot)$. For example, it is very natural to assume additionally that $g(\cdot)$ is continuous, strictly increasing as well as that there exist two numbers $t_0 > 0$ and $\delta > 0$ such that $|c|g(t) \leq g(t/\delta)$ for all $t \geq t_0$. For the Banach density $\overline{Bd}_{u;gc}$, the claimed statement then follows from the computation ($x > 0$ satisfies that $t = g^{-1}(g(x)/c) \geq t_0$):

$$\begin{aligned} \limsup_{y \rightarrow +\infty} \frac{m(cA(y, y + g(x)))}{x} &= \limsup_{y \rightarrow +\infty} \frac{cm(A(y/c, y/c + (g(x)/c)))}{x} \\ &= \limsup_{y \rightarrow +\infty} \frac{m(A(y, y + (g(x)/c)))}{x} = \limsup_{y \rightarrow +\infty} \frac{m(A(y, y + g(t)))}{g^{-1}(cg(t))} \\ &= \limsup_{y \rightarrow +\infty} \frac{m(A(y, y + g(t)))}{t} \frac{t}{g^{-1}(cg(t))} \geq \delta \limsup_{y \rightarrow +\infty} \frac{m(A(y, y + g(t)))}{t}. \end{aligned}$$

For the Banach density $\overline{Bd}_{l;gc}$ and for the densities \underline{d}_{gc} , \bar{d}_{gc} , the claimed statement follows similarly.

Remark 2.7. (see also [25, Lemma 2.1]) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a (uniformly) continuous, \odot_g -almost periodic (uniformly recurrent) function, $\epsilon > 0$, $c \in \mathbb{R}$ and $\tau \in \vartheta(f, \epsilon)$, then $\tau \in \vartheta(\min(c, f), \epsilon)$ and the function $\min(c, f(\cdot))$ is (uniformly) continuous and \odot_g -almost periodic (uniformly recurrent).

Remark 2.8. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an almost periodic function such that there exist two real numbers a and b such that $a < 0 < b$ and an analytic function $F :$

$\{z \in \mathbb{C} : a < \Re z < b\} \rightarrow \mathbb{C}$ such that $F(ix) = f(x)$ for all $x \in \mathbb{R}$. Then the function $h : \mathbb{R} \rightarrow \mathbb{R}$, defined by $h(x) := \text{sign}(f(x))$, $x \in \mathbb{R}$ is Stepanov p -almost periodic for any finite exponent $p \geq 1$. For $p = 1$, this has been proved in [33, Theorem 5.3.1, p. 210], while the general case follows from the consideration given in [29, Example 2.2.3(i)] (we feel duty bound to say that we have made small mistakes in the formulations of conditions in [29, Example 2.2.2, Example 2.2.3(ii)] by neglecting the necessary condition on the analytical extensibility of function $f((-i)\cdot)$ to the strip $\{z \in \mathbb{C} : a < \Re z < b\}$). The Bochner criterion is essentially employed in the proof of the above-mentioned theorem and we would like to observe here that the above condition on the analytical extensibility of function $f((-i)\cdot)$ can be neglected in some situations, even for the uniform recurrence and \odot_g -almost periodicity. More precisely, let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a uniformly recurrent function (an \odot_g -almost periodic function) satisfying that

$$(\exists L \geq 1) (\forall \epsilon > 0) (\forall y \in \mathbb{R}) m(\{x \in [y, y + 1] : |f(x)| \leq \epsilon\}) \leq L\epsilon.$$

Then the function $h(\cdot)$, defined above, is uniformly recurrent (\odot_g -almost periodic), which follows from the foregoing arguments.

Now we will introduce the following definition:

- Definition 2.9.** (i) Suppose that $f \in C(I : E)$. Then we say that the function $f(\cdot)$ is asymptotically uniformly recurrent iff there exist a uniformly recurrent function $h : I \rightarrow E$ and a function $\phi \in C_0(I : E)$ such that $f(t) = h(t) + \phi(t)$ for all $t \in I$.
- (ii) Suppose that $f \in C(I : E)$. Then we say that the function $f(\cdot)$ is asymptotically \odot_g -almost periodic iff there exist an \odot_g -almost periodic function $h : I \rightarrow E$ and a function $\phi \in C_0(I : E)$ such that $f(t) = h(t) + \phi(t)$ for all $t \in I$.

From this definition and previously proved results in this section, it is clear that we have the following:

- Proposition 2.10.** (i) *Any asymptotically almost periodic function is asymptotically \odot_g -almost periodic, and any asymptotically \odot_g -almost periodic function is asymptotically uniformly recurrent.*
- (ii) *Let $f : I \rightarrow E$ be continuous and $g(x) \equiv x$. Then $f(\cdot)$ is asymptotically almost periodic iff $f(\cdot)$ is asymptotically \odot_g -almost periodic.*
- (iii) *Let $f : I \rightarrow E$ be continuous and let $g : [0, \infty) \rightarrow [1, \infty)$ be an increasing mapping satisfying that there exists a finite number $L \geq 1$ such that (1.11) holds. Then $f(\cdot)$ is asymptotically almost periodic iff $f(\cdot)$ is asymptotically $\underline{Bd}_{l;gc}$ -almost periodic iff $f(\cdot)$ is asymptotically $\underline{Bd}_{u;gc}$ -almost periodic.*

Now we have an open door to introduce the concepts of (asymptotical) Stepanov p -uniform recurrence and (asymptotical) Stepanov (p, \odot_g) -almost periodicity:

- Definition 2.11.** (i) Let $1 \leq p < \infty$. A function $f \in L^p_{loc}(I : E)$ is said to be Stepanov p -uniformly recurrent iff the function $\hat{f} : I \rightarrow L^p([0, 1] : E)$, defined by (1.2), is uniformly recurrent.
- (ii) Let $1 \leq p < \infty$. A function $f \in L^p_{loc}(I : E)$ is said to be Stepanov (p, \odot_g) -almost periodic iff the function $\hat{f} : I \rightarrow L^p([0, 1] : E)$, defined by (1.2), is \odot_g -almost periodic.

