Stationary solutions to the two-dimensional Broadwell model

Leif Arkeryd, Anne Nouri

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Abstract
Existence of renormalized solutions to the two-dimensional Broadwell model with given indata in L^1 is proven. Averaging techniques from the continuous velocity case being unavailable when the velocities are discrete, the approach is based on direct L^1-compactness arguments using the Kolmogorov-Riesz theorem.

1 Introduction.

The two-dimensional stationary Broadwell model in a square is

$$\begin{aligned} &\partial_x F_1 = F_3F_4 - F_1F_2, && F_1(0,\cdot) = f_{b1},\\ &-\partial_x F_2 = F_3F_4 - F_1F_2, && F_2(1,\cdot) = f_{b2},\\ &\partial_y F_3 = F_1F_2 - F_3F_4, && F_3(\cdot,0) = f_{b3},\\ &-\partial_y F_4 = F_1F_2 - F_3F_4, && F_4(\cdot,1) = f_{b4}, \end{aligned} \tag{1.1}$$

with unknown $(F_i)_{1\le i\le 4}$ defined on $[0,1]^2$, and given $(f_{bi})_{1\le i\le 4}$ defined on $[0,1]$. It is a four velocity model for the Boltzmann equation, with $F_i(x,y) = f(x,y,v_i)$,

$$v_1 = (1,0), \quad v_2 = (-1,0), \quad v_3 = (0,1), \quad v_4 = (0,-1).$$

The boundary value problem (1.1) is considered in L^1 in one of the following equivalent forms, the exponential multiplier form:

$$F_1(x,y) = f_{b1}(y)e^{-\int_0^x F_2(s,y)ds} + \int_0^x (F_3F_4)(s,y)e^{-\int_s^x F_2(\tau)d\tau}ds, \quad \text{a.a. } (x,y)\in[0,1]^2, \tag{1.2}$$

and analogous equations for F_i, $2\le i\le 4$,
the mild form:

$$F_1(x,y) = f_{b1}(y) + \int_0^x (F_3F_4 - F_1F_2)(s,y)ds, \quad \text{a.a. } (x,y)\in[0,1]^2, \tag{1.3}$$

and analogous equations for F_i, $2\le i\le 4$,
the renormalized form:

$$\partial_x \ln(1+F_1) = \frac{F_3F_4 - F_1F_2}{1+F_1}, \quad F_1(0,\cdot) = f_{b1}, \tag{1.4}$$

in the sense of distributions, and analogous equations for F_i, $2\le i\le 4$.
The main result of the paper is the following.

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Theorem 1.1
Given a non-negative boundary value f_b with finite mass and entropy, there exists a stationary non-negative renormalized solution in L^1 with finite entropy-dissipation to the Broadwell model (1.1).

Most mathematical results for discrete velocity models of the Boltzmann equation have been performed in one space dimension. An overview is given in [9]. In two dimensions, special classes of solutions are given in [4], [5], and [10]. [4] contains a detailed study of the stationary Broadwell equation in a rectangle with comparison to a Carleman-like system, and a discussion of (in)compressibility aspects.

The existence of continuous solutions to the two-dimensional stationary Broadwell model with continuous boundary data for a rectangle, is proven in [7]. That proof starts by solving the problem with a given gain term, and uses the compactness of the corresponding twice iterated solution operator to conclude by Schaeffer's fixed point theorem.
The present paper on the Broadwell model is set in a context of physically natural quantities. Mass and entropy flow at the boundary are given, and the solutions obtained, have finite mass and finite entropy dissipation. Averaging techniques from the continuous velocity case [8] being unavailable, a direct compactness approach is used, based on the Kolmogorov-Riesz theorem.
The plan of the paper is the following. An approximation procedure for the construction of solutions to (1.1) is introduced in Section 2. The passage to the limit in the approximations is performed in Section 3. Here a compactness property of the approximated gain terms in mild form is carried over to the corresponding solutions themselves, using a particular sequence of successive alternating approximations and the Kolmogorov-Riesz theorem [11], [12].
A common approach to existence for stationary Boltzmann like equations is based on the regularizing properties of the gain term. In the continuous velocity case an averaging propery is available to keep this study of the gain term within a weak L^1 frame as in [2]. However, in the discrete velocity case, averaging is not available. Instead strong convergence of an approximating sequence is here directly proved from the regularizing properties for the gain term (cf Lemma 3.5 below). But the technique in that proof is restricted to two dimensional velocities, whereas the averaging technique in the continuous velocity case is dimension independent.

2 Approximations.

Denote by $L^1_+([0,1]^2)$ the set of non negative integrable functions on $[0,1]^2$, and by $a \wedge b$ the minimum of two real numbers a and b. Approximations to (1.1) to be used in the proof of Theorem 1, are introduced in the following lemma.

Lemma 2.1 *For any $k \in \mathbb{N}^*$, there exists a solution $F^k \in \left(L^1_+([0,1]^2)\right)^4$ to*

$$\partial_x F_1^k = \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}} - \frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}}, \tag{2.1}$$

$$-\partial_x F_2^k = \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}} - \frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}}, \tag{2.2}$$

$$\partial_y F_3^k = \frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}} - \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}, \tag{2.3}$$

$$-\partial_y F_4^k = \frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}} - \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}, \quad (x,y) \in [0,1]^2, \tag{2.4}$$

$$F_1^k(0,y) = f_{b1}(y) \wedge \frac{k}{2}, \quad F_2^k(1,y) = f_{b2}(y) \wedge \frac{k}{2}, \quad y \in [0,1], \tag{2.5}$$

$$F_3^k(x,0) = f_{b3}(x) \wedge \frac{k}{2}, \quad F_4^k(x,1) = f_{b4}(x) \wedge \frac{k}{2}, \quad x \in [0,1]. \tag{2.6}$$

Proof of Lemma 2.1.
The sequence of approximations $(F^k)_{k\in\mathbb{N}^*}$ is obtained in the limit of a further approximation with damping terms αF_j and convolutions in the collision operator.

Step I. Approximations with damping and convolutions.
Take $\alpha > 0$ and set

$$c_\alpha = \frac{1}{\alpha}\int_0^1 \sum_{i=1}^4 f_{bi}(u)du, \quad K_\alpha = \{f \in \left(L^1_+([0,1]^2)\right)^4; \sum_{i=1}^4 \int f_i(x,y)dxdy \le c_\alpha\}. \tag{2.7}$$

Let μ_α be a smooth mollifier in (x,y) with support in the ball centered at the origin of radius α. Let $\mathcal{T}$ be the map defined on K_α by $\mathcal{T}(f) = F$, where $F = (F_i)_{1\le i\le 4}$ is the solution of

$$\alpha F_1 + \partial_x F_1 = \frac{F_3}{1+\frac{F_3}{k}}\frac{f_4*\mu_\alpha}{1+\frac{f_4*\mu_\alpha}{k}} - \frac{F_1}{1+\frac{F_1}{k}}\frac{f_2*\mu_\alpha}{1+\frac{f_2*\mu_\alpha}{k}}, \tag{2.8}$$

$$\alpha F_2 - \partial_x F_2 = \frac{f_3*\mu_\alpha}{1+\frac{f_3*\mu_\alpha}{k}}\frac{F_4}{1+\frac{F_4}{k}} - \frac{f_1*\mu_\alpha}{1+\frac{f_1*\mu_\alpha}{k}}\frac{F_2}{1+\frac{F_2}{k}}, \tag{2.9}$$

$$\alpha F_3 + \partial_y F_3 = \frac{F_1}{1+\frac{F_1}{k}}\frac{f_2*\mu_\alpha}{1+\frac{f_2*\mu_\alpha}{k}} - \frac{F_3}{1+\frac{F_3}{k}}\frac{f_4*\mu_\alpha}{1+\frac{f_4*\mu_\alpha}{k}}, \tag{2.10}$$

$$\alpha F_4 - \partial_y F_4 = \frac{f_1*\mu_\alpha}{1+\frac{f_1*\mu_\alpha}{k}}\frac{F_2}{1+\frac{F_2}{k}} - \frac{f_3*\mu_\alpha}{1+\frac{f_3*\mu_\alpha}{k}}\frac{F_4}{1+\frac{F_4}{k}}, \quad (x,y) \in [0,1]^2, \tag{2.11}$$

$$F_1(0,y) = f_{b1}(y) \wedge \frac{k}{2}, \quad F_2(1,y) = f_{b2}(y) \wedge \frac{k}{2}, \quad y \in [0,1], \tag{2.12}$$

$$F_3(x,0) = f_{b3}(x) \wedge \frac{k}{2}, \quad F_4(x,1) = f_{b4}(x) \wedge \frac{k}{2}, \quad x \in [0,1]. \tag{2.13}$$

$F = \mathcal{T}(f)$ is obtained as the limit in $L^1([0,1]^2)$ of the sequence $(F^n)_{n\in\mathbb{N}}$ defined by $F^0 = 0$ and

$$
\begin{aligned}
&\alpha F_1^{n+1} + \partial_x F_1^{n+1} = \frac{F_3^n}{1+\frac{F_3^n}{k}} \frac{f_4 * \mu_\alpha}{1+\frac{f_4*\mu_\alpha}{k}} - \frac{F_1^{n+1}}{1+\frac{F_1^n}{k}} \frac{f_2 * \mu_\alpha}{1+\frac{f_2*\mu_\alpha}{k}},\\
&\alpha F_2^{n+1} - \partial_x F_2^{n+1} = \frac{f_3 * \mu_\alpha}{1+\frac{f_3*\mu_\alpha}{k}} \frac{F_4^n}{1+\frac{F_4^n}{k}} - \frac{f_1 * \mu_\alpha}{1+\frac{f_1*\mu_\alpha}{k}} \frac{F_2^{n+1}}{1+\frac{F_2^n}{k}},\\
&\alpha F_3^{n+1} + \partial_y F_3^{n+1} = \frac{F_1^n}{1+\frac{F_1^n}{k}} \frac{f_2 * \mu_\alpha}{1+\frac{f_2*\mu_\alpha}{k}} - \frac{F_3^{n+1}}{1+\frac{F_3^n}{k}} \frac{f_4 * \mu_\alpha}{1+\frac{f_4*\mu_\alpha}{k}},\\
&\alpha F_4^{n+1} - \partial_y F_4^{n+1} = \frac{f_1 * \mu_\alpha}{1+\frac{f_1*\mu_\alpha}{k}} \frac{F_2^n}{1+\frac{F_2^n}{k}} - \frac{f_3 * \mu_\alpha}{1+\frac{f_3*\mu_\alpha}{k}} \frac{F_4^{n+1}}{1+\frac{F_4^n}{k}},\\
&F_1^{n+1}(0,y) = f_{b1}(y) \wedge \frac{k}{2}, \quad F_2^{n+1}(1,y) = f_{b2}(y) \wedge \frac{k}{2}, \quad y \in [0,1],\\
&F_3^{n+1}(x,0) = f_{b3}(x) \wedge \frac{k}{2}, \quad F_4^{n+1}(x,1) = f_{b4}(x) \wedge \frac{k}{2}, \quad x \in [0,1], \quad n \in \mathbb{N}.
\end{aligned}
$$

