



HAL
open science

An unfitted hybrid high-order method for the Stokes interface problem

Erik Burman, Guillaume Delay, Alexandre Ern

► **To cite this version:**

Erik Burman, Guillaume Delay, Alexandre Ern. An unfitted hybrid high-order method for the Stokes interface problem. 2020. hal-02519896v1

HAL Id: hal-02519896

<https://hal.science/hal-02519896v1>

Preprint submitted on 26 Mar 2020 (v1), last revised 22 Jan 2021 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

An unfitted hybrid high-order method for the Stokes interface problem

Erik Burman*

Guillaume Delay[†]

Alexandre Ern[‡]

March 26, 2020

Abstract

We design and analyze a hybrid high-order (HHO) method on unfitted meshes to approximate the Stokes interface problem. The interface can cut through the mesh cells in a very general fashion. A cell-agglomeration procedure prevents the appearance of small cut cells. Our main results are inf-sup stability and a priori error estimates with optimal convergence rates in the energy norm. Numerical simulations corroborate these results. Stokes interface problem, hybrid high-order method, unfitted meshes.

1 Introduction

Generating meshes to solve problems posed on domains with a curved interface separating subdomains with different properties can be a difficult task. The use of unfitted meshes that do not fit the interface greatly simplifies the meshing process since such meshes can be chosen in a very simple manner. We can for instance mesh the domain without taking into account the interface. The analysis of finite element methods (FEM) on unfitted meshes was started in [2, 3]. The main paradigm introduced in [20] is to double the unknowns in the cut cells and to use Nitsche’s method (see [27]) to weakly impose the interface conditions. We refer the reader to [8] for an overview. One difficulty with the penalty method appears with the presence of small cuts, i.e. cells that have only a tiny fraction of their volume on one side of the interface. Small cuts have an adverse effect on the conditioning of the system matrix and can even hamper convergence (see [15] for a recent discussion on this topic). There are essentially two ways to cure the issue of small cuts: one can either add some stabilization terms such as the ghost penalty technique devised in [6] or one can agglomerate cells in the vicinity of small cuts as in [25, 30].

In the present work we deal with the Stokes interface problem. Let Ω be a polygonal/polyhedral domain in \mathbb{R}^d , $d \in \{2, 3\}$ (open, bounded, connected, Lipschitz subset of \mathbb{R}^d) and consider a partition of Ω into two disjoint subdomains so that $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$ with the interface $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. The unit normal vector \mathbf{n}_Γ to Γ conventionally points from Ω_1 to Ω_2 . For a smooth enough function v defined on $\Omega_1 \cup \Omega_2$, we define its jump across Γ as $\llbracket v \rrbracket_\Gamma := v|_{\Omega_1} - v|_{\Omega_2}$. We denote $H^1(\Omega_1 \cup \Omega_2; \mathbb{R}^d) := \{\mathbf{v} \in L^2(\Omega; \mathbb{R}^d) \mid \mathbf{v}|_{\Omega_i} \in H^1(\Omega_i; \mathbb{R}^d), \forall i \in \{1, 2\}\}$ and $L_*^2(\Omega) := \{q \in L^2(\Omega) \mid \int_\Omega q = 0\}$. We consider the following problem: Find the velocity and pressure $(\mathbf{u}, p) \in H^1(\Omega_1 \cup \Omega_2; \mathbb{R}^d) \times L_*^2(\Omega)$ such that

$$-\nabla \cdot \sigma(\mathbf{u}, p) = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega_1 \cup \Omega_2, \quad (1a)$$

$$\llbracket \sigma(\mathbf{u}, p) \rrbracket_\Gamma \mathbf{n}_\Gamma = \mathbf{g}_N, \quad \llbracket \mathbf{u} \rrbracket_\Gamma = \mathbf{0} \quad \text{on } \Gamma, \quad (1b)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega, \quad (1c)$$

with data $\mathbf{f} \in L^2(\Omega; \mathbb{R}^d)$ and $\mathbf{g}_N \in L^2(\Gamma; \mathbb{R}^d)$, and where $\sigma(\mathbf{u}, p) := 2\nu \nabla^s \mathbf{u} - p \mathbb{I}$ is the total stress tensor, $\nabla^s \mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the linearized strain tensor, and \mathbb{I} is the identity tensor. For simplicity we consider the viscosity ν to be constant in each subdomain Ω_i and we set $\nu_i := \nu|_{\Omega_i}$ for all $i \in \{1, 2\}$. To ensure robustness with respect to the viscosity contrast, the two subdomains play different roles in the numerical scheme. To fix the ideas, we enumerate the two subdomains so that $0 < \nu_1 \leq \nu_2$. We notice that several discretizations of the model problem (1) on unfitted meshes have already been analyzed, see for instance [19, 29, 4, 21, 12, 32, 26, 24, 10, 1] for finite element discretizations and [30, 22] for discontinuous Galerkin. However among these works only [10] analyzes the approximation of (1) with high order polynomials.

*Department of Mathematics, University College London, London, UK–WC1E 6BT, UK. Email: e.burman@ucl.ac.uk

[†]Sorbonne Université, CNRS, Université de Paris, LJLL, F-75005 Paris, France and Université Paris-Est, CERMICS (ENPC) and INRIA, Paris, France. Email: guillaume.delay@sorbonne-universite.fr

[‡]Université Paris-Est, CERMICS (ENPC), 77455 Marne-la-Vallée cedex 2, and INRIA, Paris, France. Email: alexandre.ern@enpc.fr

The goal of the present work is to devise and analyze a hybrid high-order (HHO) method on unfitted meshes for the Stokes interface problem. Robustness with respect to small cuts is achieved by using a cell-agglomeration procedure. HHO methods have been introduced and analyzed for diffusion and locking-free linear elasticity problems on fitted meshes in [16, 17]. As shown in [14], these methods are closely related to hybridizable discontinuous Galerkin (HDG) methods and to nonconforming virtual element methods (ncVEM). HHO methods have already been used to solve Stokes and Navier–Stokes problems on fitted meshes in [18, 5]. We consider an unfitted setting for HHO that has been first introduced and analyzed in [9] for an elliptic interface problem. This study has been continued in [7] where novel gradient reconstructions enabled the use of a parameter-free Nitsche’s method to impose the interface conditions. Moreover, the unfitted HHO method has been run on several elliptic interface problems, including high contrast of coefficients and jumps of the solution across the interface. In the present work we extend the unfitted HHO method from [7] to the Stokes interface problem. The novelties herein are the devising of linearized strain and divergence reconstruction operators and the inf-sup stability analysis where we track the dependency on the viscosity coefficients on both sides of the interface and show that the method is robust in the highly contrasted case.

This work is organized as follows. We introduce the discretization of the model problem (1) by an unfitted HHO method in Section 2. We present the key analysis tools related to the discrete unfitted setting in Section 3. We perform the numerical analysis of the unfitted HHO method in Section 4, where we establish in particular inf-sup stability and a priori error estimates with optimal convergence rates. We then present some numerical examples in Section 5. Finally, conclusions are drawn in Section 6.

2 The unfitted HHO method

The goal of this section is to introduce some important definitions concerning unfitted meshes and to present the unfitted HHO method to discretize the Stokes interface problem.

2.1 Unfitted meshes

Let $(\mathcal{T}_h)_{h>0}$ be a family of matching meshes of Ω . The meshes can have cells that are polyhedra in \mathbb{R}^d with planar faces, and hanging nodes are also possible. For all $T \in \mathcal{T}_h$, h_T denotes the diameter of the cell T and \mathbf{n}_T is the unit normal on ∂T pointing outward T . We set $h := \max_{T \in \mathcal{T}_h} h_T$. The mesh faces are collected in the set \mathcal{F}_h . Assumptions on the mesh regularity and how the interface cuts the mesh cells are stated in Section 3.1.

Let us define the partition $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^\Gamma \cup \mathcal{T}_h^2$, where the subsets

$$\mathcal{T}_h^i := \{T \in \mathcal{T}_h \mid T \subset \Omega_i\} \quad (\forall i \in \{1, 2\}), \quad \mathcal{T}_h^\Gamma := \{T \in \mathcal{T}_h \mid T \cap \Gamma \neq \emptyset\}, \quad (2)$$

collect respectively the mesh cells inside the subdomain Ω_i (the uncut cells), and those cut by the interface Γ (the cut cells). For every cut cell $T \in \mathcal{T}_h^\Gamma$ and all $i \in \{1, 2\}$, we define

$$T^i := T \cap \Omega_i, \quad T^\Gamma := T \cap \Gamma. \quad (3)$$

For all $T \in \mathcal{T}_h^\Gamma$ and all $i \in \{1, 2\}$, the boundary $\partial(T^i)$ of the subcell T^i is decomposed as

$$\partial(T^i) = (\partial T)^i \cup T^\Gamma, \quad (\partial T)^i := \partial T \cap \Omega_i. \quad (4)$$

In order to unify the notation, for all $i \in \{1, 2\}$ and $T \in \mathcal{T}_h^i$, we also set

$$T^i := T, \quad T^{\bar{i}} := \emptyset, \quad (\partial T)^i := \partial T, \quad (\partial T)^{\bar{i}} := \emptyset, \quad T^\Gamma := \emptyset, \quad (5)$$

where $\bar{i} := 3 - i$ (so that $\bar{1} := 2$ and $\bar{2} := 1$). In a similar way, for all $F \in \mathcal{F}_h$ and all $i \in \{1, 2\}$, we set

$$F^i := F \cap \Omega_i. \quad (6)$$

2.2 The local discrete problem

In this section, we describe the unfitted HHO method for the Stokes interface problem. Let $k \geq 0$ be a polynomial degree. The discrete unknowns for the velocity are piecewise polynomials of degree k attached to the mesh faces and of degree $(k + 1)$ attached to the mesh cells, whereas the discrete unknowns for the pressure are piecewise polynomials of degree k attached to the mesh cells. For $\ell \in \mathbb{N}$, we denote $\mathbb{P}^\ell(S)$ (resp. $\mathbb{P}^\ell(S; \mathbb{R}^d)$, $\mathbb{P}^\ell(S; \mathcal{S}^{d \times d})$) the space of scalar-valued (resp. vector-valued, symmetric matrix-valued) polynomials in S of degree at most ℓ . We also denote $(\cdot, \cdot)_S$ the L^2 -scalar product on S and $\|\cdot\|_S$ the associated norm. Whenever $S = \emptyset$, we abuse the notation by writing $\mathbb{P}^\ell(S) := \{0\}$ and $(\cdot, \cdot)_S := 0$.

Let $T \in \mathcal{T}_h$. For all $i \in \{1, 2\}$, we set $\mathbb{P}^k(\mathcal{F}_{(\partial T)^i}; \mathbb{R}^d) := \times_{F \in \mathcal{F}_{(\partial T)^i}} \mathbb{P}^k(F; \mathbb{R}^d)$ and $\mathcal{F}_{(\partial T)^i} := \{F^i \mid F \in \mathcal{F}_{\partial T}\}$ where $\mathcal{F}_{\partial T} := \{F \in \mathcal{F}_h \mid F \subset \partial T\}$. We define the local discrete unknowns as

$$\begin{aligned}\hat{\mathbf{v}}_T &:= (\mathbf{v}_{T^1}, \mathbf{v}_{(\partial T)^1}, \mathbf{v}_{T^2}, \mathbf{v}_{(\partial T)^2}) \in \hat{\mathbf{U}}_T^k, \\ p_T &:= (p_{T^1}, p_{T^2}) \in P_T^k,\end{aligned}$$

with $\hat{\mathbf{U}}_T^k := \hat{\mathbf{U}}_T^{k,1} \times \hat{\mathbf{U}}_T^{k,2}$, $P_T^k := P_T^{k,1} \times P_T^{k,2}$, and

$$\hat{\mathbf{U}}_T^{k,i} := \mathbb{P}^{k+1}(T^i; \mathbb{R}^d) \times \mathbb{P}^k(\mathcal{F}_{(\partial T)^i}; \mathbb{R}^d), \quad P_T^{k,i} := \mathbb{P}^k(T^i), \quad \forall i \in \{1, 2\}. \quad (7)$$

Note that there are no discrete unknowns attached to T^Γ . For all $T \in \mathcal{T}_h^\Gamma$ and all $\hat{\mathbf{v}}_T \in \hat{\mathbf{U}}_T$, we denote $\llbracket \mathbf{v}_T \rrbracket_\Gamma := \mathbf{v}_{T^1} - \mathbf{v}_{T^2}$.

For all $T \in \mathcal{T}_h$ and all $i \in \{1, 2\}$, we define a symmetric gradient reconstruction operator $\mathbb{E}_{T^i}^k : \hat{\mathbf{U}}_T^k \rightarrow \mathbb{P}^k(T^i; \mathcal{S}^{d \times d})$ such that for all $\hat{\mathbf{v}}_T := (\mathbf{v}_{T^1}, \mathbf{v}_{(\partial T)^1}, \mathbf{v}_{T^2}, \mathbf{v}_{(\partial T)^2}) \in \hat{\mathbf{U}}_T^k$, we have

$$(\mathbb{E}_{T^1}^k(\hat{\mathbf{v}}_T), \mathbf{q})_{T^1} := (\nabla^\mathbf{s} \mathbf{v}_{T^1}, \mathbf{q})_{T^1} + (\mathbf{v}_{(\partial T)^1} - \mathbf{v}_{T^1}, \mathbf{q} \mathbf{n}_T)_{(\partial T)^1} - (\llbracket \mathbf{v}_T \rrbracket_\Gamma, \mathbf{q} \mathbf{n}_\Gamma)_{T^\Gamma}, \quad (8a)$$

$$(\mathbb{E}_{T^2}^k(\hat{\mathbf{v}}_T), \mathbf{q})_{T^2} := (\nabla^\mathbf{s} \mathbf{v}_{T^2}, \mathbf{q})_{T^2} + (\mathbf{v}_{(\partial T)^2} - \mathbf{v}_{T^2}, \mathbf{q} \mathbf{n}_T)_{(\partial T)^2}, \quad (8b)$$

for all $\mathbf{q} \in \mathbb{P}^k(T^1; \mathcal{S}^{d \times d})$ in (8a) and all $\mathbf{q} \in \mathbb{P}^k(T^2; \mathcal{S}^{d \times d})$ in (8b). Similarly, for all $T \in \mathcal{T}_h$ and all $i \in \{1, 2\}$, we define a divergence reconstruction operator $D_{T^i}^k : \hat{\mathbf{U}}_T^k \rightarrow \mathbb{P}^k(T^i)$ such that for all $\hat{\mathbf{v}}_T := (\mathbf{v}_{T^1}, \mathbf{v}_{(\partial T)^1}, \mathbf{v}_{T^2}, \mathbf{v}_{(\partial T)^2}) \in \hat{\mathbf{U}}_T^k$, we have

