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# A convergence criterion for systems of point processes from the convergence of their stochastic intensities.

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**Abstract:** We define a metric  $d$  on the space  $\mathcal{N}$  of locally finite simple point measures. By construction, the convergence of a sequence of point measures with respect to  $d$  means the convergence of their atoms. The probabilistic and topological properties of this metric allow us to prove that the convergence in distribution of the stochastic intensities of a system of point processes implies the convergence of the system.

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## 1. Introduction

In this paper we study topological and probabilistic properties of the space  $\mathcal{N}$  of locally finite point measures on  $\mathbb{R}_+^2$ . More precisely, we introduce an explicit metric  $d$  on  $\mathcal{N}$  designed to compute the distance between the atoms of the point measures. In particular, the space  $(\mathcal{N}, d)$  will be shown to be separable but not complete. However, we will show that the law of Poisson measures with Lebesgue intensity is tight on  $(\mathcal{N}, d)$ . Another significant property of this space is that, for realizations of Poisson measures with Lebesgue intensity, the convergence for the metric  $d$  is equivalent to the vague convergence (see Proposition 2.6). These properties allow us to prove our main result Theorem 1.1, where  $D(\mathbb{R}_+, \mathbb{R})$  denotes Skorohod space.

**Theorem 1.1.** *Let  $\bar{Y}^k$  and  $Y^{N,k}$  ( $N, k \in \mathbb{N}^*$ ) be  $D(\mathbb{R}_+, \mathbb{R}_+)$ -valued random variables. Let  $(\pi^k)_{k \in \mathbb{N}^*}$  and  $(\bar{\pi}^k)_{k \in \mathbb{N}^*}$  be i.i.d. families of Poisson measures on  $\mathbb{R}_+ \times \mathbb{R}_+$  having Lebesgue intensity. Let  $Z^{N,k}$  and  $\bar{Z}^k$  be the point processes defined as follows*

$$Z_t^{N,k} := \int_{[0,t] \times \mathbb{R}_+} \mathbb{1}_{\{z \leq Y_{s-}^{N,k}\}} d\pi^k(s, z), \quad \bar{Z}_t^k := \int_{[0,t] \times \mathbb{R}_+} \mathbb{1}_{\{z \leq \bar{Y}_{s-}^k\}} d\bar{\pi}^k(s, z), \quad k \geq 1.$$

*Assume that, for every  $k \geq 1$ ,  $(Y^{N,1}, \pi^1, \dots, Y^{N,k}, \pi^k)$  converges in distribution to  $(\bar{Y}^1, \bar{\pi}^1, \dots, \bar{Y}^k, \bar{\pi}^k)$  in  $(D(\mathbb{R}_+, \mathbb{R}) \times \mathcal{N})^k$ , and that, for each  $k \geq 1$ ,  $\bar{Y}^k$  is independent of  $\bar{\pi}^k$ .*

*Then, for any  $n \geq 1$ ,  $(Z^{N,k})_{1 \leq k \leq n}$  converges to  $(\bar{Z}^k)_{1 \leq k \leq n}$  in distribution in  $D(\mathbb{R}_+, \mathbb{R}^n)$ . Hence  $(Z^{N,k})_{k \geq 1}$  converges to  $(\bar{Z}^k)_{k \geq 1}$  in distribution in  $D(\mathbb{R}_+, \mathbb{R})^{\mathbb{N}^*}$  endowed with the product topology.*

**Remark 1.2.** *In the statement of Theorem 1.1, we need to guarantee the following property: Poisson random measures are  $(\mathcal{N}, d)$ -valued random variables. This is a consequence of Proposition 3.4.*

**Remark 1.3.** *According to Lemma 4 of [3], a point process  $Z$  having stochastic intensity  $(Y_{s-})_{s \geq 0}$  can always be written in the form of Theorem 1.1.*

Let us note that, in Theorem 1.1, the processes  $Y^{N,k}$  are not assumed to be independent of the Poisson measures  $\pi^k$ . Otherwise, the proof would be straightforward by conditioning by  $Y^{N,k}$ .

One can note that, if the processes  $\bar{Y}^k$  and  $Y^{N,k}$  ( $k \geq 1, N \geq 1$ ) are semimartingales, then the result of Theorem 1.1 can be proved with Theorem IX.4.15 of [8]. Theorem 1.1 allows us to consider processes that are not semimartingales such as Hawkes processes and Volterra processes. In particular, since the stochastic intensity of Hawkes processes are not, in general, semimartingales, Theorem 1.1 can be used to show the convergence of Hawkes processes, provided that one can show the convergence of their stochastic intensity. Let us give an example of application of Theorem 1.1 in that direction for a one-dimensional point process (an example of application in the infinite-dimensional case would be an alternative proof of Theorem 1.7 of [6]). The example is based on Examples 7.3 and 7.4 of [1].

**Example 1.4.** *Let us consider  $K(t) := t^\gamma$  for some  $\gamma > 0$ ,  $K^N(t) := K(t/N)$  and some Poisson random measure  $\pi$  on  $\mathbb{R}_+^2$  having Lebesgue intensity. Let  $X^N$  satisfy*

$$X_t^N = \int_{[0,t] \times \mathbb{R}_+} K^N(t-s) \mathbb{1}_{\{z \leq |X_{s-}^N|\}} d\pi(s, z) - \int_0^t K^N(t-s) |X_s^N| ds.$$

Theorem 7.2 of [1] implies that the sequence of processes  $(\tilde{X}_t^N)_{t \geq 0} = (N^{-1} X_{Nt}^N)_{t \geq 0}$  has converging subsequence (in distribution in the topology  $L_{loc}^2$ ), and that the limit process  $(\bar{X}_t)_{t \geq 0}$  satisfies

$$\bar{X}_t = \int_0^t K(t-s) \sqrt{|\bar{X}_s|} dB_s,$$

for some standard Brownian motion  $B$ . Besides, one can prove with standard arguments, the tightness of  $(\tilde{X}^N)_N$  in Skorohod topology, so that we can assume that (a subsequence of)  $(\tilde{X}^N)_N$  converges to  $\bar{X}$  in Skorohod topology.

This implies the convergence of  $Y^N := |\tilde{X}^N|$  to  $\bar{Y} := |\bar{X}|$  in Skorohod topology. Moreover, the Brownian motion can be shown to be necessarily independent of the Poisson measure  $\pi$  (using Theorem II.6.3 of [7]). Then, Theorem 1.1 implies the convergence in distribution in Skorohod topology of  $Z_t^N := \int_{[0,t] \times \mathbb{R}_+} \mathbb{1}_{\{z \leq Y_{s-}^N\}} d\pi(s, z)$  to the point process  $\bar{Z}_t := \int_{[0,t] \times \mathbb{R}_+} \mathbb{1}_{\{z \leq \bar{Y}_{s-}\}} d\bar{\pi}(s, z)$ , where  $\bar{\pi}$  is independent of  $\bar{Y}$ . To the best of our knowledge, there is no classical way to prove this convergence.

Theorem 1.1 can be compared to Theorem 1 of [4] that states that the convergence of point processes is implied by the pointwise convergence in distribution of their compensators. In [4], Theorem 1 holds when the compensator of the limit point process is a deterministic function, whereas in Theorem 1.1, the limit point processes have stochastic intensities.

The idea of the proof of Theorem 1.1 is to use Skorohod representation theorem in order to show the almost sure convergence of the point processes  $Z^{N,k}$  from the convergence of the (representant of the) Poisson measures  $\pi^{N,k}$ . In this proof, the fundamental property of the convergence of Poisson measures is the convergence of their atoms. To prove Theorem 1.1, one needs to consider a metric space  $(M, m)$  that contains every realization of Poisson measures on  $\mathbb{R}_+^2$  with Lebesgue intensity and that satisfies the following conditions:

- $(M, m)$  is separable,
- the  $\sigma$ -field generated by  $m$  is the one generated by the mappings  $\pi \in M \mapsto \pi(B)$  ( $B \in \mathcal{B}(\mathbb{R}_+^2)$ ),

- the law of a Poisson measure on  $\mathbb{R}_+^2$  with Lebesgue intensity is tight on  $(M, m)$ ,
- the convergence of simple point measures on  $(M, m)$  implies the convergence of their atoms.

