# High order homogenization of the Poisson equation in a perforated periodic domain

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# HIGH ORDER HOMOGENIZATION OF THE POISSON EQUATION IN A PERFORATED PERIODIC DOMAIN

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Abstract. We derive high order homogenized models for the Poisson problem in a cubic domain periodically perforated with holes where Dirichlet boundary conditions are applied. These models have the potential to unify the three possible kinds of limit problems derived by the literature for various asymptotic regimes (namely the "unchanged" Poisson equation, the Poisson problem with a strange reaction term, and the zeroth order limit problem) of the ratio $\eta \equiv a_\varepsilon/\varepsilon$ between the size $a_\varepsilon$ of the holes and the size $\varepsilon$ of the periodic cell. The derivation relies on algebraic manipulations on formal two-scale power series in terms of $\varepsilon$ and more particularly on the existence of a "criminal" ansatz, which allows to reconstruct the oscillating solution $u_\varepsilon$ as a linear combination of the derivatives of its formal average $u_\varepsilon^*$ weighted by suitable corrector tensors. The formal average is itself the solution of a formal, infinite order homogenized equation. Classically, truncating the infinite order homogenized equation yields in general an ill-posed model. Inspired by a variational method introduced in [52, 23], we derive, for any $K \in \mathbb{N}$, well-posed corrected homogenized equations of order $2K+2$ which yields approximations of the original solutions with an error of order $O(\varepsilon^{2K+4})$ in the $L^2$ norm. Finally, we find asymptotics of all homogenized tensors in the low volume fraction regime $\eta \to 0$ and in dimension $d \geq 3$. This allows us to show that our higher order effective equations converge coefficient-wise to either of the classical homogenized regimes of the literature which arise when $\eta$ is respectively equivalent, or greater than the critical scaling $\eta_{\text{crit}} \sim \varepsilon^{2/(d-2)}$.

**Keywords.** Homogenization, higher order models, perforated Poisson problem, homogeneous Dirichlet boundary conditions, strange term.

**AMS Subject classifications.** 35B27, 76M50, 35330.

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## 1. Introduction

One of the industrial perspectives offered today by the theory of homogenization lies in the development of more and more efficient topology optimization algorithms for the design of mechanical structures; there exists a variety of homogenization based techniques [7, 24, 16], including density based (or "SIMP") methods [17]. Broadly speaking, the principle of these algorithms is to optimize one or several parameters of the micro-structures which affect the coefficients of an effective model. The latter model accounts for the constitutive physics of the mixture of two materials (one representing the solid structure and one representing void) and is mathematically obtained by homogenization of the linear elasticity system. The knowledge of the dependence between the parameters of the microstructure and the coefficient of the effective model is the key ingredient of homogenization based techniques, because it allows to numerically—and automatically—interpret "gray designs" (i.e., for which the local density of solid is not a uniformly equal to 0 or 1) into complex composite shapes characterized by multi-scale patterns and geometrically modulated micro-structures [45, 11, 34, 36].

Several works have sought extensions of these methods for topology optimization of fluid systems, where the incompressible Navier–Stokes system is involved. In this context, an effective model is needed for describing the homogenized physics of a porous medium filled with either solid obstacles, fluid, or a mixture of both. However, classical literature [25, 3, 5, 49] identifies three possible kinds of homogenized models depending on how periodic obstacles of size $a_\varepsilon$ scale within their periodic cell of size $\varepsilon$: depending on how the scaling $\eta \equiv a_\varepsilon/\varepsilon$ compares to the critical size $\sigma_\varepsilon := \varepsilon^{d/(d-2)}$ (in dimension $d \geq 3$), the fluid velocity converges as $\varepsilon \to 0$ to the solution of either a Darcy, a Brinkman or a Navier-Stokes equation. Unfortunately, there is currently no further result regarding an effective model that would be able to describe a medium featuring all possible sizes of locally periodic obstacles. The strategy that is the most commonly used in the density based topology optimization community consists in using either the Brinkman equation exclusively [20, 21, 28], or the Darcy model exclusively [54, 47]. These methodology have proved efficient in a number of works [46, 27], however, they remain inconsistent from a homogenization point of view, since these models are valid only for particular regimes of size of obstacles. In particular, this limitation makes impossible to interpret "gray" designs obtained with classical fluid topology optimization algorithms.