- Definition 2.12.** (i) Let $1 \leq p < \infty$. A function $f \in L_{loc}^p(I : E)$ is said to be asymptotically Stepanov p -uniformly recurrent iff there exist a Stepanov p -uniformly recurrent function $h(\cdot)$ and a function $q \in L_S^p(I : E)$ such that $f(t) = h(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^p([0, 1] : E))$.
- (ii) Let $1 \leq p < \infty$. A function $f \in L_{loc}^p(I : E)$ is said to be asymptotically Stepanov (p, \odot_g) -almost periodic iff there exist a Stepanov (p, \odot_g) -almost periodic function $h(\cdot)$ and a function $q \in L_S^p(I : E)$ such that $f(t) = h(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^p([0, 1] : E))$.

We can simply state the analogues of Proposition 2.3-2.5 and Proposition 2.10 for the Stepanov classes. Taking into account Proposition 2.5 and Proposition 2.10(iii), in the remainder of paper we will always assume, if not explicitly stated otherwise, that \odot_g denotes exactly one of the densities \underline{d}_{gc} , \overline{d}_{gc} , $\overline{Bd}_{l;gc}$ or $\overline{Bd}_{u;gc}$. Before proceeding any further, we would like to note that we can similarly introduce and analyze the concepts of \odot_g -almost anti-periodicity and Stepanov (p, \odot_g) -almost anti-periodicity ([29]).

The following result, which is closely related with [32, Theorem 2.5, Theorem 2.10], plays a significant role in the proof of Theorem 1.4:

- Theorem 2.13.** (i) *Suppose that the function $f : I \rightarrow E$ is asymptotically uniform recurrent and quasi-asymptotically almost periodic. Then the function $f(\cdot)$ is asymptotically almost periodic.*
- (ii) *Suppose that $1 \leq p < \infty$, the function $f \in L_p^S(I : E)$ is asymptotically Stepanov p -uniform recurrent and Stepanov p -quasi-asymptotically almost periodic. Then the function $f(\cdot)$ is asymptotically Stepanov p -almost periodic.*

Proof. The proof of theorem essentially follows from the argumentation contained in the proof of [29, Theorem 2.5]; for the sake of completeness, we will include all details of proof. Suppose that the function $f : I \rightarrow E$ satisfies the assumptions in (i). Then there exist a uniformly recurrent function $h(\cdot)$ and a function $q \in C_0(I : E)$ such that $f(t) = h(t) + q(t)$, $t \in I$ and for each $\epsilon > 0$ there exists a finite number $L(\epsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\epsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\epsilon, \tau) > 0$ such that

$$(2.1) \quad \|[h(t + \tau) - h(t)] + [q(t + \tau) - q(t)]\| \leq \epsilon, \text{ provided } t \in I \text{ and } |t| \geq M(\epsilon, \tau).$$

Since $f(\cdot)$ is bounded and $q \in C_0(I : E)$, we have that $h(\cdot)$ is bounded. The above implies the existence of a finite number $M_1(\epsilon, \tau) \geq M(\epsilon, \tau)$ such that

$$(2.2) \quad \|h(t + \tau) - h(t)\| \leq 2\epsilon, \text{ provided } t \in I \text{ and } |t| \geq M_1(\epsilon, \tau).$$

Define the function $H : I \rightarrow E$ by $H(t) := h(t + \tau) - h(t)$, $t \in I$. Then the function $H(\cdot)$ is bounded and, due to the property (xi), we have that the function $H(\cdot)$ is uniformly recurrent. Applying supremum formula clarified in Proposition 2.2 and (2.2), we get

$$\sup_{t \in I} \|H(t)\| = \sup_{t \geq M_1(\epsilon, \tau)} \|H(t)\| = \sup_{t \geq M_1(\epsilon, \tau)} \|h(t + \tau) - h(t)\| \leq 2\epsilon.$$

Hence, $\|h(t + \tau) - h(t)\| \leq 2\epsilon$ for all $t \in I$ and $h(\cdot)$ is almost periodic by definition, which completes the proof of part (i). For part (ii), observe first that there exist an Stepanov p -uniformly recurrent function $h(\cdot)$ and a function $q \in L_S^p(I : E)$ such that $f(t) = h(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^p([0, 1] : E))$. Repeating verbatim the

arguments given in the proof of part (i), with the function $f(\cdot)$ replaced therein with the function $\hat{f}(\cdot)$, we get that the function $\hat{h} : I \rightarrow L^p([0, 1] : E)$ is asymptotically almost periodic. This simply completes the proof of (ii). \square

Example 2.14. Denote by c_0 the Banach space of all numerical sequences tending to zero, equipped with the sup-norm. Define

$$f(t) := \left(\frac{4n^2 t^2}{(t^2 + n^2)^2} \right)_{n \in \mathbb{N}}, \quad t \geq 0.$$

Then $f \in Q - AAA([0, \infty) : c_0) \cap BUC([0, \infty) : c_0)$ and $f(\cdot)$ is not asymptotically almost automorphic (see [32, Example 2.6, Theorem 2.5]). Due to Theorem 2.13(ii) and Lemma 1.11(i), we have that the function $f(\cdot)$ is not asymptotically Stepanov (1-)uniform recurrent.

The results presented in the subsequent proposition are expected to a certain extent:

Proposition 2.15. *Let $p \in [1, \infty)$.*

- (i) *If $f : \mathbb{R} \rightarrow E$ is uniformly recurrent and asymptotically almost automorphic, then $f(\cdot)$ is almost automorphic.*
- (ii) *If $f : I \rightarrow E$ is uniformly recurrent and asymptotically almost periodic, then $f(\cdot)$ is almost periodic.*
- (iii) *If $f : \mathbb{R} \rightarrow E$ is Stepanov p -uniformly recurrent and asymptotically Stepanov p -almost automorphic, then $f(\cdot)$ is Stepanov p -almost automorphic.*
- (iv) *If $f : I \rightarrow E$ is Stepanov p -uniformly recurrent and asymptotically Stepanov p -almost periodic, then $f(\cdot)$ is Stepanov p -almost periodic.*

Proof. We will prove only (i) and (ii). Suppose that $f : \mathbb{R} \rightarrow E$ is uniformly recurrent and asymptotically almost automorphic. Then there exist a function $h \in AA(\mathbb{R} : E)$, a function $q \in C_0(\mathbb{R} : E)$ and a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that (1.1) holds and $f(t) = h(t) + q(t)$ for all $t \in \mathbb{R}$. Fix a number $t \in \mathbb{R}$. Then $\lim_{n \rightarrow +\infty} q(t + \alpha_n) = 0$ and, in combination with (1.1), we get

$$(2.3) \quad \lim_{n \rightarrow +\infty} h(t + \alpha_n) = f(t) \quad \text{and} \quad \lim_{n \rightarrow +\infty} f(t - \alpha_n) = f(t).$$