The sequence $(F^n)_{n\in\mathbb{N}}$ is monotone. Indeed, $F_i^0 \le F_i^1$, $1 \le i \le 4$ by the exponential form of F_i^1. Moreover, assume $F_i^n \le F_i^{n+1}$, $1 \le i \le 4$. It follows from the exponential form that $F_1^{n+1} - F_1^{n+2} \le 0$. The inequalities $F_i^{n+1} - F_i^{n+2} \le 0$, $2 \le i \le 4$ can be reached in a similar way. Moreover,

$$
\begin{aligned}
&\alpha \sum_{i=1}^4 F_i^{n+1} + \partial_x(F_1^{n+1} - F_2^{n+1}) + \partial_y(F_3^{n+1} - F_4^{n+1})\\
&= \frac{f_1 * \mu_\alpha}{1+\frac{f_1*\mu_\alpha}{k}} \frac{F_2^n - F_2^{n+1}}{1+\frac{F_2^n}{k}} + \frac{f_2 * \mu_\alpha}{1+\frac{f_2*\mu_\alpha}{k}} \frac{F_1^n - F_1^{n+1}}{1+\frac{F_1^n}{k}} + \frac{f_3 * \mu_\alpha}{1+\frac{f_3*\mu_\alpha}{k}} \frac{F_4^n - F_4^{n+1}}{1+\frac{F_4^n}{k}} + \frac{f_4 * \mu_\alpha}{1+\frac{f_4*\mu_\alpha}{k}} \frac{F_3^n - F_3^{n+1}}{1+\frac{F_3^n}{k}}\\
&\le 0,
\end{aligned}
$$

so that

$$
\sum_{i=1}^4 \int F_i^{n+1}(x,y)dxdy \le c_\alpha. \tag{2.14}
$$

By the monotone convergence theorem, $(F^n)_{n\in\mathbb{N}}$ converges in $L^1([0,1]^2)$ to some solution F of (2.8)-(2.13). The solution of (2.8)-(2.13) is unique in the set of non negative functions. Indeed, let $G = (G_i)_{1\le i\le 4}$ be a solution of (2.8)-(2.13) with $G_i \ge 0$, $1 \le i \le 4$. Let us prove by induction that

$$
\forall n \in \mathbb{N}, \quad F_i^n \le G_i, \quad 1 \le i \le 4. \tag{2.15}
$$

(2.15) holds for $n=0$, since $G_i \geq 0$, $1 \leq i \leq 4$. Assume (2.15) holds for n. Using the exponential form of F_1^{n+1} implies

$$\begin{aligned}
F_1^{n+1}(x,y) &= (f_{b1}(y) \wedge \frac{k}{2}) e^{-\alpha x - \int_0^x \frac{f_2*\mu_\alpha}{(1+\frac{F_1^n}{k})(1+\frac{f_2*\mu_\alpha}{k})}(X,y)dX} \\
&+ \int_0^x \frac{F_3^n}{1+\frac{F_3^n}{k}} \frac{f_4*\mu_\alpha}{1+\frac{f_4*\mu_\alpha}{k}}(X,y) e^{-\alpha(x-X) - \int_X^x \frac{f_2*\mu_\alpha}{(1+\frac{F_1^n}{k})(1+\frac{f_2*\mu_\alpha}{k})}(r,y)dr} dX \\
&\leq (f_{b1}(y) \wedge \frac{k}{2}) e^{-\alpha x - \int_0^x \frac{f_2*\mu_\alpha}{(1+\frac{G_1}{k})(1+\frac{f_2*\mu_\alpha}{k})}(X,y)dX} \\
&+ \int_0^x \frac{G_3}{1+\frac{G_3}{k}} \frac{f_4*\mu_\alpha}{1+\frac{f_4*\mu_\alpha}{k}}(X,y) e^{-\alpha(x-X) - \int_X^x \frac{f_2*\mu_\alpha}{(1+\frac{G_1}{k})(1+\frac{f_2*\mu_\alpha}{k})}(r,y)dr} dX \\
&= G_1(x,y), \quad (x,y) \in [0,1]^2.
\end{aligned}$$

The same argument can be applied to prove that $F_i^{n+1} \leq G_i$, $2 \leq i \leq 4$. Consequently,

$$F_i \leq G_i, \quad 1 \leq i \leq 4. \tag{2.16}$$

Moreover, substracting the partial differential equations satisfied by G_i from the partial differential equations satisfied by F_i, $1 \leq i \leq 4$, and integrating the resulting equation on $[0,1]^2$, it results

$$\begin{aligned}
\alpha \sum_{i=1}^4 \int (G_i - F_i)(x,y)dxdy &+ \int_0^1 \big((G_1-F_1)(1,y) + (G_2-F_2)(0,y)\big)dy \\
&+ \int_0^1 \big((G_3-F_3)(x,1) + (G_4-F_4)(x,0)\big)dx = 0.
\end{aligned} \tag{2.17}$$

It results from (2.16)-(2.17) that $G = F$.
The map $\mathcal{T}$ is continuous in the L^1-norm topology (cf [1] pages 124-5). Namely, let a sequence $(f_l)_{l\in\mathbb{N}}$ in K_α converge in $L^1([0,1]^2)$ to $f \in K_\alpha$. Set $F_l = T(f_l)$. Because of the uniqueness of the solution to (2.8)-(2.13), it is enough to prove that there is a subsequence of (F_l) converging to $F = T(f)$. Now there is a subsequence of (f_l), still denoted (f_l), such that decreasingly (resp. increasingly) $(G_l) = (\sup_{m\geq l} f_m)$ (resp. $(g_l) = (\inf_{m\geq l} f_m)$) converges to f in L^1. Let (S_l) (resp. (s_l)) be the sequence of solutions to

$$\alpha S_{l1} + \partial_x S_{l1} = \frac{S_{l3}}{1+\frac{S_{l3}}{k}} \frac{G_{l4}*\mu_\alpha}{1+\frac{G_{l4}*\mu_\alpha}{k}} - \frac{S_{l1}}{1+\frac{S_{l1}}{k}} \frac{g_{l2}*\mu_\alpha}{1+\frac{g_{l2}*\mu_\alpha}{k}}, \tag{2.18}$$

$$\alpha S_{l2} - \partial_x S_{l2} = \frac{G_{l3}*\mu_\alpha}{1+\frac{G_{l3}*\mu_\alpha}{k}} \frac{S_{l4}}{1+\frac{S_{l4}}{k}} - \frac{g_{l1}*\mu_\alpha}{1+\frac{g_{l1}*\mu_\alpha}{k}} \frac{S_{l2}}{1+\frac{S_{l2}}{k}}, \tag{2.19}$$

$$\alpha S_{l3} + \partial_y S_{l3} = \frac{S_{l1}}{1+\frac{S_{l1}}{k}} \frac{G_{l2}*\mu_\alpha}{1+\frac{G_{l2}*\mu_\alpha}{k}} - \frac{S_{l3}}{1+\frac{S_{l3}}{k}} \frac{g_{l4}*\mu_\alpha}{1+\frac{g_{l4}*\mu_\alpha}{k}}, \tag{2.20}$$

$$\alpha S_{l4} - \partial_y S_{l4} = \frac{G_{l1}*\mu_\alpha}{1+\frac{G_{l1}*\mu_\alpha}{k}} \frac{S_{l2}}{1+\frac{S_{l2}}{k}} - \frac{g_{l3}*\mu_\alpha}{1+\frac{g_{l3}*\mu_\alpha}{k}} \frac{S_{l4}}{1+\frac{S_{l4}}{k}}, \tag{2.21}$$

$$S_{l1}(0,y) = f_{b1}(y) \wedge \frac{k}{2}, \quad S_{l2}(1,y) = f_{b2}(y) \wedge \frac{k}{2}, \quad y \in [0,1], \tag{2.22}$$

$$S_{l3}(x,0) = f_{b3}(x) \wedge \frac{k}{2}, \quad S_{l4}(x,1) = f_{b4}(x) \wedge \frac{k}{2}, \quad x \in [0,1], \tag{2.23}$$

(resp.

$$\alpha s_{l1} + \partial_x s_{l1} = \frac{s_{l3}}{1+\frac{s_{l3}}{k}} \frac{g_{l4} * \mu_\alpha}{1+\frac{g_{l4}*\mu_\alpha}{k}} - \frac{s_{l1}}{1+\frac{s_{l1}}{k}} \frac{G_{l2} * \mu_\alpha}{1+\frac{G_{l2}*\mu_\alpha}{k}}, \tag{2.24}$$

$$\alpha s_{l2} - \partial_x s_{l2} = \frac{g_{l3} * \mu_\alpha}{1+\frac{g_{l3}*\mu_\alpha}{k}} \frac{s_{l4}}{1+\frac{s_{l4}}{k}} - \frac{G_{l1} * \mu_\alpha}{1+\frac{G_{l1}*\mu_\alpha}{k}} \frac{s_{l2}}{1+\frac{s_{l2}}{k}}, \tag{2.25}$$

$$\alpha s_{l3} + \partial_y s_{l3} = \frac{s_{l1}}{1+\frac{s_{l1}}{k}} \frac{g_{l2} * \mu_\alpha}{1+\frac{g_{l2}*\mu_\alpha}{k}} - \frac{s_{l3}}{1+\frac{s_{l3}}{k}} \frac{G_{l4} * \mu_\alpha}{1+\frac{G_{l4}*\mu_\alpha}{k}}, \tag{2.26}$$

$$\alpha s_{l4} - \partial_y s_{l4} = \frac{g_{l1} * \mu_\alpha}{1+\frac{g_{l1}*\mu_\alpha}{k}} \frac{s_{l2}}{1+\frac{s_{l2}}{k}} - \frac{G_{l3} * \mu_\alpha}{1+\frac{G_{l3}*\mu_\alpha}{k}} \frac{s_{l4}}{1+\frac{s_{l4}}{k}}, \tag{2.27}$$

$$s_{l1}(0,y) = f_{b1}(y) \wedge \frac{k}{2}, \quad s_{l2}(1,y) = f_{b2}(y) \wedge \frac{k}{2}, \quad y \in [0,1], \tag{2.28}$$

$$s_{l3}(x,0) = f_{b3}(x) \wedge \frac{k}{2}, \quad s_{l4}(x,1) = f_{b4}(x) \wedge \frac{k}{2}, \quad x \in [0,1]). \tag{2.29}$$