The main objective of this paper is to expose, accordingly, the derivation of a new class of—high order—homogenized models for perforated problems which have the potential to unify the different regimes of the literature. Our hope is that these models could permit, in future works, to develop new mathematically consistent and homogenization based topology optimization algorithms for fluid systems.

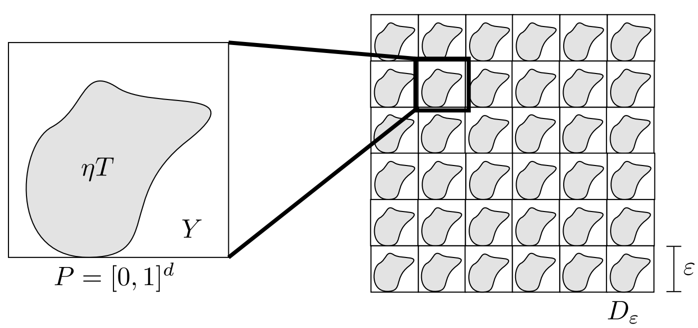


Figure 1. The perforated domain $D_\varepsilon$ and the unit cell $Y = P\backslash(\eta T)$.

This article is a preliminary study towards such purpose: we propose to investigate here the case of the Poisson problem in a perforated periodic domain with Dirichlet boundary conditions on the holes,

$$\begin{cases} -\Delta u_\varepsilon = f \text{ in } D_\varepsilon \\ u_\varepsilon = 0 \text{ on } \partial\omega_\varepsilon \\ u_\varepsilon \text{ is } D\text{–periodic,} \end{cases} \tag{1.1}$$

which can be considered to be a simplified scalar and linear version of the full Navier–Stokes system. Let us mention that the analysis of the full Navier–Stokes system is much more challenging because of the incompressibility constraint and the vectorial nature of the problem; the extension of the current work for the Stokes system—its linear counterpart—shall however be exposed in a future contribution [32, 33].

The setting considered is the classical context of periodic homogenization represented on Figure 1: $D := [0, L]^d$ is a $d$-dimensional box filled with periodic obstacles $\omega_\varepsilon := \varepsilon(\mathbb{Z}^d + \eta T) \cap D$. $P = (0,1)^d$ is the unit cell, and $Y := P \backslash \eta T$ is the unit perforated cell. The parameter $\varepsilon$ is the size of the periodic cell and is given by $\varepsilon := L/n$ where $n \in \mathbb{N}$ is an integer number assumed to be large. The parameter $\eta$ is another rescaling of the obstacle $T$ within the unit cell: the holes are therefore of size $a_\varepsilon := \eta\varepsilon$ which allow us to consider in section 5 so-called low volume fraction limits where $\eta$ converges to zero. The boundary of the obstacle $T$ is assumed to be smooth. $D_\varepsilon := D\backslash\omega_\varepsilon$ denotes the perforated domain and $f \in \mathcal{C}^\infty(D)$ is a smooth $D$–periodic right-hand side. The periodicity assumption for $u_\varepsilon$ and $f$ is classical in homogenization and is used to avoid difficulties related to the arising of boundary layers (see [39, 19, 8]).

The literature accounts for several homogenized equations depending on how the size $a_\varepsilon = \eta\varepsilon$ of the holes compares to the critical size $\sigma_\varepsilon := \varepsilon^{d/(d-2)}$ in dimension $d \geq 3$ or $\sigma_\varepsilon := \exp(-1/\varepsilon^2)$ for $d = 2$ [43, 26, 3, 5, 49, 29, 37]:

