STOKES AND NAVIER-STOKES PROBLEMS WITH NAVIER-TYPE BOUNDARY CONDITION IN L P -SPACES

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STOKES AND NAVIER–STOKES PROBLEMS WITH NAVIER–TYPE BOUNDARY CONDITION IN L^P–SPACES

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1. Introduction

A semigroup theory for the Stokes equation, in a bounded domain Ω, with Navier type boundary conditions was developed in [2, 3], that gives in particular the existence and uniqueness of strong, weak and very weak solutions in $L^p(\Omega)$ type spaces. The Navier-type boundary conditions are given by the following set of slip frictionless boundary conditions involving the tangential component of the vorticity

$$\boldsymbol{u}\cdot\boldsymbol{n}=0,\quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\mathbf{0}\quad \text{on }\Gamma, \tag{1.1}$$

The Navier-type boundary conditions can be used to simulate the flows near rough walls as well as perforated walls and in the simulation of turbulent flows (cf. [5, 9, 13, 14] and the reference therein for more details and explanation). They may be seen as limit for $\alpha\to 0$ of the Navier conditions introduced by H. Navier in 1827, [27]

$$\boldsymbol{u}\cdot\boldsymbol{n}=0,\qquad 2\,\nu\,[\mathbb{D}\boldsymbol{u}\cdot\boldsymbol{n}]_{\boldsymbol{\tau}}+\alpha\,\boldsymbol{u}_{\boldsymbol{\tau}}=\mathbf{0}\qquad \text{on }\Gamma\times(0,T), \tag{1.2}$$

where ν is the viscosity, $\alpha\geq 0$ is the coefficient of friction and $\mathbb{D}\boldsymbol{u}=\frac{1}{2}(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T)$ denotes the deformation tensor associated to the velocity field $\boldsymbol{u}$. These are nothing but a slip boundary condition with friction on the wall, based on a proportionality between the tangential components of the normal dynamic tensor and the velocit:y.

The results in [2, 3] where very natural extensions of the corresponding results with Dirichlet boundary conditions, also obtained by using semigroup theory. Our purpose is now to apply the results of [2, 3] to two different questions. The first is the maximal L^p-L^q regularity of solutions to the non homogeneous Stokes equation:

$$(\mathcal{S})\quad \begin{cases} \frac{\partial\boldsymbol{u}}{\partial t}-\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{f}, & \operatorname{div}\boldsymbol{u}=0 & \text{in }\Omega\times(0,T),\\ \boldsymbol{u}\cdot\boldsymbol{n}=0, & \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\mathbf{0} & \text{on }\Gamma\times(0,T),\\ & \boldsymbol{u}(0)=\boldsymbol{u}_0 & \text{in }\quad\Omega. \end{cases}$$

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where Ω is a bounded domain of $\mathbb{R}^3$ of class $C^{2,1}$ not necessarily simply-connected, Γ is its boundary, $\boldsymbol{n}$ is the exterior unit normal vector on Γ.

The second is to solve the Cauchy problem for the Navier Stokes equation with Navier type boundary conditions

$$(\mathcal{NS}) \quad \begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \boldsymbol{0}, & \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega\times(0,T), \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, & \boldsymbol{\operatorname{curl}}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma\times(0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega, \end{cases}$$

where $(\boldsymbol{u}\cdot\nabla) = \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j}$ and $\boldsymbol{u}_0 \in \boldsymbol{L}^p(\Omega)$ is such that $\operatorname{div}\boldsymbol{u}_0 = 0$ on Ω and $\boldsymbol{u}_0\cdot\boldsymbol{n} = 0$ on Γ. The unknowns $\boldsymbol{u}$ and π denote respectively the velocity field and the pressure of a fluid occupying the domain Ω, while $\boldsymbol{u}(0)$ and $\boldsymbol{f}$ represent respectively the given initial velocity and the external force.

1.1. Maximal Regularity.

For a Cauchy-Problem of the form:

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{A}\,u(t) = f(t)\ 0 \le t \le T \\ u(0) = 0, \end{cases} \tag{1.3}$$

where $-\mathcal{A}$ is the infinitesimal generator of a semi-group $e^{-t\mathcal{A}}$ on a Banach space X and $f \in L^p(0,T;\,X)$, we say that a solution u satisfies the maximal L^p-L^q regularity if

$$u \in W^{1,p}(0,T;\,X) \cap L^p(0,T;\,D(\mathcal{A})). \tag{1.4}$$

It is well known that the analyticity of $e^{-t\mathcal{A}}$ is not enough to ensure that property to be satisfied, although it is enough when X is a Hilbert space (cf. [12], [11]).

Under the Navier-type boundary conditions (1.1), the Stokes problem has a non trivial kernel $\boldsymbol{K}_\tau(\Omega)$ (see (2.7) below). When $1 < p, q < \infty$, the maximal L^p-L^q regularity has been proved by the authors in [3] for solutions to $(\mathcal{S})$ lying in the orthogonal of that kernel. In terms of the abstract example (1.3), the main argument of the proof, based on the use of the results of [16], was to show that the pure imaginary powers of $(I + \mathcal{A})$ are suitably bounded operators, and deduce that so where the imaginary powers of $\mathcal{A}$. That could only be done assuming the operator $\mathcal{A}$ to be invertible, but that is not the case of the Stokes operator on a non simply-connected domain, with boundary conditions (1.1). The maximal regularity result was then proved only for the restriction of the Stokes operator to the kernel's orthogonal, where it was of course invertible. The first purpose of the present work is to extend that result to the solutions of $(\mathcal{S})$ that do not necessarily lie in the orthogonal of $\boldsymbol{K}_\tau(\Omega)$. The idea is to decompose the solution as an element of the kernel and an element of its orthogonal and to apply the result of [3].

We are interested in three different types of solutions for $(\mathcal{S})$. The first, that we call strong solutions, are solutions $\boldsymbol{u}$ that belong to $L^p(0,T,\boldsymbol{L}^q(\Omega))$ type spaces. The second, called weak solutions, are solutions (in a suitable sense) $\boldsymbol{u}(t)$ that may be writen for a.e. $t > 0$, as $\boldsymbol{u}(t) = \boldsymbol{v}(t) + \nabla w(t)$ where $\boldsymbol{v}(t) \in L^p(0,T;\boldsymbol{L}^q(\Omega)))$ and

$w \in L^p(0,T;L^q(\Omega))$. The third and last, called very weak, are solutions $\boldsymbol{u}(t)$ that may be decomposed as before but where now $w \in L^p(0,T;W^{-1,\,q}(\Omega))$ (cf [3] for more details). Of course, these different types of solutions correspond to data $\boldsymbol{u}(0)$ and $\boldsymbol{f}$ with different regularity properties.

There is a wide literature on the maximal regularity for the Stokes problem with different type of boundary conditions and different domains. Among the firsts articles on this problem we may mention [33] by V. A. Solonnikov. The works by Y. Giga and H. Sohr [17, 18] consider that question for the Stokes problem with Dirichlet boundary conditions in bounded and unbounded domains; J. Saal [30] for the Stokes problem with homogeneous Robin boundary conditions in the half space $\mathbb{R}^3_+$; R. Shimada [31] for the Stokes problem with non-homogeneous Robin boundary conditions. The maximal regularity for general parabolic problems is treated in the long report [22] by P. C. Kunstmann and L. Weis. In [24], the authors proved the analyticity on L^p, for p in an interval containing 2 and depending on Ω, (Ω is a Lipschitz domain of a closed Riemannian manifold), of the semigroups associated with the Hodge Laplacian, with the linear Navier-Stokes system with Neumann boundary conditions. In [25], Mitrea et al. considered the Stokes operator in Lipschitz domains in $\mathbb{R}^N$ with a boundary conditions of Neumann-type. They established an optimal global regularity for vector fields in the domains of fractional powers of this Neumann-Stokes operator. They also studied the existence, regularity, and uniqueness of mild solutions of the Navier-Stokes system with Neumann boundary conditions. In [28], J. Neustupa and P. Penel considered the Navier-Stokes problem with inhomogeneous boundary conditions involving $\boldsymbol{u}\cdot\boldsymbol{n}$, $\mathbf{curl}\boldsymbol{u}\cdot\boldsymbol{n}$ and alternatively $\mathbf{curl}^2\boldsymbol{u}\cdot\boldsymbol{n}$ or $\partial\boldsymbol{u}/\partial\boldsymbol{n}$. They studied the existence of a steady weak solution to this problem. S. Monniaux and E. M. Ouhabaz considered in [26] the incompressible Navier-Stokes system in a $C^{1,1}$ bounded domain or a bounded convex domain of Ω of $\mathbb{R}^3$ with a non penetration condition $\boldsymbol{u}\cdot\boldsymbol{n}=0$ at the boundary $\partial\Omega$ together with a time-dependent Robin boundary condition of the type $\mathbf{curl}\boldsymbol{u}\times\boldsymbol{n}=\beta(t)\boldsymbol{u}$ on $\partial\Omega$ and proved the existence of a solution to this problem with enough regularity provided that the initial condition is small enough in an appropriate functional space.

2. Preliminaries

2.1. Stokes operator with Navier type boundary conditions.

In order to obtain strong, weak and very weak solutions to our problem $(\mathcal{S})$, we introduced in [3] three different extensions A_p, B_p, C_p, of the Stokes operators with boundary conditions (1.1), defined in different spaces of distributions with different regularity properties. Throughout this paper, if not stated otherwise, p will be a real number such that $1<p<\infty$. Let $\mathcal{D}(\Omega)$ be the set of infinitely differentiable functions with compact support in Ω and $\mathcal{D}_\sigma(\Omega)=\{\boldsymbol{\varphi}\in\mathcal{D}(\Omega);\ \operatorname{div}\boldsymbol{\varphi}=0 \text{ in } \Omega\}$.