The space  $(\mathcal{N}, d)$  is a metric space satisfying all these conditions. In this paper, we study the properties of this space. Another possible candidate for the couple  $(M, m)$  can be found in [5]. Indeed, one can define a metric  $d^\#$  on the space of locally finite measures  $\mathcal{M}^\#$  by

$$d^\#(\mu, \nu) = \int_0^\infty e^{-r} \frac{d^{(r)}(\mu^{(r)}, \nu^{(r)})}{1 + d^{(r)}(\mu^{(r)}, \nu^{(r)})} dr, \quad (1)$$

where  $\mu^{(r)}$  is the measure  $\mu$  restricted on  $[0, r]^2$ , and  $d^{(r)}$  is Prohorov metric.

In this paper, we introduce a metric  $d$  (see Definition 2.1) and use the topology of  $d$  to prove Theorem 1.1. As  $d$  is more specific to the space  $\mathcal{N}$  we are interested in than  $d^\#$ , it is also more natural and easier to compute explicitly than  $d^\#$ . Indeed, the convergence of point measures in the topology of  $d$  means exactly the convergence of the atoms of the measures, and this is the property we need in Theorem 1.1.

In Section 2, we introduce formally the space  $\mathcal{N}$  and the metric  $d$ . Section 3 is devoted to compare the  $\sigma$ -field defined by  $d$  with the smallest  $\sigma$ -field such that the functions  $\pi \in \mathcal{N} \mapsto \pi(B) \in \mathbb{N}$  are measurable ( $B \in \mathcal{B}(\mathbb{R}_+^2)$ ). In Section 4, we study a compactness criterion in the space  $(\mathcal{N}, d)$  and prove the tightness of the law of Poisson measures with Lebesgue intensity. Finally, in Section 5, we prove Theorem 1.1.

## 2. Definitions and notations

In this paper, we study the space  $\mathcal{N}$  of locally finite simple point measures on  $\mathbb{R}_+^2$ . We always identify any point measure with the set of its atoms. Consequently, we can define  $\mathcal{N}$  as

$$\mathcal{N} = \{P \subset \mathbb{R}_+ \times \mathbb{R}_+ : \forall T > 0, |P \cap [0, T]^2| < +\infty\}.$$

Besides, let  $\mathcal{N}_T$  be the set of finite subsets of  $[0, T]^2$ , and for each  $P \in \mathcal{N}$  and  $T > 0$ , let  $P_T = P \cap [0, T]^2 \in \mathcal{N}_T$ .

**Definition 2.1.** For each  $T \geq 0$ , we define the metric  $d_T$  on  $\mathcal{N}_T$  in the following way: for  $P, P' \in \mathcal{N}_T$ , we write  $P = \{(t_i, z_i) : 1 \leq i \leq n\}$  and  $P' = \{(t'_j, z'_j) : 1 \leq j \leq m\}$ , where  $n = |P|$  and  $m = |P'|$ , and where each indexing is lexicographically ordered, then

$$d_T(P, P') := \begin{cases} 1 & \text{if } |P| \neq |P'|, \\ 1 \wedge \max_{1 \leq i \leq n} (|t_i - t'_i| \vee |z_i - z'_i|) & \text{if } |P| = |P'| = n. \end{cases}$$

Then we define the metric  $d$  on  $\mathcal{N}$  by

$$d(P, P') = \int_0^{+\infty} e^{-r} d_r(P_r, P'_r) dr.$$

Let us note that the integral is well-defined because, as  $r \mapsto d_r(P_r, Q_r)$  is piecewise constant, we know that  $r \mapsto e^{-r} d_r(P_r, Q_r)$  is piecewise continuous.

We begin by proving the following fundamental property of the metric  $d$ . This property allows us to characterize  $d$  with the metrics  $d_T$  ( $T \geq 0$ ). Lemma 2.2 can be compared to Proposition A2.6.II of [5], whose proof relies on the fact that, with the notation of (1),  $r \mapsto d^{(r)}(\mu^{(r)}, \nu^{(r)})$  is non-decreasing. In the proof of Lemma 2.2, we use the fact that  $r \mapsto d_r(\mu_r, \nu_r)$  is piecewise constant.

**Lemma 2.2.** *Let  $P, P^k \in \mathcal{N}$  ( $k \in \mathbb{N}$ ). The following assertions are equivalent:*

- (i)  $d(P, P^k)$  vanishes as  $k$  goes to infinity.
- (ii) There exists some increasing sequence  $(T_n)_n$  such that  $T_n$  goes to infinity as  $n$  goes to infinity and, for each  $n$ ,  $d_{T_n}(P_{T_n}, P_{T_n}^k)$  vanishes as  $k$  goes to infinity.

*Proof. Step 1:* Let us prove that (i) implies (ii). Assume that  $d(P, P^k)$  vanishes. For each  $n \in \mathbb{N}^*$ , let  $T_n$  be any points of  $]n + 1/2, n + 1[$  that does not belong to  $\{t \vee z : (t, z) \in P \cup \bigcup_k P^k\}$  (this implies that  $T_n$  is a continuity point of  $r \mapsto d_r(P_r, P_r^k)$ ).

Now, to show that  $d_{T_n}(P_{T_n}, P_{T_n}^k)$  vanishes as  $k$  goes to infinity, we show that any subsequence admits a subsequence that vanishes. For the sake of readability, we will not write these subsequences.

Firstly, as  $d(P^k, P)$  vanishes, there exists a subsequence such that  $d_r(P_r, P_r^k)$  vanishes for almost all  $r \geq 0$ . In particular, we can fix some  $R \in ]T_n, n + 1[$  such that  $d_R(P_R, P_R^k)$  vanishes. Let us write  $P_R = \{(t_i, z_i) : 1 \leq i \leq N\}$  and  $P_R^k = \{(t_i^k, z_i^k) : 1 \leq i \leq N_k\}$ . Since  $d_R(P_R, P_R^k)$  vanishes,  $N_k$  converges to  $N$ , implying that  $N_k = N$  for  $k$  big enough, and  $(t_i^k, z_i^k)$  converges to  $(t_i, z_i)$  as  $k$  goes to infinity, and hence  $t_i^k \vee z_i^k$  converges to  $t_i \vee z_i$ . Then let us consider  $\eta > 0$  and  $i_1, i_2$  such that

$$t_{i_1} \vee z_{i_1} + \eta < T_n < t_{i_2} \vee z_{i_2} - \eta$$

and for all  $i$ ,  $t_i \vee z_i \notin ]t_{i_1} \vee z_{i_1}, t_{i_2} \vee z_{i_2}[$ .

As  $d_r(P_r, P_r^k)$  vanishes for almost all  $r \geq 0$ , we can consider such a number  $r \in ]t_{i_1} \vee z_{i_1} + \eta, t_{i_2} \vee z_{i_2} - \eta[$ . Observe that  $s \mapsto d_s(P_s, P_s^k)$  is constant on  $]t_{i_1} \vee z_{i_1} + \eta, t_{i_2} \vee z_{i_2} - \eta[$  for all  $k$  (big enough), we have that  $d_{T_n}(P_{T_n}, P_{T_n}^k) = d_r(P_r, P_r^k)$  vanishes as  $k$  goes to infinity.

