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# Theoretical Analysis of The $r^{th}$ Best Relay Selection for Amplify-and-Forward Systems

Amir Minayi Jalil\*, Vahid Meghdadi\*, Jean-Pierre Cances\*, Adel-Omar Dahmane†  
 Email: \*{amir.minayi-jalil, meghdadi, cances}@ensil.unilim.fr, †dahmane@uqtr.ca

**Abstract**—The two-hop relay network is well studied when both hops experience Rayleigh Fading channels. It is also studied when the SNR of one of the hops is the maximum among a set of i.i.d. Rayleigh Fading channels. In this paper, the two-hop relay network is analyzed when the SNR of one of the hops is the  $r^{th}$  biggest value among a set of  $N$  i.i.d. Rayleigh Fading channels. Also we have analyzed the case when the SNR of the first hop and the second hop are respectively the  $r^{th}$  and  $q^{th}$  biggest values among their set. The explicit PDF and CDF and also the average error probability for both cases are calculated in closed form. The diversity order of the resulting link is also calculated. The results have several applications in the analysis diversity schemes, especially in the analysis of relay assignment based on max-min criterion.

**Keywords**—Amplify-and-forward, cooperative networks, orders statistics, relay selection.

## I. INTRODUCTION

Amplify-and-Forward (AF) cooperative networks have attracted a lot of attention in the last decade [1]–[4]. In cooperative networks, a set of nodes act as data forwarders and relay traffic for the other nodes, therefore the overall performance can be improved [2]. Among different proposed cooperative schemes, the AF scheme is an attractive choice due to both performance [5] and simplicity, because it only requires scaling and retransmission of the received signal.

Hasna *et al.* [1] studied the basic two-hop Amplify-and-Forward link where the channels in both hops are i.i.d Rayleigh flat fading channels with different average SNRs (Scenario 1 in Table I). They calculated the probability density function (PDF) and the cumulative density function (CDF) of the resulting SNR of the equivalent two-hop link and also the average probability of error for this link. Diversity combining techniques provide higher diversity or coding gain compared to the simple two-hop cooperative model. But the advantages of these schemes come at the expense of the inefficient spectrum utilization, because of the necessity of using orthogonal channels. To remedy this, the best relay selection schemes are proposed. Krikidis *et al.* [6] considered this problem when one of the hops has the maximum instantaneous SNR among a set of available i.i.d. Rayleigh fading channels (Scenario 2 in Table I).

This paper consists of two parts: The first part is a generalization of the work of Krikidis *et al.* [6] where the instantaneous SNR of the available relay is the  $r^{th}$  magnitude among all relays (Scenario 3 in Table I). The analysis of this scenario is necessary to calculate the performance of some

TABLE I. TWO-HOP CHANNEL SCENARIOS IN AF MODE. 1) ANALYZED IN [1]; 2) ANALYZED IN [6]; 3) AND 4) ANALYZED IN THIS PAPER.

Scenario	Channel distribution in the first hop	Channel distribution in the second hop
1	Rayleigh	Rayleigh
2	Rayleigh	Maximum of $N$ Rayleigh channels
3	Rayleigh	$r^{th}$ order statistic of $N$ Rayleigh channels
4	$q^{th}$ order statistic of $N$ Rayleigh channels	$r^{th}$ order statistic of $N$ Rayleigh channels

relay assignment schemes such as relay assignment based on max-min criterion [7] and sequential relaying [8]. In relay assignment based on max-min criterion (sometimes called the linear bottleneck assignment) the minimum SNR value for each relay assignment permutation is calculated and then the permutation with the highest minimum SNR is selected. It is shown that the performance of the results of this scheme can be expressed in terms of order-statistics of the individual relaying channels [9, p. 174], [10]. In sequential relaying, for each realization of the channels, the sources are sequentially assigned their relays from the available relays [8]. The priority of the sources for relay assignment is according to the quality of their direct channel to the destination, i.e. the priority is given to the source node that has the weakest direct channel to the destination. In this scenario, it is assumed that the resource allocator is located in the destination and has the SNR information of only the incoming channels. The results of our analysis are also applicable to the case where among different relays, the best relay is not necessarily available for the intended source due to scheduling or load balancing conditions. The results of this analysis are also applicable to any application where among different relays, the best relay is not necessarily available for the intended source.

Statistical analysis of the relative magnitude among a set of random variables (RVs) leads to order statistics. We use order statistics to analyze the performance of different relays in our set and calculate the PDF and CDF of the received SNR in analytic form. Then we propose a closed form expression for the average probability of error. In the second part of the paper, we extend the model of the first part in order to include order-statistics in both hops (Scenario 4 in Table I).