We first consider A_p, the Stokes operator with the boundary conditions (1.1)

on the space $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ given by

$$\boldsymbol{L}^p_{\sigma,\tau}(\Omega) = \Big\{ \boldsymbol{f} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div} \boldsymbol{f} = 0 \text{ in } \Omega,\ \boldsymbol{f} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \Big\}. \tag{2.1}$$

By [3, Corollary 3.7], this is a well defined subspace of

$$\boldsymbol{H}^p(\operatorname{div}, \Omega) \ = \ \big\{ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div} \boldsymbol{v} \in \boldsymbol{L}^p(\Omega) \big\}, \tag{2.2}$$

equipped with the graph norm. As described in [2, Section 3], A_p is a closed linear densely defined operator on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ defined as follows

$$D(A_p) = \Big\{ \boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega);\ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u} \cdot \boldsymbol{n} = 0,\ \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma \Big\} \tag{2.3}$$

$$\forall\, \boldsymbol{u} \in D(A_p), \quad A_p \boldsymbol{u} = -P \Delta \boldsymbol{u} \quad \text{in } \Omega. \tag{2.4}$$

The operator P in (2.4), is the Helmholtz projection defined as follows:

$$P : \boldsymbol{L}^p(\Omega) \longmapsto \boldsymbol{L}^p_{\sigma,\tau}(\Omega); \quad \forall\, \boldsymbol{f} \in \boldsymbol{L}^p(\Omega) : \quad P\, \boldsymbol{f} \ = \ \boldsymbol{f} \ - \ \mathbf{grad}\, \pi, \tag{2.5}$$

where $\pi \in W^{1,p}(\Omega)/\mathbb{R}$ is the unique solution of the following weak Neuman problem (cf. [32]):

$$\operatorname{div} (\mathbf{grad}\, \pi \ - \ \boldsymbol{f}) = 0 \ \text{ in } \Omega, \quad (\mathbf{grad}\, \pi \ - \ \boldsymbol{f}) \cdot \boldsymbol{n} = 0, \ \text{ on } \Gamma. \tag{2.6}$$

It is known that, due to the slipping frictionless boundary condition (1.1), the pressure gradient disappears in the Stokes operator (cf. [2, Proposition 3.1]). As a result the Stokes problem with the boundary condition (1.1) is reduced to the study of a vectorial Laplace like problem under a free-divergence condition and the boundary conditions (1.1).

$$\forall\ \boldsymbol{u} \in D(A_p), \qquad A_p \boldsymbol{u} \ = \ -\Delta \boldsymbol{u} \quad \text{in } \Omega.$$

We also recall that the operator $-A_p$ is sectorial and generates a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, for all $1 < p < \infty$ (cf. [2, Theorem 4.12]). We denote by e^{-tA_p} the analytic semi-group associated to the operator A_p in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.

When Ω is not simply-connected, the Stokes operator with boundary condition (1.1) has a non trivial kernel included in all the L^p spaces for $p \in (1, \infty)$. It may be characterized as follows (see [7])

$$\boldsymbol{K}_\tau(\Omega) \ = \ \big\{ \boldsymbol{v} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega);\ \operatorname{div} \boldsymbol{v} = 0,\ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0} \text{ in } \Omega \big\}. \tag{2.7}$$

The restriction of the Stokes operator A_p to the subspace

$$\boldsymbol{X}_p \ = \ \big\{ \boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega);\ \int_\Omega \boldsymbol{f} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}\, x = 0,\ \forall\ \boldsymbol{v} \in \boldsymbol{K}_\tau(\Omega) \big\}, \tag{2.8}$$

gives a sectorial operator which is invertible, with bounded inverse. Notice that

$$\boldsymbol{L}^p_{\sigma,\tau}(\Omega) = \boldsymbol{K}_\tau(\Omega) \oplus \boldsymbol{X}_p. \tag{2.9}$$

The authors proved in [3] that the operator $(I + A_p)$ and the restriction of the operators A_p to $\boldsymbol{X}_p$ enjoy the property of uniformly bounded imaginary powers on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and $\boldsymbol{X}_p$ respectively. This allows us to characterize the domains of fractional powers of the Stokes operator A_p through complex interpolation argument and to prove the following embedding of Sobolev type: for all $1 < p < \infty$ and for all $\alpha \in \mathbb{R}$ such that $0 < \alpha < 3/2p$ one has

$$D(A_p^\alpha) \hookrightarrow \boldsymbol{L}^q(\Omega), \qquad \frac{1}{q} = \frac{1}{p} - \frac{2\alpha}{3}, \tag{2.10}$$

$$\forall \boldsymbol{u} \in D(A_p^\alpha), \quad \|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega, p)\, \|(I + A_p)^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{2.11}$$

Using (2.11) we obtain an estimate of type $L^p - L^q$ for the Stokes semi-group e^{-tA_p}. More precisely, for all $1 < p \leq q < \infty$ and for all $\boldsymbol{f} \in L^q(0, T; \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$, the following estimates holds:

$$\|e^{-tA_p} \boldsymbol{f}\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega, p, t)\, t^{-3/2(1/p - 1/q)} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{2.12}$$

Furthermore $D(A_p^{1/2}) = \boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$, where

$$\boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega) = \{\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega);\, \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega \text{ and } \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}.$$

with equivalent norms.

We consider now the extension of A_p to the following subspace of $[\boldsymbol{H}^p_0(\operatorname{div}, \Omega)]'$ (the dual space of $\boldsymbol{H}^p_0(\operatorname{div}, \Omega)$):

$$[\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]'_{\sigma,\tau} = \{\boldsymbol{f} \in [\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]';\, \operatorname{div} \boldsymbol{f} = 0 \,\text{in}\, \Omega,\, \boldsymbol{f} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}. \tag{2.13}$$

By [3, Corollary 3.7], that space is well defined, and the extended operator, denoted B_p, is a closed linear densely defined operator such as:

$$D(B_p) \subset [\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]'_{\sigma,\tau} \longmapsto [\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]'_{\sigma,\tau},$$

$$D(B_p) = \left\{\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega); \operatorname{div} \boldsymbol{u} = 0 \text{ in}\, \Omega, \boldsymbol{u} \cdot \boldsymbol{n} = 0, \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\right\} \tag{2.14}$$

and

$$\forall\ \boldsymbol{u} \in D(B_p), \quad B_p \boldsymbol{u} = -\Delta \boldsymbol{u} \quad \text{in } \Omega. \tag{2.15}$$

By [8, Corollary 4.2], the domain $D(B_p)$ is well defined and, by [3, Theorem 4.15] the operator $-B_p$ generates a bounded analytic semi-group on $[\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]'_{\sigma,\tau}$, for all $1 < p < \infty$, whose restriction to

$$\boldsymbol{Y}_p = \left\{\boldsymbol{f} \in [\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]'_{\sigma,\tau};\ \forall\, \boldsymbol{v} \in \boldsymbol{K}_\tau(\Omega),\ \langle \boldsymbol{f}, \boldsymbol{v} \rangle_\Omega = 0\right\}, \tag{2.16}$$

where $\langle .\,, .\rangle_\Omega = \langle .\,, .\rangle_{[\boldsymbol{H}^{p'}_0(\operatorname{div},\Omega)]' \times \boldsymbol{H}^{p'}_0(\operatorname{div},\Omega)}$, is a sectorial operator, invertible with bounded inverse. Notice also that:

$$[\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]'_{\sigma,\tau} = \boldsymbol{K}_\tau(\Omega) \oplus \boldsymbol{Y}_p. \tag{2.17}$$

In order to introduce our third operator we first need the following space:

$$\boldsymbol{T}^p(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{H}_0^p(\mathrm{div}, \Omega);\ \ \mathrm{div}\, \boldsymbol{v} \in W_0^{1,p}(\Omega)\right\} \tag{2.18}$$

and consider the following subspace

$$[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau} = \{\boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))';\ \mathrm{div}\, \boldsymbol{f} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{f} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma\},$$

that is well defined by [3, Corollary 3.12].

The Stokes operator A_p can be extended to the space $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ (cf. [3, Section 3.2.3]). This extension is a densely defined closed linear operator, denoted C_p:

$$D(C_p) \subset [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau} \longmapsto [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}, \quad \text{where}$$

$$D(C_p) = \left\{\boldsymbol{u} \in \boldsymbol{L}^p(\Omega);\, \mathrm{div}\, \boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u} \cdot \boldsymbol{n} = 0,\ \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\right\} \tag{2.19}$$

and for all $\boldsymbol{u} \in D(C_p)$, $C_p\boldsymbol{u} = -\Delta\boldsymbol{u}$ in Ω. The domain $D(C_p)$ is well defined by [8, Lemma 4.14]. The operator $-C_p$ generates a bounded analytic semi-group on $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ for all $1 < p < \infty$ (see [3, Theorem 4.18]). If we define now

$$\boldsymbol{Z}_p = \left\{\boldsymbol{f} \in [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau};\ \forall\, \boldsymbol{v} \in \boldsymbol{K}_\tau(\Omega),\ \langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega = 0\right\}, \tag{2.20}$$

where $\langle .\,, .\rangle_\Omega = \langle .\,, .\rangle_{[\boldsymbol{T}^{p'}(\Omega)]' \times \boldsymbol{T}^{p'}(\Omega)}$, then

$$[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau} = \boldsymbol{K}_\tau(\Omega) \oplus \boldsymbol{Z}_p \tag{2.21}$$

and the restriction of the Stokes operator to the space $\boldsymbol{Z}_p$, gives a sectorial operator, invertible with bounded inverse.

2.2. Y. Giga's abstract theory for semilinear parabolic problem

The existence and uniqueness of a local in time mild solution in $L^q(0, T_*, \boldsymbol{L}^p(\Omega))$ for the Problem

$$\frac{\partial \boldsymbol{u}}{\partial t} + \mathcal{A}\boldsymbol{u} = \boldsymbol{F}\boldsymbol{u} \qquad u(0) = \boldsymbol{a}, \qquad \text{in} \quad \Omega\times]0, T[, \tag{2.22}$$

where Ω is an arbitrary domain of $\mathbb{R}^3$, $\boldsymbol{F}\boldsymbol{u}$ represents the non-linear term and $-\mathcal{A}$ is the infinitesimal generator of a strongly continuous semi-group $e^{-t\mathcal{A}}$ in some closed subspace $\boldsymbol{E}^p$ of $\boldsymbol{L}^p(\Omega)$ equipped with the norm of $\boldsymbol{L}^p(\Omega)$ is obtained in [19] for initial datas $\boldsymbol{u}_0 \in \boldsymbol{E}^p$ under the following assumptions:
(i) There exists a continuous projection P from $L^p(\Omega)$ to $\boldsymbol{E}^p$ for all $1 < p < \infty$ such that the restriction of P on $\mathcal{D}(\Omega)$ is independent of p and $\mathcal{D}(\Omega) \cap \boldsymbol{E}^p$ is dense in $\boldsymbol{E}^p$.

