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THE ESSENTIAL SPECTRUM OF THE DISCRETE LAPLACIAN ON KLAUS-SPARSE GRAPHS

SYLVAIN GOLÉNIA AND FRANÇOISE TRUC

ABSTRACT. In 1983, Klaus studied a class of potentials with bumps and computed the essential spectrum of the associated Schrödinger operator with the help of some localisations at infinity. A key hypothesis is that the distance between two consecutive bumps tends to infinity at infinity. In this article, we introduce a new class of graphs (with patterns) that mimics this situation, in the sense that the distance between two patterns tends to infinity at infinity. These patterns tend, in some way, to asymptotic graphs. They are the localisations at infinity. Our result is that the essential spectrum of the Laplacian acting on our graph is given by the union of the spectra of the Laplacian acting on the asymptotic graphs. We also discuss the question of the stability of the essential spectrum in the appendix.

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1. INTRODUCTION

The computation of the essential spectrum of an operator is a standard question in spectral theory. For a large family of Schrödinger operators, it is well-known that the essential spectrum is characterised by the behaviour at infinity of the potential. In 1983, Klaus introduces in his article [Kla] a type of potential with bumps with a crucial feature, that is, the distance between two such bumps tends to infinity. He computes the essential spectrum of this $1d$ (continuous) Schrödinger operator in terms of the union of the spectrum of some simpler operators. The common patterns, that define the *localisations at infinity*, are given by the behaviour of the potential. In [LaSi], this notion is illustrated by the use of *R-limits*, see also [CWL] for this concept and references therein. The example of Klaus was generalised and encoded in some C^* -algebraic context in [GeIf1, GeIf2], see also [MaPuRi]. We refer to [Ge] for more general results and historical references. We mention also [NaTa1, NaTa2] for recent developments in a sparse context.

In the context of graphs, the computation of essential spectra is done in many places e.g., [Kel, Ra, SaSu]. In [BrDeEl, El] they extend the *R-limit* technique to discrete graphs. We refer to [Gol1, GeGo1] for a C^* -algebra approach.

Our motivation in this paper is to analyse a graph analog of the example of Klaus where the “bumps” are no more due to a potential but to patterns coming from the structure of the graph. We call this family of graphs *Klaus-sparse graphs*. The approaches of *R-limits* and C^* -algebras do not seem to apply here. We rely directly on the construction of Weyl sequences.

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We start with some definitions and fix our notation for graphs. We refer to [CdV, Chu] for surveys on the matter. Let \mathcal{V} be a countable set. Let $\mathcal{E} := \mathcal{V} \times \mathcal{V} \rightarrow [0, \infty)$ and assume that

$$\mathcal{E}(x, y) = \mathcal{E}(y, x), \quad \text{for all } x, y \in \mathcal{V}.$$

We say that $\mathcal{G} := (\mathcal{V}, \mathcal{E}, m)$ is an unoriented weighted graph with *vertices* \mathcal{V} and *weighted edges* \mathcal{E} and where m is a positive weight on the vertices

$$m : \mathcal{V} \rightarrow (0, \infty).$$

In the setting of electrical networks, the weights correspond to the conductances. We say that $x, y \in \mathcal{V}$ are *neighbours* if $\mathcal{E}(x, y) \neq 0$ and denote it by $x \sim y$. We say that there is a *loop* in $x \in \mathcal{V}$ if $\mathcal{E}(x, x) \neq 0$. The set of *neighbours* of $x \in \mathcal{E}$ is denoted by

$$\mathcal{N}_{\mathcal{G}}(x) := \{y \in \mathcal{E}, x \sim y\}.$$

A graph is *locally finite* if $|\mathcal{N}_{\mathcal{G}}(x)|$ is finite for all $x \in \mathcal{V}$. A graph is *connected*, if for all $x, y \in \mathcal{V}$, there exists an *x-y-path*, i.e., there is a finite sequence

$$(x_1, \dots, x_{N+1}) \in \mathcal{V}^{N+1} \text{ such that } x_1 = x, x_{N+1} = y \text{ and } x_n \sim x_{n+1},$$

We recall that a graph \mathcal{G} is *simple* if \mathcal{E} has values in $\{0, 1\}$, $m = 1$, and if it has no loop.

In the sequel, all graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E}, m)$ are locally finite, connected and have no loop.

We now associate a certain Hilbert space and some operators on it to a given graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, m)$. Let $\ell_m^2(\mathcal{V}) := \ell^2(\mathcal{V}, m; \mathbb{C})$ be the set of functions $f : \mathcal{V} \rightarrow \mathbb{C}$, such that $\|f\|_{\ell_m^2(\mathcal{V})}^2 := \sum_{x \in \mathcal{V}} m(x)|f(x)|^2$ is finite. The associated scalar product is given by

$$\langle f, g \rangle_{\mathcal{G}} := \langle f, g \rangle_{\ell_m^2(\mathcal{V})} := \sum_{x \in \mathcal{V}} m(x) \overline{f(x)} g(x), \text{ for } f, g \in \ell_m^2(\mathcal{V}).$$

We also denote by $\mathcal{C}_c(\mathcal{V})$ the set of functions $f : \mathcal{V} \rightarrow \mathbb{C}$, which have finite support. We define the quadratic form:

$$(1.1) \quad \mathcal{Q}_{\mathcal{G}}(f, f) := \frac{1}{2} \sum_{x, y \in \mathcal{V}} \mathcal{E}(x, y) |f(x) - f(y)|^2 \geq 0, \text{ for } f \in \mathcal{C}_c(\mathcal{V}).$$

It is closable and there exists a unique self-adjoint operator $\Delta_{\mathcal{G}}$, such that

$$\mathcal{Q}_{\mathcal{G}}(f, f) = \langle f, \Delta_{\mathcal{G}} f \rangle_{\mathcal{G}}, \text{ for } f \in \mathcal{C}_c(\mathcal{V})$$

and $\mathcal{D}(\Delta_{\mathcal{G}}^{1/2}) = \mathcal{D}(\mathcal{Q}_{\mathcal{G}})$, where the latter is the completion of $\mathcal{C}_c(\mathcal{V})$ under $\|\cdot\|^2 + \mathcal{Q}_{\mathcal{G}}(\cdot, \cdot)$. This operator is the *Friedrichs extension* associated to the form $\mathcal{Q}_{\mathcal{G}}$ (e.g., [Gol3]). It acts as follows:

$$(1.2) \quad \Delta_{\mathcal{G}} f(x) := \frac{1}{m(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y) (f(x) - f(y)), \text{ for } f \in \mathcal{C}_c(\mathcal{V}).$$

When $m = 1$ and \mathcal{E} has values in $\{0, 1\}$, the operator is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V})$, c.f., [Woj]. A large literature is devoted to this subject.

We define the *degree* associated to $\mathcal{G} = (\mathcal{V}, \mathcal{E}, m)$ by

$$\text{deg}_{\mathcal{G}}(x) := \frac{1}{m(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y), \text{ for } x \in \mathcal{V}.$$

Given a function $V : \mathcal{V} \rightarrow \mathbb{C}$, we denote by $V(\cdot)$ the operator of multiplication by V . It is elementary that $\mathcal{D}(\text{deg}_{\mathcal{G}}^{1/2}(\cdot)) \subset \mathcal{D}(\Delta_{\mathcal{G}}^{1/2})$. Indeed, one has:

$$(1.3) \quad \begin{aligned} 0 \leq \langle f, \Delta_{\mathcal{G}} f \rangle_{\mathcal{G}} &= \frac{1}{2} \sum_{x \in \mathcal{V}} \sum_{y \sim x} \mathcal{E}(x, y) |f(x) - f(y)|^2 \\ &\leq \sum_{x \in \mathcal{V}} \sum_{y \sim x} \mathcal{E}(x, y) (|f(x)|^2 + |f(y)|^2) = 2 \langle f, \text{deg}_{\mathcal{G}}(\cdot) f \rangle_{\mathcal{G}}, \end{aligned}$$

for $f \in \mathcal{C}_c(\mathcal{V})$. Moreover, setting $\tilde{\delta}_x(y) := m^{-1/2}(x) \mathbf{1}_{\{x\}}(y)$ for any $x, y \in \mathcal{V}$, $\langle \tilde{\delta}_x, \Delta_{\mathcal{G}} \tilde{\delta}_x \rangle = \text{deg}_{\mathcal{G}}(x)$, so $\Delta_{\mathcal{G}}$ is bounded if and only if $\sup_{x \in \mathcal{V}} \text{deg}_{\mathcal{G}}(x)$ is finite, e.g. [KL2, Gol3]. Here we have used the following standard notation: for any given set X , $\mathbf{1}_X(x) := 1$ is $x \in X$ and 0 otherwise.

The aim of our work is to study the essential spectrum of $\Delta_{\mathcal{G}}$ for Klaus-sparse graphs. The precise definition is given in Section 2 but let us give a rough description of such a graph: it consists of a (double infinite) family of finite graphs $\{\mathcal{G}_{i,k}\}_{(i,k)}$ which are patterns that are connected by a ‘‘medium graph’’ $\mathcal{G}_{\mathcal{M}}$. The latter has a uniformly bounded degree. For any given k , there is a kind of increasing limit of $\mathcal{G}_{i,k}$ when $i \rightarrow \infty$ which is an infinite graph $\mathcal{G}_{\infty,k}$. The graphs $\{\mathcal{G}_{\infty,k}\}_k$ are the localisations at infinity of \mathcal{G} . Moreover the distance in \mathcal{G} between the $\mathcal{G}_{i,k}$ goes to infinity as $\lim(i,k) \rightarrow \infty$ with respect to the Fréchet filter. We prove the following theorem :

Theorem 1.1. *Assume that $\mathcal{G} := (\mathcal{V}, \mathcal{E}, m)$ is a Klaus-sparse graph. Using notation of Definition 2.1, we have*

- 1) $\Delta_{\mathcal{G}}$ is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V})$.
- 2) The essential spectrum of $\Delta_{\mathcal{G}}$ is given by the union of the spectra of $\Delta_{\mathcal{G}_{\infty,k}}$, the localisations at infinity, namely:

$$\sigma_{\text{ess}}(\Delta_{\mathcal{G}}) = \overline{\bigcup_{k \in \mathcal{J}} \sigma(\Delta_{\mathcal{G}_{\infty,k}})}.$$

First of all, to simplify the presentation we do not include any potential and stick to perturbation of graphs. The proof of 1) is inspired by [Gol2]. Let us mention that the inclusion \subset in 2) holds for a wider class of graphs and is the easy part of 2). The reverse inclusion is the interesting and difficult part. To our knowledge, in the context of C^* -algebra, the only result that could tackle this issue is in [Ge]. However it is complicated to compare his results to ours, since they are given in terms of some abstract ultra-filters. Moreover, in our result the union that we obtain is minimal, e.g., Section 3.4.1. We now compare to [El]. In his chapter 4, the author’s more general results overlap with ours but they do not contain ours. They are complementary. Let us explain in which sense they do not contain our result. First we point out that the author is dealing only with bounded operators and with $m = 1$. We do not believe that it is a strong obstacle for his method. However, his approach relies fundamentally on a reverse Shnol’s Theorem that is fulfilled under a uniform sub-exponential growth: $\forall \gamma > 1, \exists C > 0$ so that $\forall r \in \mathbb{N}^*$

$$(1.4) \quad \sup_{x \in \mathcal{V}} |\{y \in \mathcal{V}, d_{\mathcal{G}}(x, y) = r\}| \leq C\gamma^r.$$

We construct in Section 3.4.2 a general family of Klaus-sparse graphs that do not have a uniform exponential growth.