$$D_{T^i}^k(\hat{\mathbf{v}}_T) := \text{trace}(\mathbb{E}_{T^i}^k(\hat{\mathbf{v}}_T)), \quad (9)$$

i.e.,

$$(D_{T^1}^k(\hat{\mathbf{v}}_T), q)_{T^1} := (\nabla \cdot \mathbf{v}_{T^1}, q)_{T^1} + (\mathbf{v}_{(\partial T)^1} - \mathbf{v}_{T^1}, q \mathbf{n}_T)_{(\partial T)^1} - (\llbracket \mathbf{v}_T \rrbracket_\Gamma, q \mathbf{n}_\Gamma)_{T^\Gamma}, \quad (10a)$$

$$(D_{T^2}^k(\hat{\mathbf{v}}_T), q)_{T^2} := (\nabla \cdot \mathbf{v}_{T^2}, q)_{T^2} + (\mathbf{v}_{(\partial T)^2} - \mathbf{v}_{T^2}, q \mathbf{n}_T)_{(\partial T)^2}, \quad (10b)$$

for all $q \in \mathbb{P}^k(T^1)$ in (10a) and all $q \in \mathbb{P}^k(T^2)$ in (10b). Furthermore, the stabilization bilinear form is defined for all $\hat{\mathbf{v}}_T, \hat{\mathbf{w}}_T \in \hat{\mathbf{U}}_T^k$ as

$$s_T(\hat{\mathbf{v}}_T, \hat{\mathbf{w}}_T) := \nu_1 h_T^{-1} (\llbracket \mathbf{v}_T \rrbracket_\Gamma, \llbracket \mathbf{w}_T \rrbracket_\Gamma)_{T^\Gamma} + \sum_{i \in \{1, 2\}} \nu_i h_T^{-1} (\Pi_{(\partial T)^i}^k(\mathbf{v}_{T^i}) - \mathbf{v}_{(\partial T)^i}, \Pi_{(\partial T)^i}^k(\mathbf{w}_{T^i}) - \mathbf{w}_{(\partial T)^i})_{(\partial T)^i}, \quad (11)$$

where $\Pi_{(\partial T)^i}^k$ denotes the L^2 -orthogonal projector onto $\mathbb{P}^k(\mathcal{F}_{(\partial T)^i}; \mathbb{R}^d)$.

The local HHO bilinear and linear forms act as follows: For all $(\hat{\mathbf{v}}_T, r_T), (\hat{\mathbf{w}}_T, q_T) \in Y_T^k := \hat{\mathbf{U}}_T^k \times P_T^k$,

$$A_T((\hat{\mathbf{v}}_T, r_T), (\hat{\mathbf{w}}_T, q_T)) := a_T(\hat{\mathbf{v}}_T, \hat{\mathbf{w}}_T) - b_T(\hat{\mathbf{w}}_T, r_T) + b_T(\hat{\mathbf{v}}_T, q_T) \quad (12a)$$

$$- \chi \nu_2^{-1} h_T (\llbracket \sigma(\mathbf{v}_T, r_T) \rrbracket_\Gamma \mathbf{n}_\Gamma, \llbracket \sigma(\mathbf{w}_T, -q_T) \rrbracket_\Gamma \mathbf{n}_\Gamma)_{T^\Gamma},$$

$$L_T(\hat{\mathbf{w}}_T, q_T) := \ell_T(\hat{\mathbf{w}}_T) - \chi \nu_2^{-1} h_T (\mathbf{g}_N, \llbracket \sigma(\mathbf{w}_T, -q_T) \rrbracket_\Gamma \mathbf{n}_\Gamma)_{T^\Gamma}, \quad (12b)$$

with $\llbracket \sigma(\mathbf{v}_T, r_T) \rrbracket_\Gamma := \sigma(\mathbf{v}_{T^1}, r_{T^1}) - \sigma(\mathbf{v}_{T^2}, r_{T^2})$ and

$$a_T(\hat{\mathbf{v}}_T, \hat{\mathbf{w}}_T) := \sum_{i \in \{1, 2\}} 2\nu_i (\mathbb{E}_{T^i}^k(\hat{\mathbf{v}}_T), \mathbb{E}_{T^i}^k(\hat{\mathbf{w}}_T))_{T^i} + s_T(\hat{\mathbf{v}}_T, \hat{\mathbf{w}}_T), \quad (13a)$$

$$b_T(\hat{\mathbf{w}}_T, r_T) := \sum_{i \in \{1, 2\}} (r_{T^i}, D_{T^i}^k(\hat{\mathbf{w}}_T))_{T^i}, \quad (13b)$$

$$\ell_T(\hat{\mathbf{w}}_T) := \sum_{i \in \{1, 2\}} (\mathbf{f}, \mathbf{w}_{T^i})_{T^i} + (\mathbf{g}_N, \mathbf{w}_{T^2})_{T^\Gamma}, \quad (13c)$$

and $\chi > 0$ is a penalty parameter that has to be chosen small enough (see Lemma 10). Our numerical results indicate that the stabilization term associated with χ can be omitted (i.e., one can set $\chi := 0$), but this term is needed in the present stability analysis.

Remark 1 (Alternative gradient reconstruction). *One can also consider a gradient reconstruction operator mapping to $\nabla^\mathbf{s} \mathbb{P}^k(T^i; \mathbb{R}^d)$ defined in a similar way to the original HHO method for the Stokes equations in [18] for all $i \in \{1, 2\}$. In this case, the specific divergence reconstruction defined in (10) has to be computed separately so as to evaluate the local bilinear form b_T in (13b).*

2.3 The global discrete problem

For all $i \in \{1, 2\}$, we define the discrete spaces

$$\hat{U}_h^{k,i} := \left(\chi_{T \in \mathcal{T}_h} \mathbb{P}^{k+1}(T^i; \mathbb{R}^d) \right) \times \left(\chi_{F \in \mathcal{F}_h} \mathbb{P}^k(F^i; \mathbb{R}^d) \right), \quad P_h^{k,i} := \chi_{T \in \mathcal{T}_h} \mathbb{P}^k(T^i), \quad (14)$$

and we set

$$\hat{U}_h^k := \hat{U}_h^{k,1} \times \hat{U}_h^{k,2}, \quad P_h^k := P_h^{k,1} \times P_h^{k,2}, \quad (15)$$

as well as $P_{h*}^k := \{q_h \in P_h^k \mid (q_h, 1)_\Omega = 0\}$. We also denote \hat{U}_{h0}^k the subspace of \hat{U}_h^k where all the degrees of freedom attached to the faces composing $\partial\Omega$ are null. Let $\hat{v}_h \in \hat{U}_h^k$ and let $q_h \in P_h^k$. For every cell $T \in \mathcal{T}_h$, we denote $\hat{v}_T := (\mathbf{v}_{T^1}, \mathbf{v}_{(\partial T)^1}, \mathbf{v}_{T^2}, \mathbf{v}_{(\partial T)^2}) \in \hat{U}_T^k$ the components of \hat{v}_h attached to T^1, T^2 and the faces composing $(\partial T)^1$ and $(\partial T)^2$, and we denote $q_T := (q_{T^1}, q_{T^2}) \in P_T^k$ the components of q_h attached to T^1 and T^2 (see (7)).

The discrete problem reads as follows: Letting $Y_h^k := \hat{U}_{h0}^k \times P_{h*}^k$, find $(\hat{u}_h, p_h) \in Y_h^k$ such that

$$A_h((\hat{u}_h, p_h), (\hat{w}_h, q_h)) = L_h(\hat{w}_h, q_h), \quad \forall (\hat{w}_h, q_h) \in Y_h^k, \quad (16)$$

where for all $(\hat{v}_h, r_h), (\hat{w}_h, q_h) \in Y_h^k$,

$$A_h((\hat{v}_h, r_h), (\hat{w}_h, q_h)) := \sum_{T \in \mathcal{T}_h} A_T((\hat{v}_T, r_T), (\hat{w}_T, q_T)), \quad L_h(\hat{w}_h, q_h) := \sum_{T \in \mathcal{T}_h} L_T(\hat{w}_T, q_T). \quad (17)$$

It is also convenient to define

$$a_h(\hat{v}_h, \hat{w}_h) := \sum_{T \in \mathcal{T}_h} a_T(\hat{v}_T, \hat{w}_T), \quad b_h(\hat{v}_h, q_h) := \sum_{T \in \mathcal{T}_h} b_T(\hat{v}_T, q_T). \quad (18)$$

The discrete problem (16) can be solved efficiently by eliminating locally all the cell velocity unknowns and all the non constant pressure unknowns using static condensation. This local elimination leads to a global transmission problem involving only the velocity unknowns on the mesh skeleton and the mean pressure in every cell (one pressure degree of freedom per cell even in cut cells). The resulting stencil couples velocity unknowns attached to neighboring faces (in the sense of cells) and mean pressure values attached to neighboring cells (in the sense of faces). Once this global transmission problem is solved, the cell velocity and pressure unknowns are recovered by local solves (see e.g. [13]).

3 Analysis tools for unfitted meshes

In this section we establish the key analysis tools on unfitted meshes regarding discrete inverse and multiplicative trace inequalities, as well as polynomial approximation properties.

3.1 Admissible mesh sequences

We consider a shape-regular polyhedral mesh sequence $(\mathcal{T}_h)_{h>0}$ in the sense of [16]. The mesh regularity is quantified by a parameter $\rho > 0$. Three additional assumptions on the meshes are introduced. The first one quantifies how well the interface cuts the mesh cells (and provides some discrete inverse inequalities), the second one quantifies how well the mesh resolves the interface (and provides a multiplicative trace inequality) and the third one requires the meshes to be not too graded.

Assumption 1 (Cut cells). *There is $\delta \in (0, 1)$ such that, for all $T \in \mathcal{T}_h^\Gamma$ and all $i \in \{1, 2\}$, there is $\tilde{\mathbf{x}}_{T^i} \in T^i$ such that $B(\tilde{\mathbf{x}}_{T^i}, \delta h_T) \subset T^i$.*

Assumption 2 (Resolving Γ). *There is $\gamma \in (0, 1)$ such that, for all $T \in \mathcal{T}_h^\Gamma$, there is a point $\tilde{\mathbf{x}}_T \in \mathbb{R}^d$ such that setting $T^\dagger := B(\tilde{\mathbf{x}}_T, \gamma^{-1} h_T)$ we have the following properties: (i) $T \subset T^\dagger$; (ii) for all $\mathbf{s} \in T^\dagger$ $\|\tilde{\mathbf{x}}_T - \mathbf{s}\|_{\ell^2} \leq \gamma^{-1} h_T$ and $d(\tilde{\mathbf{x}}_T, T_s \Gamma) \geq \gamma h_T$, where $T_s \Gamma$ is the tangent plane to Γ at the point \mathbf{s} ; (iii) For all $F \in \mathcal{F}_{\partial T}$, there is $\mathbf{x}_F \in T^\dagger$ such that $d(\mathbf{x}_F, F) \geq \gamma h_T$.*

It is shown in [9, Lem. 6.4] that if the mesh is fine enough, it is possible to devise a two-step cell-agglomeration procedure so that, choosing the parameter δ small enough (depending on the shape-regularity parameter ρ), Assumption 1 is fulfilled. In [7, Section 4.3], this procedure has been improved by adding a third step that guarantees that there is no propagation of the cell-agglomeration. Moreover it is shown in [9, Lem. 6.1] that if the mesh is fine enough compared to the curvature of the interface Γ , the statements (i) and (ii) in Assumption 2 are fulfilled, whereas the statement (iii) can be fulfilled by invoking the shape-regularity of the mesh sequence. Furthermore, on uncut cells, the shape regularity of the mesh sequence implies the existence of balls $B(\tilde{\mathbf{x}}_{T^i}, \delta h_T)$ (with $T^i = T$) and $T^\dagger := B(\tilde{\mathbf{x}}_T, \gamma^{-1} h_T)$ satisfying the assertions of Assumptions 1 and 2.

Finally, as in [7], we introduce a third assumption on the mesh sequence which is reasonable if the meshes are not excessively graded.

Assumption 3 (Mild mesh grading). For all $T \in \mathcal{T}_h$, let the neighboring layers $\Delta_j(T) \subset \mathbb{R}^d$ be defined by induction as $\Delta_0(T) := T$ and $\Delta_{j+1}(T) := \{T' \in \mathcal{T}_h \mid \overline{T'} \cap \overline{\Delta_j(T)} \neq \emptyset\}$ for all $j \in \mathbb{N}$. Then there is $n_0 \in \mathbb{N}$ such that for all $T \in \mathcal{T}_h$, the ball T^\dagger introduced in Assumption 2 satisfies $T^\dagger \subset \Delta_{n_0}(T)$.

3.2 Discrete inverse and multiplicative trace inequalities

The role of Assumption 1 is to provide the following discrete (inverse) inequalities.

Lemma 4 (Discrete (inverse) inequalities). Let Assumption 1 be fulfilled. Let $\ell \in \mathbb{N}$. There is $c_{\text{disc}} > 0$, depending on ρ , δ , and ℓ , such that, for all $T \in \mathcal{T}_h$, all $i \in \{1, 2\}$ and all $v_{T^i} \in \mathbb{P}^\ell(T^i)$, the following inequalities hold true:

- (Discrete trace inequality) $\|v_{T^i}\|_{(\partial T)^i \cup T^\Gamma} \leq c_{\text{disc}} h_T^{-\frac{1}{2}} \|v_{T^i}\|_{T^i}$.
- (Discrete inverse inequality) $\|\nabla v_{T^i}\|_{T^i} \leq c_{\text{disc}} h_T^{-1} \|v_{T^i}\|_{T^i}$.
- (Discrete Poincaré inequality) $\|v_{T^i}\|_{T^i} \leq c_{\text{disc}} h_T \|\nabla v_{T^i}\|_{T^i}$ whenever $(v_{T^i}, 1)_{B(\tilde{\mathbf{x}}_{T^i}, h_T)} = 0$.
- (Discrete Korn's inequality) $\|\nabla v_{T^i}\|_{T^i} \leq c_{\text{disc}} \|\nabla^s v_{T^i}\|_{T^i}$ whenever $(v_{T^i}, r)_{B(\tilde{\mathbf{x}}_{T^i}, h_T)} = 0$ for all $r \in \text{RM} := \{r \in \mathbb{P}^1(\mathbb{R}^d; \mathbb{R}^d) \mid \nabla^s r = 0\}$.