Throughout this paper, the PDF and CDF of random variable  $X$  are denoted by  $f_X(x)$  and  $F_X(x)$  respectively. The rest of this paper is organized as follows. Section II reviews the basic formulas for order statistics. Section III offers the statistical analysis of two-hop relaying link when the SNR

<sup>1</sup>Amir Minayi Jalil and Adel-Omar Dahmane are with ECE Department, Universite du Quebec a Trois-Rivieres, Canada. Vahid Meghdadi and Jean-Pierre Cances are with XLIM/C2S2, University of Limoges, France.

information for only one hop is available. In section IV, same analysis is performed when the SNR of both hops follow order statistics. Some simulations are performed in section V in order to validate the results and section VI offers the concluding remarks.

## II. PRELIMINARY: PDF AND CDF OF ORDER STATISTICS

If random variables  $X_1, X_2, \dots, X_N$  are arranged in order of magnitude and then written as  $X_{1:N} \leq X_{2:N} \leq \dots \leq X_{N:N}$  then  $X_{r:N}$  is called the  $r^{\text{th}}$  order statistic ( $r = 1, \dots, N$ ). Although random variables  $X_i$  are assumed to be i.i.d.,  $X_{r:N}$  are necessarily dependent because of the inequality relations among them. If  $F_{r:N}(x)$  denotes the CDF of the  $r^{\text{th}}$  order statistic, then

$$\begin{aligned} F_{r:N}(x) &= Pr\{X_{r:N} \leq x\} \\ &= Pr\{\text{at least } r \text{ of the } X_i \text{ are less than or equal to } x\} \\ &= \sum_{i=r}^N \binom{N}{i} [F_X(x)]^i [1 - F_X(x)]^{N-i}. \end{aligned} \quad (1)$$

We replace  $F_X(x)$  by the exponential PDF (SNR for Rayleigh fading channel) to derive the CDF of the  $r^{\text{th}}$  order statistic:

$$F_{r:N}(x) = \sum_{i=r}^N \binom{N}{i} (1 - e^{-\lambda x})^i (e^{-\lambda x})^{N-i}. \quad (2)$$

A useful formula to calculate the order statistics comes from the well-known relation between binomial sums and the incomplete beta function [11]:

$$F_{r:N}(x) = I_{F(x)}(r, N - r + 1). \quad (3)$$

where  $I_p(a, b)$  is the incomplete beta function. By expanding and calculating the derivative of (3) we have the following simpler formula:

$$\begin{aligned} f_{r:N}(x) &= \frac{(1 - e^{-\lambda x})^{r-1} (e^{-\lambda x})^{N-r} \lambda e^{-\lambda x}}{B(r, N - r + 1)} \\ &= \sum_{i=0}^{r-1} \frac{1}{B} \binom{r-1}{i} (-1)^i \lambda e^{-\lambda x(N-r+1+i)} = \sum_{i=0}^{r-1} \Lambda_i e^{-\lambda_i x}, \end{aligned}$$

where the constant  $B(r, N - r + 1)$  is the beta function which is replaced by  $B$  for simplicity. Also  $\frac{1}{B} \binom{r-1}{i} (-1)^i \lambda$  is denoted by  $\Lambda_i$  and  $(N - r + 1 + i)\lambda$  is denoted by  $\lambda_i$ .

## III. SNR INFORMATION AVAILABLE FOR ONE HOP

Assume a simple relay configuration of one source  $S$ , one destination  $D$  and  $N$  relays  $R_i$ ,  $i = 1, \dots, N$  (Fig. 1). The source has no direct link to the destination and the transmission is performed only via relays. We make the following assumptions:

We assume that the relays near to each other are gathered into a cluster (location-based clustering). This clustered structure is a common model in the literature and have been selected by a long-term routing process. The related routing scheme can track variations in path-loss and shadowing, hence guarantees equivalent average SNR for the terminals in one cluster ( $S \rightarrow R_i$ ). Then our relay-assignment algorithm should combat the effects of small scale fading [6], [12]. The channels in each cluster are assumed to be independent and

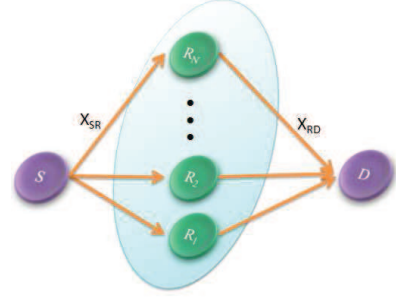


Fig. 1. A set of one source,  $N$  relays and one destination.