- if $a_\varepsilon = o(\sigma_\varepsilon)$, then the holes are "too small" and $u_\varepsilon$ converges as $\varepsilon \to 0$ to the solution $u$ of the Poisson equation in the homogeneous domain $D$ (without holes):
$$\begin{cases} -\Delta u = f \text{ in } D \\ \quad u \text{ is } D\text{–periodic.} \end{cases} \tag{1.2}$$
- if $a_\varepsilon = \sigma_\varepsilon$, then $u_\varepsilon$ converges as $\varepsilon \to 0$ to the solution $u$ of the modified Poisson equation
$$\begin{cases} -\Delta u + Fu = f \text{ in } D \\ \qquad\quad u \text{ is } D\text{–periodic,} \end{cases} \tag{1.3}$$
where the so-called *strange* reaction term $Fu$ involves a positive constant $F > 0$ which can be computed by means of an exterior problem in $\mathbb{R}^d\backslash T$ when $d \geq 3$ (see (5.10) below), and which is equal to $2\pi$ if $d = 2$ (see [3, 25, 49, 37]).
- if $\sigma_\varepsilon = o(a_\varepsilon)$ and $a_\varepsilon = \eta\varepsilon$ with $\eta \to 0$ as $\varepsilon \to 0$, then the holes are "large" and $a_\varepsilon^{d-2}\varepsilon^{-d}u_\varepsilon$ converges to the solution $u$ of the zeroth order equation
$$\begin{cases} Fu = f \text{ in } D \\ \; u \text{ is } D\text{–periodic,} \end{cases} \tag{1.4}$$
where $F$ is the same positive constant as in (1.3).
- if $a_\varepsilon = \eta\varepsilon$ with the ratio $\eta$ *fixed*, then $\varepsilon^{-2}u_\varepsilon$ converges to the solution $u$ of the zeroth order equation
$$\begin{cases} M^0 u = f \text{ in } D \\ \quad u \text{ is } D\text{–periodic,} \end{cases} \tag{1.5}$$
where $M^0$ is another positive constant (which depends on $\eta$). Furthermore it can be shown that $M^0/|\log(\eta)| \to F$ if $d = 2$, and $M^0/\eta^{d-2} \to F$ (if $d \geq 3$) when $\eta \to 0$, so that there is a continuous transition from (1.5) to (1.4); see [4] and corollary 6 below.

The different regimes (1.2) to (1.5) occur because the heterogeneity of the problem comes from the zero Dirichlet boundary condition on the holes $\omega_\varepsilon$ in (1.1): this is the major difference with the setting commonly assumed in linear elasticity, where the heterogeneity induced by the mixture of two materials is instead inscribed in the coefficients of the physical state equation [53, 7]. (1.2) and (1.3) are the respectively the analogous of the Navier–Stokes and Brinkman regimes in the context of the homogenization of the Navier-Stokes equation, while the zeroth order equations (1.4) and (1.5) are analogous to Darcy models. As stressed above, the existence of these regimes raises practical difficulties in view of applying the homogenization method for shape optimization: the previous considerations show that one should use (1.2) and (1.3) in regions featuring none or very tiny obstacles, however one should use the zeroth order model (1.5) when the obstacles become large enough.

Our goal is to propose an enlarged vision of the homogenization of (1.1) through the construction, for any $K \in \mathbb{N}$, of a homogenized equation of order $2K+2$,

$$\begin{cases} \displaystyle\sum_{k=0}^{K+1} \varepsilon^{2k-2} \mathbb{D}_K^{2k} \cdot \nabla^{2k} v_{\varepsilon,K}^* = f \text{ in } D, \\ v_{\varepsilon,K}^* \text{ is } D\text{–periodic,} \end{cases} \tag{1.6}$$

which yields an approximation of $u_\varepsilon$ of order $O(\varepsilon^{2K+4})$ in the $L^2(D_\varepsilon)$ norm (for a fixed given scaling of the obstacles $\eta$). The function $v_{\varepsilon,K}^*$ denotes the higher order homogenized approximation of $u_\varepsilon$ and $\mathbb{D}_K^{2k} \cdot \nabla^{2k}$ is a differential operator of order $2k$ with constant coefficients (the notation is defined in (2.3) below). Equation (1.6) is a "corrected" version of the zeroth order model (1.5) (for any $K$, it holds $\mathbb{D}_K^0 = M^0$), which yields a more accurate solution when $\varepsilon$ is "not so small".