*Step 2:* Now, we prove that (ii) implies (i). The main part of the proof consists in showing that, for all  $\varepsilon > 0, n \in \mathbb{N}^*$ , for  $k$  big enough (depending on  $n$  and  $\varepsilon$ ),

$$\int_{T_{n-1}}^{T_n} e^{-r} d_r(P_r, P_r^k) dr \leq \varepsilon. \quad (2)$$

To show (2), let us fix some  $n \in \mathbb{N}^*$ , and write  $P_{T_n} = \{(t_i, z_i) : 1 \leq i \leq N\}$  and  $P_{T_n}^k = \{(t_i^k, z_i^k) : 1 \leq i \leq N_k\}$ . As  $d_{T_n}(P_{T_n}, P_{T_n}^k)$  vanishes as  $k$  goes to infinity, we know that  $N_k = N$  for  $k$  big enough. Let us note  $r_i := t_i \vee z_i$ ,  $r_i^k := t_i^k \vee z_i^k$  ( $1 \leq i \leq N$ ),  $r_0 = r_0^k = 0$  and  $r_{N+1} = r_{N+1}^k = T_n$ . Let us consider  $\sigma \in \mathfrak{S}_N$  such that

$$(r_{\sigma(1)}, t_{\sigma(1)}, z_{\sigma(1)}) <_{lex} \dots <_{lex} (r_{\sigma(N)}, t_{\sigma(N)}, z_{\sigma(N)}). \quad (3)$$

To simplify, we assume in a first time that the numbers  $t_i \vee z_i$  ( $1 \leq i \leq N$ ) are all distinct. As  $d_{T_n}(P_{T_n}, P_{T_n}^k)$  vanishes, we know that  $t_i^k$  and  $z_i^k$  converge respectively to  $t_i$  and  $z_i$  ( $1 \leq i \leq N$ ). Consequently, for  $k$  big enough,

$$r_{\sigma(1)}^k < \dots < r_{\sigma(N)}^k.$$

Then, for any  $r \in [0, T_n]$ , there are two cases:

- either  $r \in ]r_i \vee r_i^k, r_{i+1} \wedge r_{i+1}^k[$ , which means that  $|P_r| = |P_r^k|$ , implying  $d_r(P_r, P_r^k) \leq d_{T_n}(P_{T_n}, P_{T_n}^k)$ ,
- or  $r \in [r_i \wedge r_i^k, r_i \vee r_i^k]$ , which means that  $|P_r| \neq |P_r^k|$  implying  $d_r(P_r, P_r^k) = 1$ .

Let us note  $I_n^k = \bigcup_{i=0}^N [r_i \wedge r_i^k, r_i \vee r_i^k]$ . The Lebesgue measure of  $I_n^k$  is

$$\sum_{i=1}^N |r_i - r_i^k| \leq N \cdot d_{T_n}(P_{T_n}, P_{T_n}^k),$$

which vanishes as  $k$  goes to infinity. As a consequence (2) is proved.

To prove (2) without the hypothesis that the  $r_i := t_i \vee z_i$  ( $1 \leq i \leq N$ ) are distinct in (3), instead of writing  $[0, T_n]$  as the union of  $[r_i \wedge r_i^k, r_i \vee r_i^k]$  and  $]r_i \vee r_i^k, r_{i+1} \wedge r_{i+1}^k[$  where  $0 \leq i \leq N$ , one has to write this union for a subset of  $i$  for which the  $r_i$  are distinct, and for each of these index  $i$ , one has to consider  $j_{i,1}^k$  (resp.  $j_{i,2}^k$ ) such that  $r_{j_{i,1}^k}^k$  is the smallest (resp. biggest)  $r_j^k$  for the  $j$  satisfying  $r_j = r_i$ . Then, it is sufficient to write  $[0, T_n]$  as the union of  $[r_i \wedge r_{j_{i,1}^k}^k, r_i \vee r_{j_{i,2}^k}^k]$  and  $]r_i \vee r_{j_{i,2}^k}^k, r_{i+1} \wedge r_{j_{i+1,1}^k}^k[$ . Then (2) is still true because  $|r_i - r_{j_{i,2}^k}^k|$  and  $|r_i - r_{j_{i,1}^k}^k|$  are both non-bigger than  $d_{T_n}(P_{T_n}, P_{T_n}^k)$ .

Consequently, (2) is proved. Now we show that

$$d(P, P^k) = \sum_{n=0}^{+\infty} \int_{T_n}^{T_{n+1}} e^{-r} d_r(P_r, P_r^k) dr$$

vanishes as  $k$  goes to infinity. Let us note that, we can assume that for all  $n$ ,  $T_n \geq n$  (possibly considering a subsequence). Hence, we assume in the rest of the proof that  $\sum_{n \geq 0} e^{-T_n} < +\infty$ .

Let  $\eta > 0$ , let  $N_\eta \in \mathbb{N}^*$  such that  $\sum_{n > N_\eta} e^{-T_n} < \eta$ , and  $k_\eta \in \mathbb{N}^*$  such that, for every  $0 \leq n \leq N_\eta$  and  $k \geq k_\eta$ , (2) holds with  $\varepsilon = \eta/N_\eta$ . Hence, for  $k \geq k_\eta$ ,  $d(P, P^k) \leq 2\eta$ .  $\square$

**Remark 2.3.** In Lemma 2.2, there does not exist a universal sequence  $(T_n)_n$ . However, if  $(P^i, P^{i,k})_k$  ( $i \in I$ ) is a finite or countable family of sequences such that, for each  $i \in I$ ,  $d(P^i, P^{i,k})$  vanishes as  $k$  goes to infinity, there exists an increasing sequence  $(T_n)_n$  that goes to infinity such that, for every  $i \in I$ , for all  $n \in \mathbb{N}$ ,  $d_{T_n}(P_{T_n}^i, P_{T_n}^{i,k})$  vanishes as  $k$  goes to infinity.

Now, the separability of  $(\mathcal{N}, d)$  is a mere consequence of Lemma 2.2.

**Lemma 2.4.** The space  $(\mathcal{N}, d)$  is separable.

*Proof.* Let  $\mathcal{D} = \bigcup_{T \in \mathbb{N}^*} \mathcal{D}_T$  where  $\mathcal{D}_T = \{P \in \mathcal{N}_T : P \subset \mathbb{Q}^2\}$ . It is classical that  $\mathcal{D}$  is countable. It is then sufficient to prove that  $\mathcal{D}$  is dense in  $(\mathcal{N}, d)$ .

Let  $P \in \mathcal{N}$ . Let us consider an increasing sequence  $T_n$  such that  $T_n$  goes to infinity, and  $P \cap \{T_n\} \times \mathbb{R}_+ = \emptyset$ . The facts that  $P_{T_n}$  is finite and that the elements of  $P_{T_n}$  can be approximated by rational numbers complete the proof.  $\square$

**Remark 2.5.** The space  $(\mathcal{N}, d)$  is separable, but not complete. Indeed, let us consider

$$\pi^n := \left\{ (1, 1), \left( 1 + \frac{1}{n}, 1 \right) \right\}.$$

For all  $1 \leq q \leq p$ ,

$$d(\pi^q, \pi^p) = \int_{1/p}^{1/q} e^{-r} dr + \left( \frac{1}{q} - \frac{1}{p} \right) \int_{1/q}^{+\infty} e^{-r} dr < \frac{2}{q}.$$

Consequently,  $(\pi^n)_n$  is a Cauchy sequence, but it does not converge. Indeed, the limit  $\pi$ , if it would exist, should have exactly two points, since every  $\pi^n$  has exactly two points (by definition of  $d$ ). But the two points of  $\pi^n$  merge as  $n$  goes to infinity.

Lemma 2.2 also allows us to prove an interesting property of the metric  $d$ , using Proposition A2.6.II of [5] that let us compare  $d$  with the metric  $d^\#$  defined in (1).

**Proposition 2.6.** On the subspace  $\{P \in \mathcal{N} : \forall T \geq 0, |P \cap \{T\} \times \mathbb{R}_+| \in \{0, 1\}\}$ , the topology of the metric  $d$ , is the topology of the vague convergence.

*Proof. Step 1:* We assume that  $d(P, P^k)$  vanishes as  $k$  goes to infinity.