Finally in the appendix, Theorem A.1 ensures the stability of the essential spectrum under a perturbation of the metric that is small at infinity.

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2. KLAUS-SPARSE GRAPHS

2.1. Further notation. We introduce some further notation and definitions. Given $a, b \in \mathbb{Z}$, we denote by $\llbracket a, b \rrbracket := [a, b] \cap \mathbb{Z}$ and $\llbracket a, \infty \rrbracket := [a, \infty[\cap \mathbb{Z}$. Given $X \subset Y$, we denote by $X^c := Y \setminus X$ the complementary set of X , when no confusion can arise. Given \mathcal{H}, \mathcal{K} Hilbert spaces, we denote by $\mathcal{B}(\mathcal{H}, \mathcal{K})$ and $\mathcal{K}(\mathcal{H}, \mathcal{K})$ the set of bounded and compact operators from \mathcal{H} to \mathcal{K} . Set also $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$ and $\mathcal{K}(\mathcal{H}) := \mathcal{K}(\mathcal{H}, \mathcal{H})$.

Given $\mathcal{G}_i := (\mathcal{V}_i, \mathcal{E}_i, m_i)$ and $x_i \in \mathcal{V}_i$ for $i = 1, 2$, we say that \mathcal{G}_1 is *induced* by \mathcal{G}_2 and denote it by $\mathcal{G}_1 \subset \mathcal{G}_2$ if there is an injection $f : \mathcal{V}_1 \rightarrow \mathcal{V}_2$ such that

$$f(\mathcal{V}_1) \subset \mathcal{V}_2, \quad \mathcal{E}_1(x, y) = \mathcal{E}_2(f(x), f(y)), \quad \text{and } m_1(x) = m_2(f(x)), \quad \forall x, y \in \mathcal{V}_1.$$

We shall write $(\mathcal{G}_1, x_1) \subset (\mathcal{G}_2, x_2)$ if we have in addition $f(x_1) = x_2$. To simplify, we shall often simply write

$$\mathcal{V}_1 \subset \mathcal{V}_2, \quad \mathcal{E}_1 = \mathcal{E}_2|_{\mathcal{V}_1, \mathcal{V}_1}, \quad m_1 = m_2|_{\mathcal{V}_1}, \quad \text{and } x_1 = x_2.$$

Moreover, given a graph $\mathcal{G} = (\mathcal{E}, \mathcal{V}, m)$ and $X \subset \mathcal{V}$, we denote by $[\mathcal{G}]^X := (X, \mathcal{E}|_{X \times X}, m|_X)$ the induced graph of \mathcal{G} by X .

We denote by $d_{\mathcal{G}}$ the (unweighted) distance for \mathcal{G} (over \mathcal{V}) given by

$$d_{\mathcal{G}}(x, y) := \min(n, x = x_0 \sim x_1 \sim \dots \sim x_n = y, \text{ with } x_i \in \mathcal{V}),$$

for $x, y \in \mathcal{V}$ when $x \neq y$ and $d_{\mathcal{G}}(x, x) := 0$. It is a distance on \mathcal{V} when \mathcal{G} is connected. Given $r \geq 0$, we set

$$B_{\mathcal{G}}(x, r) := \{y \in \mathcal{V}, d_{\mathcal{G}}(x, y) \leq r\}.$$

2.2. The Klaus-sparse graph. In the introduction we explained by hand-waving the notion of Klaus-sparse graph that we present in this article. Here is the precise definition.

Definition 2.1. A graph $\mathcal{G} := (\mathcal{V}, \mathcal{E}, m)$ is called a Klaus-sparse graph if the following is satisfied.

- *Set of patterns:* Let \mathcal{J} be at most countable such that $|\mathcal{J}| \geq 1$. There exist a family of connected subgraphs $\mathcal{G}_{i,k} := (\mathcal{V}_{i,k}, \mathcal{E}_{i,k}, m_{i,k})$ with $i \in \mathbb{N}$ and $k \in \mathcal{J}$, that are induced by \mathcal{G} .
- *Localisations at infinity:* There are $\mathcal{G}_{\infty,k} := (\mathcal{V}_{\infty,k}, \mathcal{E}_{\infty,k}, m_{\infty,k})$, for all $k \in \mathcal{J}$, and $\mathcal{G}_{\mathcal{M}} := (\mathcal{V}_{\mathcal{M}}, \mathcal{E}_{\mathcal{M}}, m_{\mathcal{M}})$ and $x_{\mathcal{M}} \in \mathcal{V}_{\mathcal{M}}$, with uniformly bounded degree:

$$(2.1) \quad \sup_{y \in \mathcal{V}_{\mathcal{M}}} \deg_{\mathcal{G}_{\mathcal{M}}}(y) < \infty.$$

Here \mathcal{M} stands for medium.

These two families verify the following compatibilities:

- (a) For all $i \in \mathbb{N}$ and $k \in \mathcal{J}$, there are $0 < r_{i,k}^{\text{int}} < r_{i,k}^{\text{ext}}$ such that

$$\lim_{(i,k) \rightarrow \infty} r_{i,k}^{\text{int}} = \infty \quad \text{and} \quad \lim_{(i,k) \rightarrow \infty} r_{i,k}^{\text{ext}} - r_{i,k}^{\text{int}} = \infty,$$

where the limit is taken with respect to the Fréchet filter, i.e., the one given by the complementary of finite sets.

- (b) For all $i \in \mathbb{N}$ and $k \in \mathcal{J}$, there exist $x_{i,k} \in \mathcal{V}_{i,k}$ and $x_{\infty,k} \in \mathcal{V}_{\infty,k}$, so that

$$([B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}})]^{\mathcal{G}}, x_{i,k}) \subset (\mathcal{G}_{\infty,k}, x_{\infty,k}),$$

and for all $r > 0$ and $k \in \mathcal{J}$, there is $i \in \mathbb{N}$ such that

$$([B_{\mathcal{G}_{\infty,k}}(x_{\infty,k}, r)]^{\mathcal{G}_{\infty,k}}, x_{\infty,k}) \subset ([B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{int}} - 1)]^{\mathcal{G}}, x_{i,k}).$$

- (c) We set

$$\{\mathcal{C}_l\}_{l \in \mathcal{L}} := \{\text{connected component of } [\mathcal{V} \setminus \cup_{i \in \mathbb{N}, k \in \mathcal{J}} B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{int}} - 1)]^{\mathcal{G}}\}.$$

Here $\mathcal{L} \subset \mathbb{N}^*$. It is finite or not. We suppose that for all $l \in \mathcal{L}$, there are a vertex x_l of \mathcal{C}_l and an order \triangleleft on \mathcal{L} such that

$$\forall l, m \in \mathcal{L}, l \triangleleft m \implies (\mathcal{C}_l, x_l) \subset (\mathcal{C}_m, x_m) \subset (\mathcal{G}_{\mathcal{M}}, x_{\mathcal{M}}).$$

- (d) For all $i, j \in \mathbb{N}$ and $k, k' \in \mathcal{J}$, such that $(i, k) \neq (j, k')$, we have

$$B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}}) \cap B_{\mathcal{G}}(x_{j,k'}, r_{j,k'}^{\text{ext}}) = \emptyset.$$

- (e) For all $r > 0$, there is $(i, k) \in \mathbb{N} \times \mathcal{J}$ such that

$$(2.2) \quad [B_{\mathcal{G}_{\mathcal{M}}}(x_{\mathcal{M}}, r)]^{\mathcal{G}_{\mathcal{M}}} \subset [B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}} - 1) \setminus B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{int}} + 1)]^{\mathcal{G}}.$$

At first sight, it looks a bit abstruse but will be more intuitive by relying on the figures. In Figure 1, $\mathcal{C}_i \simeq [-a_i, b_i]$ where a_i and b_i are integers that tend to ∞ . The medium graph is $\mathcal{G}_{\mathcal{M}} = \mathbb{Z}$, see Figure 2. Note that we could have chosen $\mathcal{G}_{\mathcal{M}} = \mathbb{N}$.

In Figure 3, $[\mathcal{V} \setminus \cup_{i \in \mathbb{N}} B_{\mathcal{G}}(x_i, r_i^{\text{int}})]^{\mathcal{G}}$ is connected. Therefore \mathcal{L} is reduced to an element and $\mathcal{G}_{\mathcal{M}} = \mathbb{Z}^2$, see Figure 4.

3. PROOF OF THE MAIN THEOREM

3.1. Essential self-adjointness. In this section we prove the first part of Theorem 1.1. We rely on a perturbative approach. We assume that $\mathcal{G} := (\mathcal{V}, \mathcal{E}, m)$ is a Klaus-sparse graph and we use notation of Definition 2.1. We set:

$$\mathcal{V}^{\#} := \bigcup_{i,k} B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}}), \quad \mathcal{E}^{\#} := \mathcal{E} \times \mathbf{1}_{\mathcal{V}^{\#} \times \mathcal{V}^{\#}}, \quad \text{and} \quad \mathcal{G}^{\#} := (\mathcal{V}^{\#}, \mathcal{E}^{\#}, m|_{\mathcal{V}^{\#}}).$$

Due to Definition 2.1 (d), the balls that are constituting $\mathcal{V}^{\#}$ are two-by-two disjoint. Using the notation of induced graph given in Section 2.1, we deduce that

$$\Delta_{\mathcal{G}^{\#}} = \bigoplus_{i,k} \Delta_{[B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}})]^{\mathcal{G}}}.$$

Since $[B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}})]^{\mathcal{G}}$ is a finite graph for all i, k , we infer that $\Delta_{\mathcal{G}^{\#}}$ is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V}^{\#})$. We extend it to \mathcal{V} by 0, we obtain that $\Delta_{\mathcal{G}^{\#}} \oplus 0$ is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V})$.