Our mathematical methodology is inspired from the works of Bakhvalov and Panasenko [15], Smyshlyaev and Cherednichenko [52], and Allaire et. al. [12]; it starts with the identification of a "classical" two-scale ansatz

$$u_\varepsilon(x) = \sum_{i=0}^{+\infty} \varepsilon^{i+2} u_i(x, x/\varepsilon), \qquad x \in D_\varepsilon, \tag{1.7}$$

expressed in terms of $Y$–periodic functions $u_i : D \times Y \to \mathbb{R}$ which do not depend on $\varepsilon$. Our procedure involves then formal operations on related power series which give rise to several families of tensors and homogenized equations for approximating the formal, infinite order homogenized average $u_\varepsilon^*$:

$$u_\varepsilon^*(x) := \sum_{i=0}^{+\infty} \varepsilon^{i+2} \int_Y u_i(x,y)\mathrm{d}y, \qquad x \in D. \tag{1.8}$$

In proposition 5 below, we obtain that $u_\varepsilon^*$ in (1.11) is the solution of a formal, "infinite order" homogenized equation,

$$\sum_{k=0}^{+\infty} \varepsilon^{2k-2} M^{2k} \cdot \nabla^{2k} u_\varepsilon^* = f, \tag{1.9}$$

where $M^0$ is the positive constant of (1.5) and $(M^k)_{k\geq 1}$ is a family of (constant) tensors or order $k$. From a computational point of view, one needs a well-posed finite order model. As it can be expected from other physical contexts [9, 12], the effective model obtained from a naive truncation of (1.9), say at order $2K$,

$$\sum_{k=0}^{K} \varepsilon^{2k-2} M^{2k} \cdot \nabla^{2k} v_{\varepsilon,K}^* = f \tag{1.10}$$

is in general not well-posed [9, 1]. Several techniques have been proposed in the literature to obtain well-posed homogenized models of finite order in the context of the conductivity or of the wave equation [13, 10, 1, 2, 12]. The derivation of the *well-posed* homogenized equation (1.6) relies on a minimization principle inspired from Smyshlyaev and Cherednichenko [52] and is marked by two surprising facts.

The first surprising result is the existence of a somewhat remarkable identity which expresses the oscillating solution $u_\varepsilon$ in terms of its non-oscillating average $u_\varepsilon^*$:

$$u_\varepsilon(x) = \sum_{k=0}^{+\infty} \varepsilon^k N^k(x/\varepsilon) \cdot \nabla^k u_\varepsilon^*(x). \tag{1.11}$$

The functions $N^k$ are $P$–periodic corrector tensors of order $k$ (definition 2) depending only on the shape of the obstacles $\eta T$ and that vanish on $\partial(\eta T)$. Furthermore, $N^0$ is of average $\int_Y N^0(y)\mathrm{d}y = 1$ and $N^k$ is of average $\int_Y N^k(y)\mathrm{d}y = 0$ for $k \geq 1$ (proposition 8): $u_\varepsilon^*$ is consistently the average of $u_\varepsilon$ with respect to the fast variable $x/\varepsilon$. While the derivation of (1.7) is very standard in periodic homogenization [50, 41, 18, 14], the existence of such a relation (1.11) (when compared to (1.7)) between the oscillating solution is less obvious; it has been noticed for the first time by Bakhvalov and Panasenko [15] for the conductivity equation, and then in further homogenization contexts in [51, 52, 23, 12, 2, 1]. Following the denomination of [12], we call the ansatz (1.11) "criminal" because the function $u_\varepsilon^*$ has the structure of a formal power series in $\varepsilon$.

The second surprise lies in that our higher order homogenized equation (1.6) is obtained by adding to (1.10) a *single* term $\varepsilon^{2K}\mathbb{D}_K^{2K+2}\cdot\nabla^{2K+2}$; in other words $\mathbb{D}_K^{2k}=M^{2k}$ for any $0\le 2k\le 2K$. This fact, which does not seem to have been noticed in previous works, is quite surprising because following [52, 23], the derivation of (1.6) is based on a minimization principle for the truncation of (1.11) at order $K$,

$$W_{\varepsilon,K}(v^*_{\varepsilon,K})(x):=\sum_{k=0}^{K}\varepsilon^k N^k(x/\varepsilon)\cdot\nabla^k v^*_{\varepsilon,K}(x),\qquad x\in D_\varepsilon \tag{1.12}$$