identically distributed (i.i.d) Rayleigh-fading channels. Thus, their SNR follows exponential distribution and their CDF is  $F(x) = (1 - e^{-\delta x})$  where  $1/\delta$  specifies SNR mean value. The considered links have an average SNR equal to  $\bar{X}_{SR} = 1/\delta$  for the links  $S \rightarrow R_i$  and  $\bar{X}_{RD} = 1/\lambda$  for the links  $R_i \rightarrow D$ . A two-slot relay mode is employed (as in [2]). In the first time slot, the source node transmits and the relays receive. In the second time slot, the relay nodes transmit and the destination receives. All the wireless nodes are assumed to work in half-duplex mode, i.e. they can not transmit and receive at the same time.

The resource allocator continuously monitors the quality of relay-destinations channels. Due to this information, the best available relay link among  $R_i \rightarrow D$  ( $i = 1, \dots, N$ ) is assigned to the intended source. Without loss of generality, we assume that the relays are sorted in order of their SNR magnitude and  $R_1$  shows the relay with the smallest received SNR at the destination. We denote the index of the best available relay by  $r$ , i.e. the PDF of the SNR in the second hop follows  $r^{\text{th}}$  order-statistic of exponential distribution ( $F_{r:N}(x)$ ).

### A. Statistical expressions

*Theorem 1:* The CDF of the equivalent SNR received at the destination for the relaying link under consideration ( $S \rightarrow R_r \rightarrow D$ ) can be approximated as

$$F_X(x) \approx 1 - \sum_{i=0}^{r-1} 2x \Lambda_i \sqrt{\frac{\delta}{\lambda_i}} e^{-x(\delta + \lambda_i)} K_1 \left( 2x \sqrt{\delta \lambda_i} \right) U(x) \quad (4)$$

where, same as before  $\frac{1}{B} \binom{r-1}{i} (-1)^i \lambda$  and  $(N - r + i + 1)\lambda$  are respectively denoted by  $\Lambda_i$  and  $\lambda_i$  and  $U(\cdot)$  is the unit step function.  $K_i(x)$  denotes the  $i^{\text{th}}$  order modified Bessel function of the second kind.

*Proof:* See appendix A. ■

*Theorem 2:* The PDF of the equivalent SNR for the relaying link under consideration ( $S \rightarrow R_r \rightarrow D$ ) can be written as

$$\begin{aligned} f_X(x) &\approx \sum_{i=0}^{r-1} 2x \delta \Lambda_i e^{-x(\delta + \lambda_i)} \left[ \frac{(\delta + \lambda_i)}{\sqrt{\delta \lambda_i}} \right. \\ &\quad \left. K_1 \left( 2x \sqrt{\delta \lambda_i} \right) U(x) + 2K_0 \left( 2x \sqrt{\delta \lambda_i} \right) \right] U(x). \end{aligned} \quad (5)$$

*Proof:* See appendix B. ■

### B. Average probability of error

*Theorem 3:* The average probability of error for the relaying link under consideration ( $S \rightarrow R_r \rightarrow D$ ) can be approximated as

$$P_E \approx \frac{c}{2} - \sum_{i=0}^{r-1} \frac{c}{B} \binom{r-1}{i} \frac{(-1)^i}{(N-r+1+i)} \times \frac{1}{2\sqrt{1+2(\delta+\lambda_i)/M}}. \quad (6)$$

In this expression,  $M$  is a constant which specifies the type of modulation and  $P_E = cQ(\sqrt{M}\gamma)$  is the bit error probability of this modulation for Gaussian channel. For example, for BPSK,  $M$  is equal to 2 and  $c$  is equal to 1.

*Proof:* See appendix C. ■

Although equation (6) does not reveal the diversity order explicitly, the diversity order of the equivalent link is one. We will express the diversity order offered by 6 in theorem 7 in the next section.

### IV. SNR INFORMATION AVAILABLE FOR BOTH HOPS

In section III, since there were no information available about the first hop, its SNR was assumed to be exponentially distributed. In this section, we assume that there are  $N$  orthogonal channels available for the first hop and each source has the possibility of using the best available channel to its corresponding relay. We assume that the SNR distribution for this channel ( $S \rightarrow R_r$ ) follows the  $q^{\text{th}}$  order-statistic of exponential distribution ( $F_{q:N}(x)$ ). The statistical expressions (PDF and CDF), the average error probability and the diversity order for this scenario are calculated in this section.