Since $h(\cdot)$ is almost automorphic, we can extract a subsequence (β_n) of (α_n) such that there exists a mapping $f_1 : \mathbb{R} \rightarrow E$ satisfying

$$(2.4) \quad \lim_{n \rightarrow +\infty} h(t + \beta_n) = f_1(t) \quad \text{and} \quad \lim_{n \rightarrow +\infty} f_1(t - \beta_n) = h(t) \quad \text{for all } t \in \mathbb{R}.$$

The uniqueness of the first limits in (2.3) and (2.4) yields $f_1(t) = f(t)$. Using the uniqueness of the second limits in (2.3) and (2.4), we get $f(t) = h(t)$, which completes the proof of (i). Since any almost periodic function $f : \mathbb{R} \rightarrow E$ is almost automorphic, part (ii) follows from the proof of (i). If $I = [0, \infty)$, then we can use the above result with $I = \mathbb{R}$ since any almost periodic function $f : [0, \infty) \rightarrow E$ has a unique almost periodic extension to the whole real line and any function $q \in C_0([0, \infty) : E)$ can be extended to an even function $q_1 \in C_0(\mathbb{R} : E)$. \square

In the following theorem, we reconsider the statements given in Lemma 1.11 for the (asymptotical) Stepanov p -uniform recurrence and (asymptotical) Stepanov (p, \odot_g) -almost periodicity:

Theorem 2.16. *Let $p \in [1, \infty)$.*

- (i) *If the function $h : I \rightarrow E$ is uniformly recurrent, $\phi \in C_0(I : E)$ and $f(t) = h(t) + \phi(t)$ for all $t \in I$, then*

$$(2.5) \quad \{h(t) : t \in I\} \subseteq \overline{\{f(t) : t \in I\}}.$$

- (ii) *If $h : I \rightarrow E$ is uniformly continuous and Stepanov p -uniformly recurrent (Stepanov (p, \odot_g) -almost periodic), then the function $h(\cdot)$ is uniformly recurrent (\odot_g -almost periodic).*
- (iii) *If $f : I \rightarrow E$ is uniformly continuous and asymptotically Stepanov p -uniformly recurrent (asymptotically Stepanov (p, \odot_g) -almost periodic), then the function $f(\cdot)$ is asymptotically uniformly recurrent (asymptotically \odot_g -almost periodic).*

Proof. Part (i) can be simply deduced as follows. Let the numbers $t \in \mathbb{R}$ and $\epsilon > 0$ be fixed. It is clear that there exists a strictly increasing sequence (α_n) of positive real numbers such that $\|h(t) - h(t + \alpha_n)\| < \epsilon/2$, $n \in \mathbb{N}$. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\|h(t) - f(t + \alpha_n)\| \leq \|h(t) - h(t + \alpha_n)\| + \|q(t + \alpha_n)\| \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

This, in turn, implies (2.5). If the function $h : I \rightarrow E$ satisfies the requirements of (ii), then for each $\sigma \in (0, 1)$ the function $h_\sigma : I \rightarrow E$, given by

$$(2.6) \quad h_\sigma(t) := \frac{1}{\sigma} \int_t^{t+\sigma} h(s) ds, \quad t \in I,$$

is continuous and, due to the uniform continuity of $h(\cdot)$, we have the existence of a number $\delta \in (0, 1)$ such that $\|h(t') - h(t'')\| < \epsilon$, provided $t', t'' \in I$ and $|t' - t''| < \delta$. Therefore, if $\sigma \in (0, \delta)$, then we have

$$(2.7) \quad \|h_\sigma(t) - h(t)\| \leq \frac{1}{\sigma} \int_t^{t+\sigma} \|h(s) - h(t)\| ds < \epsilon, \quad t \in \mathbb{R},$$

and $\lim_{\sigma \rightarrow 0+} h_\sigma(t) = h(t)$ uniformly in $t \in I$. By property (iv) from the beginning of section, it suffices to show that for each fixed number $\sigma \in (0, 1)$ the function $h_\sigma(\cdot)$ is uniformly recurrent (\odot_g -almost periodic). But, this follows from the argumentation given on [5, p. 80], where it has been proved that for each number $\epsilon > 0$ we have $\vartheta(\hat{h}, \sigma\epsilon) \subseteq \vartheta(h_\sigma, \epsilon)$. This completes the proof of (ii). To deduce (iii), observe that there exist a Stepanov p -uniformly recurrent (Stepanov (p, \odot_g) -almost periodic) function $h(\cdot)$ and a function $q \in L_S^p(I : E)$ such that $f(t) = h(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^p([0, 1] : E))$. Using (i) and the arguments contained in the proof of [28, Proposition 3.1], we get that the both functions $h(\cdot)$ and $q(\cdot)$ are uniformly continuous. This yields that $q \in C_0(I : E)$ and, due to part (ii), $h(\cdot)$ is uniformly recurrent (\odot_g -almost periodic). The proof of the theorem is thereby completed. \square

In [39, Proposition 12], R. Xie and C. Zhang have proved that any uniformly continuous function $f \in S^p SAP_\omega(I : E)$ belongs to the space $AP_\omega(I : E)$; see [39] for the notion. Due to [32, Proposition 2.11], we have $S^p SAP_\omega(I : E) \subseteq S^p Q - AAP(I : E)$ and it is reasonable to ask whether we can extend the above result by showing that any uniformly continuous function $f \in S^p Q - AAP(I : E)$ belongs to the space $Q - AAP(I : E)$. This is actually the case, as the next proposition shows:

Proposition 2.17. *Let $p \in [1, \infty)$, and let $f \in S^p Q - AAP(I : E)$ be uniformly continuous. Then $f \in Q - AAP(I : E)$.*