By Lemma 2.2, there exists an increasing sequence  $(T_n)_n$  that goes to infinity and that satisfies:  $d_{T_n}(P_{T_n}, P_{T_n}^k)$  vanishes as  $k$  goes to infinity, for every  $n \in \mathbb{N}$ . To show the vague convergence of  $P^k$  to  $P$ , we just have to show that, for each  $n$ ,  $d^{(T_n)}(P_{T_n}, P_{T_n}^k)$  vanishes, where  $d^{(T_n)}$  is Prohorov metric on  $\mathcal{N}_{T_n}$  (see Proposition A2.6.II.(ii) of [5]).

Let us fix some  $n \in \mathbb{N}$  and note  $P_{T_n} = \{(t_i, z_i) : 1 \leq i \leq N\}$  and  $P_{T_n}^k = \{(t_i^k, z_i^k) : 1 \leq i \leq N^k\}$ . As  $d_{T_n}(P_{T_n}, P_{T_n}^k)$  vanishes, we know that  $N^k$  converges to  $N$ , that is  $N_k = N$  for  $k$  big enough. Now, let  $\varepsilon > 0$  and consider  $k_\varepsilon \in \mathbb{N}$  such that, for all  $k \geq k_\varepsilon$ ,  $d_{T_n}(P_{T_n}, P_{T_n}^k) < \varepsilon$ . Hence, for  $k \geq k_\varepsilon$ ,  $|t_i - t_i^k| < \varepsilon$  and  $|z_i - z_i^k| < \varepsilon$ . This implies that, for any closed  $A \in \mathcal{B}([0, T_n]^2)$ ,  $P \cap A \subseteq P^k \cap A^\varepsilon$  and  $P^k \cap A \subseteq P \cap A^\varepsilon$  (where  $A^\varepsilon = \{x \in A : \exists y \in A, \|x - y\|_\infty \leq \varepsilon\}$ ). Consequently,  $d^{(T_n)}(P_{T_n}, P_{T_n}^k) < \varepsilon$  for all  $k \geq k_\varepsilon$ .

*Step 2:* We assume that  $P^k$  converges vaguely to  $P$ . Using Proposition A2.6.II.(iv) of [5], this implies that, for any bounded  $A \in \mathcal{B}(\mathbb{R}_+^2)$  such that  $P \cap \partial A = \emptyset$ ,  $|P^k \cap A|$  converges to  $|P \cap A|$ . To prove that  $d(P, P^k)$  vanishes, we use Lemma 2.2.

Let us consider some increasing sequence  $(T_n)_n$  that goes to infinity such that, for every  $n \in \mathbb{N}$ ,  $P \cap (\{T_n\} \times [0, T_n] \cup [0, T_n] \times \{T_n\}) = \emptyset$ . In the rest of the proof, we fix some  $n \in \mathbb{N}$  and prove that  $d_{T_n}(P_{T_n}, P_{T_n}^k)$  vanishes as  $k$  goes to infinity. For this purpose, let us note  $P_{T_n} = \{(t_i, z_i) : 1 \leq i \leq N\}$  and  $P_{T_n}^k = \{(t_i^k, z_i^k) : 1 \leq i \leq N^k\}$ .

To begin with, we know that  $N^k := |P^k \cap [0, T_n]^2|$  converges to  $N := |P \cap [0, T_n]^2|$ , hence  $N^k = N$  for  $k$  big enough. Now, we just have to show that, for all  $1 \leq i \leq N$ ,  $t_i^k$  (resp.  $z_i^k$ ) converges to  $t_i$  (resp.  $z_i$ ). Let  $\varepsilon > 0$  and  $\delta := \varepsilon \wedge \min_{1 \leq i \leq N-1} (t_{i+1} - t_i)$ . The reason why we work on a subspace of  $\mathcal{N}$  in this proposition is to guarantee that  $\delta > 0$ .

Then, by Proposition A2.6.II.(iv) of [5], we know that, for each  $1 \leq i \leq N$ ,  $|P^k \cap B((t_i, z_i), \delta/3)|$  converges to  $|P \cap B((t_i, z_i), \delta/3)| = 1$ . As a consequence, for each  $1 \leq i \leq N$  and for  $k$  big enough, there exists a unique  $j_i^k \in \llbracket 1, N \rrbracket$  such that  $(t_{j_i^k}^k, z_{j_i^k}^k)$  belongs to  $B((t_i, z_i), \delta/3)$ . Whence, for  $k$  big enough,

$$t_{j_i^k}^k < t_{i-1} + \frac{\delta}{3} < t_i - \frac{\delta}{3} < t_{j_i^k}^k,$$

and, since the indexing are lexicographically ordered, we know that  $j_i^k = i$  for  $k$  big enough. Finally, as  $(t_i^k, z_i^k)$  belongs to  $B((t_i, z_i), \delta/3)$ , we know that  $|t_i^k - t_i| < \varepsilon/3$  and  $|z_i^k - z_i| < \varepsilon/3$  for all  $1 \leq i \leq N$ , for  $k$  big enough.  $\square$

One could extend the previous proposition to the whole space  $\mathcal{N}$  by modifying the definition of the metrics  $d_T$ . In the definition of  $d_T(P, Q)$ , instead of comparing pairwise the atoms of  $P$  with those of  $Q$  when they are lexicographically ordered, one should consider an optimal ordering.

### 3. Measurability in $(\mathcal{N}, d)$

In the proof of Theorem 1.1, we need to identify any Poisson measure that has Lebesgue intensity with an  $(\mathcal{N}, d)$ -valued random variable. Indeed, we will apply Skorohod representation theorem to the Poisson measures  $\pi^k$ , seen as  $(\mathcal{N}, d)$ -valued random variables, to obtain some copy  $\tilde{\pi}^k$ , which are also  $(\mathcal{N}, d)$ -valued random variables. Then, we will need to guarantee that  $\tilde{\pi}^k$  can be seen as Poisson measures.

In order to prove it formally, let us recall the usual definition of Poisson random measure.

**Definition 3.1.** Let us note  $\mathcal{B}^m(\mathcal{N})$  the  $\sigma$ -algebra generated by the functions  $\pi \in \mathcal{N} \mapsto \pi(B)$  ( $B \in \mathcal{B}(\mathbb{R}_+^2)$ ), and  $\mathcal{B}^d(\mathcal{N})$  the  $\sigma$ -algebra generated by the metric  $d$ .

In the following, we call random measure any  $(\mathcal{N}, \mathcal{B}^m(\mathcal{N}))$ -valued random variable.

A Poisson random measure is a random measure  $\pi$  satisfying:

- for all  $B \in \mathcal{B}(\mathbb{R}_+^2)$ ,  $\pi(B)$  follows a Poisson distribution,
- for all disjoint  $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R}_+^2)$ , the variables  $\pi(B_i)$  ( $1 \leq i \leq n$ ) are independent.

The function  $\mu : B \in \mathcal{B}(\mathbb{R}_+^2) \mapsto \mathbb{E}[\pi(B)]$  is a measure on  $\mathbb{R}_+^2$  that we call the intensity of  $\pi$ .

Formally, this section is dedicated to prove that  $\mathcal{B}^m(\mathcal{N}) = \mathcal{B}^d(\mathcal{N})$ . For this purpose, let us introduce  $\mathcal{B}^m(\mathcal{N}_T)$  the  $\sigma$ -algebra generated by the functions  $\pi \in \mathcal{N} \mapsto \pi(B)$  ( $B \in \mathcal{B}([0, T])$ ), and  $\mathcal{B}^d(\mathcal{N}_T)$  the  $\sigma$ -algebra generated by the metric  $d_T$ . In a first time, we prove that  $\mathcal{B}^m(\mathcal{N}_T) = \mathcal{B}^d(\mathcal{N}_T)$  for all  $T \geq 0$ .