#### A. Statistical expressions

*Theorem 4:* The CDF of the equivalent SNR received at the destination for the relaying link under consideration ( $S \rightarrow R_r \rightarrow D$ ) can be approximated as

$$F_X(x) \approx 1 - \sum_{i=0}^{q-1} \sum_{j=0}^{r-1} 2x \Delta_i \Lambda_j \frac{1}{\sqrt{\delta_i \lambda_j}} e^{-x(\delta_i + \lambda_j)} \times K_1 \left( 2x \sqrt{\delta_i \lambda_j} \right) U(x) \quad (7)$$

where the constants  $\frac{1}{B(q, N-q+1)} \binom{q-1}{i} (-1)^i \delta$  and  $(N-q+1+i)\delta$  are respectively denoted by  $\Delta_i$  and  $\delta_i$ .

*Proof:* See appendix D. ■

*Theorem 5:* The PDF of the equivalent SNR received at the destination for the relaying link under consideration ( $S \rightarrow R_r \rightarrow D$ ) can be written as

$$f_X(x) \approx \sum_{i=0}^{q-1} \sum_{j=0}^{r-1} 2x \Delta_i \Lambda_j e^{-x(\delta_i + \lambda_j)} \left[ \frac{(\delta_i + \lambda_j)}{\sqrt{\delta_i \lambda_j}} K_1 \left( 2x \sqrt{\delta_i \lambda_j} \right) U(x) + 2K_0 \left( 2x \sqrt{\delta_i \lambda_j} \right) \right] U(x). \quad (8)$$

*Proof:* The proof is straightforwardly similar to the proof of Theorem 2. ■

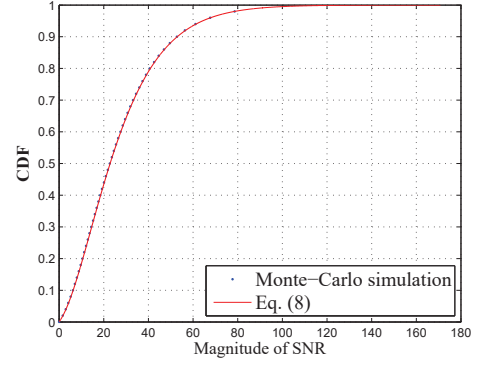


Fig. 2. Comparison of Eq. (4) with Monte-Carlo simulation for  $N = 4$  and  $r = 2$ .

### B. Average probability of error

*Theorem 6:* The average probability of error for the relaying link under consideration ( $S \rightarrow R_r \rightarrow D$ ) can be approximated as

$$P_E \approx \frac{c}{2} - \sum_{i=0}^{q-1} \sum_{j=0}^{r-1} \frac{c}{B_1 B_2} \binom{q-1}{i} \binom{r-1}{j} (-1)^{i+j} \frac{1}{\delta_i \lambda_j} \frac{1}{2\sqrt{1+\delta_i + \lambda_j}} \quad (9)$$

where the constants  $B(q, N-q+1)$  and  $B(r, N-r+1)$  are denoted by  $B_1$  and  $B_2$  respectively.

*Proof:* See appendix E. ■

*Theorem 7:* The diversity order for the relaying link under consideration is  $\min\{q, r\}$

*Proof:* See appendix F. ■

### V. SIMULATIONS AND DISCUSSIONS

Computer simulations were performed in order to validate the proposed analytical expressions. Figures (2) and (3) compare the expressions in Eq. (4) and Eq. (7) with Monte-Carlo simulations. The perfect match between these curves and the Monte-Carlo simulation is obvious from these figures. In both simulations, we have assumed  $\text{SNR} = 20$  dB and  $N = 4$ . Fig. (4) shows the average probability of error when SNR information is available for just one hop (Section III). It is assumed that  $N = 7$  and both hops have the same average SNR value  $1/\delta = 1/\lambda$ . The simulation is performed by using BPSK modulation and contains the results for five values of  $r$ . It is seen that for high SNRs, there is a significant difference between  $r = 1$  and  $r = 2$ . This is because the first hop for  $r = 2$  achieves diversity 2, however this diversity is dumped since the second hop (presenting a first order diversity) plays the role of a bottleneck, but still shows itself as an improvement in coding gain (a horizontal shift in the BER curve). Another interesting observation is that for good signal to noise ratios, the performance of the relaying link for  $r = 2$  converges to that of  $r = N$ . It means that for good SNR region, we can select the best between two randomly selected relays and the performance will be almost the same as selection of the best relay among all the available relays.