(ii) For a fixed $0 < T < \infty$ the following estimate holds

$$(\mathbf{A}) \quad \|e^{-t\mathcal{A}}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \le M\, t^{-\frac{3}{2}(\frac{1}{s}-\frac{1}{p})} \|\boldsymbol{f}\|_{\boldsymbol{L}^r(\Omega)}, \qquad \boldsymbol{f} \in \boldsymbol{E}^r, \quad 0 < t < T,$$

with $p \geq r > 1$ and the constant $M = M(p, r, T)$ depends only on p, r and T.

(iii) The non-linear $\boldsymbol{F}\boldsymbol{u}$ can be written in the form

$$\boldsymbol{F}\boldsymbol{u} = \boldsymbol{L}\,\boldsymbol{G}\boldsymbol{u}, \tag{2.23}$$

where $\boldsymbol{L}$ is the linear part and $\boldsymbol{G}$ is the non-linear part.

(iv) We suppose also that the following estimate holds

$$\textbf{(N1)} \quad \|e^{-t\mathcal{A}}\boldsymbol{L}\,\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \leq N_1 t^{-1/2}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \qquad \boldsymbol{f} \in \boldsymbol{E}^p, \quad 0 < t < T,$$

where the constant $N_1 = N_1(p, T)$ depends only on p and T.

(v) The operator $\boldsymbol{G}$ satisfies the following estimate

$$\textbf{(N2)} \quad \|\boldsymbol{G}\boldsymbol{v} - \boldsymbol{G}\boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} \leq N_2\|\boldsymbol{v} - \boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)}(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)}), \qquad \boldsymbol{G}\boldsymbol{0} = \boldsymbol{0},$$

for all $\boldsymbol{v}$, $\boldsymbol{w} \in \boldsymbol{E}^p$, for $s = \frac{p}{2} > 1$ and $N_2 = N_2(p)$ depends only on p, $1 < p < \infty$.

The precise result [19, Theorem 1, Theorem 2] is the following. In what follows BC denotes the space of bounded and continuous functions and C denotes positive constant whose value may change from one line to the next.

THEOREM 2.1. (Giga's abstract existence and uniqueness theorem) *Assume conditions (i)–(v) are satisfied. Then there is $T_0 > 0$ and a unique mild solution of (2.22) on $[0, T_0)$ such that*

$$\boldsymbol{u} \in BC([0, T_0);\, \boldsymbol{E}^{\,p}) \cap L^q(0, T_0;\, \boldsymbol{E}^{\,r})$$

with

$$q > p,\, r > p, \qquad \frac{2}{q} + \frac{3}{r} = \frac{3}{p}.$$

Moreover there is a positive constant ε such that if $\|\boldsymbol{u}_0\|_{\boldsymbol{E}^{\,p}} \leq \varepsilon$, then T_0 can be taken as infinity for $p = 3$.

We recall now the following Lemma, where the family of all Hölder continuous functions with exponent ϑ on I is denoted $C^{\vartheta}(I;\, X)$.

LEMMA 2.2. *Let $-\mathcal{A}$ be the infinitesimal generator of a bounded analytic semi-group on a Banach space X and let $e^{-t\mathcal{A}}$ be the semi-group generated by $\mathcal{A}$. Suppose that $f \in C^{\vartheta}([0, T];\, X)$ for some exponent $0 < \vartheta < 1$ and satisfies*

$$\sup_{0<s\leq t} s^{\lambda}\|f(s)\|_X \leq M(t) < \infty, \qquad (0 < t \leq T),$$

for some constant $0 \leq \lambda < 1$ and a real valued function M. If we define v as

$$v(t) = \int_0^t e^{-(t-s)\mathcal{A}} f(s)\, ds, \qquad 0 \leq t \leq T, \quad T > 0,$$

then, $v \in C^{1+\vartheta}((0,T];\, X)$, $\mathcal{A}v \in C^{\vartheta}((0,T];\, X)$ *and*

$$\frac{\partial v}{\partial t} + \mathcal{A}\, v \,=\, f.$$

(cf. [15, Lemma 2.14]). We will also need the following,

LEMMA 2.3. *Let* $-\mathcal{A}$ *be the infinitesimal generator of a bounded analytic semi-group* $e^{-t\mathcal{A}}$ *on a Banach space* X *such that* $0 \in \rho(\mathcal{A})$. *Then for all* $0 < \alpha < 1$

$$\|(I - e^{-h\mathcal{A}})\mathcal{A}^{-\alpha}\|_X \leq \frac{C}{\alpha} h^{\alpha}.$$

Proof. First recall that since $-\mathcal{A}$ is the infinitesimal generator of a bounded analytic semi-group on X then for all $x \in X$, $e^{-t\mathcal{A}}x \in D(\mathcal{A})$ and

$$\frac{\mathrm{d}e^{-t\mathcal{A}}x}{\mathrm{d}t} = -\mathcal{A}e^{-t\mathcal{A}}x \qquad \text{and} \qquad \frac{\mathrm{d}(I - e^{-t\mathcal{A}})x}{\mathrm{d}t} = -\mathcal{A}e^{-t\mathcal{A}}x.$$

As a result

$$\forall x \in X, \qquad (I - e^{-h\mathcal{A}})x = -\int_0^h \mathcal{A}e^{-t\mathcal{A}}x \,\mathrm{d}\, t.$$

Moreover in the particular case where $x \in D(\mathcal{A}^{\alpha})$ one has

$$(I - e^{-h\mathcal{A}})x = -\int_0^h \mathcal{A}e^{-t\mathcal{A}}x \,\mathrm{d}\, t = -\int_0^h \mathcal{A}^{1-\alpha}e^{-t\mathcal{A}}\mathcal{A}^{\alpha}x \,\mathrm{d}\, t.$$

Thus

$$\begin{aligned}
\|(I - e^{-h\mathcal{A}})x\|_X &\leq \int_0^h \|\mathcal{A}^{1-\alpha}e^{-t\mathcal{A}}\|_{\mathcal{L}(X)} \|\mathcal{A}^{\alpha}x\|_X \,\mathrm{d}\, t \\
&\leq \int_0^h t^{\alpha-1} \|\mathcal{A}^{\alpha}x\|_X \,\mathrm{d}\, t \\
&\leq \frac{C}{\alpha} h^{\alpha} \|\mathcal{A}^{\alpha}x\|_X.
\end{aligned}$$

Finally observe that for all $x \in X$, $\mathcal{A}^{-\alpha}x \in D(\mathcal{A}^{\alpha})$ and

$$\|(I - e^{-h\mathcal{A}})\mathcal{A}^{-\alpha}x\|_X \leq \frac{C}{\alpha} h^{\alpha} \|\mathcal{A}^{\alpha}\mathcal{A}^{-\alpha}x\|_X \leq \frac{C}{\alpha} h^{\alpha} \|x\|_X$$

and the result is proved. □

REMARK 2.4. Lemma 2.3 extends Lemma 2.11 in [15] for strictly positive self-adjoint operators $\mathcal{A}$ is on a Hilbert space H such that $-\mathcal{A}$ generates a strongly continuous semi-group on H.

Finally we recall the following proposition:

PROPOSITION 2.5. *Let* $-\mathcal{A}$ *be the infinitesimal generator of a bounded analytic semi-group* $e^{-t\mathcal{A}}$ *on a Banach space* X *such that* $0 \in \rho(\mathcal{A})$. *Then for all* $0 < \alpha \leq 1$, *and for all* $x \in X$ $\mathcal{A}^{\alpha}e^{-t\mathcal{A}}x$ *is Hölder continuous on every interval* $[\varepsilon, T]$ *for all* $\varepsilon > 0$.

3. Maximal Regularity of solutions to the Stokes Problem.

We consider in this Section the problem $(\mathcal{S})$ under different conditions of the external force f. In our first result we assume $\boldsymbol{f} \in L^q(0,T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ and $1 < p, q < \infty$.

THEOREM 3.1. *Let $1 < p, q < \infty$, $0 < T \leq \infty$ and $\boldsymbol{u}_0 = \boldsymbol{0}$. Then for every $\boldsymbol{f} \in L^q(0,T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ there exists a unique solution $\boldsymbol{u}$ of $(\mathcal{S})$ satisfying*

$$\boldsymbol{u} \in L^q(0,T_0;\, \boldsymbol{W}^{2,p}(\Omega)), \quad T_0 \leq T \text{ if } T < \infty \text{ and } T_0 < T \text{ if } T = \infty, \tag{3.1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)) \tag{3.2}$$

and

$$\int_0^T \Big\| \frac{\partial \boldsymbol{u}}{\partial t} \Big\|^q_{\boldsymbol{L}^p(\Omega)} dt + \int_0^T \|\Delta \boldsymbol{u}(t)\|^q_{\boldsymbol{L}^p(\Omega)} dt \leq C(p,q,\Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{\boldsymbol{L}^p(\Omega)} dt. \tag{3.3}$$

Proof. Since the operator $-A_p$ generates a bounded analytic semi-group in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, and $\boldsymbol{f} \in L^q(0,T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$, problem $(\mathcal{S})$ has a unique solution $\boldsymbol{u} \in C([0,T[;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$. To prove the maximal L^p-L^q regularity (3.1)-(3.3) we proceed as follows.