Proof. The proof of proposition is very similar to the proof of Theorem 2.16(ii). Clearly, it suffices to consider the case $p = 1$. Define, for every number $\sigma \in (0, 1)$, the function $f_\sigma(\cdot)$ by replacing the function $h(\cdot)$ in (2.6) with the function $f(\cdot)$. Then the function $f_\sigma(\cdot)$ is bounded and continuous ($\sigma \in (0, 1)$). Furthermore, (2.7) holds with the functions $h_\sigma(\cdot)$ and $h(\cdot)$ replaced therein with the functions $f_\sigma(\cdot)$ and $f(\cdot)$. Due to [32, Theorem 2.13(ii)], it suffices to show that the function $f_\sigma(\cdot)$ is quasi-asymptotically almost periodic for each number $\sigma \in (0, 1)$. But, this simply follows from the estimate

$$\|f_\sigma(t + \tau) - f_\sigma(t)\| \leq \frac{1}{\sigma} \int_t^{t+1} \|f(s + \tau) - f(s)\| ds, \quad t \in I, \tau \in I, \sigma \in (0, 1),$$

which can be proved as on [5, p. 80]. \square

Remark 2.18. The proof of Proposition 2.17 considerably shortens the proof of [39, Proposition 12]. Therefore, the word ‘‘Stepanov’’ in the formulations of Theorem 1.4 and Theorem 1.5 can be encompassed with the round brackets.

The following proposition will be important in the sequel:

Proposition 2.19. *Suppose that the function $f : I \rightarrow E$ is uniformly continuous and (asymptotically) uniformly recurrent. Then there exist a finite number $L \geq 1$ and an increasing mapping $g : [0, \infty) \rightarrow [1, \infty)$ such that (1.11) holds and $f(\cdot)$ is (asymptotically) \cdot_g -almost periodic for $\cdot_g \in \{\underline{d}_{gc}, \bar{d}_{gc}\}$.*

Proof. Without loss of generality, we may assume that the equation (1.1) holds with the sequence (α_n) satisfying $\alpha_{n+1} - \alpha_n \geq 1$. It suffices to prove the proposition for uniformly recurrent functions. Let $\epsilon > 0$ be fixed. Due to the uniform continuity of $f(\cdot)$, we have that there exist an integer $n_0 \in \mathbb{N}$ and a finite real number $\delta > 0$ such that the set $\vartheta(f, \epsilon)$ contains the union of disjunct intervals $[\alpha_n - \delta, \alpha_n + \delta]$ for $n \geq n_0$. Let $g : [0, \infty) \rightarrow [1, \infty)$ be any increasing mapping such that $g(n) > \alpha_{n+1}$ for all $n \in \mathbb{N}$. Hence, (1.11) holds with some finite number $L \geq 1$. Furthermore, if $x \in [n, n+1]$, then the interval $[0, g(x)]$ contains at least $(n - n_0)$ disjunct intervals of length δ whose union belongs to $\vartheta(f, \epsilon)$. This simply implies that $m([\vartheta(f, \epsilon)](0, g(x))) \geq \delta(n - n_0)$ and therefore $m([\vartheta(f, \epsilon)](0, g(x)))/x \geq \delta(n - n_0)/(n + 1)$. This simply implies $\underline{d}_c(\vartheta(f, \epsilon)) > 0$, so that $f(\cdot)$ is \underline{d}_{gc} -almost periodic and therefore \bar{d}_{gc} -almost periodic. \square

Remark 2.20. The proof of Proposition 2.19 does not work for the upper $l; gc$ -Banach density $\overline{Bd}_{l;gc}(\cdot)$ and the upper $u; gc$ -Banach density $\overline{Bd}_{u;gc}(\cdot)$. In general, these densities differ from the densities

$$\overline{Bd}_{l;gc}(A) := \liminf_{x \rightarrow +\infty} \sup_{y \geq 0} \frac{m(A(y, y + g(x)))}{x}$$

and

$$\overline{Bd}_{u;gc}(A) := \limsup_{x \rightarrow +\infty} \sup_{y \geq 0} \frac{m(A(y, y + g(x)))}{x},$$

respectively. Repeating verbatim the above arguments, it can be simply proved that for any uniformly continuous, uniformly recurrent function $f : I \rightarrow E$ there exist a finite number $L \geq 1$ and an increasing mapping $g : [0, \infty) \rightarrow [1, \infty)$ such that (1.11) holds and $f(\cdot)$ is \cdot_g -almost periodic for $\cdot_g \in \{\overline{Bd}_{l;gc}, \overline{Bd}_{u;gc}\}$.

Remark 2.21. By the proof of Proposition 2.19, it follows that, for every uniformly continuous, uniformly recurrent functions $f_i : I \rightarrow E$ ($1 \leq i \leq n$), we can find a finite number $L \geq 1$ and an increasing mapping $g : [0, \infty) \rightarrow [1, \infty)$ such that (1.11) holds and $f_i(\cdot)$ is \cdot_g -almost periodic for all $1 \leq i \leq n$ and $\cdot_g \in \{\underline{d}_{gc}, \bar{d}_{gc}\}$.

Keeping in mind the corresponding definitions and Proposition 2.19, the next result follows immediately (the previous two remarks can be reformulated in this context, as well):

Proposition 2.22. *Suppose that $1 \leq p < \infty$, $f : I \rightarrow E$ is (asymptotically) Stepanov p -uniformly recurrent and $\hat{f} : I \rightarrow L^p([0, 1] : E)$ is uniformly continuous. Then there exist a finite number $L \geq 1$ and an increasing mapping $g : [0, \infty) \rightarrow [1, \infty)$ such that (1.11) holds and $f(\cdot)$ is (asymptotically) Stepanov (p, \odot_g) -almost periodic for $\cdot_g \in \{\underline{d}_{gc}, \bar{d}_{gc}\}$.*

It is worth noticing that Proposition 2.19 cannot be applied to the compactly almost automorphic functions which are not asymptotically uniform recurrent, in general. Concerning this problematic, we would like to present the following illustrative example:

Example 2.23. Any almost periodic function has to be compactly almost automorphic, while the converse statement is not true, however. The first example of a scalar-valued compactly almost automorphic function which is not almost periodic has been constructed by A. M. Fink (see [17, p. 521]). Set $a_n := \text{sign}(\cos(n\pi\sqrt{2}))$, $n \in \mathbb{Z}$ and define after that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(t) := \alpha a_n + (1 - \alpha)a_{n+1}$ if $t \in [n, n + 1)$ for some integer $n \in \mathbb{Z}$ and $t = \alpha n + (1 - \alpha)(n + 1)$ for some number $\alpha \in (0, 1)$. As verified in [17], this function is compactly almost automorphic (therefore, uniformly continuous) but not almost periodic. We will extend this result by showing that the function $f(\cdot)$ is not asymptotically uniformly recurrent. If we suppose the contraposition, then there exists a strictly increasing sequence (τ_n) of positive real numbers tending to plus infinity such that, for every $\epsilon > 0$, we have the existence of two finite numbers $M > 0$ and $n_0 \in \mathbb{N}$ such that

$$\|f(x + \tau_n) - f(x)\| \leq 2\epsilon, \quad |x| \geq M, \quad n \geq n_0.$$

Let $\epsilon \in (0, 1/2)$ and $n \geq n_0$. Then it is clear that there exists $l \in \mathbb{N}$, as large as we want, such that $a_l > 0$ and $a_{l+1} < 0$. Then $f(l + (1/2)) = 0$ and therefore $|f(l + (1/2) + \tau_n)| \leq 2\epsilon$. This clearly implies the existence of an integer $k \in \mathbb{Z}$ such that the number $l + (1/2) + \tau_n$ lies in a certain small neighborhood of number $k + (1/2)$; more precisely, since the linear function connecting the points $(k, -1)$ and $(k + 1, 1)$ is given by $y = 2x - 2k - 1$, we get from the above that $|2(l + (1/2) + \tau_n) - 2k - 1| \leq 2\epsilon$, which simply implies $|\tau_n - (k - l)| \leq \epsilon$ and therefore $\tau_n \in (0, \epsilon] \cup \bigcup_{k \in \mathbb{N}} [k - \epsilon, k + \epsilon]$. Fix now an integer $k \in \mathbb{N}$. We will show that the inclusion $\tau_n \in [k - \epsilon, k + \epsilon]$ cannot be true. Otherwise, for each real number $t \in \mathbb{R}$ we have $|f(t + \tau_n) - f(t + k)| \leq 2 \cdot \epsilon = 2\epsilon$, which can be easily approved, so that

$$\begin{aligned} |f(t + k) - f(t)| &\leq |f(t + k) - f(t + \tau_n)| + |f(t + \tau_n) - f(t)| \\ &\leq 2\epsilon + \epsilon = 3\epsilon, \quad |t| \geq M. \end{aligned}$$

This contradicts Lemma 1.8. Notice also that the argumentation given above shows that, for every $\epsilon \in (0, 1)$, we have $\vartheta(f, \epsilon) \cap (\epsilon/2, +\infty) = \emptyset$. Furthermore, for every $\epsilon \in (0, 1)$ and $\tau \in (0, \epsilon/2]$, we have $|f(t + \tau) - f(t)| \leq 2\tau \leq \epsilon$ so that, actually,

$$\forall \epsilon \in (0, 1) : \quad \vartheta(f, \epsilon) = (0, \epsilon/2].$$

See also [16, Example 6.1], which will not be reconsidered here.

Before providing the proofs of Theorem 1.2, Theorem 1.4, Theorem 1.5 and Theorem 1.7, we would like to address one more problem to our readers:

Question 2.24. *Define*

$$f(t) := \frac{2 + e^{it} + e^{it\sqrt{2}}}{|2 + e^{it} + e^{it\sqrt{2}}|}, \quad t \in \mathbb{R}.$$

It is well known that $f(\cdot)$ is an almost automorphic function which is not compactly almost automorphic (see e.g., the papers by W. A. Veech [37]-[38]). We would like to ask whether for each number $\epsilon \in (0, 1)$ we have that $\vartheta(f, \epsilon) \neq \emptyset$ ($\vartheta(f, \epsilon)$ is unbounded)?

PROOF OF THEOREM 1.2. We will first prove that for each fixed number $\tau \in \mathbb{R}$ we have that the function $f(\cdot + \tau) - f(\cdot)$ belongs to the space $ANP(\mathbb{R} : \mathbb{C})$. Towards this end, note that

$$\begin{aligned} f(t + \tau) - f(t) &= \sum_{n=1}^{\infty} \frac{1}{n} \left[\sin^2 \frac{t + \tau}{2^n} - \sin^2 \frac{t}{2^n} \right] \\ &= \sum_{n=1}^{\infty} \frac{1}{2n} \left[\cos \frac{t}{2^{n-1}} - \cos \frac{t + \tau}{2^{n-1}} \right] \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2t + \tau}{2^n} \sin \frac{\tau}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \left[\sin \frac{t}{2^{n-1}} \cos \frac{\tau}{2^n} + \cos \frac{t}{2^{n-1}} \sin \frac{\tau}{2^n} \right] \sin \frac{\tau}{2^n}, \quad t \in \mathbb{R}. \end{aligned}$$

Since the functions $t \mapsto \sin \frac{t}{2^{n-1}}$, $t \in \mathbb{R}$ and $t \mapsto \cos \frac{t}{2^{n-1}}$, $t \in \mathbb{R}$ are anti-periodic of anti-period $T = 2^{n-1}\pi$, it follows that the function

$$f_k(t) := \sum_{n=1}^k \frac{1}{n} \left[\sin \frac{t}{2^{n-1}} \cos \frac{\tau}{2^n} + \cos \frac{t}{2^{n-1}} \sin \frac{\tau}{2^n} \right] \sin \frac{\tau}{2^n}, \quad t \in \mathbb{R}$$

belongs to the space $ANP_0(\mathbb{R} : \mathbb{C})$. Moreover, $\lim_{k \rightarrow +\infty} f_k(t) = f(t + \tau) - f(t)$ uniformly on \mathbb{R} since

$$\left| \sum_{n=k+1}^{\infty} \frac{1}{n} \left[\sin \frac{t}{2^{n-1}} \cos \frac{\tau}{2^n} + \cos \frac{t}{2^{n-1}} \sin \frac{\tau}{2^n} \right] \sin \frac{\tau}{2^n} \right| \leq |\tau| \sum_{n=k+1}^{\infty} \frac{1}{n2^{n-1}}, \quad t \in \mathbb{R}.$$

Especially, due to the fact that $ANP(\mathbb{R} : \mathbb{C}) = AP_{\mathbb{R} \setminus \{0\}}(\mathbb{R} : \mathbb{C})$, we have $0 \notin \sigma(f(\cdot + \tau) - f(\cdot))$, i.e.,

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t |f(s + \tau) - f(s)| ds = 0.$$