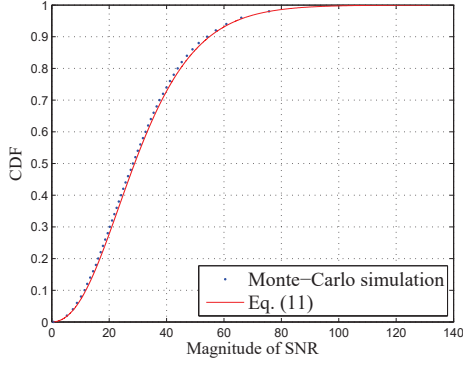


Fig. 3. Comparison of Eq. (7) with Monte-Carlo simulation for  $N = 4$ ,  $r = 3$  and  $q = 2$ .

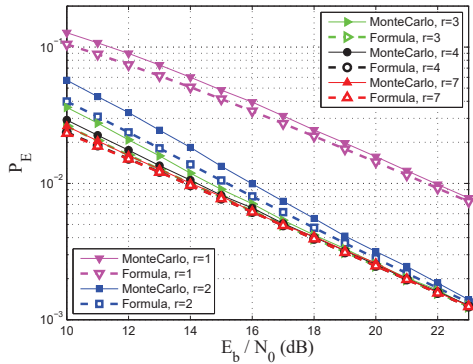


Fig. 4.  $P_E$  of equation (6) for  $N = 7$  and different values of  $r$ .

Fig. (5) shows the average probability of error when SNR information is available for both hops (Section IV). Again it is assumed that  $1/\delta = 1/\lambda$ ,  $N = 7$ ,  $q = 4$  and the simulation is performed by using BPSK modulation. From this figure, we can see that by increasing  $r$ , as far as  $r \leq q$  there is an increase in the diversity order. The amount of this diversity order is exactly what we expected in theorem 7. We can not increase the diversity order by increasing  $r$  more than  $r = q$ , but it will be an improvement in the coding gain. Another interesting observation is that for high SNR region, when  $r = q + 1$ , the performance of the relaying link converges to that of  $r = N$ . This result shows that, when there is a bottleneck in one of the hops, ( $q$  is fixed), it is almost enough to have  $r = q + 1$  in order to achieve the highest possible performance.

## VI. CONCLUSION

This paper dealt with the statistical behavior of the two-hop Amplify-and-Forward relaying link when the magnitude of SNR in one of the hops is the  $r^{th}$  order statistic among a set of  $N$  i.i.d Rayleigh fading channels. Then the analysis was extended to involve the case when the magnitude of SNR in both hops are order statistics ( $q^{th}$  and  $r^{th}$  order statistics respectively). For each case, PDF, CDF, the average probability of error and the diversity order are expressed in analytical form and confirmed by simulation. We concluded that when the minimum between  $q$  and  $r$  is fixed (for example  $q$ ), it is almost enough to have  $r = q + 1$  in order to achieve the

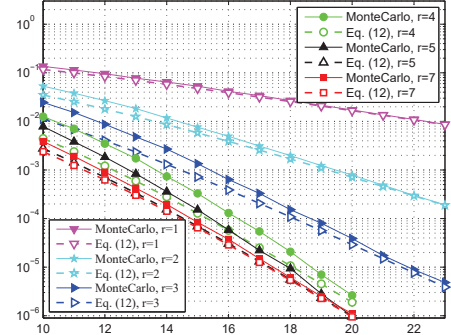


Fig. 5.  $P_E$  of equation (9) for  $N = 7$ ,  $q = 4$  and different values of  $r$ .

highest possible performance; i.e. we can not further improve the performance by increasing the number of choices in this hop.