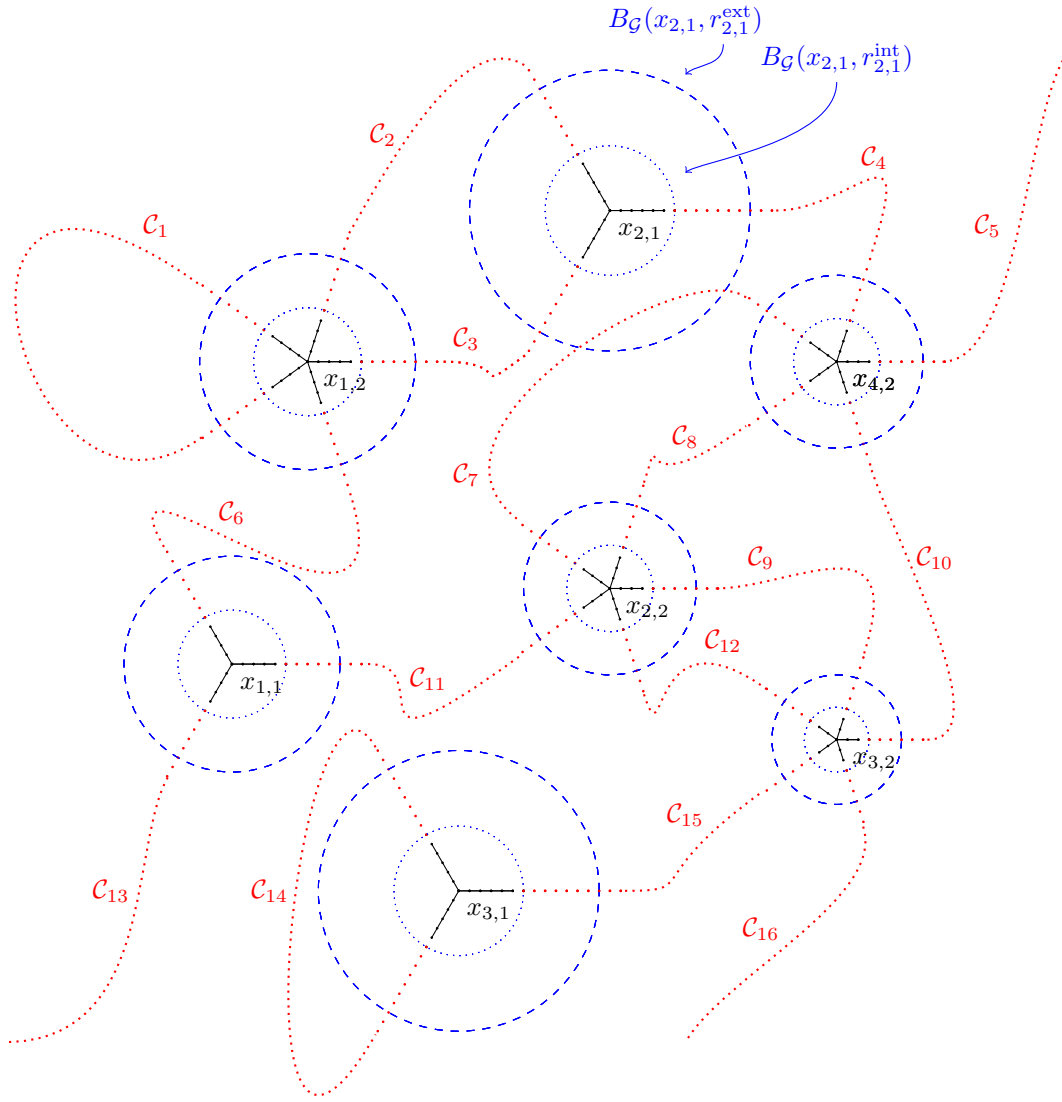


FIGURE 1. \mathcal{G} is a Klaus-sparse star-like graph. Here $\mathcal{L} = \mathbb{N}$.

Then, mimicking the proof of (1.3), we have:

$$(3.1) \quad |\langle f, (\Delta_{\mathcal{G}} - \Delta_{\mathcal{G}^{\#}} \oplus 0)f \rangle_{\mathcal{G}}| \leq 2 \langle f, W(\cdot)f \rangle_{\mathcal{G}},$$

where

$$W(x) := \frac{1}{m(x)} \sum_{y \in \mathcal{V}} |\mathcal{E}(x, y) - \mathcal{E}^{\#}(x, y)|, \text{ for } x \in \mathcal{V}.$$

Note that

$$W(x) = \begin{cases} \frac{1}{m(x)} \sum_{y \notin \mathcal{V}^{\#}} \mathcal{E}(x, y), & \text{when } x \in \mathcal{V}^{\#}, \\ \frac{1}{m(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y), & \text{when } x \notin \mathcal{V}^{\#}. \end{cases}$$

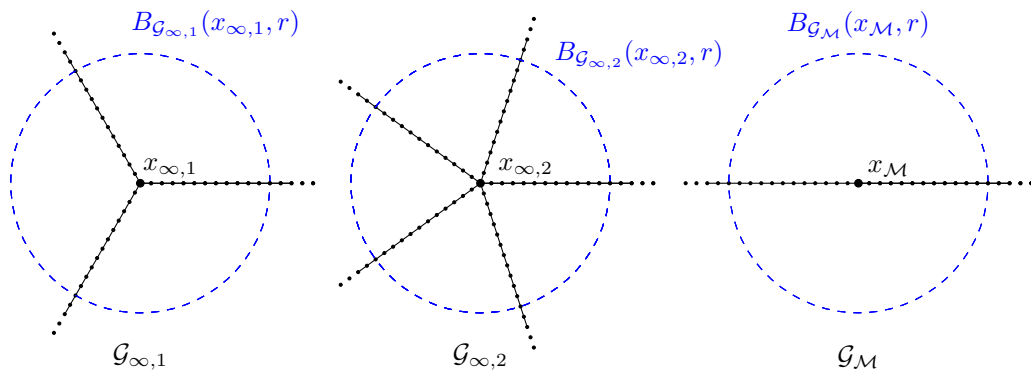
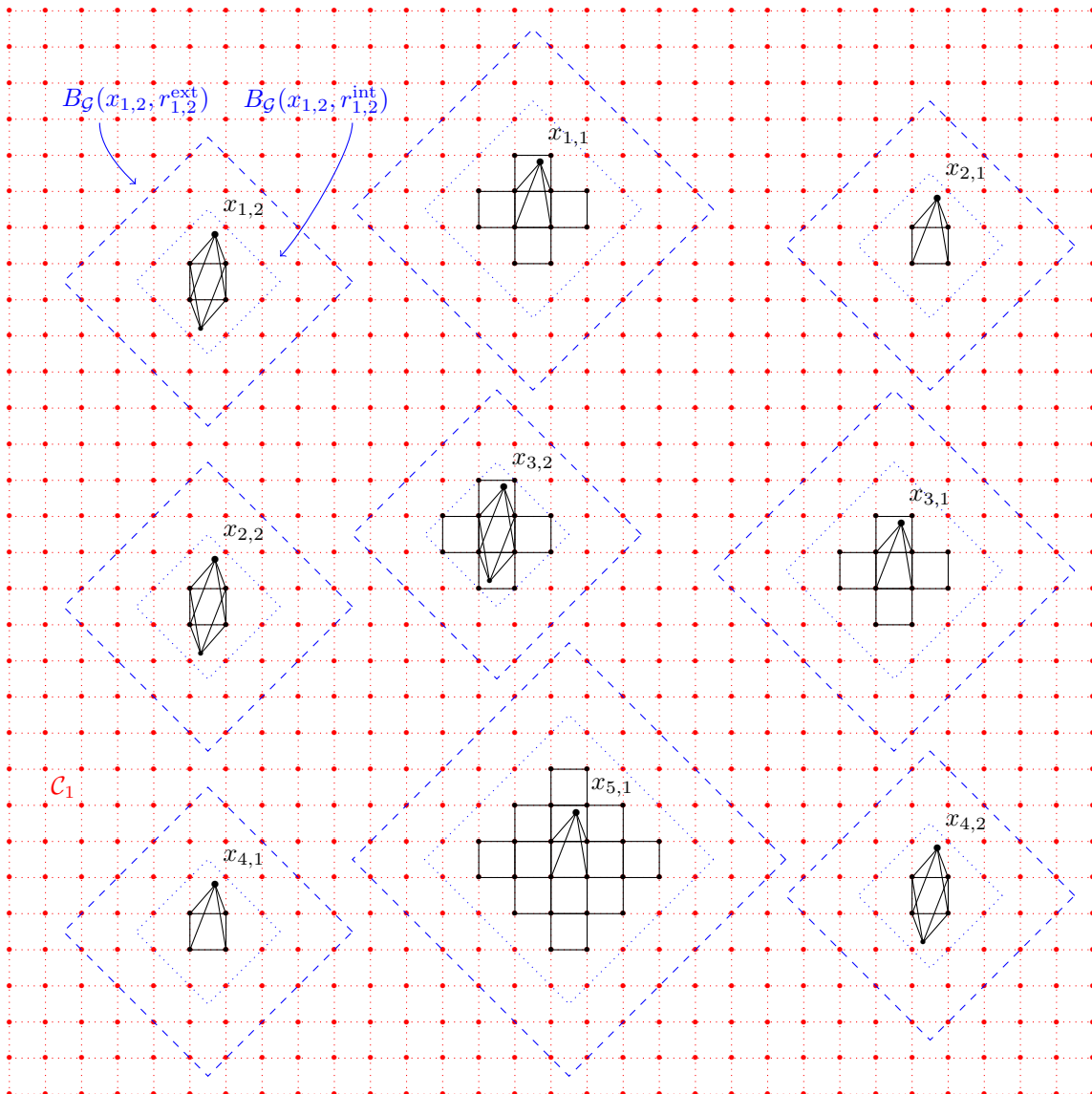
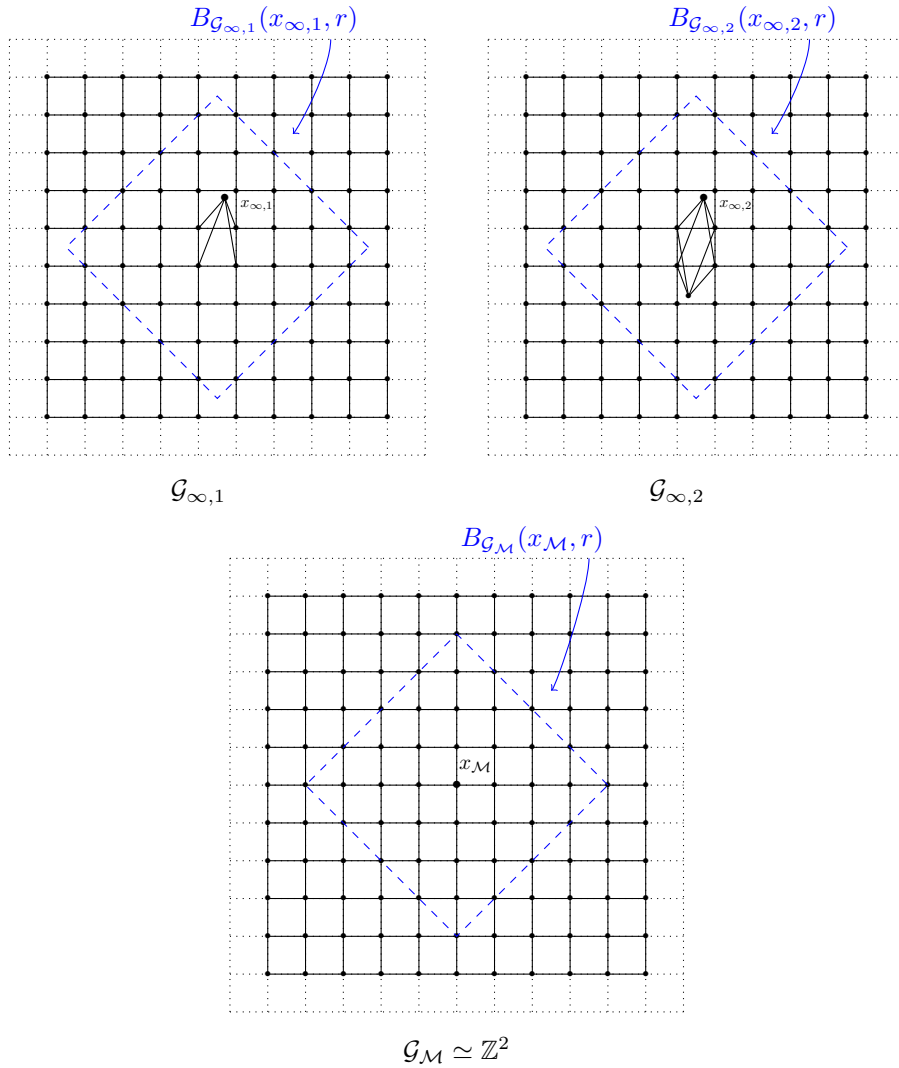