which is expected to yield an approximation of order $O(\varepsilon^{K+3})$ only in the $L^2(D_\varepsilon)$ norm ($u^*_\varepsilon$ and $v^*_{\varepsilon,K}$ are of order $O(\varepsilon^2)$). This is the order of accuracy stated in our previous work [32] and similarly obtained in the conductivity case by [52], or for the Maxwell equations by [23]; it is related to the observation that that the first half of the coefficients of (1.6) and (1.9) coincide: $\mathbb{D}_K^{2k}=M^{2k}$ for any $0\le 2k\le K$ (proposition 13). In fact, it turns out that *all coefficients* $\mathbb{D}_K^{2k}$ *and* $M^{2k}$ *coincide* except the one of the leading order; $\mathbb{D}_K^{2K+2}\neq M^{2K+2}$. As a result, we are able to show in the present paper that the reconstructed function obtained by adding more correctors,

$$W_{\varepsilon,2K+1}(v^*_{\varepsilon,K})(x):=\sum_{k=0}^{2K+1}\varepsilon^k N^k(x/\varepsilon)\cdot\nabla^k v^*_{\varepsilon,K}(x),\qquad x\in D_\varepsilon, \tag{1.13}$$

yields an approximation of $u_\varepsilon$ of order $O(\varepsilon^{2K+4})$ (corollary 5 below):

$$||u_\varepsilon-W_{\varepsilon,2K+1}(v^*_{\varepsilon,K})||_{L^2(D_\varepsilon)}+\varepsilon||\nabla(u_\varepsilon-W_{\varepsilon,2K+1}(v^*_{\varepsilon,K}))||_{L^2(D_\varepsilon)}\le C_K(f)\varepsilon^{2K+4}. \tag{1.14}$$

Finally, we obtain in corollaries 6 and 7 (see also remark 8) that our homogenized models have the potential to "unify" the different regimes of the literature, in the sense that (1.6) and (1.9) converge formally (coefficient-wise) to either of the effective equations (1.3) and (1.4) when the scaling of the obstacle $\eta\to 0$ vanishes at rates respectively equivalent or greater than the critical size $\eta_{\text{crit}}\sim\eta^{2/(d-2)}$.

Unfortunately, we do not obtain in this work that this convergence holds for all possible rates, because the estimates of corollary 6 imply that higher order coefficients $\varepsilon^{2k-2}M^{2k}$ with $k>2$ could blow up if the rate $\eta$ vanishes faster than the critical size (i.e. when $\eta=o(\varepsilon^{2/(d-2)})$). However the coefficient-wise convergence holds for the homogenized equation (1.6) of order 2 (with $K=0$). Although (i) the derivation of (1.6) has been performed by assuming $\eta$ constant and (ii) all our error bounds feature constants $C_K(f)$ which depend *a priori* on $\eta$, these results seem to indicate that (1.6) has the potential to yield valid homogenized approximations of (1.1) in any regime of size of holes if $K=0$ (which was our initial goal), and for any size $\eta\ge\eta_{\text{crit}}$ if $K\ge 1$.

The exposure of our work outlines as follows. Section 2 introduces the notation conventions and provides a brief summary of our derivations.

Section 3 then details the procedure which allows to construct the family of tensors $M^k$ and $N^k(y)$ arising in the formal infinite order homogenized equation (1.9) and in the criminal ansatz (1.11). Additionally, we establish a number of algebraic properties satisfied by these tensors and we provide an account of the simplifications which occur in case of symmetries of the obstacle with respect to the unit cell axes.

Section 4 is devoted to the construction of the finite order homogenized equation (1.6) thanks to the method of Smyshlyaev and Cherednichenko. We prove the ellipticity of the model and we establish that $\mathbb{D}_K^{2k}=M^{2k}$ for any $0\le 2k\le 2K$ (and not only for the first half coefficients with $0\le 2k\le K$ as observed in [32, 52]). The high order homogenization process is then properly justified by establishing the error estimate (1.14).

Finally, section 5 examines the asymptotic properties of the tensors $M^k$ in the low-volume fraction limit $\eta\to 0$ (in space dimension $d\ge 3$). This allows us to retrieve formally the classical regimes and the arising of the celebrated "strange term" (see [25]) at the critical scaling $\eta\sim\varepsilon^{2/(d-2)}$.