By (2.9) we may write $\boldsymbol{f}$ in the form, $\boldsymbol{f} = \boldsymbol{f}_1 + \boldsymbol{f}_2$ where $\boldsymbol{f}_1 \in L^q(0,T;\, \boldsymbol{X}_p)$ and $\boldsymbol{f}_2 \in L^q(0,T;\, \boldsymbol{K}_\tau(\Omega))$. Thus the solution $\boldsymbol{u}$ to $(\mathcal{S})$ is such that $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$, where $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ satisfy

$$\begin{cases} \frac{\partial \boldsymbol{u}_1}{\partial t} - \Delta \boldsymbol{u}_1 = \boldsymbol{f}_1, \quad \operatorname{div} \boldsymbol{u}_1 = 0 & \text{in } \Omega \times (0,T), \\ \boldsymbol{u}_1 \cdot \boldsymbol{n} = 0, \quad \operatorname{\mathbf{curl}} \boldsymbol{u}_1 \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma \times (0,T), \\ \boldsymbol{u}_1(0) = \boldsymbol{0} & \text{in } \Omega \end{cases} \tag{3.4}$$

and

$$\begin{cases} \frac{\partial \boldsymbol{u}_2}{\partial t} - \Delta \boldsymbol{u}_2 = \boldsymbol{f}_2, \quad \operatorname{div} \boldsymbol{u}_2 = 0 & \text{in } \Omega \times (0,T), \\ \boldsymbol{u}_2 \cdot \boldsymbol{n} = 0, \quad \operatorname{\mathbf{curl}} \boldsymbol{u}_2 \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma \times (0,T), \\ \boldsymbol{u}_2(0) = \boldsymbol{0} & \text{in } \Omega \end{cases} \tag{3.5}$$

respectively.

By [3, Theorem 1.2] we know that $\boldsymbol{u}_1$ satisfies

$$\boldsymbol{u}_1 \in L^q(0,T_0;\, D(A_p)) \cap W^{1,q}(0,T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)) \tag{3.6}$$

$$\int_0^T \Big\| \frac{\partial \boldsymbol{u}_1}{\partial t} \Big\|^q_{\boldsymbol{L}^p(\Omega)} dt + \int_0^T \|\Delta \boldsymbol{u}_1(t)\|^q_{\boldsymbol{L}^p(\Omega)} dt \leq C(p,q,\Omega) \int_0^T \|\boldsymbol{f}_1(t)\|^q_{\boldsymbol{L}^p(\Omega)} dt. \tag{3.7}$$

Set $\boldsymbol{z}_2 = \operatorname{\mathbf{curl}} \boldsymbol{u}_2$. Then $\boldsymbol{z}_2$ is a solution of the problem

$$\begin{cases} \frac{\partial \boldsymbol{z}_2}{\partial t} - \Delta \boldsymbol{z}_2 = \boldsymbol{0}, \ \operatorname{div} \boldsymbol{z}_2 = 0 & \text{in } \Omega \times (0,T), \\ \boldsymbol{z}_2 \times \boldsymbol{n} = \boldsymbol{0}, & \text{on } \Gamma \times (0,T), \\ \boldsymbol{z}_2(0) = \boldsymbol{0} & \text{in } \Omega. \end{cases} \tag{3.8}$$

Thus, using [4, Theorem 4.1] we deduce that $\mathbf{curl}\, \boldsymbol{u}_2 = \boldsymbol{z}_2 = \mathbf{0}$ in Ω. This means that $\boldsymbol{u}_2 \in \boldsymbol{K}_\tau(\Omega)$ and then

$$\forall t \geq 0, \quad \frac{\partial \boldsymbol{u}_2(t)}{\partial t} = \boldsymbol{f}_2(t) \quad \text{in } \Omega. \tag{3.9}$$

As a result $\boldsymbol{u}_2$ satisfies

$$\boldsymbol{u}_2 \in L^q(0,T_0;\, D(A_p)) \cap W^{1,q}(0,T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)) \tag{3.10}$$

and

$$\int_0^T \Big\| \frac{\partial \boldsymbol{u}_2}{\partial t} \Big\|^q_{\boldsymbol{L}^p(\Omega)} dt = \int_0^T \|\boldsymbol{f}_2(t)\|^q_{\boldsymbol{L}^p(\Omega)} dt \leq C(p,q,\Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{\boldsymbol{L}^p(\Omega)} dt. \tag{3.11}$$

Thus putting together (3.6)-(3.7) and (3.10)-(3.11) we deduce our result. □

We now extend the previous result to the more general case where the external force $\boldsymbol{f} \in L^q(0,T;\, \boldsymbol{L}^p(\Omega))$ is not necessarily divergence free. It is used that the pressure can be decoupled, using the weak Neumann Problem (2.6). The following theorem gives the strong solution to the inhomogeneous Stokes Problem $(\mathcal{S})$. By the inhomogeneous problem we mean the case where the external force is nonzero.

THEOREM 3.2. (Strong Solution to the inhomogeneous Stokes Problem) *Let $T \in (0,\infty]$, $1 < p,q < \infty$, $\boldsymbol{f} \in \boldsymbol{L}^q(0,T;\boldsymbol{L}^p(\Omega))$ and $\boldsymbol{u}_0 = \mathbf{0}$. The Problem $(\mathcal{S})$ has a unique solution $(\boldsymbol{u},\pi)$ such that*

$$\boldsymbol{u} \in L^q(0,T_0;\, \boldsymbol{W}^{2,p}(\Omega)), \quad T_0 \leq T \ \text{ if } T < \infty \ \text{ and } \ T_0 < T \ \text{ if } T = \infty, \tag{3.12}$$

$$\pi \in L^q(0,T;\, W^{1,p}(\Omega)/\mathbb{R}), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\, \boldsymbol{L}^p(\Omega)) \tag{3.13}$$

and

$$\int_0^T \Big\| \frac{\partial \boldsymbol{u}}{\partial t} \Big\|^q_{\boldsymbol{L}^p(\Omega)} dt + \int_0^T \|\Delta \boldsymbol{u}(t)\|^q_{\boldsymbol{L}^p(\Omega)} dt + \int_0^T \|\pi(t)\|^q_{W^{1,p}(\Omega)/\mathbb{R}} dt \tag{3.14}$$

$$\leq C(p,q,\Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{\boldsymbol{L}^p(\Omega)} dt. \tag{3.15}$$

Proof. As we saw in Section 2.1 when defining the Helmholtz projection P, for every $\boldsymbol{f} \in \boldsymbol{L}^q(0,T;\, \boldsymbol{L}^p(\Omega))$, and almost every $0 < t < T$, the problem

$$\operatorname{div}(\mathbf{grad}\, \pi(t) - \boldsymbol{f}(t)) = 0 \text{ in } \Omega, \quad (\mathbf{grad}\, \pi(t) - \boldsymbol{f}(t)) \cdot \boldsymbol{n} = 0 \text{ on } \Gamma, \tag{3.16}$$

has a unique solution $\pi(t) \in W^{1,p}(\Omega)/\mathbb{R}$ that satisfies the estimate

$$\text{for a.e. } t \in (0,T) \qquad \|\pi(t)\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(\Omega) \|\boldsymbol{f}(t)\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.17}$$

It follows that $\pi \in L^q(0,T;\, W^{1,p}(\Omega)/\mathbb{R})$ and $(\boldsymbol{f} - \mathbf{grad}\, \pi) \in L^q(0,T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$. As a result, thanks to Theorem 3.1, Problem $(\mathcal{S})$ has a unique solution $(\boldsymbol{u},\pi)$ satisfying (3.12)-(3.14). □

Similar results hold for weak and very weak solutions.

THEOREM 3.3. (Weak Solution to the inhomogeneous Stokes Problem) *Let* $1 < p, q < \infty$, $\boldsymbol{u}_0 = 0$ *and let* $\boldsymbol{f} \in L^q(0,T;\, [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]')$, $0 < T \leq \infty$. *The Problem* $(\mathcal{S})$ *has a unique solution* $(\boldsymbol{u}, \pi)$ *satisfying*

$$\boldsymbol{u} \in L^q(0,T_0;\ \boldsymbol{W}^{1,p}(\Omega)),\ \ T_0 \leq T \ \textit{if}\ T < \infty \ \text{ and } \ T_0 < T \ \textit{if}\ T = \infty, \tag{3.18}$$

$$\pi \in L^q(0,T;\ L^p(\Omega)/\mathbb{R}), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\in [\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'_{\sigma,\tau}) \tag{3.19}$$

and

$$\int_0^T \Big\| \frac{\partial \boldsymbol{u}}{\partial t} \Big\|^q_{[\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'} dt + \int_0^T \|\Delta \boldsymbol{u}(t)\|^q_{[\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'} dt + \int_0^T \|\pi(t)\|^q_{L^p(\Omega)/\mathbb{R}} dt \leq C(p,q,\Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{[\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'} dt. \tag{3.20}$$

Proof. Suppose first $\boldsymbol{f} \in L^q(0,T;\, [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau})$. Using that $-B_p$ generates a bounded analytic semigroup in $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$ we deduce the existence of a unique weak solution $\boldsymbol{u}$ to Problem $(\mathcal{S})$ that belongs to $C([0,T[;[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau})$.

By (2.17) we may write now $\boldsymbol{f}$ as, $\boldsymbol{f} = \boldsymbol{f}_1 + \boldsymbol{f}_2$ where $\boldsymbol{f}_1 \in L^q(0,T;\, \boldsymbol{Y}_p)$ and $\boldsymbol{f}_2 \in L^q(0,T;\, \boldsymbol{K}_\tau(\Omega))$. Proceeding as in the proof of Theorem 3.1 we deduce that the solution $\boldsymbol{u}$ to problem $(\mathcal{S})$ is such that $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$, where $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are weak solutions of (3.4) and (3.5) respectively and that $\boldsymbol{u}_2 \in \boldsymbol{K}_\tau(\Omega)$ for almost all $0 < t \leq T$. Using [3, Proposition 6.4, Remark 7.15] we deduce that the solution $\boldsymbol{u}$ satisfies the maximal regularity (3.18)-(3.20). Suppose now $\boldsymbol{f} \in L^q(0,T;\, [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]')$. Then, for almost every $t \in (0,T)$, there exists a unique solution $\pi(t) \in L^p(\Omega)/\mathbb{R}$ such that:

$$\|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C_2(\Omega,p) \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}, \tag{3.21}$$

(cf. [6]), and then also:

$$\pi \in L^q(0,T;\, L^p(\Omega)/\mathbb{R}) \ \text{ and } \ (\boldsymbol{f} - \mathbf{grad}\, \pi) \in L^q(0,T;\, [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}).$$

We deduce from the previous step that $(\boldsymbol{u}, \pi)$ satisfies (3.18)-(3.20). □