FIGURE 2. Localisations at infinity for the Star-like graph given in Figure 1

FIGURE 3. \mathcal{G} is a Klaus-sparse \mathbb{Z}^2 -like graph. Here $\mathcal{L} = \{1\}$.

FIGURE 4. Localisations at infinity for the \mathbb{Z}^2 -like graph given in Figure 3

Therefore, recalling (c) of Definition 2.1, we have that the support of W is contained in $\{\mathcal{C}_l\}_{l \in \mathcal{L}}$. Moreover, using again (c), we obtain that

$$\sup_{x \in \mathcal{V}} W(x) \leq \sup_{x \in \mathcal{V}_M} \deg_{G_M}(x).$$

In particular, this implies that $\Delta_G - \Delta_{G_M} \oplus 0$ is a bounded operator. Hence by the Kato-Rellich Theorem, e.g., [RS, Proposition X.12], we conclude that Δ_G is also essentially self-adjoint on $\mathcal{C}_c(\mathcal{V})$. \square

3.2. General facts about the essential spectrum. For the proof of the second point of the main theorem, we rely extensively on the properties of approximate eigenfunctions. For the convenience of the reader, we recall these results.

We start by characterising the spectrum and the essential spectrum of a general self-adjoint operator, e.g., [RS, p. 268].

Theorem 3.1. *Let H be a self-adjoint operator acting in a Hilbert space $(\mathcal{H}, \|\cdot\|)$. We have:*

- 1) $\lambda \in \sigma(H)$ if and only if there are $\varphi_n \in \mathcal{H}$ such that $\|\varphi_n\| = 1$ and $\lim_{n \rightarrow \infty} (H - \lambda)\varphi_n = 0$. The functions $(\varphi_n)_{n \in \mathbb{N}}$ are called approximate eigenfunctions.
- 2) $\lambda \in \sigma_{\text{ess}}(H)$ if and only if there are $\varphi_n \in \mathcal{H}$ such that
 - (a) $\|\varphi_n\| = 1$,
 - (b) $w - \lim_{n \rightarrow \infty} \varphi_n = 0$,
 - (c) $\lim_{n \rightarrow \infty} (H - \lambda)\varphi_n = 0$.

The sequence of functions $(\varphi_n)_{n \in \mathbb{N}}$ is called a Weyl sequence.

For the next theorem, the first point is a one-line proof. The second one relies on Persson's Lemma, e.g., [KL, Proposition 18].

Theorem 3.2. *Let $\mathcal{G} := (\mathcal{V}, \mathcal{E}, m)$ be such that $\Delta_{\mathcal{G}}$ is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V})$. It follows that:*

- 1) $\lambda \in \sigma(\Delta_{\mathcal{G}})$ if and only if there are $\varphi_n \in \mathcal{C}_c(\mathcal{V})$ such that $\|\varphi_n\| = 1$ and $\lim_{n \rightarrow \infty} (\Delta_{\mathcal{G}} - \lambda)\varphi_n = 0$.
- 2) $\lambda \in \sigma_{\text{ess}}(H)$ if and only if there are $\varphi_n \in \mathcal{C}_c(\mathcal{V})$ such that
 - (a) $\|\varphi_n\| = 1$,
 - (b) for all finite set $K \subset \mathcal{V}$, we have $\text{supp}(\varphi_n) \cap K = \emptyset$, for n large enough,
 - (c) $\lim_{n \rightarrow \infty} (\Delta_{\mathcal{G}} - \lambda)\varphi_n = 0$.

3.3. Computation of the essential spectrum. In this last section, we finish the proof of the main theorem. We start with two lemmas.

Lemma 3.1. *Let $\mathcal{G} := (\mathcal{V}, \mathcal{E}, m)$ be such that $\Delta_{\mathcal{G}}$ is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V})$. Let $\chi : \mathcal{V} \rightarrow \mathbb{R}$ be bounded, and define*

$$G(x) := \frac{1}{m(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y) |\chi(y) - \chi(x)|.$$

We have:

- (a) For all $f \in \mathcal{C}_c(\mathcal{V})$, we have

$$|\langle f, [\Delta_{\mathcal{G}}, \chi(\cdot)]f \rangle_{\mathcal{G}}| \leq \langle f, G(\cdot)f \rangle_{\mathcal{G}}.$$

- (b) Assuming that G is bounded, $[\Delta_{\mathcal{G}}, \chi(\cdot)]$ extends to a bounded operator that we denote by $[\Delta_{\mathcal{G}}, \chi(\cdot)]_{\circ}$.
- (c) The operator $[\Delta_{\mathcal{G}}, \chi(\cdot)]_{\circ}$ is compact if

$$(3.2) \quad \lim_{|x| \rightarrow \infty} G(x) = 0.$$

Proof. Take $f \in \mathcal{C}_c(\mathcal{V})$. Using the Cauchy-Schwartz inequality, we have:

$$\begin{aligned} |\langle f, [\Delta, \chi(\cdot)]f \rangle_{\mathcal{G}}| &\leq \sum_{x \in \mathcal{V}} |f(x)| \cdot \sum_{y \sim x} \mathcal{E}(x, y) |\chi(y) - \chi(x)| \cdot |f(y)| \\ &\leq \frac{1}{2} \sum_{x \in \mathcal{V}} \sum_{y \sim x} \mathcal{E}(x, y) |\chi(y) - \chi(x)| \cdot |f(x)|^2 + \mathcal{E}(x, y) |\chi(y) - \chi(x)| \cdot |f(y)|^2 \\ &= \sum_{x \in \mathcal{V}} G(x) m(x) |f(x)|^2 = \langle f, G(\cdot)f \rangle_{\mathcal{G}}. \end{aligned}$$

The boundedness is immediate and the compactness follows from the min-max theory, e.g., [Gol3][Proposition 2.8]. \square

Lemma 3.2. *Assume that $\mathcal{G} := (\mathcal{V}, \mathcal{E}, m)$ is Klaus-sparse. Using the notation of Definition 2.1, there exists $\chi : \mathcal{V} \rightarrow [0, 1]$ such that*

- 1) $\chi(x) = 1$ when $x \in \cup_{i \in \mathbb{N}, k \in \mathcal{J}} B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{int}})$ and $\chi(x) = 0$ when $x \in \left(\cup_{i,k} B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}} - 2) \right)^c$.
- 2) The operator $[\Delta_{\mathcal{G}}, \chi(\cdot)]_{\circ}$ is compact.

Proof. Set

$$\chi(x) := \begin{cases} 1 - \frac{d_{\mathcal{G}}(x, B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{int}}))}{r_{i,k}^{\text{ext}} - r_{i,k}^{\text{int}} - 1}, & \text{if } x \in B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}} - 1) \text{ for some } i \in \mathbb{N} \text{ and } k \in \mathcal{J}, \\ 0, & \text{otherwise.} \end{cases}$$

Recalling Definition 2.1 (d), this can be rewritten as follows:

$$\chi(x) = \sum_{i \in \mathbb{N}, k \in \mathcal{J}} \max \left(1 - \frac{d_{\mathcal{G}}(x, B(x_{i,k}, r_{i,k}^{\text{int}}))}{r_{i,k}^{\text{ext}} - r_{i,k}^{\text{int}} - 1}, 0 \right)$$

Given $x \in \mathcal{V}$, we have

$$G(x) := \frac{1}{m(x)} \sum_{y \sim x} \mathcal{E}(x, y) |\chi(y) - \chi(x)| \leq \text{deg}_{\mathcal{G}}(x) \times F(x),$$

where

$$F(x) := \begin{cases} \frac{1}{r_{i,k}^{\text{ext}} - r_{i,k}^{\text{int}} - 1}, & \text{if } x \in \cup_{i \in \mathbb{N}, k \in \mathcal{J}} B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}} - 1) \setminus B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{int}} - 1), \\ 0, & \text{otherwise.} \end{cases}$$

Thanks to the support of F and recalling Definition 2.1 (c), we refine the estimate as follows:

$$G(x) \leq \sup_{y \in \mathcal{V}_{\mathcal{M}}} \deg_{\mathcal{G}_{\mathcal{M}}}(y) \times F(x).$$

Moreover, we have that $F(x) \rightarrow 0$ as $|x| \rightarrow \infty$ due to Definition 2.1 (a) and the Fréchet convergence. This implies (3.2), which ensures that $[\Delta, \chi(\cdot)]_{\circ}$ is a compact operator. \square