**Lemma 3.2.** For all  $T \geq 0$ ,  $\mathcal{B}^d(\mathcal{N}_T) \subseteq \mathcal{B}^m(\mathcal{N}_T)$ .

*Proof.* For  $P \in \mathcal{N}_T$ , let us define  $n(P) := |P| = P([0, T]^2)$  and  $(t_i(P), z_i(P))$  ( $1 \leq i \leq n(P)$ ) lexicographically ordered such that  $P = \{(t_i(P), z_i(P)) : 1 \leq i \leq n(P)\}$ . By definition of  $\mathcal{B}^m(\mathcal{N}_T)$ ,  $P \in (\mathcal{N}_T, \mathcal{B}^m(\mathcal{N}_T)) \mapsto n(P) \in (\mathbb{N}, \mathcal{P}(\mathbb{N}))$  is measurable. We also have that  $P \in (\mathcal{N}_T, \mathcal{B}^m(\mathcal{N}_T)) \mapsto t_i(P) \in (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  ( $i \geq 1$ ) are measurable since  $\{t_i(P) \leq t\} = \{P([0, t] \times [0, T]) \geq i\}$ . This also holds for  $z_i(P)$  ( $i \geq 1$ ) as

$$\begin{aligned} \{z_i(P) \leq z\} &= \bigcup_{j=0}^i \{t_i(P) = \dots = t_{i-j}(P) \neq t_{i-j-1}(P)\} \\ &\quad \cap \bigcap_{n \in \mathbb{N}^*} \left\{ P \left( \left[ t_i(P) - \frac{1}{n}, t_i(P) + \frac{1}{n} \right] \times [0, z] \right) \geq j+1 \right\}. \end{aligned}$$

Finally, denoting by  $B_{d_T}$  the open balls for  $d_T$ , we have, for  $\eta > 1$ ,  $B_{d_T}(P, \eta) = \mathcal{N}_T$ , and for  $\eta \leq 1$ ,

$$B_{d_T}(P, \eta) = n^{-1}(n(P)) \cap \bigcap_{i=1}^{n(P)} t_i^{-1}([t_i(P) - \eta, t_i(P) + \eta]) \cap z_i^{-1}([z_i(P) - \eta, z_i(P) + \eta]).$$

□

**Lemma 3.3.** For all  $T \geq 0$ ,  $\mathcal{B}^m(\mathcal{N}_T) \subseteq \mathcal{B}^d(\mathcal{N}_T)$ .

*Proof.* Let us fix  $a \leq b, c \leq d$ . We have to show that, for any  $n \in \mathbb{N}$ ,  $\{Q \in \mathcal{N}_T : Q([a, b] \times [c, d]) = n\}$  belongs to  $\mathcal{B}^d(\mathcal{N}_T)$ . This is true for  $n = 0$  since

$$\{Q \in \mathcal{N}_T : Q([a, b] \times [c, d]) = 0\} = \bigcup_{k \geq 1} \bigcap_{n \in \mathbb{N}} \bigcap_{\substack{P = \{(t_1, z_1), \dots, (t_n, z_n)\} \\ P \subset \mathbb{Q}_+^2 \cap [a-1/k, b+1/k] \times [c-1/k, d+1/k]}} \left( \mathcal{N}_T \setminus B_{d_T} \left( P, \frac{1}{k} \right) \right).$$



Then, the case  $n = 1$  is a consequence of

$$\begin{aligned} \{Q([a, b] \times [c, d]) = 1\} &= \bigcap_{k \in \mathbb{N}^*} \bigcup_{\substack{q_1 \in \mathbb{Q} \cap [a, b] \\ q_2 \in \mathbb{Q} \cap [c, d]}} \left\{ Q \left( \left[ q_1, q_1 + \frac{1}{k} \right] \times \left[ q_2, q_2 + \frac{1}{k} \right] \right) \neq 0 \right\} \\ &\quad \cap \left\{ Q \left( [a, b] \times [c, d] \setminus \left[ q_1, q_1 + \frac{1}{k} \right] \times \left[ q_2, q_2 + \frac{1}{k} \right] \right) = 0 \right\}. \end{aligned}$$

Finally, we can prove that, for any  $n \geq 2$ ,  $\{Q \in \mathcal{N}_T : Q([a, b] \times [c, d]) = n\}$  belongs to  $\mathcal{B}^d(\mathcal{N}_T)$  by induction, using

$$\begin{aligned} \{Q([a, b] \times [c, d]) = n\} &= \bigcup_{\substack{n_1, n_2, n_3 \in [0, n-1] \\ 1+n_1+n_2+n_3=n}} \bigcup_{\substack{q_1 \in \mathbb{Q} \cap [a, b] \\ q_2 \in \mathbb{Q} \cap [c, d]}} \{Q([a, q_1] \times [c, q_2]) = 1\} \\ &\quad \cap \{Q([q_1, b] \times [q_2, d]) = n_1\} \cap \{Q([a, q_1] \times [q_2, d]) = n_2\} \cap \{Q([q_1, b] \times [c, q_2]) = n_3\}. \end{aligned}$$

□

Let us end this section with

**Proposition 3.4.**  $\mathcal{B}^m(\mathcal{N}) = \mathcal{B}^d(\mathcal{N})$ .

*Proof.* This follows from Lemmas 3.2 and 3.3 as in the proof of Theorem A2.6.III.(ii) of [5]. Let us mimic this proof. In a first time, let us note that  $\mathcal{B}^d(\mathcal{N}) = \sigma(\bigcup_{T \geq 0} \pi_T^{-1}(\mathcal{B}^d(\mathcal{N}_T)))$ , where  $\pi_T : P \in \mathcal{N} \mapsto P_T \in \mathcal{N}_T$ . The inclusion from the left to the right holds true because the functions  $\pi_T : P \in (\mathcal{N}, d) \mapsto P_T \in (\mathcal{N}_T, d_T)$  are measurable ( $T \geq 0$ ), and the other inclusion is a consequence of the fact that, for any  $P \in \mathcal{N}$ , the function  $Q \in \mathcal{N} \mapsto d(P, Q) \in (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  is  $\sigma(\bigcup_{T \geq 0} \pi_T^{-1}(\mathcal{B}^d(\mathcal{N}_T)))$  measurable.

To show that  $\mathcal{B}^m(\mathcal{N}) \subseteq \mathcal{B}^d(\mathcal{N})$ , it is sufficient to show that, for all  $A \in \mathcal{B}(\mathbb{R}_+^2)$ , the mapping  $\Phi_A : \pi \in (\mathcal{N}, \mathcal{B}^d(\mathcal{N})) \mapsto \pi(A)$  is measurable. This is a consequence of the facts that the set  $\mathcal{A} = \{A \in \mathcal{B}(\mathbb{R}_+^2) : \Phi_A \text{ is } \mathcal{B}^d(\mathcal{N}) \text{ measurable}\}$  is a monotone class containing the bounded measurable sets of  $\mathbb{R}_+^2$  (see Lemma 3.3).

Conversely, let  $\mathcal{R}$  be a  $\sigma$ -algebra on  $\mathcal{N}$  such that, for any  $A \in \mathcal{B}(\mathbb{R}_+^2)$ ,  $\Phi_A$  is  $\mathcal{R}$ -measurable. Then, Lemma 3.2 implies that  $\mathcal{B}^d(\mathcal{N}_T) \subseteq \mathcal{R}$  for every  $T \geq 0$ . By definition of  $d$ ,  $\mathcal{B}^d(\mathcal{N})$  is the smallest  $\sigma$ -algebra containing every  $\mathcal{B}^d(\mathcal{N}_T)$ , thus  $\mathcal{B}^d(\mathcal{N}) \subseteq \mathcal{R}$ .

Hence  $\mathcal{B}^d(\mathcal{N})$  is the smallest  $\sigma$ -field such that  $\Phi_A$  is measurable for all  $A \in \mathcal{B}(\mathbb{R}_+^2)$ . □

#### 4. Tightness and compactness in $(\mathcal{N}, d)$

Now, we prove a compactness criterion in  $(\mathcal{N}, d)$ , that relies on compactness in the spaces  $(\mathcal{N}_T, d_T)$ .