Proof. The discrete trace inequality is shown in [9, Lemma 3.4], and the discrete inverse and Poincaré inequalities are shown in [7, Lemma 3.1]. Let us now prove the discrete Korn's inequality. Let $T \in \mathcal{T}_h$ be a mesh cell. Invoking Korn's inequality in the ball $B(\tilde{\mathbf{x}}_{T^i}, h_T)$ (with constant c_0 , see [23]) followed by the inverse inequality $\|\nabla^s v_{T^i}\|_{B(\tilde{\mathbf{x}}_{T^i}, h_T)} \leq c_1 \|\nabla^s v_{T^i}\|_{B(\tilde{\mathbf{x}}_{T^i}, \delta h_T)}$ leads to $\|\nabla v_{T^i}\|_{T^i} \leq \|\nabla v_{T^i}\|_{B(\tilde{\mathbf{x}}_{T^i}, h_T)} \leq c_0 \|\nabla^s v_{T^i}\|_{B(\tilde{\mathbf{x}}_{T^i}, h_T)} \leq c_0 c_1 \|\nabla^s v_{T^i}\|_{B(\tilde{\mathbf{x}}_{T^i}, \delta h_T)} \leq c_0 c_1 \|\nabla^s v_{T^i}\|_{T^i}$ since $B(\tilde{\mathbf{x}}_{T^i}, \delta h_T) \subset T^i \subset B(\tilde{\mathbf{x}}_{T^i}, h_T)$. \square

The role of Assumption 2 is to provide the following multiplicative trace inequality (see [9] for the proof).

Lemma 5 (Multiplicative trace inequality). There is $c_{\text{mtr}} > 0$, depending on ρ and γ , such that for all $T \in \mathcal{T}_h$, all $v \in H^1(T^\dagger)$, and all $i \in \{1, 2\}$,

$$\|v\|_{(\partial T)^i \cup T^\Gamma} \leq c_{\text{mtr}} \left(h_T^{-\frac{1}{2}} \|v\|_{T^\dagger} + \|v\|_{T^\dagger}^{\frac{1}{2}} \|\nabla v\|_{T^\dagger}^{\frac{1}{2}} \right). \quad (19)$$

In what follows we use the convention $A \lesssim B$ to abbreviate the inequality $A \leq CB$ for positive real numbers A and B , where the constant C only depends on the polynomial degree $k \geq 0$ used in the unfitted HHO method, the mesh parameters ρ , δ , γ , n_0 and the above constants c_{disc} and c_{mtr} , but does not depend neither on the viscosity coefficients $0 < \nu_1 \leq \nu_2$ nor on the mesh size $h > 0$.

3.3 Polynomial approximation

Let $\mathbf{v} \in H^1(\Omega; \mathbb{R}^d)$ and $q \in L^2(\Omega)$. For all $T \in \mathcal{T}_h$, recalling the spaces $\hat{\mathbf{U}}_T^k$ and P_T^k from (7), we define for all $i \in \{1, 2\}$

$$I_{T^i}^{k+1}(\mathbf{v}) := \Pi_{T^i}^{k+1}(\mathbf{E}_i(\mathbf{v}))|_{T^i}, \quad \hat{I}_T^k(\mathbf{v}) := (I_{T^1}^{k+1}(\mathbf{v}), \Pi_{(\partial T)^1}^k(\mathbf{v}), I_{T^2}^{k+1}(\mathbf{v}), \Pi_{(\partial T)^2}^k(\mathbf{v})) \in \hat{\mathbf{U}}_T^k, \quad (20a)$$

$$J_{T^i}^k(q) := \Pi_{T^i}^k(E_i(q))|_{T^i}, \quad (20b)$$

where $\mathbf{E}_i : H^1(\Omega_i; \mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d; \mathbb{R}^d)$ and $E_i : H^1(\Omega_i; \mathbb{R}) \rightarrow H^1(\mathbb{R}^d; \mathbb{R})$ are stable extension operators (see [11, 31]), $\Pi_{T^i}^{k+1}$ and $\Pi_{T^i}^k$ denote the L^2 -orthogonal projectors onto $\mathbb{P}^{k+1}(T^i; \mathbb{R}^d)$ and $\mathbb{P}^k(T^i)$, respectively, T^\dagger is defined in Assumption 2, and $\Pi_{(\partial T)^i}^k$ is defined below (11). We also define $\hat{I}_h^k(\mathbf{v}) \in \hat{\mathbf{U}}_h^k$ such that, for all $T \in \mathcal{T}_h$, the local components of $\hat{I}_h^k(\mathbf{v})$ in T are $\hat{I}_T^k(\mathbf{v}) \in \hat{\mathbf{U}}_T^k$. Note that $\hat{I}_h^k(\mathbf{v}) \in \hat{\mathbf{U}}_{h_0}^k$ whenever $\mathbf{v} \in H_0^1(\Omega; \mathbb{R}^d)$.

Lemma 6 (Local approximation and global stability). Let $\mathbf{v} \in H^1(\Omega; \mathbb{R}^d)$ and $q \in L^2(\Omega)$. For all $i \in \{1, 2\}$, we assume that $\mathbf{v}|_{\Omega_i} \in H^{m+1}(\Omega_i)$ and $q|_{\Omega_i} \in H^m(\Omega_i)$ for some $m \geq 0$. Let $\ell := \min(k+1, m)$. For all $T \in \mathcal{T}_h$ and all $i \in \{1, 2\}$, we have

$$\|\mathbf{v} - I_{T^i}^{k+1}(\mathbf{v})\|_{T^i} + h_T^{\frac{1}{2}} \|\mathbf{v} - I_{T^i}^{k+1}(\mathbf{v})\|_{(\partial T)^i \cup T^\Gamma} + h_T \|\mathbf{v} - I_{T^i}^{k+1}(\mathbf{v})\|_{H^1(T^i)} \lesssim h_T^{\ell+1} |\mathbf{E}_i(\mathbf{v})|_{H^{\ell+1}(T^i)}, \quad (21a)$$

$$\|J_{T^i}^k(q) - q\|_{T^i} + h_T^{\frac{1}{2}} \|J_{T^i}^k(q) - q\|_{(\partial T)^i \cup T^\Gamma} \lesssim h_T^\ell |E_i(q)|_{H^\ell(T^i)}. \quad (21b)$$

Moreover we have

$$\sum_{T \in \mathcal{T}_h} |I_{T^i}^{k+1}(\mathbf{v})|_{H^1(T^i)}^2 \lesssim |\mathbf{v}|_{H^1(\Omega_i)}^2, \quad (22a)$$

$$\sum_{T \in \mathcal{T}_h} \nu_i h_T^{-1} \|[\mathbf{v} - I_{T^i}^{k+1}(\mathbf{v})]_\Gamma\|_{T^\Gamma}^2 + \sum_{i \in \{1, 2\}} \nu_i h_T^{-1} \|\Pi_{(\partial T)^i}^k(\mathbf{v}) - I_{T^i}^{k+1}(\mathbf{v})\|_{(\partial T)^i}^2 \lesssim \sum_{i \in \{1, 2\}} \nu_i |\mathbf{v}|_{H^1(\Omega_i)}^2. \quad (22b)$$

Proof. The local approximation properties (21a)-(21b) follow from the approximation properties in L^2 and H^1 of the projectors $\Pi_{T^\dagger}^{k+1}$ and $\Pi_{T^\dagger}^k$ together with the multiplicative trace inequality from Lemma 5. The global bound (22a) is a consequence of the H^1 -stability of $\Pi_{T^\dagger}^{k+1}$, Assumption 3, and the H^1 -stability of the extension operator \mathbf{E}_i since we have

$$\sum_{T \in \mathcal{T}_h} |I_{T^i}^{k+1}(\mathbf{v})|_{H^1(T^i)}^2 \leq \sum_{T \in \mathcal{T}_h} |I_{T^i}^{k+1}(\mathbf{v})|_{H^1(T^\dagger)}^2 \lesssim \sum_{T \in \mathcal{T}_h} |\mathbf{E}_i(\mathbf{v})|_{H^1(T^\dagger)}^2 \lesssim |\mathbf{E}_i(\mathbf{v})|_{H^1(\mathbb{R}^d)}^2 \lesssim |\mathbf{v}|_{H^1(\Omega_i)}^2.$$

Finally the global bound (22b) follows by using similar arguments as above and $\nu_1 \leq \nu_2$. \square

Lemma 7 (Approximation property of symmetric gradient reconstructions). *Let $\mathbf{v} \in H^1(\Omega; \mathbb{R}^d)$ be such that for all $i \in \{1, 2\}$, $\mathbf{v}|_{\Omega_i} \in H^{k+2}(\Omega_i; \mathbb{R}^d)$. For all $T \in \mathcal{T}_h$, we have*

$$\|\mathbb{E}_{T^1}^k(\hat{I}_T^k(\mathbf{v})) - \nabla^s \mathbf{v}\|_{T^1} + h_T^{\frac{1}{2}} \|\mathbb{E}_{T^1}^k(\hat{I}_T^k(\mathbf{v})) - \nabla^s \mathbf{v}\|_{(\partial T)^1 \cup T^\Gamma} \lesssim h_T^{k+1} \sum_{i \in \{1, 2\}} |\mathbf{E}_i(\mathbf{v})|_{H^{k+2}(T^\dagger)}, \quad (23a)$$

$$\|\mathbb{E}_{T^2}^k(\hat{I}_T^k(\mathbf{v})) - \nabla^s \mathbf{v}\|_{T^2} + h_T^{\frac{1}{2}} \|\mathbb{E}_{T^2}^k(\hat{I}_T^k(\mathbf{v})) - \nabla^s \mathbf{v}\|_{(\partial T)^2 \cup T^\Gamma} \lesssim h_T^{k+1} |\mathbf{E}_2(\mathbf{v})|_{H^{k+2}(T^\dagger)}. \quad (23b)$$

Proof. We have

$$\begin{aligned} \|\mathbb{E}_{T^1}^k(\hat{I}_T^k(\mathbf{v})) - \nabla^s I_{T^1}^{k+1}(\mathbf{v})\|_{T^1}^2 &= - \left(\mathbf{v} - I_{T^1}^{k+1}(\mathbf{v}), (\nabla^s I_{T^1}^{k+1}(\mathbf{v}) - \mathbb{E}_{T^1}^k(\hat{I}_T^k(\mathbf{v}))) \mathbf{n}_T \right)_{(\partial T)^1} \\ &\quad + \left([I_{T^1}^{k+1}(\mathbf{v})]_\Gamma, (\nabla^s I_{T^1}^{k+1}(\mathbf{v}) - \mathbb{E}_{T^1}^k(\hat{I}_T^k(\mathbf{v}))) \mathbf{n}_\Gamma \right)_{T^\Gamma}, \end{aligned}$$

where we used the definition of $\mathbb{E}_{T^1}^k$ and the fact that $\mathbb{E}_{T^1}^k(\hat{I}_T^k(\mathbf{v})) - \nabla^s I_{T^1}^{k+1}(\mathbf{v})$ belongs to $\mathbb{P}^k(T^1; \mathcal{S}^{d \times d})$ to replace $\Pi_{(\partial T)^1}^k(\mathbf{v})$ by \mathbf{v} . Invoking the Cauchy–Schwarz, the inverse trace, and the triangle inequalities and since $[\mathbf{v}]_\Gamma = \mathbf{0}$ because $\mathbf{v} \in H^1(\Omega; \mathbb{R}^d)$, we infer that

$$\|\mathbb{E}_{T^1}^k(\hat{I}_T^k(\mathbf{v})) - \nabla^s I_{T^1}^{k+1}(\mathbf{v})\|_{T^1} \lesssim h_T^{-\frac{1}{2}} \left(\|\mathbf{v} - I_{T^1}^{k+1}(\mathbf{v})\|_{(\partial T)^1} + \|\mathbf{v} - I_{T^1}^{k+1}(\mathbf{v})\|_{T^\Gamma} + \|\mathbf{v} - I_{T^2}^{k+1}(\mathbf{v})\|_{T^\Gamma} \right).$$

The local approximation property (21a) implies that

$$\|\mathbb{E}_{T^1}^k(\hat{I}_T^k(\mathbf{v})) - \nabla^s I_{T^1}^{k+1}(\mathbf{v})\|_{T^1} \lesssim h_T^{k+1} \sum_{i \in \{1, 2\}} |\mathbf{E}_i(\mathbf{v})|_{H^{k+2}(T^\dagger)}.$$

The discrete trace inequality from Lemma 4 also gives

$$h_T^{\frac{1}{2}} \|\mathbb{E}_{T^1}^k(\hat{I}_T^k(\mathbf{v})) - \nabla^s I_{T^1}^{k+1}(\mathbf{v})\|_{(\partial T)^1 \cup T^\Gamma} \lesssim h_T^{k+1} \sum_{i \in \{1, 2\}} |\mathbf{E}_i(\mathbf{v})|_{H^{k+2}(T^\dagger)}.$$

Finally the triangle inequality applied to $\mathbb{E}_{T^1}^k(\hat{I}_T^k(\mathbf{v})) - \nabla^s \mathbf{v} = \mathbb{E}_{T^1}^k(\hat{I}_T^k(\mathbf{v})) - \nabla^s I_{T^1}^{k+1}(\mathbf{v}) + \nabla^s (I_{T^1}^{k+1}(\mathbf{v}) - \mathbf{v})$ together with the local H^1 -approximation property (21a) yields (23a). The proof of (23b) uses similar arguments, but is simpler since we do not need to consider the jump across Γ . \square

4 Stability and error analysis

In this section we analyze the convergence of the unfitted HHO method for the Stokes interface problem. The proof consists in establishing stability together with consistency and boundedness properties. Note that most of the results in this section hold under the condition $k \geq 1$. This condition is indeed needed to utilize the discrete Korn's inequality which invokes an orthogonality property with respect to rigid body motions.

4.1 Stability and well-posedness

First we establish the coercivity and boundedness of the bilinear form a_h related to the viscous term. The proofs are only sketched since they dwell on the arguments from [17] in the fitted case and [9] in the unfitted one. We consider the following local semi-norm: For all $T \in \mathcal{T}_h$ and all $\hat{\mathbf{v}}_T := (\mathbf{v}_{T^1}, \mathbf{v}_{(\partial T)^1}, \mathbf{v}_{T^2}, \mathbf{v}_{(\partial T)^2}) \in \hat{\mathbf{U}}_T^k$,

$$|\hat{\mathbf{v}}_T|_{\mathbf{U}_T}^2 := \sum_{i \in \{1, 2\}} \nu_i \left(\|\nabla^s \mathbf{v}_{T^i}\|_{T^i}^2 + h_T^{-1} \|\mathbf{v}_{T^i} - \mathbf{v}_{(\partial T)^i}\|_{(\partial T)^i}^2 \right) + \nu_1 h_T^{-1} \|[\mathbf{v}_T]_\Gamma\|_{T^\Gamma}^2, \quad (24)$$

and $\|\hat{\mathbf{v}}_h\|_{\mathbf{U}_h}^2 := \sum_{T \in \mathcal{T}_h} |\hat{\mathbf{v}}_T|_{\mathbf{U}_T}^2$. Note that $\|\cdot\|_{\mathbf{U}_h}$ defines a norm on $\hat{\mathbf{U}}_{h0}^k$. Indeed, if $\|\hat{\mathbf{v}}_h\|_{\mathbf{U}_h} = 0$, then $|\hat{\mathbf{v}}_T|_{\mathbf{U}_T} = 0$ for all $T \in \mathcal{T}_h$, which implies that \mathbf{v}_{T^i} and $\mathbf{v}_{(\partial T)^i}$ take the same constant value for all $i \in \{1, 2\}$ and that $[\mathbf{v}_T]_\Gamma = \mathbf{0}$; we can then propagate the constant value of $\mathbf{v}_{T^{i_0}}$ up to the boundary $\partial\Omega$ (where i_0 is the index of the subdomain touching the boundary of Ω) where $\mathbf{v}_{T^{i_0}}$ is zero by definition of $\hat{\mathbf{U}}_{h0}^k$; finally, we use the zero jump condition across Γ to propagate the zero value to the other subdomain.