Lemma 3.3. *Assume that $\mathcal{G} := (\mathcal{V}, \mathcal{E}, m)$ is Klaus-sparse. Using the notation of Definition 2.1, we have $\sigma(\Delta_{\mathcal{M}}) \subset \sigma_{\text{ess}}(\oplus_{k \in \mathcal{J}} \Delta_{\mathcal{G}_{\infty,k}})$.*

Proof. Let $\lambda \in \sigma(\Delta_{\mathcal{M}})$. First note that $\Delta_{\mathcal{M}}$ is bounded by (1.3) and (2.1). In particular, $\mathcal{C}_c(\mathcal{V}_{\mathcal{M}})$ is a core for $\Delta_{\mathcal{M}}$. Then, by Theorem 3.2, there are $\varphi_n \in \mathcal{C}_c(\mathcal{V}_{\mathcal{M}})$ so that $\|\varphi_n\|_{\mathcal{G}_{\mathcal{M}}} = 1$ and $\lim_{n \rightarrow \infty} (\Delta_{\mathcal{M}} - \lambda)\varphi_n = 0$. Let $r_n > 0$ be chosen such that $\text{supp}(\varphi_n) \subset B_{\mathcal{G}_{\mathcal{M}}}(x_{\mathcal{M}}, r_n - 1)$. Using properties (d) and (e) of Definition 2.1, we see that there exist an isometric graph embedding T_n and an injective function $\phi : \mathbb{N} \rightarrow \mathbb{N} \times \mathcal{J}$ such that

$$T_n : [B_{\mathcal{G}_{\mathcal{M}}}(x_{\mathcal{M}}, r_n)]^{\mathcal{G}_{\mathcal{M}}} \rightarrow [B_{\mathcal{G}}(x_{\phi(n)}, r_{\phi(n)}^{\text{ext}} - 1) \setminus B_{\mathcal{G}}(x_{\phi(n)}, r_{\phi(n)}^{\text{int}} + 1)]^{\mathcal{G}} \stackrel{(b)}{\subset} \mathcal{G}_{\infty,k(n)},$$

for all $n \in \mathbb{N}$ and $\phi(n) = (i(n), k(n))$. Since ϕ is injective, note that $\phi(n) \rightarrow \infty$ in the Fréchet sense, as n goes to infinity. The last inclusion is due to Definition 2.1 (b).

Since it is a graph embedding and recalling that $\text{supp}(\varphi_n) \subset B_{\mathcal{G}_{\mathcal{M}}}(x_{\mathcal{M}}, r_n - 1)$, we have

$$(\oplus_{k \in \mathcal{J}} (\Delta_{\mathcal{G}_{\infty,k}} - \lambda)) T_n \varphi_n = (\Delta_{\mathcal{G}_{\infty,k(n)}} - \lambda) T_n \varphi_n \stackrel{(b)}{=} (\Delta_{\mathcal{G}} - \lambda) T_n \varphi_n = T_n ((\Delta_{\mathcal{M}} - \lambda)\varphi_n) \rightarrow 0,$$

as $n \rightarrow \infty$. Moreover, since it is an isometry we have $\|T_n \varphi_n\|_{\cup_k \mathcal{G}_{\infty,k}} = 1$. Since $\lim_{(i,k) \rightarrow \infty} r_{i,k}^{\text{int}} = \infty$, up to a subsequence, recalling the support of $T_n \varphi_n$, we see that $w - \lim_{n \rightarrow \infty} T_n \varphi_n = 0$. This is a Weyl sequence for $\oplus_{k \in \mathcal{J}} \Delta_{\mathcal{G}_{\infty,k}}$. This yields that $\lambda \in \sigma_{\text{ess}}(\oplus_{k \in \mathcal{J}} \Delta_{\mathcal{G}_{\infty,k}})$. \square

We turn to the proof of the main result.

Proof of Theorem 1.1: We prove the two inclusions.

\supset : Let $\lambda \in \sigma(\Delta_{\mathcal{G}_{\infty,k}})$ for some $k \in \mathcal{J}$. One can find functions $\varphi_n \in \mathcal{C}_c(\mathcal{V}_{\mathcal{G}_{\infty,k}})$ such that $\|\varphi_n\|_{\mathcal{G}_{\infty,k}} = 1$ and $\lim_{n \rightarrow \infty} (\Delta_{\mathcal{G}_{\infty,k}} - \lambda)\varphi_n = 0$ by Theorem 3.2. Let $r_n > 0$ be chosen so that $\text{supp}(\varphi_n) \subset B_{\mathcal{G}_{\infty,k}}(x_{\infty,k}, r_n - 1)$. By Definition 2.1 (b) and (d), we see that there exist an isometric graph embedding T_n and a strictly increasing function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$T_n : [B_{\mathcal{G}_{\infty,k}}(x_{\infty,k}, r_n)]^{\mathcal{G}_{\infty,k}} \rightarrow [B_{\mathcal{G}}(x_{\phi(n),k}, r_{\phi(n),k}^{\text{int}} - 1)]^{\mathcal{G}},$$

for all $n \in \mathbb{N}$. Since it is a graph embedding, we have

$$(\Delta_{\mathcal{G}} - \lambda) T_n \varphi_n = T_n ((\Delta_{\mathcal{G}_{\infty,k}} - \lambda)\varphi_n) \rightarrow 0,$$

as $n \rightarrow \infty$ and since it is an isometry we have $\|T_n \varphi_n\|_{\mathcal{G}} = 1$.

Recalling (d), we see that the supports of $(T_n \varphi_n)_{n \in \mathbb{N}}$ are two by two disjoint. In particular, we obtain $w - \lim_{n \rightarrow \infty} T_n \varphi_n = 0$. We infer that $T_n \varphi_n$ is a Weyl sequence for $(\Delta_{\mathcal{G}}, \lambda)$. In particular $\lambda \in \sigma_{\text{ess}}(\Delta_{\mathcal{G}})$. This implies that $\cup_{k \in \mathcal{J}} \sigma(\Delta_{\infty,k}) \subset \sigma_{\text{ess}}(\Delta_{\mathcal{G}})$. Since $\sigma_{\text{ess}}(\Delta_{\mathcal{G}})$ is closed, we obtain the first inclusion.

\subset : Let $\lambda \in \sigma_{\text{ess}}(\Delta_{\mathcal{G}})$. There are $\varphi_n \in \mathcal{C}_c(\mathcal{V})$ verifying 2 (a)-(c) in Theorem 3.2. Take χ as in Lemma 3.2 and distinguish two cases.

i) Suppose that $\liminf_{n \rightarrow \infty} \|(1 - \chi(\cdot))\varphi_n\|_{\mathcal{G}} > 0$. Set

$$\Psi_n := \frac{1}{\|(1 - \chi(\cdot))\varphi_n\|_{\mathcal{G}}} (1 - \chi(\cdot))\varphi_n.$$

We have $\|\Psi_n\|_{\mathcal{G}} = 1$, for all $n \in \mathbb{N}$ with support in $\cap_{i,k} B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{int}})^c$. By Definition 2.1 (c), we have the following direct sums:

$$(3.3) \quad \Psi_n = \sum_{l \in \mathcal{L}} \mathbf{1}_{C_l} \Psi_n \text{ and } \Delta_{\mathcal{G}} \Psi_n = \sum_{l \in \mathcal{L}} \mathbf{1}_{C_l} \Delta_{\mathcal{G}} \Psi_n.$$

For each $l \in \mathcal{L}$, we inject $\mathbf{1}_{C_l} \Psi_n$ into $\mathcal{G}_{\mathcal{M}}$ and denote it by $\Psi_{n,\mathcal{M}}^l$. Since $\Psi_{n,\mathcal{M}}^l$ is with finite support, there is $p = p(l, n)$ such that

$$1 = \sum_{l \in \mathcal{L}} \|\Psi_{n,\mathcal{M}}^l\|_{\mathcal{G}_{\mathcal{M}}}^2 = \sum_{l=0}^p \|\Psi_{n,\mathcal{M}}^l\|_{\mathcal{G}_{\mathcal{M}}}^2.$$

By Lemma 3.4 (b), there exist $(\theta_{l,n})_{l \in \mathcal{L}, n \in \mathbb{N}}$ such that

$$(3.4) \quad \left\| \sum_{l \in \mathcal{L}} e^{i\theta_{l,n}} \Psi_{n,\mathcal{M}}^l \right\|_{\mathcal{G}_{\mathcal{M}}} \geq 1.$$

Set now

$$\tilde{\Psi}_{n,\mathcal{M}} := \frac{1}{\left\| \sum_{l \in \mathcal{L}} e^{i\theta_{l,n}} \Psi_{n,\mathcal{M}}^l \right\|_{\mathcal{G}_{\mathcal{M}}}} \sum_{l \in \mathcal{L}} e^{i\theta_{l,n}} \Psi_{n,\mathcal{M}}^l.$$

Then,

$$\begin{aligned} \|(\Delta_{\mathcal{M}} - \lambda) \tilde{\Psi}_{n,\mathcal{M}}\|_{\mathcal{G}_{\mathcal{M}}} &= \frac{1}{\left\| \sum_{l \in \mathcal{L}} e^{i\theta_{l,n}} \Psi_{n,\mathcal{M}}^l \right\|_{\mathcal{G}_{\mathcal{M}}}} \left\| (\Delta_{\mathcal{G}} - \lambda) \sum_{l \in \mathcal{L}} e^{i\theta_{l,n}} \mathbf{1}_{C_l} \Psi_n \right\|_{\mathcal{G}} \\ &\stackrel{(3.4)}{\leq} \left\| \sum_{l \in \mathcal{L}} e^{i\theta_{l,n}} (\Delta_{\mathcal{G}} - \lambda) \mathbf{1}_{C_l} \Psi_n \right\|_{\mathcal{G}} = \sum_{l \in \mathcal{L}} \|(\Delta_{\mathcal{G}} - \lambda) \mathbf{1}_{C_l} \Psi_n\|_{\mathcal{G}} \\ &\stackrel{(3.3)}{=} \|(\Delta_{\mathcal{G}} - \lambda) \Psi_n\|_{\mathcal{G}} = \frac{\|(\Delta_{\mathcal{G}} - \lambda)(1 - \chi(\cdot))\varphi_n\|_{\mathcal{G}}}{\|(1 - \chi(\cdot))\varphi_n\|_{\mathcal{G}}} \\ &\leq \frac{1}{\|(1 - \chi(\cdot))\varphi_n\|_{\mathcal{G}}} (\|1 - \chi\|_{\infty} \cdot \|(\Delta_{\mathcal{G}} - \lambda)\varphi_n\|_{\mathcal{G}} + \|[\Delta_{\mathcal{G}}, \chi(\cdot)]\varphi_n\|_{\mathcal{G}}) \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, since $[\Delta_{\mathcal{G}}, \chi(\cdot)]_{\circ}$ is compact by Lemma 3.2 and φ_n tends weakly to 0. This implies that $\lambda \in \sigma(\Delta_{\mathcal{M}}) \subset \sigma_{\text{ess}}(\Delta_{\oplus_{k \in \mathcal{J}} \mathcal{G}_{\infty,k}})$, by Lemma 3.3.