(S_l) is a non-increasing sequence, since that holds for the successive iterates defining the sequence. Then (S_l) decreasingly converges in L^1 to some S. Similarly (s_l) increasingly converges in L^1 to some s. The limits S and s satisfy (2.8)-(2.13). It follows by uniqueness that $s = F = S$, hence that (F_l) converges in L^1 to F.
The map $\mathcal{T}$ is also compact in the L^1-norm topology. Indeed, let $(f_l)_{l\in\mathbb{N}}$ be a sequence in K_α and $(F_l)_{l\in\mathbb{N}} = (\mathcal{T}(f_l))_{l\in\mathbb{N}}$. For any $|h| < 1$, denote by $G_{l1}(x,y) = F_{l1}(x,y+h) - F_{l1}(x,y)$ and

$$\begin{aligned} H_{l1}(x,y) =& \frac{F_{l3}}{1+\frac{F_{l3}}{k}} \frac{f_{l4} * \mu_\alpha}{1+\frac{f_{l4}*\mu_\alpha}{k}}(x,y+h) - \frac{F_{l3}}{1+\frac{F_{l3}}{k}} \frac{f_{l4} * \mu_\alpha}{1+\frac{f_{l4}*\mu_\alpha}{k}}(x,y) \\ & - \frac{F_{l1}}{1+\frac{F_{l1}}{k}}(x,y+h)\Big(\frac{f_{l2} * \mu_\alpha}{1+\frac{f_{l2}*\mu_\alpha}{k}}(x,y+h) - \frac{f_{l2} * \mu_\alpha}{1+\frac{f_{l2}*\mu_\alpha}{k}}(x,y)\Big) \end{aligned}$$

They satisfy

$$\big(\alpha + \frac{f_{l2} * \mu_\alpha}{1+\frac{f_{l2}*\mu_\alpha}{k}}\big) G_{l1} + \partial_x G_{l1} = H_{l1}, \quad G_{l1}(0,\cdot) = 0,$$

so that

$$G_{l1}(x,y) = \int_0^x H_{l1}(X,y) e^{-\alpha(x-X) - \int_X^x \frac{f_{l2}*\mu_\alpha}{1+\frac{f_{l2}*\mu_\alpha}{k}}(u,y)du} dX, \quad (x,y) \in [0,1]^2.$$

The boundedness by k^2 of the integrands in the r.h.s. of (2.8) and (2.10) induces uniform L^1-equicontinuity of $(F_{l1})_{l\in\mathbb{N}}$ (resp. $(F_{l3})_{l\in\mathbb{N}}$) w.r.t. the x (resp. y) variable. Together with the L^1-compactness of $(f_l * \mu_\alpha)_{l\in\mathbb{N}}$, this implies uniform L^1-equicontinuity w.r.t. the y variable of $(H_{l1})_{l\in\mathbb{N}}$, then of $(F_{l1})_{l\in\mathbb{N}}$. This proves the L^1 compactness of $(F_{l1})_{l\in\mathbb{N}}$. The L^1 compactness of $(F_{li})_{l\in\mathbb{N}}$, $2 \le i \le 4$ can be proven similarly.

Hence by the Schauder fixed point theorem there is a fixed point $\mathcal{T}(F) = F$, i.e. a solution F to

$$\alpha F_1 + \partial_x F_1 = \frac{F_3}{1+\frac{F_3}{k}} \frac{F_4 * \mu_\alpha}{1+\frac{F_4 * \mu_\alpha}{k}} - \frac{F_1}{1+\frac{F_1}{k}} \frac{F_2 * \mu_\alpha}{1+\frac{F_2 * \mu_\alpha}{k}}, \tag{2.30}$$

$$\alpha F_2 - \partial_x F_2 = \frac{F_3 * \mu_\alpha}{1+\frac{F_3 * \mu_\alpha}{k}} \frac{F_4}{1+\frac{F_4}{k}} - \frac{F_1 * \mu_\alpha}{1+\frac{F_1 * \mu_\alpha}{k}} \frac{F_2}{1+\frac{F_2}{k}}, \tag{2.31}$$

$$\alpha F_3 + \partial_y F_3 = \frac{F_1}{1+\frac{F_1}{k}} \frac{F_2 * \mu_\alpha}{1+\frac{F_2 * \mu_\alpha}{k}} - \frac{F_3}{1+\frac{F_3}{k}} \frac{F_4 * \mu_\alpha}{1+\frac{F_4 * \mu_\alpha}{k}}, \tag{2.32}$$

$$\alpha F_4 - \partial_y F_4 = \frac{F_1 * \mu_\alpha}{1+\frac{F_1 * \mu_\alpha}{k}} \frac{F_2}{1+\frac{F_2}{k}} - \frac{F_3 * \mu_\alpha}{1+\frac{F_3 * \mu_\alpha}{k}} \frac{F_4}{1+\frac{F_4}{k}}, \quad (x,y) \in [0,1]^2 \tag{2.33}$$

$$F_1(0,y) = f_{b1}(y) \wedge \frac{k}{2}, \quad F_2(1,y) = f_{b2}(y) \wedge \frac{k}{2}, \quad y \in [0,1], \tag{2.34}$$

$$F_3(x,0) = f_{b3}(x) \wedge \frac{k}{2}, \quad F_4(x,1) = f_{b4}(x) \wedge \frac{k}{2}, \quad x \in [0,1]. \tag{2.35}$$

Step II. Removal of the damping and the convolutions in (2.30)-(2.35).

Let $k > 1$ be fixed. Denote by F^α the solution to (2.30)-(2.35) defined in Step I. Each component of F^α being bounded by a multiple of k^2, $(F^\alpha)_{\alpha \in]0,1[}$ is weakly compact in $L^1([0,1]^2)$. Denote by F^k a limit of a subsequence for the weak topology of $L^1([0,1]^2)$. Let us prove that the convergence is strong in $L^1([0,1]^2)$. Consider the approximation scheme $(f_1^{\alpha,l}, f_2^{\alpha,l})_{l \in \mathbb{N}}$ of (F_1^α, F_2^α),

$$\begin{aligned}
&f_1^{\alpha,0} = f_2^{\alpha,0} = 0,\\
&\alpha f_1^{\alpha,l+1} + \partial_x f_1^{\alpha,l+1} = \frac{F_3^\alpha}{1+\frac{F_3^\alpha}{k}} \frac{F_4^\alpha * \mu_\alpha}{1+\frac{F_4^\alpha * \mu_\alpha}{k}} - \frac{f_1^{\alpha,l+1}}{1+\frac{f_1^{\alpha,l+1}}{k}} \frac{f_2^{\alpha,l} * \mu_\alpha}{1+\frac{f_2^{\alpha,l} * \mu_\alpha}{k}}, \quad f_1^{\alpha,l+1}(0,y) = f_{b1}(y) \wedge \frac{k}{2},\\
&\alpha f_2^{\alpha,l+1} - \partial_x f_2^{\alpha,l+1} = \frac{F_3^\alpha}{1+\frac{F_3^\alpha}{k}} \frac{F_4^\alpha * \mu_\alpha}{1+\frac{F_4^\alpha * \mu_\alpha}{k}} - \frac{f_1^{\alpha,l} * \mu_\alpha}{1+\frac{f_1^{\alpha,l} * \mu_\alpha}{k}} \frac{f_2^{\alpha,l+1}}{1+\frac{f_2^{\alpha,l+1}}{k}}, \quad f_2^{\alpha,l+1}(1,y) = f_{b2}(y) \wedge \frac{k}{2},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad l \in \mathbb{N}.
\end{aligned} \tag{2.36}$$

By induction on l it holds that

$$\begin{aligned}
&f_1^{\alpha,2l} \le f_1^{\alpha,2l+2} \le F_1^\alpha \le f_1^{\alpha,2l+3} \le f_1^{\alpha,2l+1},\\
&f_2^{\alpha,2l} \le f_2^{\alpha,2l+2} \le F_2^\alpha \le f_2^{\alpha,2l+3} \le f_2^{\alpha,2l+1}, \quad \alpha \in]0,1[, \quad l \in \mathbb{N}.
\end{aligned} \tag{2.37}$$