## 2. Notation and summary of the derivation

The full derivation of higher order homogenized equations involves the construction of a number of families of tensors such as $\mathcal{X}^k, M^k, N^k, \mathbb{D}_K^{2k}$. For the convenience of the reader, the notation conventions related to two-scale functions and tensor operations are summarized in section 2.1. We then provide a short synthesis of our main results and of the key steps of our derivations in section 2.2.

### 2.1. Notation conventions

Below and further on, we consider scalar functions such as

$$\begin{aligned} u: \quad & D \times P \to \mathbb{R} \\ & (x, y) \mapsto u(x, y) \end{aligned} \tag{2.1}$$

which are both $D$ and $P$–periodic with respect to respectively the first and the second variable, and which vanish on the hole $D \times (\eta T)$. The arguments $x$ and $y$ of $u(x, y)$ are respectively called the "slow" and the "fast" or "oscillating" variable. With a small abuse of notation, the partial derivative with respect to the variable $y_j$ (respectively $x_j$) is simply written $\partial_j$ instead of $\partial_{y_i}$ (respectively $\partial_{x_j}$) where the context is clear, *i.e.* when the function to which it is applied depends only on $y$ (respectively only on $x$).

The star–"$*$"– symbol is used to indicate that a quantity is "macroscopic" in the sense it does not depend on the fast variable $x/\varepsilon$; e.g. $v^*_{\varepsilon,K}$ in (1.6), $u^*_\varepsilon$ in (1.8) or $J^*_K$ in (2.16) below. In the particular case where a two-variable quantity $u(x, y)$ is given such as (2.1), $u^*(x)$ always denotes the average of $y \mapsto u(x, y)$ with respect to the $y$ variable:

$$u^*(x) := \int_P u(x, y)\mathrm{d}y = \int_Y u(x, y)\mathrm{d}y, \qquad x \in D,$$

where the last equality is a consequence of $u$ vanishing on $P\backslash Y = \overline{\eta T}$.

When a function $\mathcal{X} : P \to \mathbb{R}$ depends only on the $y$ variable, we find sometimes more convenient (especially in section 5) to write its cell average with the usual angle bracket symbols:

$$\langle \mathcal{X} \rangle := \int_P \mathcal{X}(y)\mathrm{d}y.$$

In all what follows and unless otherwise specified, the Einstein summation convention over repeated *subscript* indices is assumed (but never on *superscript* indices). Vectors $\boldsymbol{b} \in \mathbb{R}^d$ are written in bold face notation.

The notation conventions including those used related to tensor are summarized in the nomenclature below.

$\boldsymbol{b}$ — Vector of $\mathbb{R}^d$

$(b_j)_{1\le j\le d}$ — Coordinates of the vector $\boldsymbol{b}$.

$b^k$ — Tensor of order $k$ ($b^k_{i_1\dots i_k} \in \mathbb{R}$ for $1 \le i_1, \dots, i_k \le d$)

$b^p \otimes c^{k-p}$ — Tensor product of tensors of order $p$ and $k - p$:

$$(b^p \otimes c^{k-p})_{i_1\dots i_k} := b^p_{i_1\dots i_p} c^{k-p}_{i_{p+1}\dots i_k}. \tag{2.2}$$

$b^k \cdot \nabla^k$ — Differential operator of order $k$ associated with a tensor $b^k$:

$$b^k \cdot \nabla^k := b^k_{i_1\dots i_k} \partial^k_{i_1\dots i_k}, \tag{2.3}$$

with implicit summation over the repeated indices $i_1 \dots i_k$.

$\delta_{ij}$ — Kronecker symbol: $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

$I$ — Identity tensor of order 2:

$$I_{i_1 i_2} = \delta_{i_1 i_2} = e_j \otimes e_j.$$

Note that the identity tensor is another notation for the Kronecker tensor.

$J^{2k}$ — Tensor of order $2k$ defined by:

$$J^{2k} := \overbrace{I \otimes I \otimes \cdots \otimes I}^{k \text{ times}}. \tag{2.4}$$

$(\boldsymbol{e}_j)_{1\le j\le d}$ — Vectors of the canonical basis of $\mathbb{R}^d$.