THEOREM 3.4. (Very weak solution to the inhomogeneous Stokes Problem) *Let* $T \in (0,\infty]$, $1 < p, q < \infty$, $\boldsymbol{u}_0 = 0$ *and* $\boldsymbol{f} \in L^q(0,T;\, [\boldsymbol{T}^{p'}(\Omega)]')$. *Then the time dependent Stokes Problem* $(\mathcal{S})$ *has a unique solution* $(\boldsymbol{u}, \pi)$ *satisfying*

$$\boldsymbol{u} \in L^q(0,T_0;\, \boldsymbol{L}^p(\Omega)),\ \ T_0 \leq T \ \textit{if}\ T < \infty \ \text{ and } \ T_0 < T \ \textit{if}\ T = \infty, \tag{3.22}$$

$$\pi \in L^q(0,T;\ W^{-1,p}(\Omega)/\mathbb{R}), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\in [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}) \tag{3.23}$$

and

$$\int_0^T \Big\| \frac{\partial \boldsymbol{u}}{\partial t} \Big\|^q_{[\boldsymbol{T}^{p'}(\Omega)]'} dt + \int_0^T \|\Delta \boldsymbol{u}(t)\|^q_{[\boldsymbol{T}^{p'}(\Omega)]'} dt + \int_0^T \|\pi(t)\|^q_{W^{-1,p}(\Omega)/\mathbb{R}} dt \quad (3.24)$$

$$\leq C(p,q,\Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{[\boldsymbol{T}^{p'}(\Omega)]'} dt. \quad (3.25)$$

Proof. The proof follows the same arguments as those in the proof of Theorem 3.3. In a first step one uses C_p, the analytic semigroup on $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ and (2.21) to prove that (3.22)-(3.24) are satisfied when $f \in [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$. In the general case, one uses the results in [6] to obtain $\pi \in L^q(0,T;\ W^{-1,p}(\Omega)/\mathbb{R})$ such that $(\boldsymbol{f} - \mathbf{grad}\,\pi) \in L^q(0,T;\ [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau})$, and the result follows using the first step. □

4. Existence and uniqueness of solutions for Navier-Stokes equations

In this section we apply Giga's abstract existence and uniqueness result to the Navier-Stokes problem $(\mathcal{NS})$ to get the existence and uniqueness of a local in time mild and classical solutions. To this end, we apply the Helmholtz projection P to the first equation of system $(\mathcal{NS})$, and obtain

$$\frac{\partial \boldsymbol{u}}{\partial t} + A_p \boldsymbol{u} = -P(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \boldsymbol{f}, \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega), \quad (4.1)$$

where the operator A_p is the Stokes operator with Navier-type boundary conditions. Let us verify assumptions $(\mathbf{A})$, $(\mathbf{N1})$ and $(\mathbf{N2})$ described above for the Stokes operator A_p, in our case $\boldsymbol{E}^p = \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. First, observe that the assumption $(\mathbf{A})$ is the $L^p - L^q$ estimate proved in [3]. Thus assumption $(\mathbf{A})$ holds. We next verify the assumptions for the non-linear term

$$\boldsymbol{F}\boldsymbol{u} = -P(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}. \quad (4.2)$$

Since $\operatorname{div}\boldsymbol{u} = 0$ in Ω, we can easily verify that

$$\forall\ 1 \leq i \leq 3, \qquad (\boldsymbol{u}\cdot\nabla\boldsymbol{u})_i = \sum_{j=1}^3 \frac{\partial(u_j u_i)}{\partial x_j}.$$

As in [19] let $(g_{ij})_{1\leq i,\,j\leq 3}$ be a matrix and for all $1 \leq i \leq 3$ we set $\boldsymbol{g}_i = (g_{ij})_{1\leq j\leq 3}$. We define $\boldsymbol{L}$ by

$$\boldsymbol{L}\,\boldsymbol{g}_i = P\operatorname{div}\boldsymbol{g}_i. \quad (4.3)$$

The non-linear term $\boldsymbol{F}\boldsymbol{u}$ is expressed by $\boldsymbol{L}\,\boldsymbol{G}\boldsymbol{u}$, where $(\boldsymbol{G}\boldsymbol{u})(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x}))$ and

$$\boldsymbol{g}(\boldsymbol{u}) :\quad \mathbb{R}^3 \longmapsto \mathbb{R}^9, \qquad (\boldsymbol{g}(\boldsymbol{u}))_{ij} = -u_i u_j, \quad 1 \leq i,j \leq 3.$$

It is easy to see that for all $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^3$ $\boldsymbol{g}$ satisfies

$$|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{z})| \leq N_2\,|\boldsymbol{y}-\boldsymbol{z}|(|\boldsymbol{y}| + |\boldsymbol{z}|), \qquad \boldsymbol{g}(\mathbf{0}) = \mathbf{0},$$

with $|\cdot|$ denotes the norm on $\mathbb{R}^k$, $k \in \{3, 9\}$. Thus $\boldsymbol{G}$ satisfies $(\mathbf{N2})$.
It remains to verify the assumption $(\mathbf{N1})$. To this end we prove the following lemmas and propositions

PROPOSITION 4.1. *Consider the Helmholtz projection $P : \boldsymbol{L}^p(\Omega) \longmapsto \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ defined in (2.5). The adjoint P^* of P is equal to the continuous embedding $I : \boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega) \longmapsto \boldsymbol{L}^{p'}(\Omega)$.*

Proof. First we recall that for all $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $P\boldsymbol{f} = \boldsymbol{f} - \operatorname{grad}\pi$ in Ω, where π is the unique solution of Problem (2.6). We recall also that for all $1 < p < \infty$ $(\boldsymbol{L}^p(\Omega))' \simeq \boldsymbol{L}^{p'}(\Omega)$ and $(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))' \simeq \boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)$. Let $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{v} \in \boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)$ we have

$$\langle P\boldsymbol{u}\,,\,\boldsymbol{v}\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)} = \int_\Omega (\boldsymbol{u} - \mathbf{grad}\pi)\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x$$

with $\pi \in \boldsymbol{W}^{1,p}(\Omega)/\mathbb{R}$ is the unique solution of the problem:

$$\operatorname{div}(\mathbf{grad}\,\pi - \boldsymbol{u}) = 0 \qquad \text{in } \Omega, \qquad (\mathbf{grad}\,\pi - \boldsymbol{u})\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma.$$

As a result,

$$\langle P\boldsymbol{u}\,,\,\boldsymbol{v}\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)} = \int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x + \int_\Omega \pi\operatorname{div}\overline{\boldsymbol{v}}\,\mathrm{d}\,x - \langle\pi\,,\,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma,$$

where $\langle\cdot\,,\,\cdot\rangle_\Gamma = \langle\cdot\,,\,\cdot\rangle_{\boldsymbol{W}^{1/p',p}(\Gamma)\times\boldsymbol{W}^{-1/p',p'}(\Gamma)}$. This means that

$$\langle P\boldsymbol{u}\,,\,\boldsymbol{v}\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)} = \langle\boldsymbol{u}\,,\,\boldsymbol{v}\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)} = \langle\boldsymbol{u}\,,\,P^*\boldsymbol{v}\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)}.$$

□

LEMMA 4.2. *For all $1 \le j \le 3$, the operator $(I + A_p)^{-1/2}P\frac{\partial}{\partial x_j}$ is a linear bounded operator from $\boldsymbol{L}^p(\Omega)$ to $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, for all $1 < p < \infty$.*

Proof. The proof is similar to the proof of [20, Lemma 2.1]. First observe that the operator

$$\frac{\partial}{\partial x_j} I\,(I + A_p)^{-1/2} : \quad \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \longmapsto D((I + A_p)^{1/2}) \longmapsto \boldsymbol{W}^{1,p}(\Omega) \longmapsto \boldsymbol{L}^p(\Omega)$$

is continuous for each p, $1 < p < \infty$, where I denotes the continuous embedding of $D((I+A_p)^{1/2}) = \boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$ in $\boldsymbol{W}^{1,p}(\Omega)$. As a result the adjoint operator $\Big[\frac{\partial}{\partial x_j} I\,(I + A_p)^{-1/2}\Big]^*$ is continuous from $\boldsymbol{L}^{p'}(\Omega)$ to $\boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)$. Let us prove that

$$\Big[\frac{\partial}{\partial x_j} I\,(I + A_p)^{-1/2}\Big]^* = (I + A_{p'})^{-1/2}P\frac{\partial}{\partial x_j}. \tag{4.4}$$

We know that the adjoint operator of $(I+A_p)^{1/2}$ is equal to $(I+A_{p'})^{1/2}$ thus the adjoint operator of $(I+A_p)^{-1/2}$ is equal to $(I+A_{p'})^{-1/2}$. Now let $\boldsymbol{u} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and let $\boldsymbol{v} \in \mathcal{D}(\Omega)$, one has

$$\begin{aligned}
\Big\langle \frac{\partial}{\partial x_j} I\,(I+A_p)^{-1/2}\boldsymbol{u}\,,\,\boldsymbol{v}\Big\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)} &= -\Big\langle I\,(I+A_p)^{-1/2}\boldsymbol{u}\,,\,\frac{\partial \boldsymbol{v}}{\partial x_j}\Big\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)} \\
&= -\Big\langle (I+A_p)^{-1/2}\boldsymbol{u}\,,\,P\,\frac{\partial \boldsymbol{v}}{\partial x_j}\Big\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)} \qquad (4.5)\\
&= -\Big\langle \boldsymbol{u}\,,\,(I+A_{p'})^{-1/2}P\,\frac{\partial \boldsymbol{v}}{\partial x_j}\Big\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)}.
\end{aligned}$$

The equality (4.5) comes from the fact that the adjoint of the Helmoltz projection P is equal to I (see Proposition 4.1). As a result for all $\boldsymbol{v} \in \mathcal{D}(\Omega)$ one has

$$\Big[\frac{\partial}{\partial x_j} I\,(I+A_p)^{-1/2}\Big]^*\boldsymbol{v} \;=\; (I+A_{p'})^{-1/2}P\frac{\partial \boldsymbol{v}}{\partial x_j}.$$

Since $\Big[\frac{\partial}{\partial x_j} I\,(I+A_p)^{-1/2}\Big]^*$ is continuous from $\boldsymbol{L}^{p'}(\Omega)$ to $\boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)$, then for all $\boldsymbol{v} \in \mathcal{D}(\Omega)$ one has

$$\Big\|(I+A_{p'})^{-1/2}P\frac{\partial \boldsymbol{v}}{\partial x_j}\Big\|_{\boldsymbol{L}^{p'}(\Omega)} = \Big\|\Big[\frac{\partial}{\partial x_j} I\,(I+A_p)^{-1/2}\Big]^*\boldsymbol{v}\Big\|_{\boldsymbol{L}^{p'}(\Omega)} \le C\|\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Thus the operator $(I+A_{p'})^{-1/2}P\frac{\partial}{\partial x_j} : \mathcal{D}(\Omega) \subset \boldsymbol{L}^{p'}(\Omega) \longmapsto \boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)$ is continuous for the norm of $\boldsymbol{L}^{p'}(\Omega)$. As a result using the density of $\mathcal{D}(\Omega)$, the operator $(I+A_{p'})^{-1/2}P\frac{\partial}{\partial x_j}$ can be extended to a linear continuous operator from $\boldsymbol{L}^{p'}(\Omega)$ to $\boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)$. Moreover (4.4) holds. □

As a consequence of Lemma 4.2 we have the following corollary

COROLLARY 4.3. *Let $\boldsymbol{L}$ be the operator defined in (4.3). The following estimate holds*

$$\forall\ \boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \qquad \|e^{-tA_p}\boldsymbol{L}\,\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \le \frac{C(p,T)}{t^{1/2}}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \qquad (4.6)$$

where e^{-tA_p} is the semi-group generated by the Stokes operator with Navier-type boundary conditions on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and $C(p,T)$ is a constant depending on p and T.