For every $l \in \mathbb{N}$, $(f_1^{\alpha,l})_{\alpha \in]0,1[}$ (resp. $(f_2^{\alpha,l})_{\alpha \in]0,1[}$) is translationnaly equicontinuous in the x-direction, since all integrands in its exponential form are bounded. It is translationnaly L^1-equicontinuous in the y-direction by induction on l. Indeed, it is so for (F_3^α) (resp. (F_4^α)) since $\partial_y(e^{\alpha y} F_3^\alpha)$ (resp. $\partial_y(e^{\alpha y} F_4^\alpha)$) is bounded by ek^2, and $(\frac{F_i^\alpha}{1+\frac{F_i^\alpha}{k}})_{\alpha \in]0,1[}$, $i \in \{3,4\}$, is bounded by k. Consequently, it is so for $(\frac{F_3^\alpha}{1+\frac{F_3^\alpha}{k}} \frac{F_4^\alpha * \mu_\alpha}{1+\frac{F_4^\alpha * \mu_\alpha}{k}})_{\alpha \in]0,1[}$. There is a limit sequence (g_1^l, g_2^l) in $(L^1([0,1]^2))^2$ such that up to

subsequences $(f_1^{\alpha,l})$ (resp. $(f_2^{\alpha,l})$) converges to g_1^l (resp. g_2^l) in $L^1([0,1]^2)$ when $\alpha \to 0$. They satisfy

$$
\begin{aligned}
&0 \le g_1^{2l} \le g_1^{2l+2} \le F_1^k \le g_1^{2l+3} \le g_1^{2l+1},\\
&0 \le g_2^{2l} \le g_2^{2l+2} \le F_2^k \le g_2^{2l+3} \le g_2^{2l+1}, \quad l \in \mathbb{N},\\
&\partial_x g_1^{2l+1} = G - \frac{g_1^{2l+1}}{1+\frac{g_1^{2l+1}}{k}} \frac{g_2^{2l}}{1+\frac{g_2^{2l}}{k}}, \quad \partial_x g_1^{2l} = G - \frac{g_1^{2l}}{1+\frac{g_1^{2l}}{k}} \frac{g_2^{2l-1}}{1+\frac{g_2^{2l-1}}{k}},\\
&-\partial_x g_2^{2l+1} = G - \frac{g_1^{2l}}{1+\frac{g_1^{2l}}{k}} \frac{g_2^{2l+1}}{1+\frac{g_2^{2l+1}}{k}}, \quad -\partial_x g_2^{2l} = G - \frac{g_1^{2l-1}}{1+\frac{g_1^{2l-1}}{k}} \frac{g_2^{2l}}{1+\frac{g_2^{2l}}{k}},\\
&g_1^l(0,y) = f_{b1}(y) \wedge \frac{k}{2}, \quad g_2^l(1,y) = f_{b2}(y) \wedge \frac{k}{2}, \quad y \in [0,1],
\end{aligned}
$$

where G is the weak L^1 limit of $(\frac{F_3^\alpha}{1+\frac{F_3^\alpha}{k}} \frac{F_4^\alpha * \mu_\alpha}{1+\frac{F_4^\alpha * \mu_\alpha}{k}})_{\alpha \in]0,1[}$ when $\alpha \to 0$. In particular, $(g_1^{2l})_{l\in\mathbb{N}}$ and $(g_2^{2l})_{l\in\mathbb{N}}$ (resp $(g_1^{2l+1})_{l\in\mathbb{N}}$ and $(g_2^{2l+1})_{l\in\mathbb{N}}$) non decreasingly (resp. non increasingly) converge in L^1 to some g_1 and g_2 (resp. h_1 and h_2) when $l \to +\infty$. The limits satisfy

$$
\begin{aligned}
&0 \le g_1 \le F_1^k \le h_1, \quad 0 \le g_2 \le F_2^k \le h_2,\\
&\partial_x h_1 = G - \frac{h_1}{1+\frac{h_1}{k}} \frac{g_2}{1+\frac{g_2}{k}}, \quad \partial_x g_1 = G - \frac{g_1}{1+\frac{g_1}{k}} \frac{h_2}{1+\frac{h_2}{k}},\\
&-\partial_x h_2 = G - \frac{g_1}{1+\frac{g_1}{k}} \frac{h_2}{1+\frac{h_2}{k}}, \quad -\partial_x g_2 = G - \frac{h_1}{1+\frac{h_1}{k}} \frac{g_2}{1+\frac{g_2}{k}},\\
&(h_1-g_1)(0,y) = 0, \quad (h_2-g_2)(1,y) = 0, \quad y \in [0,1].
\end{aligned}
$$

Hence,

$$
(h_2-g_2)(x,y) = (h_1-g_1)(x,y) - (h_1-g_1)(1,y), \quad (x,y) \in [0,1]^2,
$$

and

$$
\begin{aligned}
(h_1-g_1)(x,y) = &-(h_1-g_1)(1,y) \int_0^x \frac{h_1}{(1+\frac{h_1}{k})(1+\frac{g_2}{k})(1+\frac{h_2}{k})}(X,y)\\
&\exp\Big(-\int_X^x \frac{h_2(1+\frac{g_2}{k}) - h_1(1+\frac{g_1}{k})}{(1+\frac{g_1}{k})(1+\frac{h_1}{k})(1+\frac{g_2}{k})(1+\frac{h_2}{k})}(r,y)dr\Big)dX.
\end{aligned}
$$

The non negativity of h_1-g_1, g_1, g_2, h_1 and h_2 implies that $h_1-g_1=0$. The same holds for h_2-g_2. Consequently

$$
g_1 = h_1 = F_1^k, \quad g_2 = h_2 = F_2^k.
$$

$(F_1^\alpha)_{\alpha\in]0,1[}$ converges to F_1^k in $L^1([0,1]^2)$ when $\alpha \to 0$. Indeed, given $\eta > 0$, choose l_0 big enough so that $\| g_1^{2l_0+1} - g_1^{2l_0} \|_{L^1} < \eta$ and $\| g_1^{2l_0} - F_1^k \|_{L^1} < \eta$, then α_0 small enough so that

$$
\| f_1^{\alpha,2l_0+1} - g_1^{2l_0+1} \|_{L^1} \le \eta \quad \text{and} \quad \| f_1^{\alpha,2l_0} - g_1^{2l_0} \|_{L^1} \le \eta, \quad \alpha \in]0,\alpha_0[.
$$

Then split $\| F_1^\alpha - F_1^k \|_{L^1}$ as follows,

$$
\begin{aligned}
\| F_1^\alpha - F_1^k \|_{L^1} \le &\| F_1^\alpha - f_1^{\alpha,2l_0} \|_{L^1} + \| f_1^{\alpha,2l_0} - g_1^{2l_0} \|_{L^1} + \| g_1^{2l_0} - F_1^k \|_{L^1}\\
\le &\| f_1^{\alpha,2l_0+1} - f_1^{\alpha,2l_0} \|_{L^1} + 2\eta \quad \text{by} \quad (2.37)\\
\le &\| f_1^{\alpha,2l_0+1} - g_1^{2l_0+1} \|_{L^1} + \| g_1^{2l_0+1} - g_1^{2l_0} \|_{L^1} + \| g_1^{2l_0} - f_1^{\alpha,2l_0} \|_{L^1} + 2\eta\\
\le &\, 5\eta, \quad \alpha \in]0,\alpha_0[.
\end{aligned}
$$

The L^1 convergence of $(F_i^\alpha)_{k\in\mathbb{N}}$ to F_i^k, $2\le i\le 4$, can be proven similarly. Passing to the limit when $\alpha\to 0$ in (2.30)-(2.35) is straightforward. And so, F^k is a non negative solution to (2.1)-(2.6). ■

3 Passage to the limit when $k\to+\infty$.

The study of the passage to the limit when $k\to+\infty$ in (2.1)-(2.6) is split into six lemmas. In Lemma 3.1, uniform bounds are obtained for mass, entropy and entropy production term of the approximations. Lemma 3.2 splits $[0,1]^2$ into 'large' sets of type $0\le x\le 1$ times a 'large' set in y for (F_1^k,F_2^k) (resp. a 'large' set in x times $0\le y\le 1$ for (F_3^k,F_4^k)), where the approximations are uniformly bounded in L^∞, and their complements where the mass of the approximations is small. Lemma 3.3 proves uniform equicontinuity with respect to the x (resp. y) variable of the two first (resp. last) components of the approximations. In Lemma 3.4, L^1-compactness of a truncated gain term of the approximations is proven. Lemma 3.5 proves that the approximations form a Cauchy sequence in $L^1([0,1]^2)$. Their limit is proven to be a renormalized solution to the Broadwell model in Lemma 3.6 .
In this section, c_b denotes constants that only depend on the given boundary value f_b.

Lemma 3.1
There are constants c_b such that

$$\int F_i^k(x,y)dxdy\le c_b, \tag{3.1}$$

$$\int_{F_i^k(x,y)>k} F_i^k(x,y)dxdy\le \frac{c_b}{\ln k},\quad i\in\{1,\cdots,4\}, \tag{3.2}$$

$$\int\Big(\frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}}-\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}\Big)\ln\frac{F_1^kF_2^k(1+\frac{F_3^k}{k})(1+\frac{F_4^k}{k})}{(1+\frac{F_1^k}{k})(1+\frac{F_2^k}{k})F_3^kF_4^k}(x,y)dxdy\le c_b,\quad k>2. \tag{3.3}$$

Proof of Lemma 3.1.
Adding (2.1)-(2.4), integrating the resulting equation on $[0,1]^2$ and taking (2.5)-(2.6) into account, implies that total outflow equals total inflow. Also using $\partial_x(F_1^k+F_2^k)=\partial_y(F_3^k+F_4^k)=0$ implies boundedness of the total mass $\sum_{i=1}^4\int F_i^k(x,y)dxdy$. Multiply (2.1) (resp. (2.2), resp. (2.3), resp. (2.4)) by $\ln\frac{F_1^k}{1+\frac{F_1^k}{k}}$ (resp. $\ln\frac{F_2^k}{1+\frac{F_2^k}{k}}$, resp. $\ln\frac{F_3^k}{1+\frac{F_3^k}{k}}$, resp. $\ln\frac{F_4^k}{1+\frac{F_1^k}{4}}$), add the corresponding equations, and integrate the resulting equation on $[0,1]^2$. Denoting by D^k the entropy production term for the approximation F^k,

$$D^k=\int\Big(\frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}}-\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}\Big)\ln\frac{F_1^kF_2^k(1+\frac{F_3^k}{k})(1+\frac{F_4^k}{k})}{(1+\frac{F_1^k}{k})(1+\frac{F_2^k}{k})F_3^kF_4^k}(x,y)dxdy,$$

leads to

$$\int\Big(F_1^k\ln F_1^k-k(1+\frac{F_1^k}{k})\ln(1+\frac{F_1^k}{k})\Big)(1,y)dy+\int\Big(F_2^k\ln F_2^k-k(1+\frac{F_2^k}{k})\ln(1+\frac{F_2^k}{k})\Big)(0,y)dy$$
$$+\int\Big(F_3^k\ln F_3^k-k(1+\frac{F_3^k}{k})\ln(1+\frac{F_3^k}{k})\Big)(x,1)dx+\int\Big(F_4^k\ln F_4^k-k(1+\frac{F_4^k}{k})\ln(1+\frac{F_4^k}{k})\Big)(x,0)dx$$
$$+D^k\le c_b.$$