Lemma 8 (Coercivity of a_h). *Let $k \geq 1$. There exists $c_{\text{coer}} > 0$ such that for all $\hat{\mathbf{v}}_h \in \hat{\mathbf{U}}_{h0}^k$, the following holds true:*

$$c_{\text{coer}} \|\hat{\mathbf{v}}_h\|_{\mathbf{U}_h}^2 \leq a_h(\hat{\mathbf{v}}_h, \hat{\mathbf{v}}_h). \quad (25)$$

Proof. Using the usual arguments one proves that

$$\sum_{T \in \mathcal{T}_h} \nu_1 h_T^{-1} \|[\mathbf{v}_T]_{\Gamma}\|_{T^{\Gamma}}^2 + \sum_{i \in \{1,2\}} \nu_i (\|\nabla^s \mathbf{v}_{T^i}\|_{T^i}^2 + h_T^{-1} \|\Pi_{(\partial T)^i}^k(\mathbf{v}_{T^i}) - \mathbf{v}_{(\partial T)^i}\|_{(\partial T)^i}^2) \lesssim a_h(\hat{\mathbf{v}}_h, \hat{\mathbf{v}}_h).$$

For $T \in \mathcal{T}_h$ and $i \in \{1,2\}$, to prove that $h_T^{-1} \|(I - \Pi_{(\partial T)^i}^k)(\mathbf{v}_{T^i})\|_{(\partial T)^i}^2 \lesssim \|\nabla^s \mathbf{v}_{T^i}\|_{T^i}^2$, one observes that since $k \geq 1$,

$$\begin{aligned} h_T^{-1} \|(I - \Pi_{(\partial T)^i}^k)(\mathbf{v}_{T^i})\|_{(\partial T)^i}^2 &\leq h_T^{-1} \|(I - \Pi_{(\partial T)^i}^1)(\mathbf{v}_{T^i})\|_{(\partial T)^i}^2 \leq h_T^{-1} \|(I - \Pi_{(\partial T)^i}^1)(\mathbf{v}_{T^i} - \Pi_{\text{RM}}(\mathbf{v}_{T^i}))\|_{(\partial T)^i}^2 \\ &\leq h_T^{-1} \|\mathbf{v}_{T^i} - \Pi_{\text{RM}}(\mathbf{v}_{T^i})\|_{(\partial T)^i}^2 \lesssim h_T^{-2} \|\mathbf{v}_{T^i} - \Pi_{\text{RM}}(\mathbf{v}_{T^i})\|_{T^i}^2 \\ &\lesssim \|\nabla(\mathbf{v}_{T^i} - \Pi_{\text{RM}}(\mathbf{v}_{T^i}))\|_{T^i}^2 \lesssim \|\nabla^s(\mathbf{v}_{T^i} - \Pi_{\text{RM}}(\mathbf{v}_{T^i}))\|_{T^i}^2 \leq \|\nabla^s \mathbf{v}_{T^i}\|_{T^i}^2, \end{aligned}$$

where $\Pi_{\text{RM}}(\mathbf{v}_{T^i})$ is the L^2 -orthogonal projection of \mathbf{v}_{T^i} onto RM defined in Lemma 4. The above argument used the discrete inverse, Poincaré and Korn's inequalities from Lemma 4. \square

Lemma 9 (Boundedness of a_h). *Let $k \geq 0$. For all $\hat{\mathbf{v}}_h, \hat{\mathbf{w}}_h \in \hat{\mathbf{U}}_h^k$, the following holds true:*

$$a_h(\hat{\mathbf{v}}_h, \hat{\mathbf{w}}_h) \lesssim \|\hat{\mathbf{v}}_h\|_{\mathbf{U}_h} \|\hat{\mathbf{w}}_h\|_{\mathbf{U}_h}. \quad (26)$$

Proof. It suffices to prove that $a_T(\hat{\mathbf{v}}_T, \hat{\mathbf{w}}_T) \lesssim |\hat{\mathbf{v}}_T|_{\mathbf{U}_T} |\hat{\mathbf{w}}_T|_{\mathbf{U}_T}$ for all $T \in \mathcal{T}_h$. The Cauchy–Schwarz inequality and the L^2 -stability of $\Pi_{(\partial T)^i}^k$ imply that $|s_T(\hat{\mathbf{v}}_T, \hat{\mathbf{w}}_T)| \leq |\hat{\mathbf{v}}_T|_{\mathbf{U}_T} |\hat{\mathbf{w}}_T|_{\mathbf{U}_T}$. Moreover the definition (8a) of the reconstructed symmetric gradient yields

$$\begin{aligned} \|\mathbb{E}_{T^1}^k(\hat{\mathbf{v}}_T)\|_{T^1}^2 &= (\mathbb{E}_{T^1}^k(\hat{\mathbf{v}}_T), \mathbb{E}_{T^1}^k(\hat{\mathbf{v}}_T))_{T^1} \\ &= (\nabla^s \mathbf{v}_{T^1}, \mathbb{E}_{T^1}^k(\hat{\mathbf{v}}_T))_{T^1} + (\mathbf{v}_{(\partial T)^1} - \mathbf{v}_{T^1}, \mathbb{E}_{T^1}^k(\hat{\mathbf{v}}_T) \mathbf{n}_T)_{(\partial T)^1} - ([\mathbf{v}_T]_{\Gamma}, \mathbb{E}_{T^1}^k(\hat{\mathbf{v}}_T) \mathbf{n}_{\Gamma})_{T^{\Gamma}} \\ &\leq \|\mathbb{E}_{T^1}^k(\hat{\mathbf{v}}_T)\|_{T^1} \|\nabla^s \mathbf{v}_{T^1}\|_{T^1} + h_T^{\frac{1}{2}} \|\mathbb{E}_{T^1}^k(\hat{\mathbf{v}}_T)\|_{(\partial T)^1} h_T^{-\frac{1}{2}} \|\mathbf{v}_{(\partial T)^1} - \mathbf{v}_{T^1}\|_{(\partial T)^1} \\ &\quad + h_T^{\frac{1}{2}} \|\mathbb{E}_{T^1}^k(\hat{\mathbf{v}}_T)\|_{T^{\Gamma}} h_T^{-\frac{1}{2}} \|[\mathbf{v}_T]_{\Gamma}\|_{T^{\Gamma}}. \end{aligned}$$

Invoking the discrete trace inequality from Lemma 4 shows that $\nu_1 \|\mathbb{E}_{T^1}^k(\hat{\mathbf{v}}_T)\|_{T^1}^2 \lesssim |\hat{\mathbf{v}}_T|_{\mathbf{U}_T}^2$. A similar inequality is obtained for $\|\mathbb{E}_{T^2}^k(\hat{\mathbf{v}}_T)\|_{T^2}^2$. \square

We now define, for all $T \in \mathcal{T}_h$ and all $(\hat{\mathbf{v}}_T, p_T) \in Y_T^k$,

$$\begin{aligned} |(\hat{\mathbf{v}}_T, r_T)|_{Y_T}^2 &:= |\hat{\mathbf{v}}_T|_{\mathbf{U}_T}^2 + \sum_{i \in \{1,2\}} \nu_i^{-1} \|r_{T^i}\|_{T^i}^2 \\ &= \sum_{i \in \{1,2\}} \left(\nu_i \|\nabla^s \mathbf{v}_{T^i}\|_{T^i}^2 + \nu_i h_T^{-1} \|\mathbf{v}_{T^i} - \mathbf{v}_{(\partial T)^i}\|_{(\partial T)^i}^2 + \nu_i^{-1} \|r_{T^i}\|_{T^i}^2 \right) \\ &\quad + \nu_1 h_T^{-1} \|[\mathbf{v}_T]_{\Gamma}\|_{T^{\Gamma}}^2. \end{aligned} \quad (27)$$

Summing the local semi-norms over the cells, we define, for all $(\hat{\mathbf{v}}_h, r_h) \in Y_h^k$,

$$\|(\hat{\mathbf{v}}_h, r_h)\|_{Y_h}^2 := \sum_{T \in \mathcal{T}_h} |(\hat{\mathbf{v}}_T, r_T)|_{Y_T}^2 = \|\hat{\mathbf{v}}_h\|_{\mathbf{U}_h}^2 + \sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} \nu_i^{-1} \|r_{T^i}\|_{T^i}^2. \quad (28)$$

Note that $\|\cdot\|_{Y_h}$ defines a norm on Y_h^k .

Lemma 10 (Inf-sup stability of A_h). *Let $k \geq 1$. Assume that the penalty parameter $\chi > 0$ is small enough so that $\chi \leq \frac{\min(1, c_{\text{coer}})}{16} c_{\text{disc}}^{-2}$, where c_{disc} is the constant in the discrete trace inequality from Lemma 4 and c_{coer} is the coercivity constant from Lemma 8. There exists $\beta > 0$ such that for all $(\hat{\mathbf{v}}_h, r_h) \in Y_h^k$, we have*

$$\beta \|(\hat{\mathbf{v}}_h, r_h)\|_{Y_h} \leq \sup_{(\hat{\mathbf{w}}_h, q_h) \in Y_h^k \setminus \{0\}} \frac{A_h((\hat{\mathbf{v}}_h, r_h), (\hat{\mathbf{w}}_h, q_h))}{\|(\hat{\mathbf{w}}_h, q_h)\|_{Y_h}}. \quad (29)$$

Moreover the discrete problem (16) is well-posed.

Proof. Let $(\hat{\mathbf{v}}_h, r_h) \in Y_h^k$ and set $S := \sup_{(\hat{\mathbf{w}}_h, q_h) \in Y_h^k \setminus \{0\}} \frac{A_h((\hat{\mathbf{v}}_h, r_h), (\hat{\mathbf{w}}_h, q_h))}{\|(\hat{\mathbf{w}}_h, q_h)\|_{Y_h}}$.

• Control on the velocity and pressure jumps. Recalling the definitions of A_h and A_T (see (17) and (12a)), we have

$$A_h((\hat{\mathbf{v}}_h, r_h), (\hat{\mathbf{v}}_h, r_h)) = a_h(\hat{\mathbf{v}}_h, \hat{\mathbf{v}}_h) + \chi \sum_{T \in \mathcal{T}_h} \nu_2^{-1} h_T (\| [r_T]]_\Gamma \|_{T^\Gamma}^2 - \| [\sigma(\mathbf{v}_T, 0)]_\Gamma \mathbf{n}_\Gamma \|_{T^\Gamma}^2).$$

We use the coercivity of a_h (Lemma 8) to infer that

$$\begin{aligned} \min(1, c_{\text{coer}}) \left(\| \hat{\mathbf{v}}_h \|_{\mathcal{U}_h}^2 + \chi \sum_{T \in \mathcal{T}_h} \nu_2^{-1} h_T \| [r_T]]_\Gamma \|_{T^\Gamma}^2 \right) &\leq a_h(\hat{\mathbf{v}}_h, \hat{\mathbf{v}}_h) + \chi \sum_{T \in \mathcal{T}_h} \nu_2^{-1} h_T \| [r_T]]_\Gamma \|_{T^\Gamma}^2 \\ &\leq A_h((\hat{\mathbf{v}}_h, r_h), (\hat{\mathbf{v}}_h, r_h)) + \chi \sum_{T \in \mathcal{T}_h} \nu_2^{-1} h_T \| [\sigma(\mathbf{v}_T, 0)]_\Gamma \mathbf{n}_\Gamma \|_{T^\Gamma}^2. \end{aligned}$$

Since $\nu_1 \leq \nu_2$, we have $\sum_{T \in \mathcal{T}_h} \nu_2^{-1} h_T \| [\sigma(\mathbf{v}_T, 0)]_\Gamma \mathbf{n}_\Gamma \|_{T^\Gamma}^2 \leq 8c_{\text{disc}}^2 \| \hat{\mathbf{v}}_h \|_{\mathcal{U}_h}^2$ owing to the discrete trace inequality from Lemma 4. Since $\chi \leq \frac{\min(1, c_{\text{coer}})}{16} c_{\text{disc}}^{-2}$ by assumption, the rightmost term in the above right-hand side can be hidden on the left-hand side. We infer that

$$\| \hat{\mathbf{v}}_h \|_{\mathcal{U}_h}^2 + \chi \sum_{T \in \mathcal{T}_h} \nu_2^{-1} h_T \| [r_T]]_\Gamma \|_{T^\Gamma}^2 \lesssim A_h((\hat{\mathbf{v}}_h, r_h), (\hat{\mathbf{v}}_h, r_h)) \leq S \| (\hat{\mathbf{v}}_h, r_h) \|_{Y_h}. \quad (30)$$

• Control on the pressure gradient. Let $\hat{\mathbf{w}}_h \in \hat{\mathcal{U}}_{h0}^k$ be such that, for all $T \in \mathcal{T}_h$, $\hat{\mathbf{w}}_T := (\mathbf{w}_{T^1}, \mathbf{0}, \mathbf{w}_{T^2}, \mathbf{0})$ with $\mathbf{w}_{T^i} := -\nu_i^{-1} h_T^2 \nabla r_{T^i}$. We have

$$\begin{aligned} h_T^2 \sum_{i \in \{1,2\}} \nu_i^{-1} \| \nabla r_{T^i} \|_{T^i}^2 &= - \sum_{i \in \{1,2\}} (\nabla r_{T^i}, \mathbf{w}_{T^i})_{T^i} \\ &= (r_{T^2} \mathbf{n}_\Gamma, \mathbf{w}_{T^2})_{T^\Gamma} - (r_{T^1} \mathbf{n}_\Gamma, \mathbf{w}_{T^1})_{T^\Gamma} + \sum_{i \in \{1,2\}} \left((r_{T^i}, \nabla \cdot \mathbf{w}_{T^i})_{T^i} - (r_{T^i} \mathbf{n}_T, \mathbf{w}_{T^i})_{(\partial T)^i} \right) \\ &= \sum_{i \in \{1,2\}} (r_{T^i}, D_{T^i}^k(\hat{\mathbf{w}}_T))_{T^i} - ([r_T]]_\Gamma \mathbf{n}_\Gamma, \mathbf{w}_{T^2})_{T^\Gamma}. \end{aligned}$$