This readily implies

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t |f(s + \tau) - f(s)|^p ds = 0, \quad p \geq 1,$$

because

$$|f(s + \tau) - f(s)|^p \leq |f(s + \tau) - f(s)| \cdot \left(\sup_{x \geq 0} |f(x + \tau) - f(x)| \right)^{p-1}, \quad s \geq 0.$$

Taking into account [29, Proposition 2.13.4], we easily get that for each numbers $t, \tau \in \mathbb{R}$ we have

$$\begin{aligned} & \lim_{l \rightarrow +\infty} \frac{1}{2l} \int_{-l}^l |f(t + \tau + x) - f(t + x)|^p dx \\ &= \limsup_{l \rightarrow +\infty} \frac{1}{2l} \int_{-l}^l |f(t + \tau + x) - f(t + x)|^p dx = 0, \end{aligned}$$

so that the function $f(\cdot)$ is Weyl p -almost automorphic with the limit function $f^* \equiv f$. This completes the proof of Theorem 1.2. \square

PROOF OF THEOREM 1.4. Suppose that the function $h(\cdot)$ is Stepanov quasi-asymptotically almost periodic. It is clear that the function $h(\cdot)$ is asymptotically Stepanov uniform recurrent, so that Theorem 2.13(ii) implies that the function $h(\cdot)$ is asymptotically Stepanov almost periodic. Since $h(\cdot)$ is uniformly continuous, Lemma 1.11(i) implies that the function $h(\cdot)$ is asymptotically almost periodic. This cannot be true because the restriction of function $h(\cdot)$ to the non-negative real axis is not asymptotically (Stepanov) almost automorphic by Lemma 1.3. \square

PROOF OF THEOREM 1.5. The function $f(\cdot)$, given by (1.5), satisfies that for each $\epsilon > 0$ there exists a positive real number $\delta > 0$ such that the set $\vartheta(f, \epsilon)$ contains the set $\bigcup_{n \geq \lceil 1/\epsilon \rceil} [\tau_n - \delta, \tau_n + \delta]$ as well as $f(x) = f_n(x)$ for all $x \in [-\tau_{n-1}, \tau_{n-1}]$ ($n \in \mathbb{N}$). Furthermore, the function $f(\cdot)$ equals zero on arbitrarily long intervals and for each number $\epsilon \in (0, 1)$ we have that the sets $\{x \in \mathbb{R} : f(x) \notin [1 - \epsilon, 1 + \epsilon]\}$ and $\vartheta(f, \epsilon)$ are disjoint (see [18, Example 8, pp. 31-33] for more details). This essentially implies that the function $f(\cdot)$ cannot be asymptotically Stepanov almost automorphic (we will present a direct proof, without appealing to Lemma 1.11(ii) and Proposition 2.15(iii)). If we suppose the contraposition, then there exist a Stepanov almost automorphic function $h(\cdot)$ and a function $q \in C_0(\mathbb{R} : L^1([0, 1] : \mathbb{C}))$ such that $f(t) = h(t) + q(t)$ for a.e. $t \in \mathbb{R}$. Moreover, we have the existence of disjoint intervals $I_n = [b_n, b_n'] \subseteq [0, \infty)$ whose length is strictly greater than n^2 and which satisfy that $f(x) = 0$ for all $x \in I_n$ ($n \in \mathbb{N}$). Define $b_n := (b_n' + b_n)/2$ ($n \in \mathbb{N}$). Then there exist a subsequence (a_n) of (b_n) and a function $g^* \in L^1_{loc}(\mathbb{R} : \mathbb{C})$ such that

$$\lim_{n \rightarrow +\infty} \int_t^{t+1} |f(x + a_n) - q(x + a_n) - g^*(x)| dx = 0$$

for all $t \in \mathbb{R}$, and

$$\lim_{n \rightarrow +\infty} \int_t^{t+1} |g^*(x - a_n) - [f(x) - q(x)]| dx = 0$$

for all $t \geq 0$. Let $\epsilon \in (0, 1/2)$ be given. Then there exists $n_0 \in \mathbb{N}$ such that $n_0/(n_0-1) > 3\epsilon/2$ and $\int_{\tau_{n_0}}^{1+\tau_{n_0}} |q(x)| dx < \epsilon/8$. Since $1 \geq f(x) \geq f_n(x) \geq n_0/(n_0-1)$ for $x = \tau_{n_0}$, $f_n(x) = 0$ for $x = \tau_{n_0} + 1$ and the function $f_n(\cdot)$ is linear on the interval $[\tau_{n_0}, \tau_{n_0} + 1]$ (see also [6, part I, p. 115]), the second limit equality with $t = \tau_{n_0}$

easily implies the existence of an integer $n_1 \geq n_0$ such that

$$\int_{\tau_{n_0}-a_n}^{1+\tau_{n_0}-a_n} |g^*(x)| dx \geq \frac{n_0}{2(n_0-1)} - \frac{\epsilon}{2} > \frac{\epsilon}{4}, \quad n \geq n_1.$$

Returning to the first limit equation, with $t = \tau_{n_0} - a_{n_1}$, and taking into account that $\lim_{m \rightarrow \infty} \int_t^{t+1} |q(x + a_m)| dx < \epsilon/8$ for all $m \in \mathbb{N}$ sufficiently large, we obtain the existence of an integer $m_1 \geq n_1$ such that

$$\int_{\tau_{n_0}-a_{n_1}+a_m}^{1+\tau_{n_0}-a_{n_1}+a_m} |f(x)| dx = \int_{\tau_{n_0}-a_{n_1}}^{1+\tau_{n_0}-a_{n_1}} |f(x + a_m)| dx > \frac{\epsilon}{4} - \frac{\epsilon}{8} > 0$$

for all $m \geq m_1$. But, this is simply impossible because for large values of m we have that $[\tau_{n_0} - a_{n_1} + a_m, 1 + \tau_{n_0} - a_{n_1} + a_m]$ is contained in a larger interval where the function $f(\cdot)$ equals zero. If we assume that the function $f(\cdot)$ is Stepanov quasi-asymptotically almost periodic, then the first part of proof of Theorem 1.4 yields that the function $f(\cdot)$ is asymptotically Stepanov almost periodic, which cannot be true according to the first part of proof of this theorem. \square