Moreover,

$$k\int \ln(1+\frac{F_i^k}{k}) \le \int F_i^k \le c_b, \quad 1 \le i \le 4.$$

Hence

$$\int \Big(F_1^k \ln \frac{F_1^k}{1+\frac{F_1^k}{k}}(1,y) + F_2^k \ln \frac{F_2^k}{1+\frac{F_2^k}{k}}(0,y)\Big)dy + \int \Big(F_3^k \ln \frac{F_3^k}{1+\frac{F_3^k}{k}}(x,1) + F_4^k \ln \frac{F_4^k}{1+\frac{F^k 42}{k}}(x,0)\Big)dx$$
$$+ D^k \le c_b.$$

Consequently,

$$\int_{F_1^k(1,y)>\frac{k}{k-1}} F_1^k \ln \frac{F_1^k}{1+\frac{F_1^k}{k}}(1,y)dy + \int_{F_2^k(0,y)>\frac{k}{k-1}} F_2^k \ln \frac{F_2^k}{1+\frac{F_2^k}{k}}(0,y)dy$$
$$+ \int_{F_3^k(x,1)>\frac{k}{k-1}} F_3^k \ln \frac{F_3^k}{1+\frac{F_3^k}{k}}(x,1)dx + \int_{F_4^k(x,0)>\frac{k}{k-1}} F_4^k \ln \frac{F_4^k}{1+\frac{F_4^k}{k}}(x,0)dx + D_k$$
$$\le c_b, \quad k > 2.$$

And so, (3.3) holds. Moreover, for any $\Lambda > 2$ and $k > 2$,

$$\ln \frac{\Lambda}{1+\frac{\Lambda}{k}}\Big(\int_{F_1^k(1,y)>k} F_1^k(1,y)dy + \int_{F_2^k(0,y)>k} F_2^k(0,y)dy$$
$$+ \int_{F_3^k(x,1)>k} F_3^k(x,1)dx + \int_{F_4^k(x,0)>k} F_4^k(x,0)dx\Big)$$
$$\le c_b + \int_{F_1^k(1,y)<\frac{k}{k-1}} F_1^k \mid \ln \frac{F_1^k}{1+\frac{F_1^k}{k}} \mid (1,y)dy + \int_{F_2^k(0,y)<\frac{k}{k-1}} F_2^k \mid \ln \frac{F_2^k}{1+\frac{F_2^k}{k}} \mid (0,y)dy$$
$$+ \int_{F_3^k(x,1)<\frac{k}{k-1}} F_3^k \mid \ln \frac{F_3^k}{1+\frac{F_3^k}{k}} \mid (x,1)dx + \int_{F_4^k(x,0)<\frac{k}{k-1}} F_4^k \mid \ln \frac{F_4^k}{1+\frac{F_4^k}{k}} \mid (x,0)dx$$
$$\le c_b + 2, \quad k > 2. \tag{3.4}$$

In particular,

$$\int_{F_1^k(1,y)>k} F_1^k(1,y)dy + \int_{F_2^k(0,y)>k} F_2^k(0,y)dy$$
$$+ \int_{F_3^k(x,1)>k} F_3^k(x,1)dx + \int_{F_4^k(x,0)>k} F_4^k(x,0)dx \le \frac{c_b}{\ln k}, \quad k > 2. \tag{3.5}$$

Since

$$(F_1^k + F_2^k)(x,y) = F_1^k(1,y) + f_{b2}(y) \wedge \frac{k}{2}, \quad (x,y) \in [0,1]^2, \tag{3.6}$$

it holds that

$$F_1^k(x,y) > k \Rightarrow F^k(1,y) > \frac{k}{2}, \quad (x,y) \in [0,1]^2.$$

Consequently, for some subset ω_k of $[0,1]$ such that $|\omega_k| < \frac{c}{k}$,

$$\begin{aligned}\int_{F_1^k(x,y)>k} F_1^k(x,y)dxdy &\le \int_{F_1^k(1,y)>\frac{k}{2}} F_1^k(1,y)dy + \int_{\omega_k} f_{b2}(y)dy \\ &\le \frac{c}{\ln k},\end{aligned}$$

by (3.4) and the boundedness of the f_{b2} entropy. ■

Lemma 3.2
For $\epsilon > 0$, $\Lambda \ge \exp(\frac{2c_b}{\epsilon})$ and $k \ge \exp(\frac{3c_b}{\epsilon})$, there is a subset $\Omega_{k1}^{\epsilon\Lambda}$ of $[0,1]$ with measure smaller than $\frac{c_b\epsilon}{\Lambda}$ such that

$$F_1^k(x,y) \le \frac{\Lambda}{\epsilon}\exp(\frac{2\Lambda}{\epsilon}), \quad F_2^k(x,y) \le \frac{2\Lambda}{\epsilon}\exp(\frac{2\Lambda}{\epsilon}), \quad x \in [0,1], \quad y \in [0,1]\setminus\Omega_{k1}^{\epsilon\Lambda}, \tag{3.7}$$

$$\int_0^1 \Big(\int_{\Omega_{k1}^{\epsilon\Lambda}} (F_1^k + F_2^k)(x,y)dy\Big)dx \le c_b\epsilon. \tag{3.8}$$

Proof of Lemma 3.2.
Since $f_{b2} \in L^1([0,1])$ and

$$\int_0^1 (F_1^k(1,y) + F_2^k(0,y))dy + \int_0^1 (F_3^k(x,1) + F_4^k(x,0))dx \le c_b,$$

the measure of the set

$$\Omega_{k1}^{\epsilon\Lambda} := \{y \in [0,1]; f_{b2}(y) \ge \frac{\Lambda}{\epsilon} \text{ or } F_1^k(1,y) \ge \frac{\Lambda}{\epsilon}\}, \tag{3.9}$$

is smaller than $\frac{c_b\epsilon}{\Lambda}$. (F_1^k, F_2^k) is uniformly bounded on $[0,1]\times([0,1]\setminus\Omega_{k1}^{\epsilon\Lambda})$, since

$$\begin{aligned}F_1^k(x,y) &\le F_1^k(1,y)\exp(\int_0^1 F_2^k(X,y)dX) \\ &\le F_1^k(1,y)\exp(F_1^k(1,y) + f_{b2}(y)) \quad \text{by} \quad (3.6) \\ &\le \frac{\Lambda}{\epsilon}\exp(\frac{2\Lambda}{\epsilon}),\end{aligned}$$

and

$$\begin{aligned}F_2^k(x,y) &\le F_2^k(0,y)\exp(\int_0^1 F_1^k(X,y)dX) \\ &\le (F_1^k(1,y) + f_{b2}(y))\exp(F_1^k(1,y) + f_{b2}(y)) \\ &\le \frac{2\Lambda}{\epsilon}\exp(\frac{2\Lambda}{\epsilon}), \quad x \in [0,1], \quad y \in [0,1]\setminus\Omega_{k1}^{\epsilon\Lambda}.\end{aligned}$$

Moreover, for any $\Lambda \geq \exp(\frac{2c_b}{\epsilon})$ and $k \geq \exp(\frac{3c_b}{\epsilon})$,

$$\int_0^1 \Big(\int_{\Omega_{k1}^{\epsilon\Lambda}} (F_1^k + F_2^k)(x,y)dy \Big) dx = \int_{\Omega_{k1}^{\epsilon\Lambda}} (F_1^k(1,y) + f_{b2}(y))dy$$
$$\leq \int_{y\in\Omega_{k1}^{\epsilon\Lambda}; F_1^k(1,y)<\Lambda} F_1^k(1,y)dy + \int_{F_1^k(1,y)>\Lambda} F_1^k(1,y)dy$$
$$+ \int_{y\in\Omega_{k1}^{\epsilon\Lambda}, f_{b2}(y)<\Lambda} f_{b2}(y)dy + \int_{f_{b2}(y)>\Lambda} f_{b2}(y)dy$$
$$\leq 2\Lambda|\Omega_{k1}^{\epsilon\Lambda}| + \frac{c_b}{\ln \frac{\Lambda}{1+\frac{\Lambda}{k}}} + \frac{c_b}{\ln \Lambda} \quad \text{by (3.4) and the boundedness of the entropy of } f_{b2}$$
$$\leq c_b\epsilon.$$

■

Lemma 3.3
There is $c_b > 0$, and for $\epsilon > 0$ given there is $\delta > 0$ such that for $|h| < \delta$, uniformly in $k \in \mathbb{N}^*$,

$$\int_{[0,1]^2} |F_i^k(x+h,y) - F_i^k(x,y)|dxdy \leq c_b\epsilon, \quad i \in \{1,2\},$$
$$\int_{[0,1]^2} |F_i^k(x,y+h) - F_i^k(x,y)|dxdy \leq c_b\epsilon, \quad i \in \{3,4\}. \tag{3.10}$$

Proof of Lemma 3.3.
The four cases F_1^k,..., F_4^k are analogous. The detailed estimates are carried out for F_1^k. The translational L^1 equicontinuity in the x-direction for $\ln(1+F_1^k)$ is obtained as follows from the ∂_x-term in the renormalized equation. Consider $h \in [0,1[$. Write the equation for F_1^k in renormalized form (1.4) integrated on $[x, x+h]$, where the integration from $x+h > 1$ tending to zero with h uniformly in k, is being omitted from the following computations;

$$\ln(1+F_1^k(x+h,y)) - \ln(1+F_1^k(x,y))$$
$$= \int_x^{x+h} \frac{1}{1+F_1^k} \Big(\frac{F_3^k}{1+\frac{F_3^k}{k}} \frac{F_4^k}{1+\frac{F_4^k}{k}} - \frac{F_1^k}{1+\frac{F_1^k}{k}} \frac{F_2^k}{1+\frac{F_2^k}{k}} \Big)(X,y)dX. \tag{3.11}$$

Denote by sgn the sign function,

$$\text{sgn(r)} = 1 \text{ if r} > 0, \quad \text{sgn(r)} = -1 \text{ if r} < 0.$$