We infer that

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} h_T^2 \sum_{i \in \{1,2\}} \nu_i^{-1} \| \nabla r_{T^i} \|_{T^i}^2 &= b_h(\hat{\mathbf{w}}_h, r_h) - \sum_{T \in \mathcal{T}_h} ([r_T]]_\Gamma \mathbf{n}_\Gamma, \mathbf{w}_{T^2})_{T^\Gamma} \\ &= -A_h((\hat{\mathbf{v}}_h, r_h), (\hat{\mathbf{w}}_h, 0)) + a_h(\hat{\mathbf{v}}_h, \hat{\mathbf{w}}_h) - \sum_{T \in \mathcal{T}_h} ([r_T]]_\Gamma \mathbf{n}_\Gamma, \mathbf{w}_{T^2})_{T^\Gamma} \\ &\quad - \chi \sum_{T \in \mathcal{T}_h} \nu_2^{-1} h_T ([\sigma(\mathbf{v}_T, r_T)]_\Gamma \mathbf{n}_\Gamma, [\sigma(\mathbf{w}_T, 0)]_\Gamma \mathbf{n}_\Gamma)_{T^\Gamma}. \end{aligned}$$

Let us denote A_1, A_2, A_3 , and A_4 the four terms on the right-hand side. We have $|A_1| \leq S \| \hat{\mathbf{w}}_h \|_{\mathcal{U}_h}$ by definition of S , and we have $|A_2| \lesssim \| \hat{\mathbf{v}}_h \|_{\mathcal{U}_h} \| \hat{\mathbf{w}}_h \|_{\mathcal{U}_h}$ owing to the boundedness of a_h (Lemma 9). Moreover, the Cauchy–Schwarz and Young inequalities, the discrete trace inequality from Lemma 4 and the definition of \mathbf{w}_{T^2} imply that

$$|A_3| \leq \alpha \sum_{T \in \mathcal{T}_h} h_T^2 \nu_2^{-1} \| \nabla r_{T^2} \|_{T^2}^2 + c_\alpha \sum_{T \in \mathcal{T}_h} \nu_2^{-1} h_T \| [r_T]]_\Gamma \|_{T^\Gamma}^2,$$

where α can be chosen as small as needed and $c_\alpha > 0$. Finally, recalling the definition of $\sigma(\mathbf{v}_T, r_T)$, using that $\nu_1 \leq \nu_2$, and invoking again the discrete trace inequality from Lemma 4 leads to

$$|A_4| \lesssim \left(\| \hat{\mathbf{v}}_h \|_{\mathcal{U}_h}^2 + \sum_{T \in \mathcal{T}_h} \nu_2^{-1} h_T \| [r_T]]_\Gamma \|_{T^\Gamma}^2 \right)^{\frac{1}{2}} \| \hat{\mathbf{w}}_h \|_{\mathcal{U}_h}.$$

Putting everything together and choosing α small enough, we infer that

$$\sum_{T \in \mathcal{T}_h} h_T^2 \sum_{i \in \{1,2\}} \nu_i^{-1} \| \nabla r_{T^i} \|_{T^i}^2 \lesssim S^2 + \| \hat{\mathbf{v}}_h \|_{\mathcal{U}_h}^2 + \sum_{T \in \mathcal{T}_h} \nu_2^{-1} h_T \| [r_T]]_\Gamma \|_{T^\Gamma}^2.$$

Combined with (30), the above bound implies that

$$\sum_{T \in \mathcal{T}_h} h_T^2 \sum_{i \in \{1,2\}} \nu_i^{-1} \| \nabla r_{T^i} \|_{T^i}^2 \lesssim S^2 + S \| (\hat{\mathbf{v}}_h, r_h) \|_{Y_h}. \quad (31)$$

• Control on the pressure in the L^2 -norm. Letting $\kappa := \frac{1}{|\Omega|} \sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} \nu_i^{-1} (r_{T^i}, 1)_{T^i}$, there exists $\mathbf{v}_{r_h} \in H_0^1(\Omega)$ such that $\nabla \cdot \mathbf{v}_{r_h} = \nu_i^{-1} r_h - \kappa$ in Ω_i , i.e. $(\nabla \cdot \mathbf{v}_{r_h})|_{T^i} = \nu_i^{-1} r_{T^i} - \kappa$ for all $T \in \mathcal{T}_h$ and all $i \in \{1, 2\}$, and we also have $\sum_{i \in \{1,2\}} \|\mathbf{v}_{r_h}\|_{H^1(\Omega_i)} \lesssim \sum_{i \in \{1,2\}} \nu_i^{-1} (\sum_{T \in \mathcal{T}_h} \|r_{T^i}\|_{T^i}^2)^{\frac{1}{2}}$. Let $\hat{\mathbf{w}}_h := \hat{I}_h^k(\mathbf{v}_{r_h})$ and note that $\hat{\mathbf{w}}_h \in \hat{U}_{h_0}^k$. We have

$$\sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} \nu_i^{-1} \|r_{T^i}\|_{T^i}^2 = \sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} (r_{T^i}, \nabla \cdot \mathbf{v}_{r_h})_{T^i},$$

since $\kappa \sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} (r_{T^i}, 1)_{T^i} = 0$ because $r_h \in P_{h*}^k$. Therefore we can write $\sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} \nu_i^{-1} \|r_{T^i}\|_{T^i}^2 = \Psi_1 + \Psi_2$ with

$$\Psi_1 := \sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} (r_{T^i}, \nabla \cdot \mathbf{v}_{r_h} - D_{T^i}^k(\hat{\mathbf{w}}_T))_{T^i}, \quad \Psi_2 := \sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} (r_{T^i}, D_{T^i}^k(\hat{\mathbf{w}}_T))_{T^i} = b_h(\hat{\mathbf{w}}_h, r_h).$$

We observe that

$$\begin{aligned} \Psi_1 &= \sum_{T \in \mathcal{T}_h} ([[\mathbf{w}_T]]_\Gamma, r_{T^1} \mathbf{n}_\Gamma)_{T^\Gamma} + \sum_{i \in \{1,2\}} (r_{T^i}, \nabla \cdot (\mathbf{v}_{r_h} - \mathbf{w}_{T^i}))_{T^i} + (r_{T^i} \mathbf{n}_T, \mathbf{w}_{T^i} - \mathbf{w}_{(\partial T)^i})_{(\partial T)^i} \\ &= \sum_{T \in \mathcal{T}_h} -(r_{T^1} \mathbf{n}_\Gamma, \mathbf{w}_{T^1} - \mathbf{v}_{r_h})_{T^\Gamma} + (r_{T^2} \mathbf{n}_\Gamma, \mathbf{w}_{T^2} - \mathbf{v}_{r_h})_{T^\Gamma} + ([[\mathbf{w}_T]]_\Gamma, r_{T^1} \mathbf{n}_\Gamma)_{T^\Gamma} \\ &\quad + \sum_{i \in \{1,2\}} -(\nabla r_{T^i}, \mathbf{v}_{r_h} - \mathbf{w}_{T^i})_{T^i} + (r_{T^i} \mathbf{n}_T, \mathbf{v}_{r_h} - \mathbf{w}_{(\partial T)^i})_{(\partial T)^i} \\ &= \sum_{T \in \mathcal{T}_h} -(\mathbf{w}_{T^2} - \mathbf{v}_{r_h}, [[r_T]]_\Gamma \mathbf{n}_\Gamma)_{T^\Gamma} + \sum_{i \in \{1,2\}} -(\nabla r_{T^i}, \mathbf{v}_{r_h} - \mathbf{w}_{T^i})_{T^i} + (r_{T^i} \mathbf{n}_T, \mathbf{v}_{r_h} - \mathbf{w}_{(\partial T)^i})_{(\partial T)^i}. \end{aligned}$$

We have $(r_{T^i} \mathbf{n}_T, \mathbf{v}_{r_h} - \mathbf{w}_{(\partial T)^i})_{(\partial T)^i} = 0$ and the approximation properties of \mathbf{w}_{T^i} (Lemma 6) give

$$\begin{aligned} |\Psi_1| &\lesssim \sum_{T \in \mathcal{T}_h} \|\mathbf{w}_{T^2} - \mathbf{v}_{r_h}\|_{T^\Gamma} \|[[r_T]]_\Gamma\|_{T^\Gamma} + \sum_{i \in \{1,2\}} \|\nabla r_{T^i}\|_{T^i} \|\mathbf{v}_{r_h} - I_{T^i}^{k+1}(\mathbf{v}_{r_h})\|_{T^i} \\ &\lesssim ((\sum_{T \in \mathcal{T}_h} \nu_2^{-1} h_T \|[[r_T]]_\Gamma\|_{T^\Gamma}^2)^{\frac{1}{2}} + (\sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} \nu_i^{-1} h_T^2 \|\nabla r_{T^i}\|_{T^i}^2)^{\frac{1}{2}}) (\sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} \nu_i |\mathbf{E}_i(\mathbf{v}_{r_h})|_{H^1(T^i)}^2)^{\frac{1}{2}}. \end{aligned}$$

Moreover by definition of the bilinear form A_h , we have

$$\Psi_2 = -A_h((\hat{\mathbf{v}}_h, r_h), (\hat{\mathbf{w}}_h, 0)) + a_h(\hat{\mathbf{v}}_h, \hat{\mathbf{w}}_h) - \chi \sum_{T \in \mathcal{T}_h} \nu_2^{-1} h_T ([[\sigma(\mathbf{v}_T, r_T)]]_\Gamma \mathbf{n}_\Gamma, [[\sigma(\mathbf{w}_T, 0)]]_\Gamma \mathbf{n}_\Gamma)_{T^\Gamma},$$

so that

$$|\Psi_2| \lesssim S \|\hat{\mathbf{w}}_h\|_{\mathcal{U}_h} + \|\hat{\mathbf{v}}_h\|_{\mathcal{U}_h} \|\hat{\mathbf{w}}_h\|_{\mathcal{U}_h} + \chi \sum_{T \in \mathcal{T}_h} \nu_2^{-\frac{1}{2}} h_T^{\frac{1}{2}} \|[[r_T]]_\Gamma\|_{T^\Gamma} \|\hat{\mathbf{w}}_h\|_{\mathcal{U}_h},$$

where we used the boundedness of a_h (cf Lemma 9). We now use the estimates

$$\|\hat{\mathbf{w}}_h\|_{\mathcal{U}_h}^2 \lesssim \nu_1 |\mathbf{v}_{r_h}|_{H^1(\Omega_1)}^2 + \nu_2 |\mathbf{v}_{r_h}|_{H^1(\Omega_2)}^2 \lesssim \|\nu^{-\frac{1}{2}} r_h\|_{L^2(\Omega)}^2,$$

and $\sum_{T \in \mathcal{T}_h} |\mathbf{E}_i(\mathbf{v}_{r_h})|_{H^1(T^i)}^2 \lesssim |\mathbf{E}_i(\mathbf{v}_{r_h})|_{H^1(\mathbb{R}^d)}^2 \lesssim |\mathbf{v}_{r_h}|_{H^1(\Omega_i)}^2 \lesssim \|\nu^{-\frac{1}{2}} r_h\|_{L^2(\Omega)}^2$ (see Lemma 6). With Young's inequality, the estimates on Ψ_1 and Ψ_2 give

$$\sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} \nu_i^{-1} \|r_{T^i}\|_{T^i}^2 \lesssim S^2 + \|\mathbf{v}_h\|_{\mathcal{U}_h}^2 + \chi \sum_{T \in \mathcal{T}_h} \nu_2^{-1} h_T \|[[r_T]]_\Gamma\|_{T^\Gamma}^2 + \sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} \nu_i^{-1} h_T^2 \|\nabla r_{T^i}\|_{T^i}^2.$$

Combined with (30) and (31), this bound implies that $\sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} \nu_i^{-1} \|r_{T^i}\|_{T^i}^2 \lesssim S^2 + S \|(\hat{\mathbf{v}}_h, r_h)\|_{Y_h}$. Using again (30) and Young's inequality yields the inf-sup condition (29). Finally the inf-sup condition readily implies the well-posedness of (16) which amounts to a square linear system. \square

4.2 Consistency and boundedness

Let $(\mathbf{u}, p) \in H_0^1(\Omega; \mathbb{R}^d) \times L_*^2(\Omega)$ be the solution to the exact problem (1). For all $T \in \mathcal{T}_h$, let us define the discrete errors $\hat{\mathbf{e}}_T := \hat{I}_T^k(\mathbf{u}) - \hat{\mathbf{u}}_T \in \hat{U}_T^k$ and $\eta_T := J_T^k(p) - p_T \in P_T^k$, where $(\hat{\mathbf{u}}_h, p_h)$ is the solution to the discrete problem (16).