$e_j$ — Tensor of order 1 whose entries are $(\delta_{i_1 j})_{1\le i_1\le d}$ (for any $1 \le j \le d$). Note: $e_j$ and $\boldsymbol{e}_j$ are the same mathematical object when identifying tensors of order 1 to vectors in $\mathbb{R}^d$.

$\mathbb{B}^{k,l}_K$ — Tensor of order $l + m$ associated with the quadratic form

$$\mathbb{B}^{l,m}_K \nabla^l v \nabla^m w := \mathbb{B}^{l,m}_{l,i_1\dots i_l,j_1\dots j_m} \partial^l_{i_1\dots i_l} v \, \partial^m_{j_1\dots j_m} w, \tag{2.5}$$

for any smooth scalar fields $v, w \in \mathcal{C}^\infty(D)$.

With a small abuse of notation, we consider zeroth order tensors $b^0$ to be constants (i.e. $b^0 \in \mathbb{R}$) and we still denote by $b^0 \otimes c^k := b^0 c^k$ the tensor product with a $k$-th order tensor $c^k$.

In all what follows, a $k$-th order tensor $b^k$ truly makes sense when contracted with $k$ partial derivatives, as in (2.3). Therefore all the tensors considered throughout this work are identified to their symmetrization:

$$b^k_{i_1\dots i_k} \equiv \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} b_{i_{\sigma(1)}\dots i_{\sigma(k)}},$$

where $\mathfrak{S}_k$ is the permutation group of order $k$. Consequently, the order in which the indices $i_1, \dots, i_k$ are written in $b^k_{i_1\dots i_k}$ does not matter and the tensor product $\otimes$ is commutative under this identification:

$$b^k \otimes c^{k-p} = c^{k-p} \otimes b^k. \tag{2.6}$$

Finally, $C$, $C_K$ or $C_K(f)$ denote universal constants that do not depend on $\varepsilon$ but whose values may change from line to line (and which depend *a priori* on the shape of the hole $\eta T$).

2.2. **Summary of the derivation**

One of the main results of this paper is the derivation of the higher order homogenized equation (1.6) for any desired order $K \in \mathbb{N}$, and the justification of the procedure by establishing the error estimate (1.14). Before summarizing the most essential steps of our analysis, let us recall that here and in all what follows, equalities involving infinite power series such as (1.11) are formal and without a precise meaning of convergence. Our derivation outlines as follows:

(1) following classical literature [40, 41, 15, 19], we introduce a family of $k$-th order tensors $(\mathcal{X}^k(y))_{k\in\mathbb{N}}$ obtained as the solutions of cell problems (see definition 1 and proposition 1) which allows to identify the functions $u_i(x, y)$ arising in (1.11) and to rewrite the traditional ansatz more explicitly:

$$u_\varepsilon(x) = \sum_{i=0}^{+\infty} \varepsilon^{i+2} \mathcal{X}^i(x/\varepsilon) \cdot \nabla^i f(x), \qquad x \in D_\varepsilon. \tag{2.7}$$

Introducing the averaged tensors of order $i$, $\mathcal{X}^{i*} := \int_Y \mathcal{X}^i(y)\mathrm{d}y$, the formal average (1.8) reads

$$u^*_\varepsilon(x) = \sum_{i=0}^{+\infty} \varepsilon^{i+2} \mathcal{X}^{i*} \cdot \nabla^i f(x). \tag{2.8}$$

(2) We construct (in proposition 5) constant tensors $M^i$ by inversion of the formal equality

$$\left(\sum_{i=0}^{+\infty} \varepsilon^{i-2} M^i \cdot \nabla^i\right) \left(\sum_{i=0}^{+\infty} \varepsilon^{i+2} \mathcal{X}^{i*} \cdot \nabla^i\right) = I, \tag{2.9}$$

which yields, after left multiplication of (2.8) by $\sum_{i=0}^{+\infty} \varepsilon^{i-2} M^i \cdot \nabla^i$, the "infinite order homogenized equation" for $u^*_\varepsilon(x)$:

$$\sum_{i=0}^{+\infty} \varepsilon^{i-2} M^i \cdot \nabla^i u^*_\varepsilon(x) = f(x). \tag{2.10}$$

Note that (2.10) is exactly (1.9) because all tensors $\mathcal{X}^{2k+1*}$ and $M^{2k+1}$ of odd order vanish (proposition 3 and corollary 1).