Example 2.25. Without going into full details, let us only note that the function $f(\cdot)$ considered above can be Weyl p -almost automorphic ($p \geq 1$) if the sequence (τ_n) marches rapidly to plus infinity. This follows from the fact that the function $f(\cdot)$ is bounded and belongs to the space $PAP_0(\mathbb{R} : \mathbb{C})$. To explain this in more detail, let a_n denote the number of triangles appearing on the graph of function $f_n(\cdot)$. Then $a_1 = 1$ and $a_n = (2n - 1)a_{n-1}$, $n \in \mathbb{N} \setminus \{1\}$ so that $a_n = (2n - 1)!!$, $n \in \mathbb{N}$. The Lebesgue measure of each such triangle cannot exceed 1 so that $\int_{-\infty}^{+\infty} f_n(x) dx \leq (2n - 1)!!$, $n \in \mathbb{N}$. Suppose, for simplicity, that $\lim_{n \rightarrow +\infty} (2n - 1)!!/\tau_{n-2} = 0$. If $\tau_{n-1} \geq l \geq \tau_{n-2}$ for some sufficiently large integer $n \in \mathbb{N}$, then

$$\frac{1}{l} \int_{-l}^l f(x) dx = \frac{1}{l} \int_{-l}^l f_n(x) dx \leq \frac{1}{\tau_{n-2}} \int_{-\infty}^{\infty} f_n(x) dx \leq \frac{(2n-1)!!}{\tau_{n-2}},$$

so that $\lim_{l \rightarrow +\infty} (1/2l) \int_{-l}^l f(x) dx = 0$, as claimed. Needless to say that, due to Proposition 2.19, there exists a suitable function $g(\cdot)$ such that the function $f(\cdot)$ is \cdot_g -almost periodic for $\cdot_g \in \{\underline{d}_{gc}, \bar{d}_{gc}\}$ (see also [24, pp. 477-478]).

PROOF OF THEOREM 1.7. It is already known that the function $f(\cdot)$ satisfies $\lim_{i \rightarrow +\infty} \|f(\cdot + 2p_i) - f(\cdot)\|_{\infty} = 0$, so that $f(\cdot)$ is uniformly recurrent. Keeping in mind Proposition 2.17 and arguing as in the proof of Theorem 1.4, we get that $f(\cdot)$ is (Stepanov) quasi-asymptotically almost periodic iff $f(\cdot)$ is asymptotically almost periodic. By Proposition 2.15(ii), this would imply that the function $f(\cdot)$ is almost periodic; this is not the case because the function $f(\cdot)$ is not almost automorphic (asymptotically almost automorphic, equivalently, due to Proposition 2.15(i)). If we suppose the contrary, then there exist a subsequence (p_{i_k}) of (p_i) and a function $\omega : \mathbb{R} \rightarrow \mathbb{R}$ such that $\lim_{k \rightarrow +\infty} f(t + p_{i_k}) = \omega(t)$ and $\lim_{k \rightarrow +\infty} \omega(t - p_{i_k}) = f(t)$ for all $t \in \mathbb{R}$. Observe that the function $f_i(\cdot)$ satisfies $f_i(t + p_i) \geq 1 - \epsilon$, provided $|t| \leq \epsilon p_i$ and $i \in \mathbb{N}$. Let $t \in \mathbb{R}$ and $\epsilon > 0$ be given. Then there exists $i_0 \in \mathbb{N}$ such that $|t| \leq \epsilon p_i$ for all integers $i \geq i_0$. Therefore, for any integer $i \geq i_0$, we have

$$1 \geq f(t + p_i) \geq f_i(t + p_i) \geq 1 - \epsilon,$$

so that $1 = \lim_{i \rightarrow +\infty} f(t + p_i) = \lim_{k \rightarrow +\infty} f(t + p_{i_k}) = \omega(t)$. Therefore, $\omega(t) \equiv 1$ and returning to the second limit equality we get $f(t) \equiv 1$, which is a contradiction (see also [9, Figure 3.7.3, p. 208]). \square

3. APPLICATIONS TO ABSTRACT INTEGRO-DIFFERENTIAL EQUATIONS

Concerning applications, we shall mostly be concerned with the invariance of (asymptotical) uniform recurrence and (asymptotical) \odot_g -almost periodicity under the actions of convolution products. As mentioned in the introductory part, this section will be written in a semi-heuristical manner.

Let $f : \mathbb{R} \rightarrow E$. We will first investigate the uniformly recurrent and \odot_g -almost periodic properties of the function

$$(3.1) \quad F(t) := \int_{-\infty}^t R(t-s)f(s) ds, \quad t \in \mathbb{R},$$

where a strongly continuous operator family $(R(t))_{t>0} \subseteq L(E, X)$ satisfies certain assumptions. In our recent research studies regarding this question, it is commonly assumed that the function $f(\cdot)$ is Stepanov p -bounded for some finite number $p \geq 1$. If this is the case and $\tau \in \mathbb{R}$ is an ϵ -period of function $\hat{f} : \mathbb{R} \rightarrow L^p([0, 1] : E)$, then the resulting function $G(\cdot)$ satisfies, under certain conditions on $(R(t))_{t>0}$, an estimate of the type $\|F(t+\tau) - F(t)\|_X \leq L\epsilon$, $t \in \mathbb{R}$, where $L \geq 1$ is a finite constant independent of t , ϵ and τ . Hence, the assumption $\odot_g(\vartheta(\hat{f}, \epsilon)) > 0$ for all $\epsilon > 0$ implies that $\odot_g(\vartheta(F, \epsilon)) > 0$ for all $\epsilon > 0$. This fact, taken together with the argument used for proving the continuity of function $F(\cdot)$ in [29, Proposition 3.5.3], enables us to simply reformulate the statement of [29, Proposition 2.6.11] in our new framework (cf. also [40, Examples 4, 5, 7, 8; pp. 32-34], which can be simply reformulated for the uniform recurrence and \odot_g -almost periodicity):

Proposition 3.1. *Suppose that $1 \leq p < \infty$, $1/p + 1/q = 1$ and $(R(t))_{t>0} \subseteq L(E, X)$ is a strongly continuous operator family satisfying that $M := \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k, k+1]} < \infty$. If $f : \mathbb{R} \rightarrow E$ is Stepanov p -bounded and Stepanov p -uniformly recurrent (Stepanov (p, \odot_g) -almost periodic), then the function $F : \mathbb{R} \rightarrow X$, given by (3.1), is well-defined and uniformly recurrent (\odot_g -almost periodic).*