Lemma 11 (Consistency and boundedness). *Assume that $(\mathbf{u}, p) \in H^{1+s}(\Omega; \mathbb{R}^d) \times H^s(\Omega)$ for some $s > \frac{1}{2}$. For all $y_h := (\hat{\mathbf{v}}_h, q_h) \in Y_h^k := \hat{\mathbf{U}}_{h0}^k \times P_{h*}^k$, let us define*

$$\mathcal{F}(y_h) := \sum_{T \in \mathcal{T}_h} A_T((\hat{\mathbf{e}}_T, \eta_T), (\hat{\mathbf{v}}_T, q_T)). \quad (32)$$

Let $\delta_{T^i} := \mathbf{u}_i - I_{T^i}^{k+1}(\mathbf{u}_i)$, $\mathfrak{d}_{T^i} := \nabla^s \mathbf{u}_i - \mathbb{E}_{T^i}^k(\hat{I}_T^k(\mathbf{u}))$, $d_{T^i} := \nabla \cdot \mathbf{u}_i - D_{T^i}^k(\hat{I}_T^k(\mathbf{u}))$ and $\xi_{T^i} := p_i - J_{T^i}^k(p)$, for all $T \in \mathcal{T}_h$. We have, for all $y_h \in Y_h^k$,

$$|\mathcal{F}(y_h)| \lesssim \left\{ \sum_{T \in \mathcal{T}_h} \left(\|(\mathfrak{d}_T, \xi_T)\|_{*T}^2 + \|\delta_T\|_{\#T}^2 + \sum_{i \in \{1,2\}} \nu_i \|d_{T^i}\|_{T^i}^2 \right) \right\}^{\frac{1}{2}} \times \|y_h\|_{Y_h}, \quad (33)$$

where

$$\begin{aligned} \|(\mathfrak{d}_T, \xi_T)\|_{*T}^2 &:= \nu_1 h_T \|\mathfrak{d}_{T^1}\|_{T^1}^2 + \sum_{i \in \{1,2\}} \left(\nu_i (\|\mathfrak{d}_{T^i}\|_{T^i}^2 + h_T \|\mathfrak{d}_{T^i}\|_{(\partial T)^i}^2) \right. \\ &\quad \left. + \nu_i^{-1} (\|\xi_{T^i}\|_{T^i}^2 + h_T \|\xi_{T^i}\|_{(\partial T)^i}^2 + h_T \|\xi_{T^i}\|_{T^i}^2) \right), \\ \|\delta_T\|_{\#T}^2 &:= \nu_1 h_T^{-1} \|\delta_{T^1}\|_{T^1}^2 + \sum_{i \in \{1,2\}} \nu_i (h_T^{-1} \|\delta_{T^i}\|_{(\partial T)^i}^2 + h_T \|\nabla \delta_{T^i}\|_{T^i}^2). \end{aligned}$$

Proof. We observe that $\mathcal{F}(y_h) = \Psi_1 + \Psi_2 + \Psi_3$ with

$$\begin{aligned} \Psi_1 &:= \sum_{T \in \mathcal{T}_h} \left(\sum_{i \in \{1,2\}} 2\nu_i (\mathbb{E}_{T^i}^k(\hat{I}_T^k(\mathbf{u})), \mathbb{E}_{T^i}^k(\hat{\mathbf{v}}_T))_{T^i} - (J_{T^i}^k(p), D_{T^i}^k(\hat{\mathbf{v}}_T))_{T^i} + (\nabla \cdot (2\nu_i \nabla^s \mathbf{u}_i - p_i \mathbb{I}), \mathbf{v}_{T^i})_{T^i} \right) \\ &\quad - (\mathbf{g}_N, \mathbf{v}_{T^2})_{T^1}, \\ \Psi_2 &:= \sum_{T \in \mathcal{T}_h} s_T (\hat{I}_T^k(\mathbf{u}), \hat{\mathbf{v}}_T) - \chi \nu_2^{-1} h_T (\llbracket \sigma(I_T^{k+1}(\mathbf{u}), J_T^k(p)) \rrbracket_{\Gamma} \mathbf{n}_{\Gamma} - \mathbf{g}_N, \llbracket \sigma(\mathbf{v}_T, -q_T) \rrbracket_{\Gamma} \mathbf{n}_{\Gamma})_{T^1}, \\ \Psi_3 &:= \sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} (D_{T^i}^k(\hat{I}_T^k(\mathbf{u})), q_{T^i})_{T^i}. \end{aligned}$$

Using the definitions (8a)-(8b) of $\mathbb{E}_{T^i}^k(\hat{\mathbf{v}}_T)$, we infer that

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} 2\nu_i (\mathbb{E}_{T^i}^k(\hat{I}_T^k(\mathbf{u})), \mathbb{E}_{T^i}^k(\hat{\mathbf{v}}_T))_{T^i} &= \sum_{T \in \mathcal{T}_h} \left\{ -2\nu_1 (\mathbb{E}_{T^1}^k(\hat{I}_T^k(\mathbf{u})), \mathbf{n}_{\Gamma}, \llbracket \mathbf{v}_T \rrbracket_{\Gamma})_{T^1} \right. \\ &\quad \left. + \sum_{i \in \{1,2\}} 2\nu_i \left((\mathbb{E}_{T^i}^k(\hat{I}_T^k(\mathbf{u})), \nabla^s \mathbf{v}_{T^i})_{T^i} + (\mathbb{E}_{T^i}^k(\hat{I}_T^k(\mathbf{u})), \mathbf{n}_T, \mathbf{v}_{(\partial T)^i} - \mathbf{v}_{T^i})_{(\partial T)^i} \right) \right\}. \end{aligned}$$

Similarly, using the definitions (10a)-(10b) of $D_{T^i}^k(\hat{\mathbf{v}}_T)$, we infer that

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} (J_{T^i}^k(p), D_{T^i}^k(\hat{\mathbf{v}}_T))_{T^i} &= \sum_{T \in \mathcal{T}_h} \left\{ - (J_{T^1}^k(p), \mathbf{n}_{\Gamma}, \llbracket \mathbf{v}_T \rrbracket_{\Gamma})_{T^1} \right. \\ &\quad \left. + \sum_{i \in \{1,2\}} \left((J_{T^i}^k(p), \nabla \cdot \mathbf{v}_{T^i})_{T^i} + (J_{T^i}^k(p), \mathbf{n}_T, \mathbf{v}_{(\partial T)^i} - \mathbf{v}_{T^i})_{(\partial T)^i} \right) \right\}. \end{aligned}$$

We integrate by parts on all the mesh cells. Moreover, since the trace of $2\nu_i \nabla^s \mathbf{u}_i - p_i \mathbb{I}$ over the mesh faces is well-defined owing to the regularity assumption made on the exact solution, we have $\sum_{T \in \mathcal{T}_h} ((-2\nu_i \nabla^s \mathbf{u}_i + p_i \mathbb{I}) \mathbf{n}_T, \mathbf{v}_{(\partial T)^i})_{(\partial T)^i} = 0$. We infer that

$$\begin{aligned} &\sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} (\nabla \cdot (2\nu_i \nabla^s \mathbf{u}_i - p_i \mathbb{I}), \mathbf{v}_{T^i})_{T^i} \\ &= \sum_{T \in \mathcal{T}_h} \left\{ ((2\nu_1 \nabla^s \mathbf{u}_1 - p_1 \mathbb{I}) \mathbf{n}_{\Gamma}, \mathbf{v}_{T^1})_{T^1} - ((2\nu_2 \nabla^s \mathbf{u}_2 - p_2 \mathbb{I}) \mathbf{n}_{\Gamma}, \mathbf{v}_{T^2})_{T^1} \right. \\ &\quad \left. + \sum_{i \in \{1,2\}} \left((-2\nu_i \nabla^s \mathbf{u}_i + p_i \mathbb{I}, \nabla^s \mathbf{v}_{T^i})_{T^i} + ((-2\nu_i \nabla^s \mathbf{u}_i + p_i \mathbb{I}) \mathbf{n}_T, \mathbf{v}_{(\partial T)^i} - \mathbf{v}_{T^i})_{(\partial T)^i} \right) \right\}. \end{aligned}$$

Moreover we have

$$\begin{aligned} & \sum_{T \in \mathcal{T}_h} \left(((2\nu_1 \nabla^s \mathbf{u}_1 - p_1 \mathbb{I}) \mathbf{n}_\Gamma, \mathbf{v}_{T^1})_{T^\Gamma} - ((2\nu_2 \nabla^s \mathbf{u}_2 - p_2 \mathbb{I}) \mathbf{n}_\Gamma, \mathbf{v}_{T^2})_{T^\Gamma} \right) \\ &= \sum_{T \in \mathcal{T}_h} \left(([\sigma(\mathbf{u}, p)]_\Gamma \mathbf{n}_\Gamma, \mathbf{v}_{T^2})_{T^\Gamma} + (\sigma(\mathbf{u}_1, p_1) \mathbf{n}_\Gamma, [\mathbf{v}_T]_\Gamma)_{T^\Gamma} \right). \end{aligned}$$

Putting the above four identities together we infer that

$$\begin{aligned} \Psi_1 &= \sum_{T \in \mathcal{T}_h} \left\{ \sum_{i \in \{1,2\}} \left((-2\nu_i \mathbf{d}_{T^i}, \nabla^s \mathbf{v}_{T^i})_{T^i} + (\xi_{T^i}, \nabla \cdot \mathbf{v}_{T^i})_{T^i} \right. \right. \\ &\quad \left. \left. + ((-2\nu_i \mathbf{d}_{T^i} + \xi_{T^i} \mathbb{I}) \mathbf{n}_T, \mathbf{v}_{(\partial T)^i} - \mathbf{v}_{T^i})_{(\partial T)^i} \right) + ((2\nu_1 \mathbf{d}_{T^1} - \xi_{T^1} \mathbb{I}) \mathbf{n}_\Gamma, [\mathbf{v}_T]_\Gamma)_{T^\Gamma} \right\}. \end{aligned}$$

Concerning Ψ_2 , since $\Pi_{(\partial T)^i}^k$ is the L^2 -orthogonal projection and $[\mathbf{u}]_\Gamma = \mathbf{0}$ on T^Γ , we have

$$\begin{aligned} \Psi_2 &= \sum_{T \in \mathcal{T}_h} \left\{ -\nu_1 h_T^{-1} ([\delta_T]_\Gamma, [\mathbf{v}_T]_\Gamma)_{T^\Gamma} + \chi \nu_2^{-1} h_T ([\sigma(\delta_T, \xi_T)]_\Gamma \mathbf{n}_\Gamma, [\sigma(\mathbf{v}_T, -q_T)]_\Gamma \mathbf{n}_\Gamma)_{T^\Gamma} \right. \\ &\quad \left. + \sum_{i \in \{1,2\}} \nu_i h_T^{-1} (\delta_{T^i}, \mathbf{v}_{(\partial T)^i} - \Pi_{(\partial T)^i}^k(\mathbf{v}_{T^i}))_{(\partial T)^i} \right\}. \end{aligned}$$

Finally, concerning Ψ_3 , we have $\nabla \cdot \mathbf{u} = 0$, which implies that

$$\Psi_3 = - \sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} (d_{T^i}, q_{T^i})_{T^i}.$$

The final estimate follows from the Cauchy–Schwarz inequality. \square

4.3 Error estimate

Let $(\mathbf{u}, p) \in H_0^1(\Omega; \mathbb{R}^d) \times L_*^2(\Omega)$ be the solution to the exact problem (1) and $(\hat{\mathbf{u}}_h, p_h)$ be the solution to the discrete problem (16).

Theorem 12. *Let $k \geq 1$. Assume that $(\mathbf{u}, p) \in H^{1+s}(\Omega; \mathbb{R}^d) \times H^s(\Omega)$ with $s > \frac{1}{2}$. The following holds true:*

$$\begin{aligned} & \sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} \left(\nu_i \|\nabla^s(\mathbf{u}_i - \mathbf{u}_{T^i})\|_{T^i}^2 + \nu_i^{-1} \|p_i - p_{T^i}\|_{T^i}^2 \right) \\ & \lesssim \sum_{T \in \mathcal{T}_h} \left\{ \|(\mathbf{d}_T, \xi_T)\|_{*T}^2 + \|\delta_T\|_{\#T}^2 + \sum_{i \in \{1,2\}} \left(\nu_i \|\nabla^s(\mathbf{u}_i - I_{T^i}^{k+1}(\mathbf{u}))\|_{T^i}^2 + \nu_i^{-1} \|p_i - J_{T^i}^k(p)\|_{T^i}^2 + \nu_i \|d_{T^i}\|_{T^i}^2 \right) \right\}. \end{aligned}$$

Moreover, if $(\mathbf{u}, p) \in H^{k+2}(\Omega; \mathbb{R}^d) \times H^{k+1}(\Omega)$, we have

$$\sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} \left(\nu_i \|\nabla^s(\mathbf{u}_i - \mathbf{u}_{T^i})\|_{T^i}^2 + \nu_i^{-1} \|p_i - p_{T^i}\|_{T^i}^2 \right) \lesssim h^{2(k+1)} \sum_{i \in \{1,2\}} \left(\nu_i |\mathbf{u}|_{H^{k+2}(\Omega_i)}^2 + \nu_i^{-1} |p|_{H^{k+1}(\Omega_i)}^2 \right).$$

Proof. Owing to the triangle inequality, we have

$$\begin{aligned} & \sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} \left(\nu_i \|\nabla^s(\mathbf{u}_i - \mathbf{u}_{T^i})\|_{T^i}^2 + \nu_i^{-1} \|p_i - p_{T^i}\|_{T^i}^2 \right) \\ & \leq 2 \sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} \left(\nu_i \|\nabla^s(\mathbf{u}_i - I_{T^i}^{k+1}(\mathbf{u}))\|_{T^i}^2 + \nu_i^{-1} \|p_i - J_{T^i}^k(p)\|_{T^i}^2 \right) + 2 \|(\hat{\mathbf{e}}_h, \eta_h)\|_{Y_h}^2. \end{aligned}$$

According to Lemma 10, we have

$$\beta \|(\hat{\mathbf{e}}_h, \eta_h)\|_{Y_h} \leq \sup_{y_h \in Y_h^k \setminus \{0\}} \frac{A_h((\hat{\mathbf{e}}_h, \eta_h), y_h)}{\|y_h\|_{Y_h}} = \sup_{y_h \in Y_h^k \setminus \{0\}} \frac{\mathcal{F}(y_h)}{\|y_h\|_{Y_h}}.$$

Moreover, according to Lemma 11, we have

$$|\mathcal{F}(y_h)| \lesssim \left\{ \sum_{T \in \mathcal{T}_h} \left(\|(\mathbf{d}_T, \xi_T)\|_{*T}^2 + \|\delta_T\|_{\#T}^2 + \sum_{i \in \{1,2\}} \nu_i \|d_{T^i}\|_{T^i}^2 \right) \right\}^{\frac{1}{2}} \|y_h\|_{Y_h}.$$

Combining these bounds proves the first error estimate. Finally invoking the approximation properties of $\hat{I}_T^k(\mathbf{v})$ (see Lemmas 6 and 7), we obtain the second estimate. \square

5 Numerical simulations

In this section, the global domain is the unit square $\Omega := (0, 1)^2$, and the interface is a circle of center $(0.5, 0.5)$ and radius $R = 1/3$. The subdomain Ω_2 where the viscosity is higher lies inside the circular interface, and the subdomain Ω_1 lies outside. We consider Cartesian meshes of mesh size $h := \sqrt{2}/N$, where $N = 8, 16, 32$ and 64 . We consider that a cell $T \in \mathcal{T}_h^\Gamma$ has a small cut if $|T^i| \leq 0.3|T|$ for some $i \in \{1, 2\}$, where $|T^i|$ and $|T|$ are the volumes of T^i and T , respectively. Such small cut cells are agglomerated following the procedure described in Algorithm 1 of [7]. The domain, the interface and the meshes (for $h = \sqrt{2}/16$ and $h = \sqrt{2}/32$) are presented in Figure 1.