**Lemma 4.1.** *Let  $K \subseteq \mathcal{N}$  and  $(T_n)_n$  be an increasing sequences that goes to infinity. Let us note  $K_{T_n} = \{P_{T_n} : P \in K\}$ . Assume that, for all  $n \in \mathbb{N}^*$ ,  $K_{T_n}$  is a compact set of  $(\mathcal{N}_{T_n}, d_{T_n})$ . We also assume, that there exists a sequence of positive numbers  $(\delta_n)_{n \in \mathbb{N}^*}$  such that, for all  $(n, P) \in \mathbb{N}^* \times K$ ,  $P \cap (]T_n - \delta_n, T_n + \delta_n[ \times [0, T_n] \cup [0, T_n] \times ]T_n - \delta_n, T_n + \delta_n[) = \emptyset$ . Then  $K$  is a compact set of  $(\mathcal{N}, d)$ .*

*Proof.* Let  $(P^k)_k$  be a sequence of  $K$ . By induction, and using the axiom of dependent choice, we show the existence of a sequence of increasing functions  $(\varphi_n)_n$  and a sequence  $(Q^n)_n$  such that:

- $\forall n \in \mathbb{N}^*, Q^n \in \mathcal{N}_{T_n}$ ,
- $\forall n \in \mathbb{N}^*, d_{T_n} \left( Q^n, P_{T_n}^{\varphi_1 \circ \dots \circ \varphi_n(k)} \right) \xrightarrow[k \rightarrow \infty]{} 0$ ,
- $\forall m \leq n, Q_{T_m}^n = Q^m$ , that is  $Q^m \subseteq Q^n$ .

Let  $\psi(n) = \varphi_1 \circ \dots \circ \varphi_n(n)$  and  $Q = \bigcup_k Q^k$ . We just have to show that  $d(Q, P^{\psi(k)})$  goes to 0 as  $k$  goes to infinity. This is a mere consequence of Lemma 2.2, noticing that for each  $k \geq n$ ,  $d_{T_n} \left( Q_{T_n}, P_{T_n}^{\psi(k)} \right)$  is a subsequence of  $d_{T_n} \left( Q_{T_n}, P_{T_n}^{\varphi_1 \circ \dots \circ \varphi_n(k)} \right)$  that vanishes.  $\square$

For  $\varepsilon > 0$  and  $T \in \mathbb{N}^*$ , we consider some  $N(T, \varepsilon) \in \mathbb{N}^*$ ,  $\gamma(T, \varepsilon) \in \mathbb{R}_+^*$  and  $\delta(T, \varepsilon) \in \mathbb{R}_+^*$  (we will fix their values in the proof of Proposition 4.4). And we define  $K^\varepsilon \subseteq \mathcal{N}$  as follows

$$K^\varepsilon = \{P \in \mathcal{N} : \forall T \in \mathbb{N}^*, \#P_T \leq N(T, \varepsilon) \\ \text{and } P \cap ((T - \delta(T, \varepsilon), T + \delta(T, \varepsilon)) \times [0, T] \cup [0, T] \times (T - \delta(T, \varepsilon), T + \delta(T, \varepsilon))) = \emptyset \\ \text{and } \forall (t, z), (t', z') \in P_T, (t, z) \neq (t', z') \Rightarrow |t - t'| \geq \gamma(T, \varepsilon)\}.$$

Using Lemma 4.1, we prove

**Lemma 4.2.** *For all  $\varepsilon > 0$ ,  $K^\varepsilon$  is a compact set of  $(\mathcal{N}, d)$ .*

*Proof.* Thanks to Lemma 4.1, it is sufficient to show that, for every  $T \in \mathbb{N}^*$ ,  $K_T^\varepsilon$  is a compact set of  $(\mathcal{N}_T, d_T)$ .

Let  $(P^k)_k$  be a sequence of  $K_T^\varepsilon$ , we write  $P^k = \{(t_1^k, z_1^k), \dots, (t_{n_k}^k, z_{n_k}^k)\}$  where  $n_k \in \llbracket 0, N(T, \varepsilon) \rrbracket$  and the pairs  $(t_1^k, z_1^k), \dots, (t_{n_k}^k, z_{n_k}^k)$  are lexicographically ordered. Considering a subsequence, we can assume that, for each  $k$ ,  $n_k = n \in \llbracket 0, N(T, \varepsilon) \rrbracket$ .

For every  $1 \leq i \leq n$ , the sequences  $(t_i^k)_k$  and  $(z_i^k)_k$  are bounded, hence they admit converging subsequences. Extracting successively the subsequences, we can assume that  $t_i^k$  converges to some  $t_i$ , and  $z_i^k$  converges to some  $z_i$  ( $1 \leq i \leq n$ ).

Besides, for all  $1 \leq i \leq n$  and  $k \in \mathbb{N}^*$ , we know that  $t_i^k \leq t_{i+1}^k - \gamma(T, \varepsilon)$ , whence  $t_i \leq t_{i+1} - \gamma(T, \varepsilon)$ . Consequently  $(t_1, z_1), \dots, (t_n, z_n)$  are lexicographically ordered. Then, we consider  $P = \{(t_1, z_1), \dots, (t_n, z_n)\}$ , and we know that  $d_T(P, P^k)$  goes to 0 as  $k$  goes to infinity.

In short, we have shown that  $(P^k)_k$  has a converging subsequence.  $\square$

**Remark 4.3.** *The set  $K^\varepsilon$  has been defined using the sequence  $T_n = n$  (with the notation of Lemma 4.1). Obviously, with any sequence that goes to infinity, we would have defined another compact set.*

The goal of the compact set  $K^\varepsilon$  is to show that

**Proposition 4.4.** *The law of a Poisson measure  $\pi$  on  $\mathbb{R}_+ \times \mathbb{R}_+$  having Lebesgue intensity is tight on  $(\mathcal{N}, d)$ .*

*Proof.* Let  $\pi$  be a Poisson measure on  $\mathbb{R}_+ \times \mathbb{R}_+$  having Lebesgue intensity. We show that for all  $\varepsilon > 0$ ,  $\mathbb{P}(\pi \notin K^\varepsilon) \leq 2\varepsilon$  for a suitable choice of  $N(T, \varepsilon)$ ,  $\delta(T, \varepsilon)$  and  $\gamma(T, \varepsilon)$ . By definition,

$$\mathbb{P}(\pi \in K^\varepsilon) = \mathbb{P}(\forall T \in \mathbb{N}^*, \pi([0, T] \times [0, T]) \leq N(T, \varepsilon), \\ \pi((T - \delta(T, \varepsilon), T + \delta(T, \varepsilon)) \times [0, T] \cup [0, T] \times (T - \delta(T, \varepsilon), T + \delta(T, \varepsilon))) = 0, \\ \forall (\tau, \zeta), (\tau', \zeta') \in \pi \cap [0, T]^2, (\tau, \zeta) \neq (\tau', \zeta') \Rightarrow |\tau - \tau'| \geq \gamma(T, \varepsilon))$$

Then,

$$\mathbb{P}(\pi \notin K^\varepsilon) \leq \sum_{T=1}^{+\infty} \mathbb{P}(\pi([0, T] \times [0, T]) > N(T, \varepsilon)) \quad (4)$$

$$+ \sum_{T=1}^{+\infty} \mathbb{P}(\pi(\lfloor T - \delta(T, \varepsilon), T + \delta(T, \varepsilon) \rfloor \times [0, T] \cup [0, T] \times \lfloor T - \delta(T, \varepsilon), T + \delta(T, \varepsilon) \rfloor) \geq 1) \quad (5)$$

$$+ \sum_{T=1}^{+\infty} \mathbb{P}(\exists n \in \llbracket 0, \lfloor T/\gamma(T, \varepsilon) \rfloor - 1 \rrbracket, \pi([n\gamma(T, \varepsilon), (n+2)\gamma(T, \varepsilon)] \times [0, T]) \geq 2) \quad (6)$$

Now we control the terms (4), (5) and (6). Let us begin with (4), by Taylor-Lagrange inequality,

$$\mathbb{P}(\pi([0, T] \times [0, T]) > N(T, \varepsilon)) = e^{-T^2} \left( e^{T^2} - \sum_{k=0}^{N(T, \varepsilon)} \frac{T^{2k}}{k!} \right) \leq \frac{T^{2N(T, \varepsilon)+2}}{(N(T, \varepsilon) + 1)!}.$$

As for all  $x \geq 0$ ,  $x^{2n}/n!$  goes to 0, we can fix some  $N(T, \varepsilon) \in \mathbb{N}^*$  such that the expression above is bounded by  $\varepsilon/2^T$ . This implies that (4) is bounded by  $\sum_{T=1}^{\infty} \frac{\varepsilon}{2^T} = \frac{\varepsilon}{2}$ .