## APPENDIX A PROOF OF THEOREM 1: CDF

The RVs  $X_1$  and  $X_2$  denote the instantaneous SNRs of the links  $S \rightarrow R$  and  $R \rightarrow D$ , respectively, with PDFs equal to

$$\begin{cases} X_1 \sim \delta e^{-\delta x} U(x) \\ X_2 \sim \sum_{i=0}^{r-1} \Lambda_i e^{-\lambda_i x} U(x) \end{cases} \quad (10)$$

Then we have

$$\begin{cases} \frac{1}{X_1} \sim \frac{\delta}{x^2} e^{-\delta/x} U(x) \\ \frac{1}{X_2} \sim \frac{1}{x^2} \sum_{i=0}^{r-1} \Lambda_i e^{-\lambda_i/x} U(x) \end{cases} \quad (11)$$

Same as [1], the moment generating function (MGF) of the variables  $1/X_1$  and  $1/X_2$  can be evaluated by the help of [13] (eq. 3.471.9) and using the symmetry property of the modified Bessel function (i.e.,  $K_{-\nu}(z) = K_{\nu}(z)$ ) given in [13](eq. 8.486.16):

$$\begin{cases} M_1(s) = 2\sqrt{\delta s} K_1(2\sqrt{\delta s}) \\ M_2(s) = 2 \sum_{i=0}^{r-1} \Lambda_i \sqrt{\frac{s}{\lambda_i}} K_1(2\sqrt{\lambda_i s}) \end{cases} \quad (12)$$

Therefore the MGF of  $X = (1/X_1) + (1/X_2)$  is

$$\begin{aligned} M(s) &= \sum_{i=0}^{r-1} 2\sqrt{\delta s} K_1(2\sqrt{\delta s}) 2\Lambda_i \sqrt{\frac{s}{\lambda_i}} K_1(2\sqrt{\lambda_i s}) \\ &= \sum_{i=0}^{r-1} 4\Lambda_i \sqrt{\frac{\delta}{\lambda_i}} s K_1(2\sqrt{\delta s}) K_1(2\sqrt{\lambda_i s}). \end{aligned}$$

By using [14] Eq. (13.2.20) and the differentiation property of the Laplace transform, we can write the CDF of  $X$  as

$$\begin{aligned} F_X(x) &= \mathcal{L}^{-1} \left\{ \frac{M(s)}{s} \right\} \\ &= 1 - \sum_{i=0}^{r-1} 2x \Lambda_i \sqrt{\frac{\delta}{\lambda_i}} e^{-x(\lambda_i + \delta)} K_1(2x\sqrt{\lambda_i \delta}) U(x). \end{aligned} \quad (13)$$

APPENDIX B  
PROOF OF THEOREM 2: PDF

We must differentiate the CDF in (4) with respect to  $x$ . For this purpose we need the derivative of the modified Bessel function that is given in [13], eq. (8.486.12):

$$u \frac{d}{du} K_1(u) + K_1(u) = -u K_0(u) \Rightarrow \frac{d}{dx} K_1(2x\sqrt{\delta\lambda_i}) + \frac{1}{x} K_1(2x\sqrt{\delta\lambda_i}) = -2\sqrt{\delta\lambda_i} K_0(2x\sqrt{\delta\lambda_i}).$$

By replacing the above result in the derivative of  $F_X(x)$  we have:

$$\begin{aligned} f_X(x) &= \left[ \sum_{i=0}^{r-1} 2x\Lambda_i \sqrt{\frac{\delta}{\lambda_i}} (\delta + \lambda_i) e^{-x(\delta+\lambda_i)} K_1(2x\sqrt{\delta\lambda_i}) \right. \\ &\quad \left. + \sum_{i=0}^{r-1} 4x\Lambda_i \delta e^{-x(\delta+\lambda_i)} K_0(2x\sqrt{\delta\lambda_i}) \right] U(x) \\ &= \sum_{i=0}^{r-1} 2x\Lambda_i \delta e^{-x(\delta+\lambda_i)} \left[ \frac{(\delta + \lambda_i)}{\sqrt{\delta\lambda_i}} \right. \\ &\quad \left. \times K_1(2x\sqrt{\delta\lambda_i}) + 2K_0(2x\sqrt{\delta\lambda_i}) \right] U(x). \quad (14) \end{aligned}$$

APPENDIX C

PROOF OF THEOREM 3: AVERAGE PROBABILITY OF ERROR

We start with the definition of average error probability:

$$\begin{aligned} P_E &= \int_0^\infty cQ(\sqrt{Mx}) f_{r:N}(x) dx \\ &= c \int_0^\infty \int_{\sqrt{Mx}}^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} f_{r:N}(x) dy dx \end{aligned}$$

where  $M$  is determined by the type of modulation. By changing the order of integration, we have

$$\begin{aligned} P_E &= c \int_0^\infty \int_0^{y^2/M} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} f_{r:N}(x) dx dy \\ &= c \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} F_{r:N}(y^2/M) dy. \quad (15) \end{aligned}$$