Proof. First we recall that the Stokes operator with Navier-type boundary conditions generates a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ for all $< p < \infty$. We also know that for a fixed $\lambda > 0$ one has

$$\forall\, t > 0, \qquad e^{-tA_p} \;=\; e^{\lambda t}e^{-t(\lambda I + A_p)},$$

where $e^{-t(\lambda I+A_p)}$ is the analytic semi-group generated by the operator $-(\lambda\, I + A_p)$ on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. Now, let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ one has

$$\begin{aligned}\|e^{-tA_p}\boldsymbol{L}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} &= e^t\|e^{-t(I+A_p)}\boldsymbol{L}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\\ &= e^t\,\|(I+A_p)^{1/2}\,e^{-t(I+A_p)}\,(I+A_p)^{-1/2}\,\boldsymbol{L}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\\ &\le \frac{Ce^T}{t^{1/2}}\,\|(I+A_p)^{-1/2}\,\boldsymbol{L}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \le \frac{Ce^T}{t^{1/2}}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.\end{aligned}$$

The last inequality comes from the fact that the operator $(I+A_p)^{-1/2}\boldsymbol{L}$ is a bounded operator from $\boldsymbol{L}^p(\Omega)$ into $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ which is a consequence of Lemma 4.2. □

REMARK 4.4. Corollary 4.3 means the Stokes operator with Navier-type boundary conditions satisfies the assumption (**N1**)

We thus have checked all assumptions that guarantee the existence and uniqueness of local in time mild solution for the Navier-Stokes Problem (4.1). As a result applying Theorem 2.1 to the Stokes operator A_p with $\boldsymbol{E}^p = \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ we have the following theorem :

THEOREM 4.5. (Existence and uniqueness) *Let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $p \ge 3$. There is a $T_0 > 0$ and a unique mild solution of (4.1) on $[0, T_0)$ such that*

$$\boldsymbol{u} \in BC([0,T_0);\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)) \cap L^q(0,T_0;\, \boldsymbol{L}^r_{\sigma,\tau}(\Omega))$$

with

$$q > p,\ r > p, \qquad \frac{2}{q} + \frac{3}{r} = \frac{3}{p}.$$

Moreover there is a positive constant ε such that if $\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p_{\sigma,\tau}(\Omega)} \le \varepsilon$ then T_0 can be taken as infinity for $p = 3$.

Next we want to prove that the mild solution obtained above is a classical solution. For this reason we will proceed as in [20]. We start by the following lemma.

LEMMA 4.6. *Let $0 \le \delta \le \frac{1}{2} + \frac{3}{2}(1 - \frac{1}{p})$ and $1 < p < \infty$. Then*

$$\forall\ \boldsymbol{u},\ \boldsymbol{v} \in \mathcal{D}_\sigma(\Omega), \qquad \|(I+A_p)^{-\delta}P(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \ \le\ M\,\|(I+A_p)^{\theta}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\|(I+A_p)^{\rho}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}, \tag{4.7}$$

where the constant $M = M(\delta,\theta,\rho,p)$ provided that

$$\delta + \theta + \rho \ge 1/2 + 3/2p, \qquad \theta > 0, \quad \rho > 0, \qquad \rho + \delta > 1/2.$$

REMARK 4.7. By density of $\mathcal{D}_\sigma(\Omega)$ in $D((I+A_p)^\alpha)$ for all $0 \le \alpha \le 1$ one has estimate (4.7) for all $\boldsymbol{u} \in D((I+A_p)^\theta)$ and for all $\boldsymbol{v} \in D((I+A_p)^\rho)$.

Proof of Lemma 4.6. Assume that $0 \le \varepsilon \le \frac{3}{2}(1-\frac{1}{p})$. We know that the operator

$$(\lambda I + A_{p'})^{-\varepsilon} \,:\, \boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega) \longmapsto D((I+A_{p'})^\varepsilon) \hookrightarrow \boldsymbol{L}^{s'}_{\sigma,\tau}(\Omega)$$

is a bounded linear operator with

$$\frac{1}{s'} = \frac{1}{p'} - \frac{2\varepsilon}{3}.$$

By duality this implies that the operator

$$(I + A_p)^{-\varepsilon} \,:\, \boldsymbol{L}^s_{\sigma,\tau}(\Omega) \longrightarrow \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$$

extends uniquely to a bounded linear operator from $\boldsymbol{L}^s_{\sigma,\tau}(\Omega)$ to $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ with

$$\frac{1}{s} = \frac{1}{p} + \frac{2\varepsilon}{3}. \tag{4.8}$$

(i) First consider the case $\delta \geq 1/2$ and take $\varepsilon = \delta - \frac{1}{2}$ and observe that with such ε, the operator $(I + A_p)^{-\varepsilon}$ is a bounded linear operator from $\boldsymbol{L}^s_{\sigma,\tau}(\Omega)$ to $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, where s is given by (4.8). Using Lemma 4.2 one has

$$\|(I+A_p)^{-\delta} P(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} = \Big\| \sum_{j=1}^{3} (I+A_p)^{-\varepsilon-1/2} P \frac{\partial(u_j \boldsymbol{v})}{\partial x_j} \Big\|_{\boldsymbol{L}^p(\Omega)} \leq C \,\| \,|\boldsymbol{u}|\cdot|\boldsymbol{v}|\, \|_{\boldsymbol{L}^s(\Omega)}. \tag{4.9}$$

We recall that since $\operatorname{div} \boldsymbol{u} = 0$ in Ω we have

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{v} = \sum_{j=1}^{3} \frac{\partial(u_j \boldsymbol{v})}{\partial x_j}.$$

By assumption we can take r_1 and r_2 such that

$$\frac{1}{r_1} \geq \frac{1}{p} - \frac{2\theta}{3}, \qquad \frac{1}{r_2} \geq \frac{1}{p} - \frac{2\rho}{3}, \qquad \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{s}, \qquad 1 < r_1,\, r_2 < \infty. \tag{4.10}$$

As a result Holder inequality and (4.10) yield

$$\| \,|\boldsymbol{u}| \cdot |\boldsymbol{v}|\, \|_{\boldsymbol{L}^s(\Omega)} \leq \|\boldsymbol{u}\|_{\boldsymbol{L}^{r_1}(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{L}^{r_2}(\Omega)} \leq C \,\|(I + A_p)^{\theta}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \|(I + A_p)^{\rho}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.11}$$

Finally putting together (4.9) and (4.11) we obtain the required result.

(ii) The case $0 \leq \delta \leq 1/2$ is obtain in the same way as in the proof of [20, Lemma 2.2]. □

In the particular case where $p > 3$ we have the following proposition

PROPOSITION 4.8. *Let $p > 3$ then for all $\boldsymbol{u}$, $\boldsymbol{v} \in D(A_p^{1/2})$ one has*

$$\|P(\boldsymbol{u} \cdot \nabla)\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \leq C \,\|(I + A_p)^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \|(I + A_p)^{1/2}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.12}$$

Proof. First, since for $p > 3$, $\boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{\infty}(\Omega)$ one has

$$\begin{aligned}\|P(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} &\le C\,\|\boldsymbol{u}\|_{\boldsymbol{L}^{\infty}(\Omega)}\|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\\ &\le C\,\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}\\ &\le C\,\|(I+A_p)^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\|(I+A_p)^{1/2}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\end{aligned}$$

The last inequality comes from the fact that $D(A_p^{1/2}) = \boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$ and for all $\boldsymbol{u} \in D(A_p^{1/2})$ the norm $\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}$ is equivalent to the norm $\|(I+A_p)^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$. □

Consider now the non-linear term $\boldsymbol{F}\boldsymbol{u}$ defined by (4.2), we have the following proposition

PROPOSITION 4.9. *Let δ, θ, ρ be as in Lemma 4.6 and let $1 < p < \infty$. For all $\boldsymbol{u}$, $\boldsymbol{v} \in \boldsymbol{D}_{\sigma}(\Omega)$ one has*

$$\begin{aligned}\|(I+A_p)^{-\delta}(\boldsymbol{F}\boldsymbol{u}-\boldsymbol{F}\boldsymbol{v})\|_{\boldsymbol{L}^p(\Omega)} \le C\,\|(I+A_p)^{\theta}(\boldsymbol{u}-\boldsymbol{v})\|_{\boldsymbol{L}^p(\Omega)}\|(I+A_p)^{\rho}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} +\\ \|(I+A_p)^{\theta}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\|(I+A_p)^{\rho}(\boldsymbol{u}-\boldsymbol{v})\|_{\boldsymbol{L}^p(\Omega)}. \quad (4.13)\end{aligned}$$

Moreover for $p > 3$ we have

$$\begin{aligned}\|\boldsymbol{F}\boldsymbol{u}-\boldsymbol{F}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \le C\,\|(I+A_p)^{1/2}(\boldsymbol{u}-\boldsymbol{v})\|_{\boldsymbol{L}^p(\Omega)}\|(I+A_p)^{1/2}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} +\\ \|(I+A_p)^{1/2}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\|(I+A_p)^{1/2}(\boldsymbol{u}-\boldsymbol{v})\|_{\boldsymbol{L}^p(\Omega)}. \quad (4.14)\end{aligned}$$

Proof. Just observe that

$$\boldsymbol{F}\boldsymbol{u}-\boldsymbol{F}\boldsymbol{v} \;=\; P(\boldsymbol{u}-\boldsymbol{v})\cdot\nabla\boldsymbol{u} \;+\; P\boldsymbol{v}\cdot\nabla(\boldsymbol{u}-\boldsymbol{v}).$$

As a result, estimates (4.13) and (4.14) follow directly from Lemma 4.6 and Proposition 4.8. □

The following theorem shows that the solution $\boldsymbol{u}(t)$ of Theorem 4.5 is in $D(A_p^{\alpha})$ for all $t \in (0, T_*]$ and for all $0 < \alpha < 1-\delta$, where δ satisfies the assumptions of Lemma 4.6.