ii) Suppose now that $\liminf_{n \rightarrow \infty} \|(1 - \chi(\cdot))\varphi_n\|_{\mathcal{G}} = 0$. Up to a subsequence, we can suppose that $\lim_{n \rightarrow \infty} \|\chi(\cdot)\varphi_n\|_{\mathcal{G}} = 1$. Thanks to Lemma 3.2, note that

$$\text{supp}(\chi(\cdot)\varphi_n) \subset \bigcup_{i,k} B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}} - 2).$$

Set

$$\Psi_n := \frac{1}{\|\chi(\cdot)\varphi_n\|_{\mathcal{G}}} \chi\varphi_n = \frac{1}{\|\mathbf{1}_{\bigcup_{i,k} B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}} - 2)} \chi(\cdot)\varphi_n\|_{\mathcal{G}}} \mathbf{1}_{\bigcup_{i,k} B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}} - 2)} \chi(\cdot)\varphi_n \in \bigcup_{i,k} B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}} - 2).$$

Note that $\|\Psi_n\|_{\mathcal{G}} = 1$. Using Definition 2.1 (d), we have

$$(3.5) \quad \Psi_n = \oplus_{k \in \mathcal{J}} \oplus_{i=0}^{\infty} \mathbf{1}_{B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}} - 2)} \Psi_n \text{ and } \Delta_{\mathcal{G}} \Psi_n = \oplus_{k \in \mathcal{J}} \oplus_{i=0}^{\infty} \mathbf{1}_{B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}} - 1)} \Delta_{\mathcal{G}} \Psi_n$$

where the sum is taken over a finite number since φ_n is with compact support. Moreover,

$$1 = \|\Psi_n\|_{\mathcal{G}}^2 = \sum_{k \in \mathcal{J}} \sum_{i=0}^{\infty} \|\mathbf{1}_{B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}} - 2)} \Psi_n\|_{\mathcal{G}}^2.$$

We now inject $\mathbf{1}_{B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}} - 2)} \Psi_n$ into $\mathcal{G}_{\infty,k}$ using Definition 2.1 (b). We denote by $\Psi_{n,\infty}^{i,k} : \mathcal{V}_{\infty,k} \rightarrow \mathbb{C}$ the new function. Trivially, we have

$$\sum_{k \in \mathcal{J}} \sum_{i=0}^{\infty} \|\Psi_{n,\infty}^{i,k}\|_{\mathcal{G}_{\infty,k}}^2 = 1.$$

Since $\Psi_{n,\infty}^{i,k}$ is with finite support, there is $p = p(k, n)$ such that

$$\alpha_{n,k} := \sum_{i=0}^{\infty} \|\Psi_{n,\infty}^{i,k}\|_{\mathcal{G}_{\infty,k}}^2 = \sum_{i=0}^p \|\Psi_{n,\infty}^{i,k}\|_{\mathcal{G}_{\infty,k}}^2.$$

Thanks to Lemma 3.4 b), there exist $(\theta_{i,k,n})_{i \in \mathbb{N}, k \in \mathcal{J}, n \in \mathbb{N}} \in [0, 2\pi]^{\mathbb{N} \times \mathcal{J}}$ such that

$$(3.6) \quad \sum_{k \in \mathcal{J}} \left\| \sum_{i=0}^{\infty} e^{i\theta_{i,k,n}} \Psi_{n,\infty}^{i,k} \right\|_{\mathcal{G}_{\infty,k}}^2 \geq \sum_{k \in \mathcal{J}} \sum_{i=0}^{\infty} \|\Psi_{n,\infty}^{i,k}\|_{\mathcal{G}_{\infty,k}}^2 = \sum_{k \in \mathcal{J}} \alpha_{n,k} = 1.$$

Set

$$\tilde{\Psi}_{n,\infty}^k := \frac{1}{\sum_{k' \in \mathcal{J}} \left\| \sum_{i=0}^{\infty} e^{i\theta_{i,k',n}} \Psi_{n,\infty}^{i,k'} \right\|_{\mathcal{G}_{\infty,k'}}} \sum_{i=0}^{\infty} e^{i\theta_{i,k,n}} \Psi_{n,\infty}^{i,k}.$$

Note that $\sum_{k \in \mathcal{J}} \|\tilde{\Psi}_{n,\infty}^k\|_{\mathcal{G}_{\infty,k}} = 1$ for all $n \in \mathbb{N}$. Using again Definition 2.1 (b) and (d), we have

$$\begin{aligned} & \left\| \bigoplus_{k \in \mathcal{J}} (\Delta_{\mathcal{G}_{\infty,k}} - \lambda) \tilde{\Psi}_{n,\infty}^k \right\|_{\bigcup_{k \in \mathcal{J}} \mathcal{G}_{\infty,k}} = \sum_{k \in \mathcal{J}} \left\| (\Delta_{\mathcal{G}_{\infty,k}} - \lambda) \tilde{\Psi}_{n,\infty}^k \right\|_{\mathcal{G}_{\infty,k}} \\ & \stackrel{(b)}{=} \frac{1}{\sum_{k' \in \mathcal{J}} \left\| \sum_{i=0}^{\infty} e^{i\theta_{i,k',n}} \Psi_{n,\infty}^{i,k'} \right\|_{\mathcal{G}_{\infty,k'}}} \sum_{k \in \mathcal{J}} \left\| (\Delta_{\mathcal{G}} - \lambda) \left(\sum_{i=0}^{\infty} e^{i\theta_{i,k,n}} \mathbf{1}_{B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}} - 2)} \Psi_n \right) \right\|_{\mathcal{G}} \\ & \stackrel{(3.6)}{\leq} \sum_{k \in \mathcal{J}} \left\| \sum_{i=0}^{\infty} e^{i\theta_{i,k,n}} (\Delta_{\mathcal{G}} - \lambda) \left(\mathbf{1}_{B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}} - 2)} \Psi_n \right) \right\|_{\mathcal{G}} \\ & \stackrel{(d)}{=} \sum_{k \in \mathcal{J}} \sum_{i=0}^{\infty} \left\| (\Delta_{\mathcal{G}} - \lambda) \left(\mathbf{1}_{B_{\mathcal{G}}(x_{i,k}, r_{i,k}^{\text{ext}} - 2)} \Psi_n \right) \right\|_{\mathcal{G}} \stackrel{(3.5)}{=} \|(\Delta_{\mathcal{G}} - \lambda) \Psi_n\|_{\mathcal{G}} \\ & \leq \frac{1}{\|\chi(\cdot) \varphi_n\|_{\mathcal{G}}} (\|\chi\|_{\infty} \cdot \|(\Delta_{\mathcal{G}} - \lambda) \varphi_n\|_{\mathcal{G}} + \|[\Delta_{\mathcal{G}}, \chi(\cdot)] \varphi_n\|_{\mathcal{G}}) \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, since $[\Delta_{\mathcal{G}}, \chi(\cdot)]_{\circ}$ is compact by Lemma 3.2 and φ_n tends weakly to 0. Thus $(\bigoplus_{k \in \mathcal{J}} \tilde{\Psi}_{n,\infty}^k)_{n \in \mathbb{N}}$ is a Weyl sequence for $\Delta_{\bigoplus_{k \in \mathcal{J}} \mathcal{G}_{\infty,k}}$. We conclude that $\lambda \in \sigma(\Delta_{\bigoplus_{k \in \mathcal{J}} \mathcal{G}_{\infty,k}}) = \overline{\bigcup_{k \in \mathcal{J}} \sigma(\Delta_{\mathcal{G}_{\infty,k}})}$. \square

We have used the following Lemma coming originally from [Am].

Lemma 3.4. *a) Let $(e_i)_{i \in \mathbb{N}}$ be included in a Hilbert space \mathcal{H} such that $\|e_i\| = 1$. Let $(\beta_i)_{i \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$. For all $p \in \mathbb{N}^*$, we have:*

$$(3.7) \quad \exists \theta_1, \dots, \theta_p \in [0, 2\pi] \text{ such that } \left\| \sum_{j=1}^p e^{i\theta_j} \beta_j e_j \right\|^2 \geq \sum_{j=1}^p |\beta_j|^2.$$

b) In particular, let $(f_j)_{j \in \mathbb{N}}$ be included in a Hilbert space \mathcal{H} . For all $p \in \mathbb{N}^$, set $\alpha(p) := \sum_{j=1}^p \|f_j\|^2$. There exist $(\theta_j)_{j \in [1,p]} \subset [0, 2\pi]^{[1,p]}$ such that*

$$\left\| \sum_{j=1}^p e^{i\theta_j} f_j \right\|^2 \geq \alpha(p).$$

Proof. a) We prove the result by induction. The case $p = 1$ is trivial. Suppose that we have (3.7). Set $k := \sum_{j=1}^p e^{i\theta_j} \beta_j e_j$. Let $\theta_{p+1} \in [0, 2\pi]$ such that $2\Re\langle e^{i\theta_{p+1}} \beta_{p+1} e_{p+1}, k \rangle \geq 0$. We have:

$$\left\| \sum_{j=1}^{p+1} e^{i\theta_j} \beta_j e_j \right\|^2 = \|k\|^2 + |\beta_{p+1}|^2 + 2\Re\langle e^{i\theta_{p+1}} \beta_{p+1} e_{p+1}, k \rangle \geq \|k\|^2 + |\beta_{p+1}|^2.$$

which concludes the proof.

b) Set $f_j := \beta_j e_j$ such that $\|e_j\| = 1$ and apply a). \square

3.4. Sharpness. We conclude by discussing further examples.

3.4.1. The closeness of the union is not automatic. In the context of R -limits and C^* -algebra, the union of the spectra of the localisations at infinity is always closed. A contrario, in the context of Klaus-sparse graphs, we prove in this section that this union is not always a closed set. Theorem 1.1 is sharp.