Before we replace for  $F_{r:N}(y^2/M)$  in the above expression, we can simplify it for high SNR regime, where  $K_1(x)$  can be approximated by  $1/x$ , ([15], Eq. (9.6.9)). By substitution in (4), the CDF of the considered link can be simplified as follows

$$\begin{aligned} F_X(x) &= 1 - \sum_{i=0}^{r-1} 2x\Lambda_i \sqrt{\frac{\delta}{\lambda_i}} e^{-x(\lambda_i+\delta)} K_1(2x\sqrt{\lambda_i\delta}) U(x) \\ &= 1 - \sum_{i=0}^{r-1} \Lambda_i \frac{1}{\lambda_i} e^{-x(\lambda_i+\delta)} U(x) \\ &= 1 - \sum_{i=0}^{r-1} \frac{1}{B} \binom{r-1}{i} (-1)^i \frac{e^{-x(\lambda_i+\delta)}}{(N-r+1+i)} U(x). \quad (16) \end{aligned}$$

Hence, the average error probability becomes:

$$\begin{aligned} P_E &= c \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} F_{r:N}(y^2/M) dy \\ &= c \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \left( 1 - \sum_{i=0}^{r-1} \frac{1}{B} \binom{r-1}{i} \right. \\ &\quad \left. (-1)^i \times \frac{e^{-\frac{y^2}{M}(\lambda_i+\delta)}}{(N-r+1+i)} \right) dy. \end{aligned}$$

This result is the integral of some Gaussian functions:

$$\begin{aligned} P_E &= \frac{c}{2} - \sum_{i=0}^{r-1} \frac{c}{B} \binom{r-1}{i} (-1)^i \frac{1}{(N-r+1+i)} \\ &\quad \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-(y^2/2)(1+2(\lambda_i+\delta)/M)} dy \\ &= \frac{c}{2} - \sum_{i=0}^{r-1} \frac{c}{B} \binom{r-1}{i} (-1)^i \frac{1}{(N-r+1+i)} \\ &\quad \frac{1}{2\sqrt{1+2(\lambda_i+\delta)/M}}. \end{aligned}$$

APPENDIX D

PROOF OF THEOREM 4: CDF

The proof of this theorem is straightforwardly similar to the proof of theorem 1. The MGF of  $\bar{X} = (1/X_1) + (1/X_2)$  is

$$\begin{aligned} M(s) &= \sum_{i=0}^{q-1} 2\Delta_i \sqrt{\frac{s}{\delta_i}} K_1(2\sqrt{\delta_i s}) \sum_{j=0}^{r-1} 2\Lambda_j \sqrt{\frac{s}{\lambda_j}} K_1(2\sqrt{\lambda_j s}) \\ &= \sum_{i=0}^{q-1} \sum_{j=0}^{r-1} 4\Delta_i \Lambda_j \frac{1}{\sqrt{\delta_i \lambda_j}} s K_1(2\sqrt{\delta_i s}) K_1(2\sqrt{\lambda_j s}). \end{aligned}$$

Again by using the differentiation property of the Laplace transform, similar to (13) we have

$$\begin{aligned} F_X(x) &= 1 - \sum_{i=0}^{q-1} \sum_{j=0}^{r-1} 2x\Delta_i \Lambda_j \frac{1}{\sqrt{\delta_i \lambda_j}} e^{-x(\delta_i+\lambda_j)} \\ &\quad \times K_1(2x\sqrt{\delta_i \lambda_j}) U(x). \end{aligned}$$

APPENDIX E

PROOF OF THEOREM 6: AVERAGE PROBABILITY OF ERROR

Let us start from equation (15), but this time the CDF function is  $F_{q,r:N}(x)$ . Then by using the same simplification as in equation (16) for  $F_{q,r:N}(y^2/M)$  we have

$$\begin{aligned} P_E &= c \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \left( 1 - \sum_{i=0}^{q-1} \sum_{j=0}^{r-1} \frac{\Delta_i \Lambda_j}{\delta_i \lambda_j} e^{-y^2(\delta_i+\lambda_j)/2} \right) dy \\ &= \frac{c}{2} - c \sum_{i=0}^{q-1} \sum_{j=0}^{r-1} \frac{1}{B_1 B_2} \binom{q-1}{i} \binom{r-1}{j} (-1)^{i+j} \quad (17) \end{aligned}$$

$$P_{E_2} = c \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \int_{y^2/2}^\infty f_{X_2}(x_2) \sum_{i=q}^N \binom{N}{i} \left[ \left( 1 - \exp \left( -\lambda \frac{y^2 (y^2 + 1)}{2 x_2 - \frac{y^2}{2}} \right) \right)^i \left( \exp \left( -\lambda \frac{y^2 (y^2 + 1)}{2 x_2 - \frac{y^2}{2}} \right) \right)^{N-i} - \left( 1 - \exp \left( -\lambda \frac{y^2}{2} \right) \right)^i \left( \exp \left( -\lambda \frac{y^2}{2} \right) \right)^{N-i} \right] dx_2 dy. \quad (23)$$