Multiply the previous equation by $\mathrm{sgn}\big(\ln(1+\mathrm{F}_1^k(\mathrm{x}+\mathrm{h},\mathrm{y}))-\ln(1+\mathrm{F}_1^k(\mathrm{x},\mathrm{y}))\big)$ and integrate on $[0,1]^2$. Uniformly w.r.t. $k\in\mathbb{N}^*$,

$$
\begin{aligned}
&\int |\ln(1+F_1^k(x+h,y))-\ln(1+F_1^k(x,y))|dxdy\\
&\le h\int_{[0,1]^2}\frac{1}{1+F_1^k}\Big|\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}-\frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}}\Big|(X,y)dXdy\\
&\le h\Big(\int_{\frac{\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}<\frac{\frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}}}{}}\frac{F_1^k}{(1+F_1^k)(1+\frac{F_1^k}{k})}\frac{F_2^k}{1+\frac{F_2^k}{k}}(X,y)dXdy\\
&\quad+\int_{\frac{\frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}}<\frac{\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}<2\frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}}}{}}\frac{F_3^k}{(1+F_1^k)(1+\frac{F_3^k}{k})}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dXdy\\
&\quad+\int_{\frac{\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}>2\frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}}}{}}\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dXdy\Big)\\
&\le h\Big(3\int F_2^k(X,y)dXdy+\frac{2}{\ln 2}D^k\Big)\\
&\le c_b h. \qquad (3.12)
\end{aligned}
$$

Recall that for any non negative real numbers $x_1>x_2$, there is $\theta\in]0,1[$ such that

$$
\begin{aligned}
x_1-x_2&=\exp(\ln(1+x_1))-\exp(\ln(1+x_2))\\
&=\exp\big(\theta\ln(1+x_1)+(1-\theta)\ln(1+x_2)\big)\big(\ln(1+x_1)-\ln(1+x_2)\big).
\end{aligned}
$$

And so the L^1-norms of the translation differences of F_1^k and $\ln(1+F_1^k)$, are equivalent on $[0,1]\times([0,1]\setminus\Omega_{k1}^{\epsilon\Lambda})$ since F_1^k and $(x,y)\to F_1^k(x+h,y)$ are bounded in $L^\infty([0,1]\times([0,1]\setminus\Omega_{k1}^{\epsilon\Lambda}))$. There is also the small set $[0,1]\times\Omega_{k1}^{\epsilon\Lambda}$ with masses of F_1^k and $F_1^k(\cdot+h,\cdot)$ bounded by $c\epsilon$. Together with (3.12) this proves the translational equicontinuity in the x-direction for $k\ge\exp(\frac{3c_b}{\epsilon})$. The proof for $h\in]-1,0[$ is similar. ∎

Given $\epsilon>0$, $\Lambda\ge\exp(\frac{2c_b}{\epsilon})$ and $k\ge\exp(\frac{3c_b}{\epsilon})$, let $\Omega_{k1}^{\epsilon\Lambda}\subset[0,1]$ as defined in Lemma 3.2, and take $\chi_{k1}^{\epsilon\Lambda}$ as the corresponding cutoff function,

$$
\chi_{k1}^{\epsilon\Lambda}(y)=1 \text{ if } \ y\notin\Omega_{k1}^{\epsilon\Lambda},\qquad \chi_{k1}^{\epsilon\Lambda}(y)=0 \text{ if } \ y\in\Omega_{k1}^{\epsilon\Lambda}.
$$

Lemma 3.4
Let $(\alpha^k)_{k\in\mathbb{N}}$ be a non negative sequence bounded in L^∞ and compact in L^1 . The sequences

$$
\Big(\chi_{k1}^{\epsilon\Lambda}(y)\int_0^x\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)e^{-\int_X^x\alpha^k(u,y)du}dX\Big)_{k\in\mathbb{N}^*}
$$

and $\Big(\chi_{k1}^{\epsilon\Lambda}(y)\int_x^1\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)e^{-\int_x^X\alpha^k(u,y)du}dX\Big)_{k\in\mathbb{N}^*}$,

$$
\Big(\textit{resp. } \big(\chi_{k1}^{\epsilon\Lambda}(y)\int_0^1\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dX\big)_{k\in\mathbb{N}^*}\Big),
$$

are compact in $L^1([0,1]^2)$ (resp. in $L^1([0,1])$).

Proof of Lemma 3.4. For any $\gamma > 1$, using Lemmas 3.1-3.2,

$$\begin{aligned}
&\int \chi_{k1}^{\epsilon\Lambda}(y)\Big|\int_0^{x+h} \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)e^{-\int_X^{x+h}\alpha^k(u,y)du}dX \\
&\qquad - \int_0^{x} \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)e^{-\int_X^{x}\alpha^k(u,y)du}dX\Big|dxdy \\
&\le \int \chi_{k1}^{\epsilon\Lambda}(y)\Big|\int_x^{x+h} \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dX\Big|dxdy \\
&\qquad + \int \chi_{k1}^{\epsilon\Lambda}(y)\int_0^{x} \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dX\Big|\int_x^{x+h}\alpha^k(u,y)du\Big|dxdy \\
&\le \frac{c_b}{\ln\gamma} + \gamma h\int \chi_{k1}^{\epsilon\Lambda}(y)F_1^kF_2^k(x,y)dxdy \\
&\le \frac{c_b}{\ln\gamma} + 2\gamma h\big(\frac{\Lambda}{\epsilon}\big)^2 e^{\frac{4\Lambda}{\epsilon}}.
\end{aligned}$$

Choosing γ big enough, then h small enough, proves the translational L^1 equicontinuity in the x direction of $\Big(\chi_{k1}^{\epsilon\Lambda}(y)\int_0^x \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)e^{-\int_X^x\alpha^k(u,y)du}dX\Big)_{k\in\mathbb{N}^*}$. Let us prove its translational L^1 equicontinuity in the y direction. Given $\tilde\epsilon > 0$, let

$$\gamma > \exp(\frac{3c_b}{\tilde\epsilon}), \quad \epsilon_3 < \frac{\tilde\epsilon}{6c_b\gamma}\big(\frac{\epsilon}{\Lambda}\big)^2 e^{-\frac{4\Lambda}{\epsilon}}, \quad \Lambda_3 \ge \exp(\frac{2c_b}{\epsilon_3}). \tag{3.13}$$

Let $\Omega_{k3}^{\epsilon_3\Lambda_3} \subset [0,1]$ as defined in Lemma 3.2 for (F_3^k, F_4^k), and $\chi_{k3}^{\epsilon_3\Lambda_3}$ the corresponding cutoff function,

$$\chi_{k3}^{\epsilon_3\Lambda_3}(x) = 1 \text{ if } x \notin \Omega_{k3}^{\epsilon_3\Lambda_3}, \qquad \chi_{k3}^{\epsilon\Lambda}(x) = 0 \text{ if } x \in \Omega_{k3}^{\epsilon_3\Lambda_3}.$$

First,

$$\int \Big(\int_{X\in[0,x];\frac{\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)>\gamma\frac{\frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}}(X,y)}} \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dX\Big)dxdy \le \frac{c_b}{\ln\gamma} \le \frac{\tilde\epsilon}{3}.$$

Moreover,

$$\begin{aligned}
&\int \chi_{k1}^{\epsilon\Lambda}(y)\int_{X\in[0,x];\frac{\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)<\gamma\frac{\frac{F_1^k}{1+\frac{F_1^k}{k}}\frac{F_2^k}{1+\frac{F_2^k}{k}}(X,y)} (1-\chi_{k3}^{\epsilon_3\Lambda_3}(X))\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dXdxdy \\
&\le 2c_b\gamma\Big(\frac{\Lambda}{\epsilon}\Big)^2 e^{\frac{4\Lambda}{\epsilon}}\epsilon_3 \\
&\le \frac{\tilde\epsilon}{3}, \quad \text{by definition of } \epsilon_3.
\end{aligned}$$

Given the boundedness of $(F_3^k, F_4^k)_{k\ge\exp(\frac{3c_b}{\epsilon_3})}$ on $\big(\Omega_{k3}^{\epsilon_3\Lambda_3}\big)^c \times [0,1]$, and the statements of Lemmas 3.2-3.3 for (F_3^k, F_4^k), there is $h_3 > 0$ such that

$$\int\int_0^x \chi_{k3}^{\epsilon_3\Lambda_3}(X)\Big|\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y+h) - \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)\Big|dXdxdy \le \frac{\tilde\epsilon}{3},$$

for $h \in]0, h_3[$, uniformly with respect to $k \geq \exp(\frac{3c_b}{\epsilon_3})$.
The proofs of the $L^1([0,1]^2)$ (resp. $L^1([0,1])$) compactness of

$$\Big(\chi_{k1}^{\epsilon\Lambda}(y)\int_x^1 \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)e^{-\int_x^X \alpha^k(u,y)du}dX\Big)_{k\in\mathbb{N}^*},$$

$$\Big(\text{resp.}\big(\chi_{k1}^{\epsilon\Lambda}(y)\int_0^1 \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dX\big)_{k\in\mathbb{N}^*}\Big)$$

are similar. ∎

Lemma 3.5
$(F^k)_{k\in\mathbb{N}^*}$ is compact in $L^1([0,1]^2)$.