To see this let us consider a Klaus-sparse graph $\mathcal{G} := (\mathcal{V}, \mathcal{E}, m)$ constructed as in Figure 1 where the localisations at infinity of \mathcal{G} are of the following types:

- 1) Let $\mathcal{G}_{\infty,0}(\mathcal{V}_{\infty,0}, \mathcal{E}_{\infty,0}, 1)$ be a simple 3-star infinite graph. Namely, let $\mathcal{V}_{\infty,0} := \{0\} \cup (\{1, 2, 3\} \times \mathbb{N}^*)$ and set $\mathcal{E}_{\infty,0}(0, (i, 1)) := 1$ and $\mathcal{E}_{\infty,0}((i, j), (i, k)) := 1$ if $|k - j| = 1$, for all $i \in \{1, 2, 3\}$ and $j, k \in \mathbb{N}^*$ and set $\mathcal{E}_{\infty,0}(x, y) := 0$ otherwise.

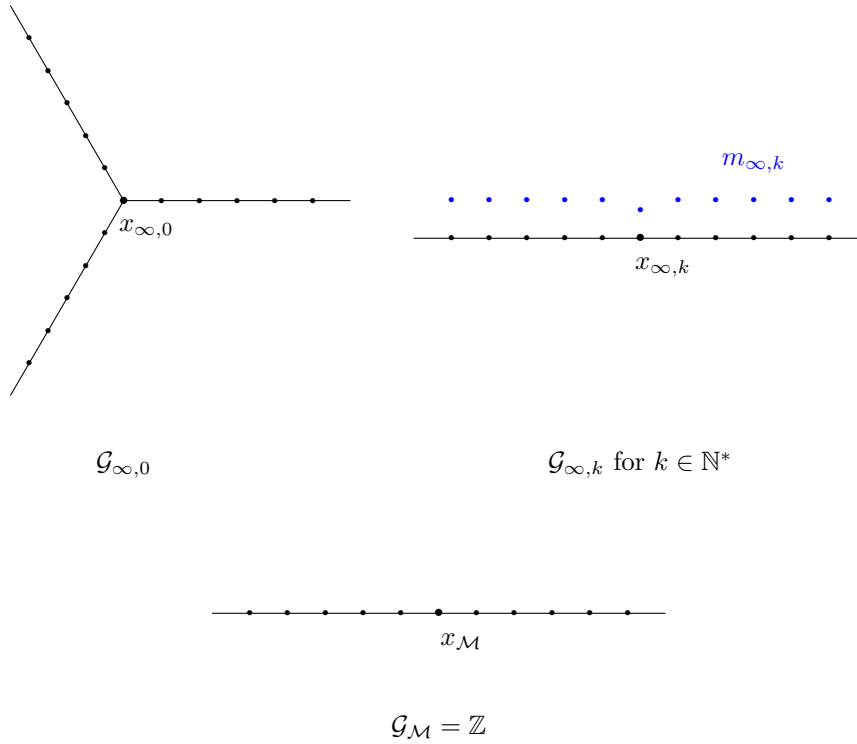


FIGURE 5. Localisations at infinity, counter example for the closeness

2) For each $k \in \mathbb{N}^*$, let $\mathcal{G}_{\infty,k} := (\mathcal{V}_{\infty,k}, \mathcal{E}_{\infty,k}, m_{\infty,k})$ be given by

$$\mathcal{V}_{\infty,k} = \mathbb{Z}, \quad \mathcal{E}_{\infty,k}(x, y) = \mathcal{E}_{\mathbb{Z}}(x, y) = \begin{cases} 1, & \text{if } |x - y| = 1, \\ 0, & \text{elsewhere,} \end{cases} \quad m_{\infty,k}(x) := \begin{cases} s(k), & \text{if } x = 0, \\ 1, & \text{elsewhere.} \end{cases}$$

where $s(k)$ is a sequence such that $0 < s(k) < 1$, to be fixed later.

3) Let $\mathcal{G}_{\mathcal{M}} = (\mathbb{Z}, \mathcal{E}_{\mathbb{Z}}, 1)$.

We refer to Figure 5 for an illustration. We have:

$$\sigma(\Delta_{\infty,0}) = [0, 4] \cup \left\{ \frac{9}{2} \right\}, \quad \sigma(\Delta_{\mathcal{G}_{\infty,k}}) = [0, 4] \cup \left\{ \frac{4}{s(k)(2-s(k))} \right\}.$$

The former is well-known, e.g. [BrDeEl, Lemma 4.4.5] with a verbatim proof. The latter comes from a direct computation: the spectrum has an absolute continuous part and a discrete one, constituted by a unique eigenvalue $\lambda_k = \frac{4}{s(k)(2-s(k))}$. Let us choose $s(k) := 1 - \frac{\sqrt{2}}{2} + \frac{1}{2k}$. When k goes to $+\infty$, $(s(k) - 1)^2$ tends to $\frac{1}{2}$, so that λ_k tends to 8, which shows that the set $\bigcup_{k \in \mathcal{J}} \sigma(\Delta_{\mathcal{G}_{\infty,k}})$ is not a closed set. A suitable sequence s_k can be chosen in order to get any limit l for λ_k , provided $4 < l < \infty$.

3.4.2. *A graph without uniform sub-exponential growth.* Take a graph $\mathcal{G}_1 := (\mathcal{V}, \mathcal{E}, 1)$ such that (1.4) is wrong and $\mathcal{G}_2 := \mathbb{Z}$. Then take $x \in \mathcal{V}$ and glue \mathcal{G}_1 with \mathcal{G}_2 in a direct way such that x is identified with 0. We denote by $\mathcal{G}_{\infty,0}$ this graph. Take a Klaus-graph constructed as in 1 where the localisation at infinity is given by $\mathcal{G}_{\infty,0}$, see Figure 6 and $\mathcal{G}_{\mathcal{M}} = \mathbb{Z}$ as medium graph. Clearly \mathcal{G} has no uniform sub-exponential growth. The result of [El] cannot apply whereas ours can.

APPENDIX A. STABILITY OF THE ESSENTIAL SPECTRUM

Stability of the essential spectrum is a wild subject. The general idea is that if a perturbation is small at infinity then the essential remains the same. To establish it, given H and H_{pertu} being self-adjoint, one usually proves that $(H_{\text{pertu}} + i)^{-1} - (H + i)^{-1}$ is compact to obtain that $\sigma_{\text{ess}}(H_{\text{pertu}}) = \sigma_{\text{ess}}(H)$. This is the Weyl's theorem, e.g. [RS, Theorem XIII.14]. The difficulty lies in proving the compactness by taking advantage of the smallness of the perturbation at infinity. We refer [GeGo2] historical references therein and also for a general abstract method. However, our situation does not fall exactly in the abstract

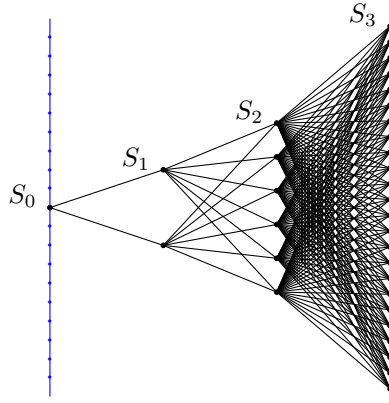


FIGURE 6. An antitree with spheres S_n of size $(n+1)!$ glued with \mathbb{Z} at 0, a counter-example with non-exponential growth

setting of [GeGo2] and we need some adaptations. We present an extended version of [Gol3, Proposition 5.2] (where $m = \tilde{m}$).

Theorem A.1. *Let $\mathcal{G} := (\mathcal{E}, \mathcal{V}, m)$ be such that $\mathcal{D}(\Delta_{\mathcal{G}}^{1/2}) = \mathcal{D}(\deg_{\mathcal{G}}^{1/2}(\cdot))$. Let $\tilde{\mathcal{G}} := (\tilde{\mathcal{E}}, \tilde{\mathcal{V}}, \tilde{m})$ with $\tilde{\mathcal{V}} := \mathcal{V}$ and such that there is c with $\tilde{\mathcal{E}}(x, y) \leq c\mathcal{E}(x, y)$, for all $x, y \in \mathcal{V}$ and*

$$(A.1) \quad \lim_{|x| \rightarrow \infty} \frac{m(x)}{\tilde{m}(x)} = 1 \text{ and } \lim_{|x|, |y| \rightarrow \infty} \mathcal{E}(x, y) - \tilde{\mathcal{E}}(x, y) = 0.$$

Then $\mathcal{D}(\Delta_{\tilde{\mathcal{G}}}^{1/2}) = \mathcal{D}(\deg_{\tilde{\mathcal{G}}}^{1/2}(\cdot))$ and

$$\sigma_{\text{ess}}(\Delta_{\mathcal{G}}) = \sigma_{\text{ess}}(\Delta_{\tilde{\mathcal{G}}}).$$

We stress that the operators $\Delta_{\tilde{\mathcal{G}}}$ are not necessarily supposed to be essentially self-adjoint $\mathcal{C}_c(\mathcal{V})$ and that we consider their Friedrichs's extension.

Proof. First we transport unitarily $\Delta_{\tilde{\mathcal{G}}}$ from $\ell^2(\mathcal{V}, \tilde{m})$ into $\ell^2(\mathcal{V}, m)$. Namely, we set $\tilde{\Delta} := \sqrt{\frac{\tilde{m}(\cdot)}{m(\cdot)}} \Delta_{\tilde{\mathcal{G}}} \sqrt{\frac{m(\cdot)}{\tilde{m}(\cdot)}}$. We shall prove that

$$(A.2) \quad \Delta_{\mathcal{G}} - \tilde{\Delta} \in \mathcal{K}(\deg_{\mathcal{G}}^{1/2}(\cdot), (\deg_{\mathcal{G}}^{1/2}(\cdot))^*),$$

here $*$ denotes the antidual and we have identified $\ell^2(\mathcal{V}, m)$ with its antidual.