Let us bound (5). Using Taylor-Lagrange inequality once again,

$$\begin{aligned} \mathbb{P}(\pi(\lfloor T - \delta(T, \varepsilon), T + \delta(T, \varepsilon) \rfloor \times [0, T] \cup [0, T] \times \lfloor T - \delta(T, \varepsilon), T + \delta(T, \varepsilon) \rfloor) \geq 1) \\ = e^{-(4\delta(T, \varepsilon)(T + \delta(T, \varepsilon)) - 4\delta(T, \varepsilon)^2)} \left( e^{4\delta(T, \varepsilon)(T + \delta(T, \varepsilon)) - 4\delta(T, \varepsilon)^2} - 1 \right) \\ \leq 4\delta(T, \varepsilon)(T + \delta(T, \varepsilon)) - 4\delta(T, \varepsilon)^2 = 4\delta(T, \varepsilon)T \end{aligned}$$

Then, we just have to choose some  $\delta(T, \varepsilon) \leq \frac{\varepsilon}{4T^2}$ , to bound (5) by  $\varepsilon/2$ .

To control (6), we first notice that the probability inside the sum is bounded by

$$\begin{aligned} \sum_{n=0}^{\lfloor T/\gamma(T, \varepsilon) \rfloor - 1} \mathbb{P}(\pi([n\gamma(T, \varepsilon), (n+2)\gamma(T, \varepsilon)] \times [0, T]) \geq 2) \\ \leq \frac{T}{\gamma(T, \varepsilon)} \mathbb{P}(\pi([0, 2\gamma(T, \varepsilon)] \times [0, T]) \geq 2) = \frac{T}{\gamma(T, \varepsilon)} e^{-2T\gamma(T, \varepsilon)} \left( e^{2T\gamma(T, \varepsilon)} - 1 - 2T\gamma(T, \varepsilon) \right) \\ \leq \frac{T}{\gamma(T, \varepsilon)} 2T^2\gamma(T, \varepsilon)^2 \leq 2T^3\gamma(T, \varepsilon) \end{aligned}$$

Fixing  $\gamma(T, \varepsilon) \leq \frac{\varepsilon}{T^3 2^T}$ , we obtain that (6) is bounded by  $\varepsilon$ .

Finally,  $\mathbb{P}(\pi \notin K^\varepsilon) \leq 2\varepsilon$ , where  $K^\varepsilon$  is a compact set of  $(\mathcal{N}, d)$ .  $\square$

## 5. Convergence of point processes

This section is dedicated to prove Theorem 1.1. Let us begin with

**Theorem 5.1.** *Let  $\Phi : D(\mathbb{R}_+, \mathbb{R}_+)^m \times \mathcal{N}^m \rightarrow D(\mathbb{R}_+, \mathbb{R}^m)$  be defined as*

$$\Phi(x, \pi)_t := \left( \int_{[0, t] \times \mathbb{R}_+} \mathbf{1}_{\{z \leq x_{s-}^j\}} d\pi^j(s, z) \right)_{1 \leq j \leq m}.$$

$\Phi$  is continuous at the point  $(x, \pi) \in D(\mathbb{R}_+, \mathbb{R}_+)^m \times \mathcal{N}^m$  if:

- for each  $1 \leq j \leq m$ , for every  $t \geq 0$  such that  $\pi^j(\{t\} \times \mathbb{R}_+) = 0$ ,  $x^j$  is continuous at  $t$ ,
- for each  $1 \leq j \leq m$ , for every  $t \geq 0$ ,  $\pi^j(\{t\} \times \mathbb{R}_+) \leq 1$ ,
- for each  $1 \leq j \leq m$ ,  $\pi^j\left(\{(t, x_{t-}^j) : t \geq 0\}\right) = 0$ .

The proof of Theorem 5.1 uses the following Lemmas about the convergence in Skorohod space. Their proofs being quite straightforward and classic, we omit them.

**Lemma 5.2.** *Let  $(x_N)_N$  be a sequence of  $D(\mathbb{R}_+, \mathbb{R})$  that converges to some  $x \in D(\mathbb{R}_+, \mathbb{R})$ , and  $(t_N)_N$  be a sequence that converges to some  $t > 0$ . If  $x$  is continuous at  $t$ , then  $x_N(t_N -) \rightarrow x(t)$ .*

**Lemma 5.3.** *Let  $T > 0$ ,  $k \in \mathbb{N}^*$ , increasing sequences  $0 = t_{i,0} < t_{i,1} < \dots < t_{i,n_i-1} < t_{i,n_i} = T$  ( $1 \leq i \leq k$ ),  $0 = t_{i,0}^N < t_{i,1}^N < \dots < t_{i,n_i^N-1}^N < t_{i,n_i^N}^N = T$  ( $1 \leq i \leq k$ ). We define the functions  $g, g_N \in D([0, T], \mathbb{R}^k)$  by, for  $t \in [0, T[$*

$$\left\{ \begin{array}{l} g(t) = \left( \sum_{j=0}^{n_i-1} \mathbb{1}_{[t_{i,j}, t_{i,j+1}[}(t)j \right)_{1 \leq i \leq k}, \\ g(T) = (n_i - 1)_{1 \leq i \leq k}, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} g_N(t) = \left( \sum_{j=0}^{n_i^N-1} \mathbb{1}_{[t_{i,j}^N, t_{i,j+1}^N[}(t)j \right)_{1 \leq i \leq k}, \\ g_N(T) = (n_i^N - 1)_{1 \leq i \leq k}. \end{array} \right.$$

We assume that there exists a dense subset  $A \subseteq [0, T]$  that contains  $T$  such that, for all  $t \in A$ ,  $g_N(t)$  converges to  $g(t)$ , and we assume that for all  $i_1 \neq i_2$  for all  $j_1 \in \llbracket 1, n_{i_1-1} \rrbracket$  and  $j_2 \in \llbracket 1, n_{i_2-1} \rrbracket$ ,  $t_{i_1, j_1} \neq t_{i_2, j_2}$ . Then  $g_N$  converges to  $g$  in  $D([0, T], \mathbb{R}^k)$ .

*Proof of Theorem 5.1.* Let  $(x^k, \pi^k) = (x^{1,k}, \dots, x^{m,k}, \pi^{1,k}, \dots, \pi^{m,k})_k$  converges in  $D(\mathbb{R}_+, \mathbb{R})^m \times \mathcal{N}^m$  to  $(x, \pi) = (x^1, \dots, x^m, \pi^1, \dots, \pi^m)$ . Let  $Z := \Phi(x, \pi)$  and  $Z^k := \Phi(x^k, \pi^k)$ . Thanks to Lemma 2.2 and Remark 2.3, we can consider an increasing sequence  $(T_n)_n$  such that  $T_n$  goes to infinity and  $d_{T_n}(\pi_{T_n}^{j,k}, \pi_{T_n}^j)$  vanishes as  $k$  goes to infinity, for every  $1 \leq j \leq m$ .

We fix  $t \geq 0$  such that for all  $1 \leq j \leq m$ ,  $\pi^j(\{t\} \times \mathbb{R}_+) = 0$  and for all  $k \in \mathbb{N}^*$ ,  $\pi^{j,k}(\{t\} \times \mathbb{R}_+) = 0$ . In particular  $t$  is a point of continuity of  $Z$  and of each  $Z^k$ . By Lemma 5.3 and Proposition 16.2 of [2], the convergence of  $Z^k$  to  $Z$  in  $D(\mathbb{R}_+, \mathbb{R}^m)$  will follow from the convergence of  $Z_t^k$  to  $Z_t$  for every such point  $t$ . Let us show the convergence of  $Z_t^{j,k}$  to  $Z_t^j$  for every  $1 \leq j \leq m$ . In the rest of the proof, we work with a fix  $j$ .