Proof of Lemma 3.5.
By (3.1)-(3.2), $(F^k)_{k\in\mathbb{N}^*}$ is weakly compact in $(L^1([0,1]^2))^4$. Denote by F the weak limit of a subsequence, still denoted by (F^k). Let us prove that $(F_1^k)_{k\in\mathbb{N}^*}$ is strongly compact in $L^1([0,1]^2)$. It is by (3.8) enough to prove that up to a subsequence, given $\epsilon > 0$, for $\Lambda \geq e^{\frac{2c_b}{\epsilon}}$, $k \geq e^{\frac{3c_b}{\epsilon}}$ and $\Omega_{k1}^{\epsilon\Lambda}$ as defined in Lemma 3.2, $(\chi_{k1}^{\epsilon\Lambda}F_1^k)_{k\in\mathbb{N}^*}$ is strongly compact in $L^1([0,1]^2)$. For every F^k in the subsequence, consider the approximation scheme $(f_1^{k,l}, f_2^{k,l})_{l\in\mathbb{N}}$ of (F_1^k, F_2^k), defined by

$$f_1^{k,-1} = f_2^{k,-1} = f_1^{k,0} = f_2^{k,0} = 0,$$

$$f_1^{k,l+1}(x,y) = f_{b1}(y) + \int_0^x \big(\chi_{k1}^{\epsilon\Lambda}(y)\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}} - \frac{f_1^{k,l+1}}{1+\frac{f_1^{k,l-1}}{k}}\frac{f_2^{k,l}}{1+\frac{f_2^{k,l}}{k}}\big)(X,y)dX, \tag{3.14}$$

$$f_2^{k,l+1}(x,y) = f_{b2}(y) + \int_x^1 \big(\chi_{k1}^{\epsilon\Lambda}(y)\frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}} - \frac{f_1^{k,l}}{1+\frac{f_1^{k,l}}{k}}\frac{f_2^{k,l+1}}{1+\frac{f_2^{k,l-1}}{k}}\big)(X,y)dX. \tag{3.15}$$

By induction on l, and using an exponential form of $(f_1^{k,l+1}, f_2^{k,l+1})$, it holds that

$$f_1^{k,2l} \leq f_1^{k,2l+2}, \quad f_1^{k,2l+3} \leq f_1^{k,2l+1},$$
$$f_2^{k,2l} \leq f_2^{k,2l+2}, \quad f_2^{k,2l+3} \leq f_2^{k,2l+1}, \quad (x,y) \in [0,1]^2, \quad k \in \mathbb{N}^*, \quad l \in \mathbb{N}, \tag{3.16}$$

and

$$f_1^{k,2l} \leq F_1^k \leq f_1^{k,2l+1}, \quad f_2^{k,2l} \leq F_2^k \leq f_2^{k,2l+1}, \quad (x,y) \in [0,1]\times(\Omega_{k1}^{\epsilon\Lambda})^c, \quad k \in \mathbb{N}^*, \quad l \in \mathbb{N}. \tag{3.17}$$

The sequence $(\chi_{k1}^{\epsilon\Lambda}f_1^{k,2l})_{k\geq e^{\frac{3c_b}{\epsilon}}}$ (resp. $(\chi_{k1}^{\epsilon\Lambda}f_2^{k,2l})_{k\geq e^{\frac{3c_b}{\epsilon}}}$) is bounded from above by $(\chi_{k1}^{\epsilon\Lambda}F_1^k)_{k\geq e^{\frac{3c_b}{\epsilon}}}$ (resp. $(\chi_{k1}^{\epsilon\Lambda}F_2^k)_{k\geq e^{\frac{3c_b}{\epsilon}}}$), hence by $\frac{2\Lambda}{\epsilon}\exp(\frac{2\Lambda}{\epsilon})$. The sequence $(\chi_{k1}^{\epsilon\Lambda}f_1^{k,2l+1})_{k\geq e^{\frac{3c_b}{\epsilon}}}$ (resp. $(\chi_{k1}^{\epsilon\Lambda}f_2^{k,2l+1})_{k\geq e^{\frac{3c_b}{\epsilon}}}$) is bounded by $\frac{2\Lambda}{\epsilon}\exp(\frac{2\Lambda}{\epsilon})(1+\frac{2\Lambda}{\epsilon}\exp(\frac{2\Lambda}{\epsilon}))$, since

$$\chi_{k1}^{\epsilon\Lambda}(y)f_1^{k,2l+1}(x,y) = \chi_{k1}^{\epsilon\Lambda}(y)F_1^k(x,y)$$
$$+\chi_{k1}^{\epsilon\Lambda}(y)\int_0^x \frac{F_1^k}{1+\frac{F_1^k}{k}}\big(\frac{F_2^k}{1+\frac{F_2^k}{k}} - \frac{f_2^{k,l}}{1+\frac{f_2^{k,l}}{k}}\big)(X,y)e^{-\int_X^x \frac{f_2^{k,l}}{(1+\frac{f_2^{k,l}}{k})(1+\frac{f_1^{k,l-1}}{k})(1+\frac{F_1^k}{k})}(r,y)dr}dX$$
$$\leq \chi_{k1}^{\epsilon\Lambda}F_1^k(x,y) + \chi_{k1}^{\epsilon\Lambda}(y)\int_0^x F_1^kF_2^k(X,y)dX.$$

By Lemma 3.4, there is a subsequence of $\big(\chi^{\epsilon\Lambda}_{k1}(y)\int_0^1 \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dX\big)_{k\in\mathbb{N}^*}$, still denoted by $\big(\chi^{\epsilon\Lambda}_{k1}(y)\int_0^1 \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dX\big)_{k\in\mathbb{N}^*}$, converging in $L^1([0,1])$ to some $\tilde{F}_1$. Given $\eta>0$, there is a subset ω_η of $[0,1]$ with measure smaller than η such that on ω_η^c the convergence of this sequence is uniform and $(\tilde{F}_1, f_{b1}, f_{b2})$ is bounded. It follows from (3.14)-(3.15) and the non-negativity of $(f_1^{k,2l}, f_2^{k,2l})_{(k,l)\in\mathbb{N}^2}$ that $(f_1^{k,2l}, f_2^{k,2l})_{(k,l)\in\mathbb{N}^2}$ is bounded on $[0,1]\times\omega_\eta^c$. Given these bounds, Lemma 3.4 and the expression of $(f_1^{k,l}, f_2^{k,l})$ in exponential form, it holds by induction that for each $l\in\mathbb{N}$, the sequence $(f_1^{k,2l}, f_2^{k,l})_{k\geq e^{\frac{3c_b}{\epsilon}}}$ is strongly compact in $L^1([0,1]\times\omega_\eta^c)$. Denote by (g_1^l, g_2^l) its limit up to a subsequence. By Lemma 3.4, let G (resp. H) with $\partial_x G = -\partial_x H$, be the limit in L^1 when $k\to+\infty$ of

$$(\chi^{\epsilon\Lambda}_{k1}(y)\int_0^x \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dX)_{k\geq e^{\frac{3c_b}{\epsilon}}}, \big(\text{resp. } (\chi^{\epsilon\Lambda}_{k1}(y)\int_x^1 \frac{F_3^k}{1+\frac{F_3^k}{k}}\frac{F_4^k}{1+\frac{F_4^k}{k}}(X,y)dX)_{k\geq e^{\frac{3c_b}{\epsilon}}}\big).$$

$(g_1^{2l}, g_2^{2l}, g_1^{2l+1}, g_2^{2l+1})$ satisfies

$$\begin{aligned}
&g_1^0 = g_2^0 = 0,\\
&g_1^{2l}(x,y) = f_{b1}(y) + G(x,y) - \int_0^x g_1^{2l}g_2^{2l-1}(X,y)dX, \quad l\in\mathbb{N}^*,\\
&g_1^{2l+1}(x,y) = f_{b1}(y) + G(x,y) - \int_0^x g_1^{2l+1}g_2^{2l}(X,y)dX, \quad l\in\mathbb{N},\\
&g_2^{2l}(x,y) = f_{b2}(y) + H(x,y) - \int_x^1 g_1^{2l-1}g_2^{2l}(X,y)dX, \quad l\in\mathbb{N}^*,\\
&g_2^{2l+1}(x,y) = f_{b2}(y) + H(x,y) - \int_x^1 g_1^{2l}g_2^{2l+1}(X,y)dX, \quad l\in\mathbb{N}, \quad (x,y)\in[0,1]\times\omega_\eta^c. \qquad (3.18)
\end{aligned}$$

By induction on l it holds that

$$\begin{aligned}
&0\leq g_1^{2l}\leq g_1^{2l+2}\leq g_1^{2l+3}\leq g_1^{2l+1},\\
&0\leq g_2^{2l}\leq g_2^{2l+2}\leq g_2^{2l+3}\leq g_2^{2l+1}, \quad l\in\mathbb{N}. \qquad (3.19)
\end{aligned}$$

Moreover,

$$\int_{[0,1]\times\omega_\eta^c} g_j^{2l}(x,y)dxdy \leq \int_0^1 f_{bj}(y)dy + \int_{[0,1]\times\omega_\eta^c}(G+H)(x,y)dxdy, \quad j\in\{1,2\}, \quad l\in\mathbb{N}.$$

By the monotone convergence theorem, $(g^{2l})_{l\in\mathbb{N}}$ (resp. $(g^{2l+1})_{l\in\mathbb{N}}$) increasingly (resp. decreasingly) converges in $L^1([0,1]\times\omega_\eta^c)$ and almost everywhere on $[0,1]\times\omega_\eta^c$ to some g (resp. h). By the dominated convergence theorem,

$$\lim_{l\to+\infty} g_1^{2l}g_2^{2l-1} = g_1h_2 \quad \text{and} \quad \lim_{l\to+\infty} g_1^{2l+1}g_2^{2l} = h_1g_2 \quad \text{in} \quad L^1([0,1]\times\omega_\eta^c).$$

Consequently,

$$
\begin{aligned}
g_1(x,y) &= f_{b1}(y) + G(x,y) - \int_0^x g_1 h_2(X,y)dX,\\
h_1(x,y) &= f_{b1}(y) + G(x,y) - \int_0^x h_1 g_2(X,y)dX,\\
g_2(x,y) &= f_{b2}(y) + H(x,y) - \int_x^1 h_1 g_2(X,y)dX = g_2(0,y) - G(x,y) + \int_0^x h_1 g_2(X,y)dX,\\
h_2(x,y) &= f_{b2}(y) + H(x,y) - \int_x^1 g_1 h_2(X,y)dX = h_2(0,y) - G(x,y) + \int_0^x g_1 h_2(X,y)dX,\\
&\qquad\qquad (x,y) \in [0,1]\times\omega_\eta^c,
\end{aligned}
$$

and

$$
h_1 \geq g_1, \quad h_2 \geq g_2, \quad (x,y) \in [0,1]\times\omega_\eta^c. \tag{3.20}
$$

Hence

$$
(h_1 - g_1)(1,y) = -(h_2 - g_2)(0,y),
$$

so that, by (3.20),

$$
g_1(1,y) = h_1(1,y), \quad g_2(0,y) = h_2(0,y).
$$

Consequently, $h_1 - g_1 = h_2 - g_2$, $g_1h_2 - h_1g_2 = (g_1 - g_2)(h_1 - g_1)$ and

$$
(h_1 - g_1)(x,y) = \int_0^x (g_1 - g_2)(h_1 - g_1)(X,y)dX. \tag{3.21}
$$

It follows from $(h_1 - g_1)(0,y) = 0$ and the boundedness of (g_1, g_2) on $[0,1]\times\omega_\eta^c$ that $h_1 - g_1 = 0$ and $(g_1, g_2) = (h_1, h_2)$ on $[0,1]\times\omega_\eta^c$. Hence the whole sequence $(g_1^l, g_2^l)_{l\in\mathbb{N}}$ converges to (g_1, g_2) in $L^1([0,1]\times\omega_\eta^c)$. Letting $\eta\to 0$ and using (2.16), the convergence holds in $L^1([0,1]^2)$.