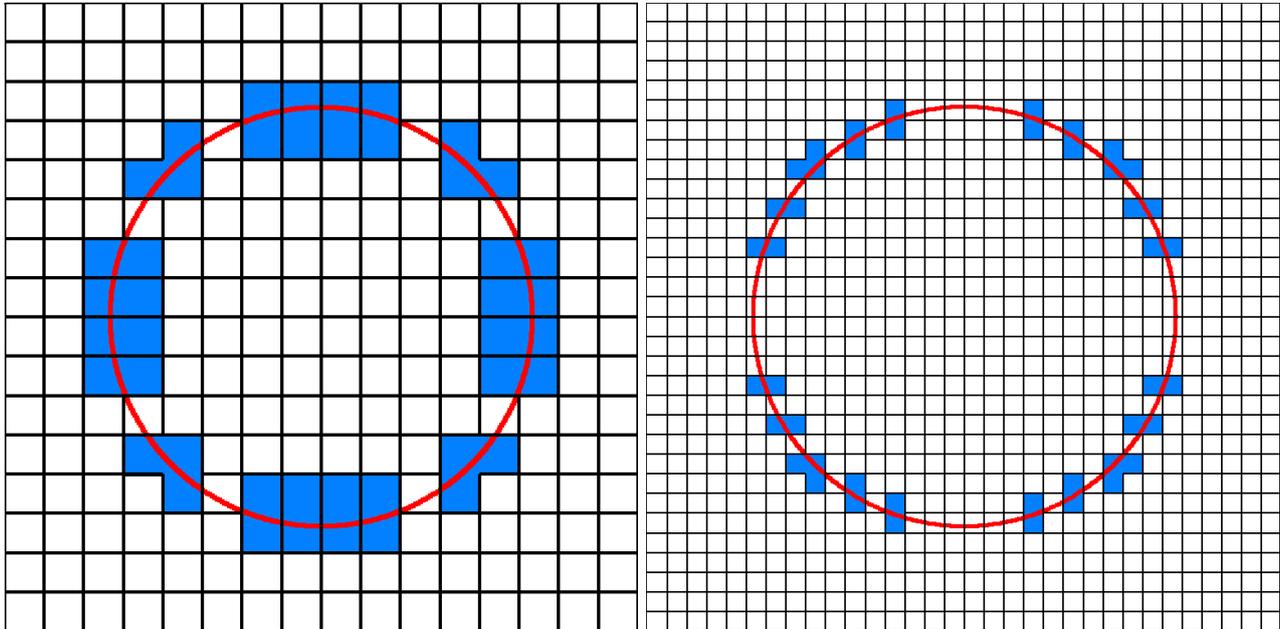


Figure 1: The domain, the mesh and the interface for $h = \sqrt{2}/16$ (left) and $h = \sqrt{2}/32$ (right). The agglomerated cells are highlighted.

The integration over the curved interface is done following the method proposed in Section 4.2 of [7]. In every cut cell $T \in \mathcal{T}_h^\Gamma$, the interface is represented by $2^{n_{\text{int}}}$ segments, where n_{int} is a positive integer. The numerical integration over the interface is the sum of the integrations over all the segments. Choosing n_{int} higher means that we approximate the interface in a better way but makes the method more expensive. In a similar way, the integrations over the cut cells are carried out by dividing the cut cells into several triangles.

We consider two test cases: a test case with a pressure jump across the interface and a test case with a contrast in the viscosity coefficient. The exact solution is denoted (\mathbf{u}, p) and the velocity components (u_1, u_2) . We report the errors $(\sum_{i \in \{1, 2\}} \sum_{T \in \mathcal{T}_h} \nu_i \|\nabla^s(\mathbf{u} - \mathbf{u}_{T^i})\|^2)^{\frac{1}{2}}$ for the velocity and $(\sum_{i \in \{1, 2\}} \sum_{T \in \mathcal{T}_h} \nu_i^{-1} \|p - p_{T^i}\|^2)^{\frac{1}{2}}$ for the pressure.

Even if the theoretical analysis of the unfitted HHO method requires $\chi > 0$, our numerical tests indicate that the method works perfectly well with $\chi = 0$. All the numerical simulations are run with $\chi = 0$. Moreover, although the theoretical analysis requires $k \geq 1$ for the polynomial degree, we also include in our numerical tests the lowest-order case $k = 0$.

5.1 Test case with pressure jump

We consider here a test case where the velocity is null in the whole domain, and the pressure is discontinuous across the circular interface. We have

$$u_1(x, y) = u_2(x, y) := 0 \quad \forall (x, y) \in \Omega, \quad (34a)$$

$$p(x, y) := -\pi RK \quad \forall (x, y) \in \Omega_1, \quad (34b)$$

$$p(x, y) := \frac{K}{R} - \pi RK \quad \forall (x, y) \in \Omega_2, \quad (34c)$$

where the pressure jump is proportional to the curvature of the interface since we have $\llbracket \sigma(u, p) \mathbf{n}_\Gamma \rrbracket_\Gamma = -\frac{K}{R}$. Moreover, $\nu_1 = \nu_2 = 1$. This test case was proposed in [19]. Note that the mean pressure is null. In our tests we consider $K := 0.05$.

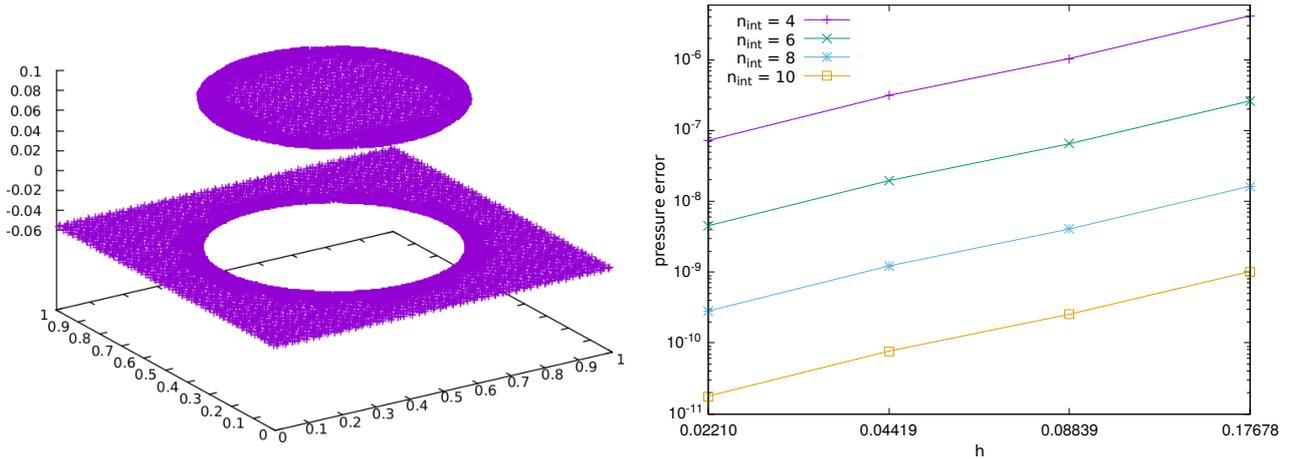


Figure 2: Test case with pressure jump. Left: discrete pressure for $h = \sqrt{2}/16$ and $k = 1$. Right: pressure error for $k = 1$.

	h	$\sqrt{2}/8$	$\sqrt{2}/16$	$\sqrt{2}/32$	$\sqrt{2}/64$
$n_{\text{int}} = 4$	$k = 0$	2.42e-05	1.03e-05	4.93e-06	1.63e-06
	1	4.97e-09	1.11e-09	3.20e-10	4.20e-11
	2	2.59e-12	3.01e-13	7.64e-14	8.85e-13
	3	6.32e-15	9.25e-15	7.29e-14	6.63e-13
$n_{\text{int}} = 6$	$k = 0$	1.51e-06	6.47e-07	3.08e-07	1.02e-07
	1	1.94e-11	4.33e-12	1.25e-12	1.40e-12
	2	1.85e-15	5.16e-15	5.61e-14	7.00e-13
	3	5.27e-15	8.79e-15	8.38e-14	6.87e-13
$n_{\text{int}} = 8$	$k = 0$	9.45e-08	4.04e-08	1.92e-08	6.38e-09
	1	7.57e-14	1.78e-14	3.69e-14	7.78e-13
	2	2.55e-15	5.12e-15	5.76e-14	7.69e-13
	3	7.63e-15	1.11e-14	7.82e-14	5.54e-13
$n_{\text{int}} = 10$	$k = 0$	5.91e-09	2.53e-09	1.20e-09	3.99e-10
	1	1.29e-15	7.98e-15	5.18e-14	7.34e-13
	2	3.72e-15	4.78e-15	5.54e-14	8.42e-13
	3	1.42e-14	1.60e-14	9.54e-14	5.31e-13

Table 1: Test case with pressure jump: velocity errors for various meshes (parameter h), polynomial degrees (parameter k), and geometric resolutions of the interface (parameter n_{int}).

A plot of the discrete pressure is shown in the left panel of Figure 2, whereas the pressure errors are reported in the right panel of Figure 2. A more comprehensive convergence overview is provided in Tables 1 and 2 for the velocity and the pressure respectively. We observe that the exact solution belongs to all the discrete spaces provided the interface is perfectly represented. This means that we expect the errors to vanish when n_{int} is increased. This is well reflected in our numerical results since we observe that the error diminishes when n_{int} is increased (see Figure 2 and Tables 1, 2). As can be expected for very small errors, increasing n_{int} is no longer beneficial. We expect that it is due to the fact that the number of operations increases with n_{int} and thus the rounding in floating point arithmetics can no longer be neglected (the unknowns are represented with double precision).

On the same token, we expect that increasing the polynomial degree k will not improve the numerical results since the error is due to the geometric error in the representation of the interface. As expected we can see in Table 2 that the pressure error is the same for $k = 1, 2, 3$. It is however larger for $k = 0$. An interesting result is that we observe numerically the convergence of the scheme even for the case $k = 0$, which is not covered by the present analysis, although the errors are somewhat larger than for $k \geq 1$. Thus we conjecture that the convergence of the scheme can also be obtained for $k = 0$ but with less favorable constants. A similar behavior has already been observed in Section 4.5.2 of [28] in the context of elasticity, where the author reported that the method remains convergent for $k = 0$ provided the cells have at least $2d$ faces, i.e. quadrangles in $2d$ and hexahedra in $3d$. We concur with these numerical observations for the present Stokes interface problem using quadrangles in $2d$.

	h	$\sqrt{2}/8$	$\sqrt{2}/16$	$\sqrt{2}/32$	$\sqrt{2}/64$
$n_{\text{int}} = 4$	$k = 0$	6.11e-06	1.57e-06	6.32e-07	1.76e-07
	1	4.19e-06	1.04e-06	3.13e-07	7.24e-08
	2	4.19e-06	1.04e-06	3.13e-07	7.24e-08
	3	4.19e-06	1.04e-06	3.13e-07	7.24e-08
$n_{\text{int}} = 6$	$k = 0$	3.82e-07	9.81e-08	3.95e-08	1.10e-08
	1	2.62e-07	6.51e-08	1.96e-08	4.53e-09
	2	2.62e-07	6.51e-08	1.96e-08	4.53e-09
	3	2.62e-07	6.51e-08	1.96e-08	4.53e-09
$n_{\text{int}} = 8$	$k = 0$	2.39e-08	6.13e-09	2.47e-09	6.88e-10
	1	1.64e-08	4.07e-09	1.22e-09	2.83e-10
	2	1.64e-08	4.07e-09	1.22e-09	2.83e-10
	3	1.64e-08	4.07e-09	1.22e-09	2.83e-10
$n_{\text{int}} = 10$	$k = 0$	1.49e-09	3.83e-10	1.54e-10	4.30e-11
	1	1.02e-09	2.54e-10	7.65e-11	1.77e-11
	2	1.02e-09	2.54e-10	7.65e-11	1.78e-11
	3	1.02e-09	2.54e-10	7.65e-11	1.77e-11

Table 2: Test case with pressure jump: pressure errors for various meshes (parameter h), polynomial degrees (parameter k), and geometric resolutions of the interface (parameter n_{int}).

5.2 Test case with contrasted viscosity

In this section, we consider the following test case:

$$\mathbf{u}(r, \theta) := \tilde{u}(r)(\sin \theta, -\cos \theta), \quad (35a)$$

$$p(r) := r^4 - \frac{7}{180}, \quad (35b)$$

where $r^2 := (x - 0.5)^2 + (y - 0.5)^2$, $x - 0.5 := r \sin \theta$, $y - 0.5 := r \cos \theta$ and

$$\tilde{u}(r) = \begin{cases} \frac{r^6}{\nu_2} & \text{if } r < R, \\ \frac{r^6 - R^6}{\nu_1} + \frac{R^6}{\nu_2} & \text{otherwise.} \end{cases} \quad (36)$$

Notice that the pressure has zero mean-value. Moreover we have $\mathbf{g}_N = (1 - \frac{\nu_1}{\nu_2})r^5 \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$ and

$$\mathbf{f} = 4r^3 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} + \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \times \begin{cases} 35r^4 & \text{if } r \leq R, \\ 35r^4 + \frac{\nu_2 - \nu_1}{\nu_2} \frac{R^6}{r^2} & \text{otherwise.} \end{cases} \quad (37)$$

The interesting feature of this test case is that $\sum_{i \in \{1, 2\}} (\nu_i |\mathbf{u}|_{H^{k+2}(\Omega_i)}^2 + \nu_i^{-1} |p|_{H^{k+1}(\Omega_i)}^2)$ remains bounded for $\nu_1 = 1$ when ν_2 becomes large. We can then study the variation of the error when the viscosity contrast, as measured by the ratio ν_2/ν_1 , increases. We expect from the theoretical analysis that the error remains bounded for high contrasts. The elevation for the velocity magnitude and the pressure is presented in Figure 3.

The errors (velocity in H^1 -seminorm, pressure in L^2 -norm) are reported in Figures 4, 5 and 6. In Figure 4 we present the errors with respect to the mesh size h for various polynomial degrees. We recover the convergence rates stated in Theorem 12. In Figure 5, we report the errors with respect to the viscosity contrast $\frac{\nu_2}{\nu_1}$ for various polynomial degrees. These results confirm that the method is robust with respect to the viscosity contrast ν_2/ν_1 (as soon as the geometric representation of the interface is fine enough). The error is somewhat larger for $\nu_2 = 10^6$ when $k = 3$. In order to study this phenomenon, we draw in Figure 6 the error with respect to the geometric resolution parameter n_{int} for various viscosity contrasts. We observe that the error tends to the same value for $\nu_2 = 1, 10^2, 10^4$ but not for $\nu_2 = 10^6$. This indicates that for very large contrasts, refining the interface is not enough and explains why we can see in Figure 5 a larger error for high contrasts.