THEOREM 4.10. *Let δ be as in Lemma 4.6 be fixed and let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $p \ge 3$. There exists a maximal interval of time $T_* \in (0, T)$ such that the unique solution $\boldsymbol{u}(t)$ of Problem (4.1) is in $C((0,T_*];\, D(A_p^{\alpha}))$ for all $0 < \alpha < 1-\delta$. Moreover the solution $\boldsymbol{u}(t)$ satisfies*

$$\|(I+A_p)^{\alpha}\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \le K_{\alpha}t^{-\alpha}, \quad (4.15)$$

for some constant $K_{\alpha} > 0$.

Proof. First, we note that, thanks to Theorem 4.5, there exists a $T_0 > 0$ such that the unique solution $\boldsymbol{u}(t)$ of Problem (4.1) is in $BC([0,T_0);\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$. Moreover for all $0 \le t \le T_0$, $\boldsymbol{u}(t)$ is given by

$$\boldsymbol{u}(t) = \boldsymbol{u}_0(t) + \boldsymbol{S}\boldsymbol{u}(t) \quad (4.16)$$

with

$$\boldsymbol{u}_0(t) = e^{-tA_p}\boldsymbol{u}_0 \qquad \text{and} \qquad \boldsymbol{S}\boldsymbol{u}(t) = \int_0^t e^{-(t-s)A_p}\boldsymbol{F}\boldsymbol{u}(s)\,\mathrm{d}\,s, \tag{4.17}$$

where $\boldsymbol{F}\boldsymbol{u} = -P(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$. In addition, thanks to [19, Theorem 1], we know that by construction there exists a sequence $(\boldsymbol{u}_m(t))_{m\geq 0}$ such that $(\boldsymbol{u}_m)_m$ converges to $\boldsymbol{u}$ in $BC([0,T_0];\boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ and $\boldsymbol{u}_m(t)$ is defined recursively by

$$\boldsymbol{u}_0(t) = e^{-tA_p}\boldsymbol{u}_0 \qquad \text{and} \quad \forall\, m\geq 1, \qquad \boldsymbol{u}_{m+1}(t) = \boldsymbol{u}_0(t) + \boldsymbol{S}\boldsymbol{u}_m(t). \tag{4.18}$$

Now, let δ be as in Lemma 4.6 and $0<\alpha<1-\delta$. Since e^{-tA_p} is a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, then $\boldsymbol{u}_0(t)\in D(A_p)\hookrightarrow D(A_p^\alpha)$ and

$$\|(I+A_p)^\alpha\boldsymbol{u}_0(t)\|_{\boldsymbol{L}^p(\Omega)} = \|(I+A_p)^\alpha e^{-tA_p}\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} = e^t\|(I+A_p)^\alpha e^{-t(I+A_p)}\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \leq K_{\alpha 0}t^{-\alpha}, \tag{4.19}$$

with

$$K_{\alpha 0} = \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}\sup_{0<t\leq T_0} e^t t^\alpha \|(I+A_p)^\alpha e^{-t(I+A_p)}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))}. \tag{4.20}$$

The factor e^t in (4.20) is irrelevant since our existence is local in time.

Suppose that for some $m\geq 1$, $\boldsymbol{u}_m(t)\in D(A_p^\alpha)$ for all $0<t\leq T_0$ and satisfies

$$\|(I+A_p)^\alpha\boldsymbol{u}_m(t)\|_{\boldsymbol{L}^p(\Omega)} \leq K_{\alpha m}t^{-\alpha}, \qquad \forall\ \alpha,\ 0<\alpha<1-\delta \tag{4.21}$$

for some constant $K_{\alpha m}>0$ and let us prove that $\boldsymbol{u}_{m+1}(t)\in D(A_p^\alpha)$ and satisfies

$$\|(I+A_p)^\alpha\boldsymbol{u}_{m+1}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq K_{\alpha m+1}t^{-\alpha}, \qquad 0<\alpha<1-\delta,$$

for some constant $K_{\alpha m+1}>0$. We shall estimate $\|(I+A_p)^\alpha\boldsymbol{u}_{m+1}(t)\|_{\boldsymbol{L}^p(\Omega)}$ by using the explicit formula (4.18). Observe that

$$(I+A_p)^\alpha\boldsymbol{u}_{m+1}(t) = (I+A_p)^\alpha\boldsymbol{u}_0(t) + (I+A_p)^\alpha\boldsymbol{S}\boldsymbol{u}_m(t), \tag{4.22}$$

where $\boldsymbol{S}\boldsymbol{u}(t)$ is given by (4.17).

$$\begin{aligned}
\|(I+A_p)^\alpha\boldsymbol{S}\boldsymbol{u}_m(t)\|_{\boldsymbol{L}^p(\Omega)} &\leq \int_0^t \|(I+A_p)^\alpha e^{-(t-s)A_p}\boldsymbol{F}\boldsymbol{u}_m(s)\|_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\,s\\
&\leq e^T\int_0^t \|(I+A_p)^\alpha e^{-(t-s)(I+A_p)}\boldsymbol{F}\boldsymbol{u}_m(s)\|_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\,s\\
&\leq e^T\int_0^t \|(I+A_p)^{\alpha+\delta}e^{-(t-s)(I+A_p)}(I+A_p)^{-\delta}\boldsymbol{F}\boldsymbol{u}_m(s)\|_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\,s\\
&\leq C_{\alpha+\delta}\,e^T\int_0^t (t-s)^{-\alpha-\delta}\|(I+A_p)^{-\delta}\boldsymbol{F}\boldsymbol{u}_m(s)\|_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\,s.
\end{aligned} \tag{4.23}$$

As in the proof of [20, Theorem 2.3] to estimate the term $\|(I+A_p)^{-\delta}\boldsymbol{F}\boldsymbol{u}_m(s)\|_{\boldsymbol{L}^p(\Omega)}$ we choose $\theta>0$ and $\rho>0$ such that

$$\theta+\rho+\delta = 1, \qquad 0<\theta<1-\delta, \qquad 1/2<\delta+\rho<1.$$

We can easily verify that θ, ρ and δ satisfy the assumptions of Lemma 4.6. Thus using Lemma 4.6 and (4.21) one has

$$\begin{aligned}\|(I+A_p)^{-\delta}\boldsymbol{F}\boldsymbol{u}_m(s)\|_{\boldsymbol{L}^p(\Omega)} &\leq M\|(I+A_p)^{\theta}\boldsymbol{u}_m(s)\|_{\boldsymbol{L}^p(\Omega)}\|(I+A_p)^{\rho}\boldsymbol{u}_m(s)\|_{\boldsymbol{L}^p(\Omega)}\\ &\leq M\,K_{\theta m}K_{\rho m}\,s^{\delta-1}. \end{aligned} \tag{4.24}$$

Now putting together (4.23) and (4.24) one has

$$\|(I+A_p)^{\alpha}\boldsymbol{S}\boldsymbol{u}_m(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C_{\alpha+\delta}\,M\,K_{\theta m}K_{\rho m}\,e^T\int_0^t (t-s)^{-\alpha-\delta}\,s^{\delta-1}\,\mathrm{d}\,s. \tag{4.25}$$

Putting together (4.22), (4.19) and (4.25) and using Lemma 2.12 [15] one has

$$\|(I+A_p)^{\alpha}\boldsymbol{u}_{m+1}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq K_{\alpha m+1}t^{-\alpha} \tag{4.26}$$

with $K_{\alpha m+1}$ defined recursively by

$$K_{\alpha m+1} = K_{\alpha 0} \,+\, M\,e^T\,C_{\alpha+\delta}B(1-\delta-\alpha,\delta)K_{\theta m}K_{\rho m} \tag{4.27}$$

and $B(\cdot\,,\,\cdot)$ denotes the beta function. Thus $\boldsymbol{u}_m(t)$ is well defined for each $m \geq 0$ as an element of $C((0,T_0];\,D(A_p^{\alpha}))$ for all $0<\alpha<1-\delta$, moreover $\boldsymbol{u}_m(t)$ satisfies (4.26) with $K_{\alpha m}$ defined recursively by (4.20) and (4.27).

As in the proof of [20, Theorem 2.3] we can show that if

$$K_0 \,<\, \frac{1}{4C_1MB_1}, \tag{4.28}$$

with $C_1 = \max(C_{\theta+\delta}\,,\,C_{\rho+\delta})$ and $B_1 = \max(B(1-\delta-\theta\,,\,\delta)\,,\,B(1-\delta-\rho\,,\,\delta))$, then

$$\|(I+A_p)^{\alpha}\boldsymbol{u}_{m+1}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq K_{\alpha}t^{-\alpha}, \tag{4.29}$$

with a constant K_{α} independent of m. As a result, for all $0<t<T_0$ the sequence $(\boldsymbol{u}_m(t))_{m\geq 0}$ is bounded in $D(A_p^{\alpha})$ and thus it converges weakly in $D(A_p^{\alpha})$ to a function denoted by $\boldsymbol{v}(t)$ and $(I+A_p)^{\alpha}\boldsymbol{u}_m(t)$ converges weakly to $(I+A_p)^{\alpha}\boldsymbol{v}(t)$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. In the other hand $\boldsymbol{u}_m(t)$ converges to $\boldsymbol{u}(t)$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ thus $\boldsymbol{u}(t)=\boldsymbol{v}(t)$ and $\boldsymbol{u}(t)\in D(A_p^{\alpha})$ for all $0<t<T_0$. As stated in the proof of [20, Theorem 2.3], if $T>0$ is chosen sufficiently small then $K_{\alpha 0}$, $0<\alpha<1-\delta$ becomes small and K_0 satisfies (4.28). This shows the existence of $T_*>0$ such that $\boldsymbol{u}\in C((0,T_*];\,D(A_p^{\alpha}))$.