## APPENDIX F

### PROOF OF THEOREM 7: DIVERSITY ORDER ANALYSIS

We introduce a new random variable  $\Upsilon$  for the SNR of this link. From [1] the equivalent SNR of the two hop system ( $X_\Upsilon$ ) is given by:

$$X_\Upsilon = \frac{X_1 X_2}{X_1 + X_2 + 1} \quad (18)$$

where  $X_1$  and  $X_2$  are per-hop SNR values. The CDF of  $\Upsilon$  can be calculated by the following integral over the region  $\Upsilon \leq x$

$$F_\Upsilon(x) = \iint_D f_{X_1}(x_1) f_{X_2}(x_2) dx_1 dx_2. \quad (19)$$

The integration surface  $D$  in (19) can be divided into two regions, namely  $D_1$  and  $D_2$ , where  $D_1$  shows the region  $\{x_1 < x\} \cup \{x_2 < x\}$  and the remaining is denoted by  $D_2$  (Fig. (6)). Let us denote the result of integral in (19) over  $D_1$  and  $D_2$  by  $F_{\Upsilon_1}(x)$  and  $F_{\Upsilon_2}(x)$ . So for  $D_1$  we have:  $F_{\Upsilon_1}(x) = \Pr\{X_1 \leq x\} + \Pr\{X_2 \leq x\} - \Pr\{X_1 \leq x, X_2 \leq x\}$ , where  $\Pr\{X_1 \leq x\} = F_{r;N}(x)$  and  $\Pr\{X_2 \leq x\} = F_{q;N}(x)$ . Derivation of this result gives the PDF of  $\Upsilon_1$  around zero. Then, we can write the Taylor expansion of  $F_{\Upsilon_1}(x)$  around zero. By using proposition 1 in [16], the result implies that the system has a diversity order of  $\min(r, q)$ . According to this proposition when the PDF of SNR can be approximated by a single polynomial term for  $x \rightarrow 0^+$  (i.e.  $p_X(x) = ax^t + O(x^{t+\epsilon})$ ), then the system has a diversity order of  $t + 1$ . In here,  $\epsilon > 0$  and  $a$  is a positive constant. For  $D_2$  we have:

$$\begin{aligned} \Pr\{\Upsilon_2 \leq x\} &= \int_x^\infty \int_x^{\frac{x(x_2+1)}{x_2-x}} f_{X_1}(x_1) f_{X_2}(x_2) dx_1 dx_2 \\ &= \int_x^\infty f_{X_2}(x_2) \sum_{i=q}^N \binom{N}{i} \left[ (1 - \Omega)^i \Omega^{N-i} - (1 - e^{-\lambda x})^i (e^{-\lambda x})^{N-i} \right] dx_2. \quad (20) \end{aligned}$$

where  $\Omega = \exp\left(-\lambda x \frac{x_2+1}{x_2-x}\right)$ . Now, let us consider the average error probability for  $\Upsilon_2$ . By using (15) and  $F_{\Upsilon_2}(x)$  as the CDF function, we arrive at (23) at the top of this page. In this expression, the Taylor expansion of  $f_{X_2}(x_2)$  in terms of  $\lambda$  is of order  $r$  (because  $X_2$  is the  $r^{\text{th}}$  order statistic). The Taylor expansion of the terms in brackets in terms of  $\lambda$  is of order  $r$ . Hence the whole result of the integral (which is a number between zero and one) at least is of order  $r + q$ . Hence, in good SNR regime, this term is negligible compared to the  $P_{E_1} = k\lambda^{\min\{r,q\}}$  (resulting from  $F_{\Upsilon_1}(x)$ ).

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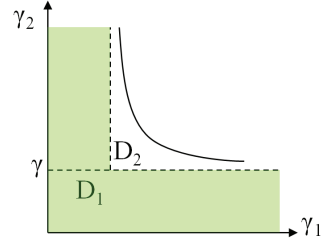


Fig. 6. Integration surface  $\Upsilon \leq x$ .

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