Basically, the case in which the function $f : \mathbb{R} \rightarrow E$ is not Stepanov p -bounded has not attracted the attention of the authors so far. Keeping in mind our previous results, we would like to state the following proposition with regards to this question (the uniform continuity of function $\hat{f} : \mathbb{R} \rightarrow L^p([0, 1] : E)$ has not been assumed above):

Proposition 3.2. *Suppose that $1 \leq p < \infty$, $1/p + 1/q = 1$, $f : \mathbb{R} \rightarrow E$ is Stepanov p -uniformly recurrent (Stepanov (p, \odot_g) -almost periodic), there exists a continuous function $P : \mathbb{R} \rightarrow [1, \infty)$ such that*

$$(3.2) \quad \left(\int_t^{t+1} \|f(s)\|^p ds \right)^{1/p} \leq P(t), \quad t \in \mathbb{R}$$

and $(R(t))_{t>0} \subseteq L(E, X)$ is a strongly continuous operator family satisfying that for each $t \in \mathbb{R}$ we have

$$\sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k, k+1]} P(t-k) < \infty.$$

If the function $\hat{f} : \mathbb{R} \rightarrow L^p([0, 1] : E)$ is uniformly continuous, then the function $F : \mathbb{R} \rightarrow X$, given by (3.1), is well-defined and uniformly recurrent (\odot_g -almost periodic).

Proof. The proof is very similar to the proof of [29, Proposition 2.6.11], so that we will only outline the most important details for Stepanov (p, \odot_g) -almost periodic functions. The function $F(\cdot)$ is well defined since

$$\begin{aligned} \int_0^\infty \|R(s)\| \|f(t-s)\| ds &= \sum_{k=0}^\infty \int_k^{k+1} \|R(s)\| \|f(t-s)\| ds \\ &\leq \sum_{k=0}^\infty \|R(\cdot)\|_{L^q[k, k+1]} P(t-k), \quad t \in \mathbb{R} \end{aligned}$$

and (3.2) holds true. It is clear that our assumptions imply

$$M := \sum_{k=0}^\infty \|R(\cdot)\|_{L^q[k, k+1]} < \infty,$$

so that $\vartheta(f, \epsilon) \subseteq \vartheta(F, M\epsilon)$. Since we have assumed that the function $\hat{f} : \mathbb{R} \rightarrow L^p([0, 1] : E)$ is uniformly continuous, the arguments contained in the proof of the above-mentioned proposition can be repeated verbatim in order to see that the function $F(\cdot)$ is continuous. This simply completes the proof of proposition. \square

Proposition 3.1 and Proposition 3.2 can be simply incorporated in the study of the existence and uniqueness of uniformly recurrent and \odot_g -almost periodic solutions of the fractional Cauchy inclusion

$$D_{t,+}^\gamma u(t) \in \mathcal{A}u(t) + f(t), \quad t \in \mathbb{R},$$

where $D_{t,+}^\gamma$ denotes the Riemann-Liouville fractional derivative of order $\gamma \in (0, 1]$, $f : \mathbb{R} \rightarrow E$ satisfies certain properties, and \mathcal{A} is a closed multivalued linear operator (see [29] for more details).

Taking into account Proposition 3.1 and Proposition 3.2, we can simply provide extensions of [29, Proposition 2.6.13, Theorem 2.9.5, Theorem 2.9.7, Theorem 2.9.15], concerning the asymptotical Stepanov p -uniform recurrence/asymptotical Stepanov (p, \odot_g) -almost periodicity of the finite convolution product

$$\mathbf{F}(t) := \int_0^t R(t-s)f(s) ds, \quad t \geq 0.$$

These results can be applied in the qualitative analysis of asymptotically uniformly recurrent/asymptotically \odot_g -almost periodic solutions (asymptotically Stepanov p -uniformly recurrent/asymptotically Stepanov (p, \odot_g) -almost periodic solutions) of the following abstract Cauchy inclusion

$$(\text{DFP})_{f,\gamma} : \begin{cases} \mathbf{D}_t^\gamma u(t) \in \mathcal{A}u(t) + f(t), \quad t \geq 0, \\ u(0) = x_0, \end{cases}$$

where \mathbf{D}_t^γ denotes the Caputo fractional derivative of order $\gamma \in (0, 1]$, $x_0 \in E$, $f : [0, \infty) \rightarrow E$ satisfies certain properties, and \mathcal{A} is a closed multivalued linear operator (see [29] for more details).

The sum of two uniformly recurrent (\odot_g -almost periodic) functions need not be uniformly recurrent (\odot_g -almost periodic), unfortunately. But, it is worth noticing that there exist many concrete situations where this difficulty can be overcome. For example, it is very simple to extend the assertions of [29, Theorem 2.14.7] and [12, Theorem 2.3] for the asymptotical Stepanov (p, \odot_g) -almost periodicity. To

explain this in more detail, let us observe that the equation appearing on [12, p. 240, l. 5] can be rewritten as

$$\int_{-\infty}^t \Gamma(t, s)f(s) ds = \lim_{k \rightarrow +\infty} \int_0^k \Gamma(t, t-s)f(t-s) ds, \quad t \in \mathbb{R};$$

arguing as in the proof of above-mentioned theorem from [12] we may conclude that for each integer $k \in \mathbb{N}$ the function $t \mapsto \int_0^k \Gamma(t, t-s)f(t-s) ds$, $t \in \mathbb{R}$ is \odot_g -almost periodic, provided that the function $f(\cdot)$ is Stepanov (p, \odot_g) -almost periodic and Stepanov p -bounded ($p > 1$), while the case $p = 1$ follows from the same arguments and the proof of [29, Theorem 2.14.6], when it is necessary to assume that $f(\cdot)$ is Stepanov $(1, \odot_g)$ -almost periodic and Stepanov 1-bounded. In both cases, $p > 1$ and $p = 1$, we need to employ the property (iv) to achieve the final results.

We close the paper with the observation that the results whose proofs lean heavily on the use of Bochner criterion cannot be really reconsidered for uniformly recurrent and \odot_g -almost periodic functions.

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FACULTY OF TECHNICAL SCIENCES, UNIVERSITY OF NOVI SAD, TRG D. OBRADOVIĆA 6, 21125 NOVI SAD, SERBIA

Email address: marco.s@verat.net