Finally observe that since $(I+A_p)^{\alpha}\boldsymbol{u}_m(t)$ converges weakly to $(I+A_p)^{\alpha}\boldsymbol{u}(t)$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ then

$$\|(I+A_p)^{\alpha}\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \,\leq\, \liminf_m \|(I+A_p)^{\alpha}\boldsymbol{u}_m(t)\|_{\boldsymbol{L}^p(\Omega)} \,\leq\, K_{\alpha}t^{-\alpha}.$$

Thus one has estimate (4.15). □

The next step is to prove that the solution $\boldsymbol{u}$ of Problem (4.1) is in $C((0,T_*];\,D(A_p))$. Since the Stokes operator generates a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ then $\boldsymbol{u}_0(t)$ defined in (4.17) is in $D(A_p)$ for all $t>0$. It remains to prove that $\boldsymbol{S}\boldsymbol{u}(t)$

defined in (4.17) is in $D(A_p)$ for all $0 < t \le T_*$. The proof is done in three steps. First we prove that $(I + A_p)^\alpha \boldsymbol{u}$, $0 < \alpha < 1 - \delta$ is Hölder continuous on every interval $[\varepsilon, T_*]$. This gives us that the non-linear term $\boldsymbol{Fu}$ is also Hölder continuous on every interval $[\varepsilon, T_*]$ and thus $\boldsymbol{u} \in D(A_p)$ for all $0 < t \le T_*$.

PROPOSITION 4.11. *Let $0 \le \delta < 1$ be as in Lemma 4.6, $0 < \alpha < 1 - \delta$, let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $p \ge 3$ and let $\boldsymbol{u}(t)$ be the unique solution of Problem (4.1). Then $(I + A_p)^\alpha \boldsymbol{u}$ is Hölder continuous on every interval $[\varepsilon, T_*]$, $(0 < \varepsilon < T_*)$.*

Proof. First we recall that for all $0 < t \le T_*$

$$(I + A_p)^\alpha \boldsymbol{u}(t) = (I + A_p)^\alpha \boldsymbol{u}_0(t) + (I + A_p)^\alpha \boldsymbol{Su}(t)$$

with $\boldsymbol{u}_0(t)$ and $\boldsymbol{Su}(t)$ are defined in (4.17). Since the operators A_p and $I + A_p$ generates bounded analytic semi-groups on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and since $e^{-tA_p} = e^t e^{-t(I+A_p)}$ then for all $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $(I + A_p)^\alpha \boldsymbol{u}_0(t)$ is Hölder continuous on every interval $[\varepsilon, T_*]$, $0 < \varepsilon < T_*$, (see Proposition 2.5).

Let us prove the Hölder continuity of $(I + A_p)^\alpha \boldsymbol{Su}(t)$. observe that

$$\begin{aligned}(I + A_p)^\alpha \boldsymbol{Su}(t+h) - (I + A_p)^\alpha \boldsymbol{Su}(t) = \\ \int_0^t (I + A_p)^\alpha \big[e^{t+h-s} e^{-(t+h-s)(I+A_p)} \boldsymbol{Fu}(s) - e^{t-s} e^{-(t-s)(I+A_p)} \boldsymbol{Fu}(s)\big] \mathrm{d}\, s + \\ \int_t^{t+h} e^{t+h-s} (I + A_p)^\alpha e^{-(t+h-s)(I+A_p)} \boldsymbol{Fu}(s)\, \mathrm{d}\, s.\end{aligned}$$

As a result,

$$\|(I + A_p)^\alpha \boldsymbol{Su}(t+h) - (I + A_p)^\alpha \boldsymbol{Su}(t)\|_{\boldsymbol{L}^p(\Omega)} \le I_1 + I_2$$

with

$$I_1 = e^T \int_0^t \|(I + A_p)^\alpha e^{-(t-s)(I+A_p)} (e^{-h(I+A_p)} - I) \boldsymbol{Fu}(s)\|_{\boldsymbol{L}^p(\Omega)}\, \mathrm{d}\, s \tag{4.30}$$

and

$$I_2 = e^T \int_t^{t+h} \|(I + A_p)^\alpha e^{-(t+h-s)(I+A_p)} \boldsymbol{Fu}(s)\|_{\boldsymbol{L}^p(\Omega)}\, \mathrm{d}\, s. \tag{4.31}$$

We recall that the factor e^T in I_1 and I_2 is irrelevant since our existence is local in time.
Now as in the proof of [20, Proposition 2.4], let $0 < \mu < 1 - \delta - \alpha$ then

$$\begin{aligned}I_1 &= e^T \int_0^t \|(I + A_p)^{\alpha+\delta+\mu} e^{-(t-s)(I+A_p)} (e^{-h(I+A_p)} - I)(I + A_p)^{-\mu-\delta} \boldsymbol{Fu}(s)\|_{\boldsymbol{L}^p(\Omega)}\, \mathrm{d}\, s \\ &\le C_\mu \|(e^{-h(I+A_p)} - I)(I + A_p)^{-\mu}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \int_0^t (t-s)^{-\alpha-\delta-\mu} s^{\delta-1}\, \mathrm{d}\, s.\end{aligned} \tag{4.32}$$

The last inequality comes from the fact that $\|(I + A_p)^{-\delta} \boldsymbol{F}\boldsymbol{u}(s)\|_{\boldsymbol{L}^p(\Omega)} \leq Cs^{-\delta-1}$ which is a consequence of estimate (4.15) and Lemma 4.6. Now using Lemma 2.3 one has

$$\|(e^{-h(I+A_p)} - I)(I + A_p)^{-\mu}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \leq \frac{C}{\mu} h^{\mu}.$$

Substituting in (4.32) one has as in the proof of [20, Proposition 2.4]

$$I_1 \leq C\, h^{\mu}, \tag{4.33}$$

with some constant C depending on ε and μ.
Next consider the integral I_2 given by (4.31) one has

$$\begin{aligned} I_2 &\leq e^T \int_t^{t+h} \|(I + A_p)^{\alpha+\delta} e^{-(t+h-s)(I+A_p)}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \|(I + A_p)^{-\delta} \boldsymbol{F}\boldsymbol{u}(s)\|_{\boldsymbol{L}^p(\Omega)} \,\mathrm{d}\, s \\ &\leq C_\varepsilon \int_t^{t+h} (t + h - s)^{-\alpha-\delta} \,\mathrm{d}\, s \leq \frac{C_\varepsilon}{1 - \delta - \alpha} h^{1-\delta-\alpha} \\ &\leq Ch^{\mu} \end{aligned} \tag{4.34}$$

with

$$C_\varepsilon = \sup_{\varepsilon \leq t \leq T_*} \|(I + A_p)^{-\delta} \boldsymbol{F}\boldsymbol{u}(s)\|_{\boldsymbol{L}^p(\Omega)}.$$

Finally putting together (4.33) and (4.34) one gets directly the Hölder continuity of $(I + A_p)^{\alpha} \boldsymbol{S}\boldsymbol{u}$ on $(0, T_*]$. □

Now we can prove the Hölder continuity of $\boldsymbol{F}\boldsymbol{u}$ given by (4.2).

PROPOSITION 4.12. *Under the same assumptions of Proposition 4.11, let $\boldsymbol{u}$ be the unique solution of Problem (4.1). Then $\boldsymbol{F}\boldsymbol{u}$ is Hölder continuous on every interval $[\varepsilon, T_*]$, $0 < \varepsilon < T_*$.*

Proof. Let $\boldsymbol{u}(t)$ be the unique solution of Problem (4.1). Thanks to Theorem 4.10 we know that $\boldsymbol{u} \in C((0, T_*]; D(A_p^{\alpha}))$ for all $0 < \alpha < 1 - \delta$, where δ is as in Lemma 4.6. Under a suitable choice of δ we can show that $\boldsymbol{u} \in C((0, T_*]; D(A_p^{1/2}))$. Now, using Proposition 2.23, estimate (4.14) one has

$$\begin{aligned} \|\boldsymbol{F}\boldsymbol{u}(t+h) - \boldsymbol{F}\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq\ & C\, \|(I+A_p)^{1/2}(\boldsymbol{u}(t+h) - \boldsymbol{u}(t))\|_{\boldsymbol{L}^p(\Omega)} \|(I+A_p)^{1/2}\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} + \\ & \|(I + A_p)^{1/2}\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \|(I + A_p)^{1/2}(\boldsymbol{u}(t + h) - \boldsymbol{u}(t))\|_{\boldsymbol{L}^p(\Omega)}. \end{aligned}$$

Next, using the fact $(I + A_p)^{1/2}\boldsymbol{u}$ is Hölder continuous on every interval $[\varepsilon, T_*]$ (see Proposition 4.11), there exists $0 < \mu < 1 - \delta - 1/2$ such that

$$\|\boldsymbol{F}\boldsymbol{u}(t + h) - \boldsymbol{F}\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C\, h^{\mu}$$

and the result is proved. □

THEOREM 4.13. *Let* $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $p \geq 3$ *and let* $\boldsymbol{u}(t)$ *be the unique solution of Problem (4.1), then*

$$\boldsymbol{u} \in C((0,T_*], D(A_p)) \cap C^1((0,T_*]; \boldsymbol{L}^p_{\sigma,\tau}(\Omega)).$$

Proof. First we recall that the solution $\boldsymbol{u}$ is given explicitly by (4.16). We recall also that since e^{-tA_p} is an analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ then $\boldsymbol{u}_0(t) \in D(A_p)$ for all $t > 0$. It suffices to verify that $\boldsymbol{Su}(t) \in D(A_p)$ for all $t \in (0, T_*]$ which is a consequence of Proposition 4.12 and [29, Chapter 4, Corollary 3.3]. Moreover, thanks to Lemma 2.2 one has $\boldsymbol{u} \in C^1((0,T]; \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ this ends the proof. □

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