Given $\bar\epsilon > 0$, choose l_0 big enough so that $\| g_1^{2l_0} - g_1^{2l_0+1} \|_{L^1} < \bar\epsilon$, then k_0 big enough so that

$$
\| f_1^{k,2l_0+1} - g_1^{2l_0+1} \|_{L^1} \leq \bar\epsilon \quad \text{and} \quad \| f_1^{k,2l_0} - g_1^{2l_0} \|_{L^1} \leq \bar\epsilon, \quad k \geq k_0.
$$

Hence $\| f_1^{k,2l_0+1} - f_1^{k,2l_0} \|_{L^1} \leq 3\bar\epsilon$ for $k \geq k_0$. Then

$$
\begin{aligned}
&\| F_1^k - F_1^{k'} \|_{L^1}\\
&\leq \| F_1^k - F_1^{k'} \|_{L^1((\Omega_{k1}^{\epsilon\Lambda})^c)} + 2c_b\epsilon \quad \text{by (3.8)}\\
&\leq \| F_1^k - f_1^{k,2l_0} \|_{L^1((\Omega_{k1}^{\epsilon\Lambda})^c)} + \| F_1^{k'} - f_1^{k',2l_0} \|_{L^1((\Omega_{k1}^{\epsilon\Lambda})^c)} + \| f_1^{k,2l_0} - f_1^{k',2l_0} \|_{L^1} + 2c_b\epsilon\\
&\leq \| f_1^{k,2l_0+1} - f_1^{k,2l_0} \|_{L^1} + \| f_1^{k',2l_0+1} - f_1^{k',2l_0} \|_{L^1} + \| f_1^{k,2l_0} - f_1^{k',2l_0} \|_{L^1} + 2c_b\epsilon \quad \text{by (3.17)}\\
&\leq 8\bar\epsilon + 2c_b\epsilon, \quad k \geq \max\{k_0, \exp(\frac{3c_b}{\epsilon})\}, \quad k' \geq \max\{k_0, \exp(\frac{3c_b}{\epsilon})\}.
\end{aligned}
$$

And so (F_1^k) is a Cauchy sequence in $L^1([0,1]^2)$ with the limit equal to the weak limit F_1. Similarly, $(F_j^k)_{2\leq j\leq 4}$ is a Cauchy sequence in $(L^1([0,1]^2))^3$ with the limit equal to the weak limit $(F_j)_{2\leq j\leq 4}$. ■

Lemma 3.6
The limit F of $(F^k)_{k\in\mathbb{N}^*}$ in $L^1([0,1]^2)$ is a renormalized solution to the Broadwell model (1.1).

Proof of Lemma 3.6.
Start from a renormalized formulation of (2.1),

$$\begin{aligned}
&\int_0^1 \varphi_1(1,y)\ln\big(1+F_1^k(1,y)\big)dy-\int_0^1 \varphi_1(0,y)\ln\big(1+f_{b1}(y)\wedge\frac{k}{2}\big)dy\\
&-\int_0^1\int_0^1 \ln\big(1+F_1^k(x,y)\big)\partial_x\varphi_1(x,y)dxdy\\
&=\int_0^1\int_0^1 \varphi_1(x,y)\frac{F_3^kF_4^k}{(1+F_1^k)(1+\frac{F_3^k}{k})(1+\frac{F_4^k}{k})}(x,y)dxdy\\
&-\int_0^1\int_0^1 \varphi_1(x,y)\frac{F_1^kF_2^k}{(1+F_1^k)(1+\frac{F_1^k}{k})(1+\frac{F_2^k}{k})}(x,y)dxdy, \qquad (3.22)
\end{aligned}$$

for test functions $\varphi\in(C^1([0,1]^2))^4$. Using the strong L^1 convergence of the sequence (F^k) to pass to the limit when $k\to+\infty$ in the left hand side of (3.22), gives in the limit,

$$\begin{aligned}
&\int_0^1 \varphi_1(1,y)\ln\big(1+F_1(1,y)\big)dy-\int_0^1 \varphi_1(0,y)\ln\big(1+f_{b1}(y)\big)dy\\
&-\int_0^1\int_0^1 \ln\big(1+F_1(x,y)\big)\partial_x\varphi_1(x,y)dxdy.
\end{aligned}$$

For the passage to the limit when $k\to+\infty$ in the right hand side of (3.22), given $\eta>0$ there is a subset A_η of $[0,1]^2$ with $|A_\eta^c|<\eta$, such that up to a subsequence, $(F^k)_{k\in\mathbb{N}^*}$ uniformly converges to F on A_η and $F\in L^\infty(A_\eta)$. Passing to the limit when $k\to+\infty$ on A_η is straightforward. Moreover,

$$\lim_{\eta\to0}\int_{A_\eta^c}\varphi\frac{F_1F_2}{1+F_1}(x,y)dxdy=0 \quad\text{and}\quad \lim_{\eta\to0}\int_{A_\eta^c}\varphi\frac{F_1^kF_2^k}{(1+F_1^k)(1+\frac{F_1^k}{k})(1+\frac{F_2^k}{k})}(x,y)dxdy=0,$$

uniformly with respect to k, since

$$\frac{F_1}{1+F_1}\le1,\quad \frac{F_1^k}{(1+F_1^k)(1+\frac{F_1^k}{k})(1+\frac{F_2^k}{k})}\le1,\quad\text{and}\quad \lim_{\eta\to0}\int_{A_\eta^c}F_2^k=0,$$

uniformly with respect to k.
The gain term can be estimated as follows. The uniform boundedness of the entropy production term of (F^k) is given in Lemma 3.1. A convexity argument together with the L^1 convergence of (F^k) to F (see [8]), imply that

$$\int(F_1F_2-F_3F_4)\ln\frac{F_1F_2}{F_3F_4}(x,y)dxdy\le c_b. \qquad (3.23)$$

It follows that, for any $\gamma>1$,

$$\begin{aligned}
\int_{A_\eta^c}|\varphi|\frac{F_3F_4}{1+F_1}(x,y)dxdy&\le\frac{c}{\ln\gamma}+c\gamma\int_{A_\eta^c}\frac{F_1F_2}{1+F_1}(x,y)dxdy\\
&\le\frac{c}{\ln\gamma}+c\gamma\int_{A_\eta^c}F_2(x,y)dxdy,
\end{aligned}$$

which tends to zero when $\eta \to 0$. Similarly, using (3.3),

$$\begin{aligned}
&\int_{A_\eta^c} |\varphi| \frac{F_3^k F_4^k}{(1+F_1^k)(1+\frac{F_3^k}{k})(1+\frac{F_4^k}{k})}(x,y)dxdy \\
&\leq c \int_{A_\eta^c} \frac{F_3^k F_4^k}{(1+F_1^k)(1+\frac{F_3^k}{k})(1+\frac{F_4^k}{k})}(x,y)dxdy \\
&\leq \frac{c}{\ln \gamma} + C\gamma \int_{A_\eta^c} \frac{F_1^k F_2^k}{(1+F_1^k)(1+\frac{F_1^k}{k})(1+\frac{F_2^k}{k})}(x,y)dxdy \\
&\leq \frac{C}{\ln \gamma} + C\gamma \int_{A_\eta^c} F_2^k(x,y)dxdy,
\end{aligned}$$

which tends to zero when $\eta \to 0$, uniformly in k. It follows that the right hand side of (3.22) converges to

$$\int_0^1 \int_0^1 \varphi(x,y) \frac{F_3 F_4}{1+F_1}(x,y)dxdy - \int_0^1 \varphi(x,y) \frac{F_1 F_2}{1+F_1}(x,y)dxdy,$$

when $k \to +\infty$. Consequently, F_1 satisfies the first equation of (1.1) in renormalized form. It can be similarly proven that $(F_j)_{2\leq j\leq 4}$ is solution to the last equations of (1.1). ∎

This completes the proof of Theorem 2.1.

References

[1] L. Arkeryd, A. Nouri, *On the stationary Povzner equation in* $\mathbb{R}^n$ J. Math. Kyoto Univ. 39 (1) (1999), 115-153.

[2] L. Arkeryd, A. Nouri, *The stationary Boltzmann equation in the slab with given weighted mass for hard and soft forces*, Annali della Scuole Normale Superiore di Pisa Cl. Sci. (4), Vol XXVII (1998), 553-556.

[3] L. Arkeryd, A. Nouri *On a Boltzmann equation for Haldane statistics*, KRM 12 (2019), 323-346.

[4] A. Bobylev, *Exact solutions of discrete kinetic models and stationary problems for the plane Broadwell model*, Math. Meth. Appl. Sci. (4) 19 (1996), 825-845.

[5] A. Bobylev, G. Toscani, *Two dimensional alf-space problems for the Broadwell discrete velocitry model*, Cont. Mechanics and Thermodyamics 8 (1996), 257-274.

[6] A. Bobylev, M. Vinerean-Bernhoff, *Discrete velocity models of the Boltzmann equation and conservation laws*, Kinetic and Related Models 3 (2008), 335-358.

[7] C. Cercignani, R. Illner, M. Shinbrot, *A boundary value problem for the 2-dimensional Broadwell model*, Comm. Math. Phys. 114 (1988), 687-698.

[8] R. J. DiPerna, P. L. Lions *On the Cauchy problem for Boltzmann equations: Global existence and weak stability*, Ann. Math 130 (1989), 321-366.

[9] R. Illner, T. Platkowski, *Discrete velocity models of the Boltzmann equation: survey on the mathematical aspects of the theory*, Siam Rev. 30 (1988), 213-255.

[10] O. V. Ilyin, *Symmetries, the current function, and exact solutions for Broadwell's two-dimensional stationary kinetic model*, Teor. Mat. Fiz. 179 (2014), 350-359.

[11] A. N. Kolmogorov, *Über Kompaktheit der Funktionenmengen bei der Konvergenz im Mittel*, Nachr. Ges. Wiss. Göttingen 9 (1931), 60-63.

[12] M. Riesz, *Sur les ensembles compacts de fonctions sommables*, Acta Szeged Sect. Math. 6 (1933), 136-142.