To show this, fix some  $n$  such that  $T_n > \max(t, \|x^j\|_{\infty, [0, t]}, \sup_k \|x^{j,k}\|_{\infty, [0, t]})$ , and write  $\pi^j \cap [0, T_n]^2 = \{(\tau_i, \zeta_i) : 1 \leq i \leq N\}$  and  $\pi^{j,k} \cap [0, T_n]^2 = \{(\tau_i^k, \zeta_i^k) : 1 \leq i \leq N_k\}$ , where the pairs are lexicographically ordered.

The convergence of  $\pi_{T_n}^{j,k}$  to  $\pi_{T_n}^j$  in  $(\mathcal{N}_{T_n}, d_{T_n})$  implies that  $N_k = N$  for all  $k$  (big enough), and that, for each  $1 \leq i \leq N$ ,  $\tau_i^k$  and  $\zeta_i^k$  converge respectively to  $\tau_i$  and  $\zeta_i$ .

Notice that

$$Z_t^{j,k} = \sum_{i=1}^N \mathbb{1}_{\left\{ \zeta_i^k \leq x_{\tau_i^k-}^{j,k} \right\}} \mathbb{1}_{\left\{ \tau_i^k \leq t \right\}}.$$

To end the proof, one has to note that  $\mathbb{1}_{\left\{ \zeta_i^k \leq x_{\tau_i^k-}^{j,k} \right\}}$  converges to  $\mathbb{1}_{\left\{ \zeta_i \leq x_{\tau_i}^j \right\}}$ , and that  $\mathbb{1}_{\left\{ \tau_i^k \leq t \right\}}$  converges to  $\mathbb{1}_{\left\{ \tau_i \leq t \right\}}$ . These convergences hold true because  $\zeta_i \neq x_{\tau_i}^j$  and  $\tau_i \neq t$ , and because of Lemma 5.2.  $\square$

Before proving Theorem 1.1, let us state two lemmas, whose proofs are omitted.

**Lemma 5.4.** *If  $(E_n)_n$  is a sequence of separable metric spaces,  $\prod_n E_n$  is a separable metric space.*

**Lemma 5.5.** *Let  $(E_n)_{n \in \mathbb{N}}$  be a sequence of topological spaces, and, for each  $n$ ,  $(X_n^k)_{k \in \mathbb{N}}$  be a tight sequence of random variables on  $E_n$ . Then, defining  $Y^k := (X_n^k)_n \in \prod_{n \in \mathbb{N}} E_n$ , the sequence  $(Y^k)_{k \in \mathbb{N}}$  is tight on  $\prod_{n \in \mathbb{N}} E_n$  for the product topology.*

*Proof of Theorem 1.1. Step 1:* Let us show that  $(Z^{N,k})_{1 \leq k \leq n}$  converges to  $(\bar{Z}^k)_{1 \leq k \leq n}$  as  $N$  goes to infinity in  $D(\mathbb{R}_+, \mathbb{R}^n)$ .

Since  $(D(\mathbb{R}_+, \mathbb{R}) \times \mathcal{N})^n$  is a separable metric space (see Theorem 16.3 of [2], and Lemma 2.4), we can apply Skorohod representation theorem to show the almost sure convergence of a sequence  $((\tilde{Y}^{N,1}, \tilde{\pi}^{N,1}), \dots, (\tilde{Y}^{N,n}, \tilde{\pi}^{N,n}))$  to  $((\tilde{Y}^1, \tilde{\pi}^1), \dots, (\tilde{Y}^n, \tilde{\pi}^n))$  in  $(D(\mathbb{R}_+, \mathbb{R}) \times \mathcal{N})^n$  as  $N$  goes to infinity, where these variables have respectively the same distribution as  $((Y^{N,1}, \pi^1), \dots, (Y^{N,n}, \pi^n))$  and  $((Y^1, \pi^1), \dots, (Y^n, \pi^n))$ .

Then Theorem 5.1 implies the almost sure convergence of the multivariate point processes  $(\tilde{Z}^{N,k})_{1 \leq k \leq n} := \Phi((\tilde{Y}^{N,k}, \tilde{\pi}^{N,k})_{1 \leq k \leq n})$  to  $(\tilde{Z}^k)_{1 \leq k \leq n} := \Phi((\tilde{Y}^k, \tilde{\pi}^k)_{1 \leq k \leq n})$  in  $D(\mathbb{R}_+, \mathbb{R}^n)$ .

*Step 2:* Now, we prove that  $(Z^{N,k})_{k \geq 1}$  converges to  $(\bar{Z}^k)_{k \geq 1}$  as  $N$  goes to infinity in  $D(\mathbb{R}_+, \mathbb{R})^{\mathbb{N}^*}$ .

A consequence of *Step 1* is that, for every  $n \in \mathbb{N}^*$ ,  $(Z^{N,k})_{1 \leq k \leq n}$  converges to  $(\bar{Z}^k)_{1 \leq k \leq n}$  in  $D(\mathbb{R}_+, \mathbb{R})^n$ . This implies that, for every  $k \in \mathbb{N}^*$ ,  $Z^{N,k}$  converges in distribution to  $\bar{Z}^k$ , hence, as  $D(\mathbb{R}_+, \mathbb{R})$  is a separable metric space, the sequence  $(\mathcal{L}(Z^{N,k}))_N$  is tight on  $D(\mathbb{R}_+, \mathbb{R})$ . Then, Lemma 5.5 implies that the sequence  $(\mathcal{L}((X^{N,k})_{k \geq 1}))_N$  is tight on  $D(\mathbb{R}_+, \mathbb{R})^{\mathbb{N}^*}$ .

Let  $P$  be a limit of a converging subsequence. Noting  $p_k : D(\mathbb{R}_+, \mathbb{R})^{\mathbb{N}^*} \rightarrow D(\mathbb{R}_+, \mathbb{R})^k$  the projection on the  $k$  first coordinates, we know that  $P \circ p_k^{-1}$  is uniquely determined for all  $k \geq 1$ . Then, as  $D(\mathbb{R}_+, \mathbb{R})^{\mathbb{N}^*}$  is a separable metric space (see Lemma 5.4),  $P$  is determined by  $P \circ p_k^{-1}$  ( $k \in \mathbb{N}^*$ ).  $\square$

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## Bibliography

- [1] Abi Jaber, E., Cuchiero, C., Larsson, M., and Pulido, S. (2019). A weak solution theory for stochastic Volterra equations of convolution type. *arXiv:1909.01166 [math]*.
- [2] Billingsley, P. (1999). *Convergence of Probability Measures*. Wiley Series In Probability And Statistics, second edition.
- [3] Brémaud, P. and Massoulié, L. (1996). Stability of Nonlinear Hawkes Processes. *The Annals of Probability*, 24(3):1563–1588.
- [4] Brown, T. (1978). A Martingale Approach to the Poisson Convergence of Simple Point Processes. *The Annals of Probability*, 6(4):615–628.
- [5] Daley, D. J. and Vere-Jones, D. (2003). *An Introduction to the Theory of Point Processes: Volume I: Elementary Theory and Methods*. Springer, second edition.
- [6] Erny, X., Löcherbach, E., and Loukianova, D. (2019). Mean field limits for interacting Hawkes processes in a diffusive regime. *arXiv:1904.06985 [math]*.
- [7] Ikeda, N. and Watanabe, S. (1989). *Stochastic Differential Equations and Diffusion Processes*. North-Holland Publishing Company, second edition.
- [8] Jacod, J. and Shiryaev, A. N. (2003). *Limit Theorems for Stochastic Processes*. Springer-Verlag BerlinHeidelberg NewYork, second edition.