6 Conclusions

In this work we designed and analyzed an unfitted HHO method for the Stokes interface problem (with symmetric gradients). Error estimates have been established for polynomial degrees $k \geq 1$ and for a penalty parameter $\chi > 0$ that is small enough. The same results can be established for the formulation with full gradients (in this case $\sigma(\mathbf{u}, p) := \nabla \mathbf{u} - p\mathbb{I}$), the only difference being that we do not need the assumption $k \geq 1$ at the

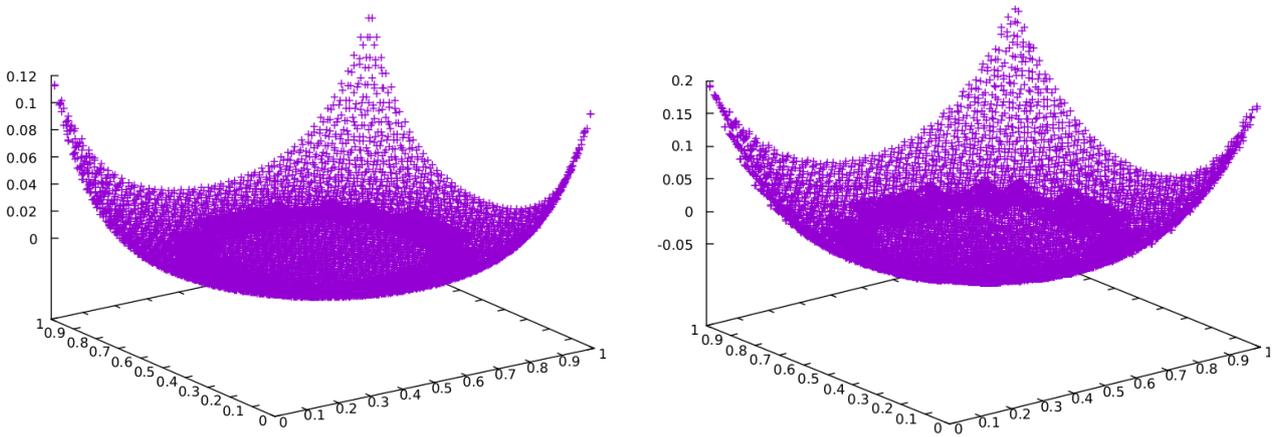


Figure 3: Test case with contrasted viscosity ($\nu_1 = 1.0$, $\nu_2 = 10^4$), $n_{\text{int}} = 4$, $k = 1$, $h = \sqrt{2}/16$. Elevation for the velocity magnitude (left) and the pressure (right).

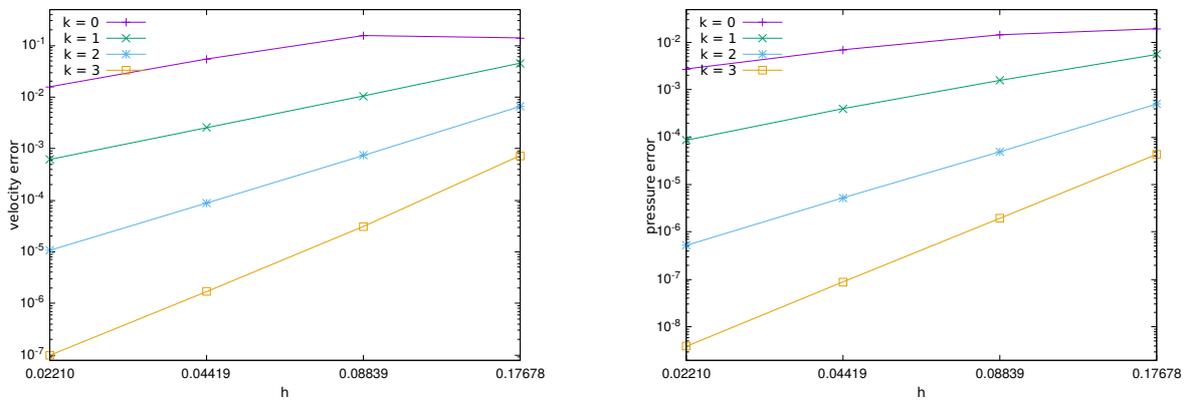


Figure 4: Test case with contrasted viscosity ($\nu_1 = 1.0$, $\nu_2 = 10^4$), $n_{\text{int}} = 10$. Errors (left: velocity in H^1 -seminorm; right: pressure in L^2 -norm) as a function of the mesh size for various polynomial degrees.

theoretical level since Korn's inequality is not used. Neglecting geometric errors in the representation of the interface and quadrature errors in the cut and uncut cells, we have proved a priori error estimates that are optimally convergent with respect to the mesh size and robust with respect to the viscosity contrast.

Our numerical results show that in practice, one can choose $\chi = 0$ as well as $k = 0$ when using quadrangles in 2d. We expect that the inf-sup stability analysis remains feasible without the stabilization term associated with χ . Such an analysis has already been carried out in [4] for a low-order finite element method. However, obtaining such results for a high-order method is, to the best of our knowledge, an open problem. Furthermore, the use of $k = 0$ eludes Korn's inequality in the stability analysis. This observation, which concurs with that in Section 4.5.2 of [28] for elasticity, still requires, to the best of our knowledge, a theoretical justification.

Finally the present integration method relies on a discretization of the interface with a step much finer than the size of the cells. This enables us to recover in numerical simulations the a priori convergence rates. However, for very high contrasts or for very fine meshes this approach is not sufficient to compute a very precise solution, and calls for the development of another integration method.

References

- [1] S. Adjerid, N. Chaabane, T. Lin, and P. Yue. An immersed discontinuous finite element method for the Stokes problem with a moving interface. *J. Comput. Appl. Math.*, 362:540–559, 2019.
- [2] J. W. Barrett and C. M. Elliott. A finite-element method for solving elliptic equations with Neumann data on a curved boundary using unfitted meshes. *IMA J. Numer. Anal.*, 4(3):309–325, 1984.
- [3] J. W. Barrett and C. M. Elliott. Fitted and unfitted finite-element methods for elliptic equations with smooth interfaces. *IMA J. Numer. Anal.*, 7(3):283–300, 1987.

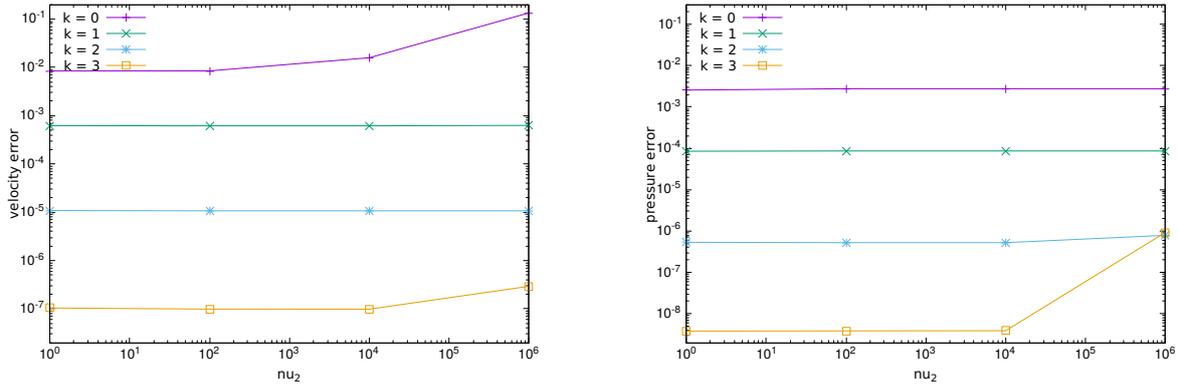


Figure 5: Test case with contrasted viscosity ($\nu_1 = 1.0$), $n_{\text{int}} = 10$, and $h = \sqrt{2}/64$. Errors (left: velocity in H^1 -seminorm; right: pressure in L^2 -norm) as a function of the viscosity contrast for various polynomial degrees.

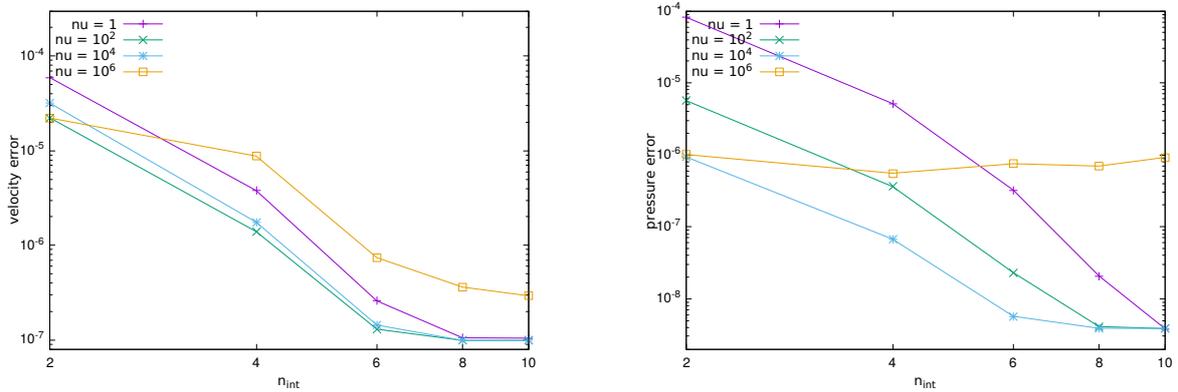


Figure 6: Test case with contrasted viscosity ($\nu_1 = 1.0$), $h = \sqrt{2}/64$, and $k = 3$. Errors (left: velocity in H^1 -seminorm; right: pressure in L^2 -norm) as a function of the geometric resolution of the interface for various viscosity contrasts.

- [4] R. Becker, E. Burman, and P. Hansbo. A Nitsche extended finite element method for incompressible elasticity with discontinuous modulus of elasticity. *Comput. Methods Appl. Mech. Engrg.*, 198(41-44):3352–3360, 2009.
- [5] L. Botti, D. A. Di Pietro, and J. Droniou. A hybrid high-order method for the incompressible Navier-Stokes equations based on Temam’s device. *J. Comput. Phys.*, 376:786–816, 2019.
- [6] E. Burman. Ghost penalty. *C. R. Math. Acad. Sci. Paris*, 348(21-22):1217–1220, 2010.
- [7] E. Burman, M. Cicuttin, G. Delay, and A. Ern. An unfitted hybrid high-order method with cell agglomeration for elliptic interface problems. 2019.
- [8] E. Burman, S. Claus, P. Hansbo, M. G. Larson, and A. Massing. CutFEM: discretizing geometry and partial differential equations. *Internat. J. Numer. Methods Engrg.*, 104(7):472–501, 2015.
- [9] E. Burman and A. Ern. An unfitted hybrid high-order method for elliptic interface problems. *SIAM J. Numer. Anal.*, 56(3):1525–1546, 2018.
- [10] E. Cáceres, J. Guzmán, and M. Olshanskii. New stability estimates for an unfitted finite element method for two-phase Stokes problem. *ArXiv e-prints*, June 2019.
- [11] A. P. Calderón. Lebesgue spaces of differentiable functions and distributions. *Proc. Sympos. Pure Math.*, 4:33–49, 1961.
- [12] L. Cattaneo, L. Formaggia, G. F. Iori, A. Scotti, and P. Zunino. Stabilized extended finite elements for the approximation of saddle point problems with unfitted interfaces. *Calcolo*, 52(2):123–152, 2015.
- [13] M. Cicuttin, D. A. Di Pietro, and A. Ern. Implementation of discontinuous skeletal methods on arbitrary-dimensional, polytopal meshes using generic programming. *J. Comput. Appl. Math.*, 344:852–874, 2018.

- [14] B. Cockburn, D. A. Di Pietro, and A. Ern. Bridging the Hybrid High-Order and Hybridizable Discontinuous Galerkin methods. *ESAIM: Math. Model. Numer. Anal. (M2AN)*, 50(3):635–650, 2016.
- [15] F. de Prenter, C. Lehrenfeld, and A. Massing. A note on the stability parameter in Nitsche’s method for unfitted boundary value problems. *Comput. Math. Appl.*, 75(12):4322–4336, 2018.
- [16] D. A. Di Pietro and A. Ern. A Hybrid High-Order locking-free method for linear elasticity on general meshes. *Comput. Meth. Appl. Mech. Engrg.*, 283:1–21, 2015.
- [17] D. A. Di Pietro, A. Ern, and S. Lemaire. An arbitrary-order and compact-stencil discretization of diffusion on general meshes based on local reconstruction operators. *Comput. Meth. Appl. Math.*, 14(4):461–472, 2014.
- [18] Daniele A. Di Pietro, Alexandre Ern, Alexander Linke, and Friedhelm Schieweck. A discontinuous skeletal method for the viscosity-dependent Stokes problem. *Comput. Methods Appl. Mech. Engrg.*, 306:175–195, 2016.
- [19] S. Groß and A. Reusken. An extended pressure finite element space for two-phase incompressible flows with surface tension. *J. Comput. Phys.*, 224(1):40–58, 2007.
- [20] A. Hansbo and P. Hansbo. An unfitted finite element method, based on Nitsche’s method, for elliptic interface problems. *Comput. Methods Appl. Mech. Engrg.*, 191(47-48):5537–5552, 2002.
- [21] P. Hansbo, M. G. Larson, and S. Zahedi. A cut finite element method for a Stokes interface problem. *Appl. Numer. Math.*, 85:90–114, 2014.
- [22] F. Heimann, C. Engwer, O. Ippisch, and P. Bastian. An unfitted interior penalty discontinuous Galerkin method for incompressible Navier-Stokes two-phase flow. *Internat. J. Numer. Methods Fluids*, 71(3):269–293, 2013.
- [23] C. O. Horgan. Korn’s inequalities and their applications in continuum mechanics. *SIAM Rev.*, 37(4):491–511, 1995.
- [24] H. Ji and Q. Zhang. A simple finite element method for Stokes flows with surface tension using unfitted meshes. *Internat. J. Numer. Methods Fluids*, 81(2):87–103, 2016.
- [25] A. Johansson and M. G. Larson. A high order discontinuous Galerkin Nitsche method for elliptic problems with fictitious boundary. *Numer. Math.*, 123(4):607–628, 2013.
- [26] M. Kirchhart, S. Gross, and A. Reusken. Analysis of an XFEM discretization for Stokes interface problems. *SIAM J. Sci. Comput.*, 38(2):A1019–A1043, 2016.
- [27] J. Nitsche. Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. *Abh. Math. Sem. Univ. Hamburg*, 36:9–15, 1971. Collection of articles dedicated to Lothar Collatz on his sixtieth birthday.
- [28] N. Pignet. *Hybrid High-Order methods for nonlinear solid mechanics*. PhD thesis, ENPC - École des Ponts ParisTech, 2019.
- [29] A. Reusken. Analysis of an extended pressure finite element space for two-phase incompressible flows. *Comput. Vis. Sci.*, 11(4-6):293–305, 2008.
- [30] W. E. H. Sollie, O. Bokhove, and J. J. W. van der Vegt. Space-time discontinuous Galerkin finite element method for two-fluid flows. *J. Comput. Phys.*, 230(3):789–817, 2011.
- [31] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton, NJ, Princeton University Press, 1970.
- [32] Q. Wang and J. Chen. A new unfitted stabilized Nitsche’s finite element method for Stokes interface problems. *Comput. Math. Appl.*, 70(5):820–834, 2015.