We have:

$$\Delta_{\mathcal{G}} - \tilde{\Delta} = \left(1 - \sqrt{\frac{\tilde{m}(\cdot)}{m(\cdot)}}\right) \Delta_{\mathcal{G}} + \sqrt{\frac{\tilde{m}(\cdot)}{m(\cdot)}} (\Delta_{\mathcal{G}} - \Delta_{\tilde{\mathcal{G}}}) \sqrt{\frac{m(\cdot)}{\tilde{m}(\cdot)}} + \sqrt{\frac{\tilde{m}(\cdot)}{m(\cdot)}} \Delta_{\mathcal{G}} \left(1 - \sqrt{\frac{m(\cdot)}{\tilde{m}(\cdot)}}\right)$$

By (1.3), $\Delta_{\mathcal{G}}$ is bounded in $\mathcal{B}(\deg_{\mathcal{G}}^{1/2}(\cdot), (\deg_{\mathcal{G}}^{1/2}(\cdot))^*)$ and $\left(1 - \sqrt{\frac{\tilde{m}(\cdot)}{m(\cdot)}}\right) \in \mathcal{K}((\deg_{\mathcal{G}}^{1/2}(\cdot))^*)$ by (A.1). The first term is compact in $\mathcal{K}(\deg_{\mathcal{G}}^{1/2}(\cdot), (\deg_{\mathcal{G}}^{1/2}(\cdot))^*)$. In a same way, since $\sqrt{\frac{\tilde{m}(\cdot)}{m(\cdot)}} \in \mathcal{B}((\deg_{\mathcal{G}}^{1/2}(\cdot))^*)$, the third term is also compact. We turn to the second one. Given $f \in \mathcal{C}_c(\mathcal{V})$, we have:

$$(A.3) \quad \begin{aligned} 0 \leq \langle f, (\Delta_{\mathcal{G}} - \Delta_{\tilde{\mathcal{G}}})f \rangle_{\mathcal{G}} &\leq \langle f, |\deg_{\mathcal{G}}(\cdot) - \deg_{\tilde{\mathcal{G}}}(\cdot)|f \rangle_{\mathcal{G}} + \sum_{x, y \in \mathcal{V}} |f(x)| \cdot |f(y)| \cdot \left| \mathcal{E}(x, y) - \frac{m(x)}{\tilde{m}(x)} \tilde{\mathcal{E}}(x, y) \right| \\ &\leq \langle f, |\deg_{\mathcal{G}}(\cdot) - \deg_{\tilde{\mathcal{G}}}(\cdot)|f \rangle_{\mathcal{G}} + \frac{1}{2} \sum_{x, y \in \mathcal{V}} |f(x)|^2 \cdot \left| \mathcal{E}(x, y) - \frac{m(x)}{\tilde{m}(x)} \tilde{\mathcal{E}}(x, y) \right| \\ &\quad + \frac{1}{2} \sum_{x, y \in \mathcal{V}} |f(x)|^2 \cdot \left| \mathcal{E}(x, y) - \frac{m(y)}{\tilde{m}(y)} \tilde{\mathcal{E}}(x, y) \right| \\ &\leq \langle f, F(\cdot)f \rangle_{\mathcal{G}}, \end{aligned}$$

where

$$(A.4) \quad F(x) = o(1 + \deg_{\mathcal{G}}(x)), \text{ as } |x| \rightarrow \infty.$$

To see this we use (A.1) and the Lebesgue convergence. We justify the domination of the first term by

$$\begin{aligned} \langle f, |\deg_{\mathcal{G}}(\cdot) - \deg_{\tilde{\mathcal{G}}}(\cdot)|f \rangle_{\mathcal{G}} &= \sum_{x \in \mathcal{V}} m(x)|f(x)|^2 \cdot \left| \frac{1}{m(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y) - \frac{1}{\tilde{m}(x)} \sum_{y \in \mathcal{V}} \tilde{\mathcal{E}}(x, y) \right| \\ &\leq \sum_{x \in \mathcal{V}} m(x)|f(x)|^2 \cdot \left(\left| \left(\frac{1}{m(x)} - \frac{1}{\tilde{m}(x)} \right) \sum_{y \in \mathcal{V}} \mathcal{E}(x, y) \right| + \left| \frac{1}{\tilde{m}(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y) - \tilde{\mathcal{E}}(x, y) \right| \right) \\ &\leq \left(\left(1 + \sup_{t \in \mathcal{V}} \frac{m(t)}{\tilde{m}(t)} \right) + \left(1 + c \sup_{t \in \mathcal{V}} \frac{m(t)}{\tilde{m}(t)} \right) \right) \langle f, \deg_{\mathcal{G}}(\cdot)f \rangle \end{aligned}$$

and that of the last term, as follows:

$$\begin{aligned} \frac{1}{2} \sum_{x, y \in \mathcal{V}} |f(x)|^2 \cdot \left| \mathcal{E}(x, y) - \frac{m(y)}{\tilde{m}(y)} \tilde{\mathcal{E}}(x, y) \right| &= \frac{1}{2} \sum_{x, y \in \mathcal{V}} m(x)|f(x)|^2 \cdot \left| \frac{1}{m(x)} \mathcal{E}(x, y) - \frac{1}{m(x)} \frac{m(y)}{\tilde{m}(y)} \tilde{\mathcal{E}}(x, y) \right| \\ &\leq \frac{1}{2} \left(1 + c \sup_{t \in \mathcal{V}} \frac{m(t)}{\tilde{m}(t)} \right) \langle f, \deg_{\mathcal{G}}(\cdot)f \rangle. \end{aligned}$$

The treatment of the second term is similar. Thanks to (A.4), we infer that the operator $F(\cdot)(1 + \deg(\cdot))$ is compact in $\mathcal{K} \left(\deg_{\mathcal{G}}^{1/2}(\cdot), \left(\deg_{\mathcal{G}}^{1/2}(\cdot) \right)^* \right)$. By the min-max principle, e.g., [Gol3, Proposition 2.4], (A.3) ensures that the second term also belongs to $\mathcal{K} \left(\deg_{\mathcal{G}}^{1/2}(\cdot), \left(\deg_{\mathcal{G}}^{1/2}(\cdot) \right)^* \right)$. We conclude that (A.2) follows.

We turn to the consequences of (A.2). Using the KLMN's theorem, e.g. [RS, Theorem X.17], we obtain that $\mathcal{D}(\tilde{\Delta}^{1/2}) = \mathcal{D}(\deg_{\mathcal{G}}^{1/2}(\cdot))$. Going back by unitary transform into $\ell^2(\mathcal{V}, \tilde{m})$, we obtain the result for $\mathcal{D}(\Delta_{\mathcal{G}}^{1/2})$. Concerning the equality of the essential spectra, the compactness of $\Delta_{\mathcal{G}} - \tilde{\Delta}$ implies that $(\Delta_{\mathcal{G}} + i)^{-1} - (\tilde{\Delta} + i)^{-1}$ is a compact operator, e.g., [GeGo2, Condition (AB)] or [Gol3, Proof of Proposition 5.2]. The Weyl's Theorem concludes, e.g., [RS, Theorem XIII.14]. \square

To apply this theorem we suppose crucially that the form-domain of $\Delta_{\mathcal{G}}$ is equal to that of $\deg_{\mathcal{G}}(\cdot)$. Recalling (1.3), we have $\mathcal{D} \left(\deg_{\mathcal{G}}^{1/2}(\cdot) \right) \subset \mathcal{D} \left(\Delta_{\mathcal{G}}^{1/2} \right)$ in general but the reverse inclusion is not automatic. We refer to [Gol3] for the beginning of this question and [BoGoKe] for an equivalence. We refer also to [BoGoKeal] for a magnetic version. In our context, it is enough to suppose the equality of the form-domains for the localisations at infinity. This is the aim of the next Proposition.

Proposition A.1. *Let $\mathcal{G} := (\mathcal{V}, \mathcal{E}, m)$ be a Klaus-sparse graph defined as in Definition 2.1. Assume that $\mathcal{D} \left(\Delta_{\oplus_{k \in \mathcal{J}} \mathcal{G}_{\infty, k}}^{1/2} \right) = \mathcal{D} \left(\deg_{\oplus_{k \in \mathcal{J}} \mathcal{G}_{\infty, k}}^{1/2}(\cdot) \right)$, for all $k \in \mathcal{J}$. Then $\mathcal{D} \left(\Delta_{\mathcal{G}}^{1/2} \right) = \mathcal{D} \left(\deg_{\mathcal{G}}^{1/2}(\cdot) \right)$.*

Proof. Recalling (1.3) it is enough to show that $\mathcal{D} \left(\Delta_{\mathcal{G}}^{1/2} \right) \subset \mathcal{D} \left(\deg_{\mathcal{G}}^{1/2}(\cdot) \right)$. By hypothesis and with the help of the uniform boundedness principle, there is $C > 0$ such that

$$\langle f, \deg_{\oplus_{k \in \mathcal{J}} \mathcal{G}_{\infty, k}}(\cdot)f \rangle_{\oplus_{k \in \mathcal{J}} \mathcal{G}_{\infty, k}} \leq C \left(\langle f, \Delta_{\oplus_{k \in \mathcal{J}} \mathcal{G}_{\infty, k}} f \rangle_{\oplus_{k \in \mathcal{J}} \mathcal{G}_{\infty, k}} + \|f\|_{\oplus_{k \in \mathcal{J}} \mathcal{G}_{\infty, k}}^2 \right),$$

for all $f \in \mathcal{C}_c(\oplus_{k \in \mathcal{J}} \mathcal{V}_{\infty, k})$. Take χ as in Lemma 3.2. For all $f \in \mathcal{C}_c(\mathcal{V})$, we have

$$\begin{aligned} \langle f, \Delta_{\mathcal{G}} f \rangle_{\mathcal{G}} &= \langle \chi f, \Delta_{\mathcal{G}} \chi f \rangle_{\mathcal{G}} + \langle \chi f, \Delta_{\mathcal{G}} (1 - \chi) f \rangle_{\mathcal{G}} + \langle (1 - \chi) f, \Delta_{\mathcal{G}} \chi f \rangle_{\mathcal{G}} + \langle (1 - \chi) f, \Delta_{\mathcal{G}} (1 - \chi) f \rangle_{\mathcal{G}} \\ &\geq \frac{1}{C} \langle \chi f, \deg_{\mathcal{G}}(\cdot) \chi f \rangle_{\mathcal{G}} - \|\chi f\|_{\mathcal{G}}^2 - \|(1 - \chi) \Delta_{\mathcal{G}}\| \cdot (2\|f\|_{\mathcal{G}} \cdot \|\chi f\|_{\mathcal{G}} + \|f\|_{\mathcal{G}} \cdot \|(1 - \chi) f\|_{\mathcal{G}}). \end{aligned}$$

Recalling that $\|\chi\| = \|1 - \chi\| = 1$ and that $\|(1 - \chi) \Delta_{\mathcal{G}}\|$ and $\|(1 - \chi) \deg_{\mathcal{G}}(\cdot)\|$ are finite by (2.1), we infer there is c such that

$$\langle f, \deg_{\mathcal{G}}(\cdot)f \rangle_{\mathcal{G}} \leq c \left(\langle f, \Delta_{\mathcal{G}} f \rangle_{\mathcal{G}} + \|f\|_{\mathcal{G}}^2 \right), \quad \text{for all } f \in \mathcal{C}_c(\mathcal{V}),$$

which ensures that $\mathcal{D} \left(\Delta_{\mathcal{G}}^{1/2} \right) \subset \mathcal{D} \left(\deg_{\mathcal{G}}^{1/2}(\cdot) \right)$ and concludes. \square

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