(3) We substitute the expression of $f(x)$ given by (2.10) into the ansatz (2.7) so as to recognize a formal series product:

$$u_\varepsilon(x) = \left(\sum_{i=0}^{+\infty} \varepsilon^i \mathcal{X}^i(x/\varepsilon) \cdot \nabla^i\right) \left(\sum_{i=0}^{+\infty} \varepsilon^i M^i \cdot \nabla^i\right) u^*_\varepsilon(x). \tag{2.11}$$

Introducing a new family of tensors $N^k(y)$ defined by the corresponding Cauchy product,

$$N^k(y) := \sum_{p=0}^{k} \mathcal{X}^p(y) \otimes M^{k-p}, \qquad y \in Y,$$

we obtain the "criminal" ansatz (1.11):

$$u_\varepsilon(x) = \sum_{k=0}^{+\infty} \varepsilon^k N^k(x/\varepsilon) \cdot \nabla^k u_\varepsilon^*(x). \tag{2.12}$$

(4) We now seek to construct well-posed effective models of finite order. Inspired by [52, 23], we consider truncated versions of functions of the form of (2.12): for any $v \in H^{K+1}(D)$, we define

$$W_{\varepsilon,K}(v)(x) := \sum_{k=0}^{K} \varepsilon^k N^k(x/\varepsilon) \cdot \nabla^k v(x), \qquad x \in D. \tag{2.13}$$

where the tensors $N^k$ are extended by zero in $P \backslash Y$ for this expression to make sense in $D \backslash D_\varepsilon$. Remembering that $u_\varepsilon$ (identified with its extension by 0 in $D \backslash D_\varepsilon$) is the solution to the energy minimization problem

$$\begin{aligned} \min_{u \in H^1(D)} \quad & J(u,f) := \int_D \left( \frac{1}{2} |\nabla u|^2 - fu \right) \mathrm{d}x \\ s.t. \quad & \begin{cases} u = 0 \text{ in } \omega_\varepsilon \\ u \text{ is } D\text{–periodic,} \end{cases} \end{aligned} \tag{2.14}$$

we formulate an analogous minimization problem for $v \in H^{K+1}(D)$ by restriction of (2.14) to the smaller space of functions $W_{\varepsilon,K}(v) \in H^1(D)$:

$$\begin{aligned} \min_{v \in H^{K+1}(D)} \quad & J(W_{\varepsilon,K}(v), f) = \int_D \left( \frac{1}{2} \left| \nabla W_{\varepsilon,K}(v) \right|^2 - f(x) W_{\varepsilon,K}(v)(x) \right) \mathrm{d}x \\ s.t. \quad & v \text{ is } D\text{–periodic.} \end{aligned} \tag{2.15}$$

Averaging over the fast variable $x/\varepsilon$ (by using lemma 3), we obtain a new minimization problem involving an approximate energy $J_K^*$ (definition 4) which does not depend on $x/\varepsilon$,

$$\begin{aligned} \min_{v \in H^{K+1}(D)} \quad & J_K^*(v, f, \varepsilon) \\ s.t. \quad & v \text{ is } D\text{–periodic.} \end{aligned} \tag{2.16}$$

Its Euler–Lagrange equation (see definition 5) defines finally our well-posed homogenized equation (1.6) and in particular the family of tensors $(\mathbb{D}_K^{2k})_{0 \le 2k \le 2K++22}$.

(5) In view of (2.13), this procedure is expected to yield by construction an approximation $W_{\varepsilon,K}(v_{\varepsilon,K}^*)$ of $u_\varepsilon$ with an error $O(\varepsilon^{K+3})$ (because $v_{\varepsilon,K}^*$ is of order $O(\varepsilon^2)$, see lemma 7). Surprisingly, we verify that all the tensors $\mathbb{D}_K^{2k}$ and the tensors $M^{2k}$ coincide for $0 \le 2k \le 2K$ (proposition 13). This allows to obtain in corollary 5 the error estimate (1.14), which states that $v_{\varepsilon,K}^*$ and the reconstruction (1.13) yield a *much better* approximation than expected, namely of order $O(\varepsilon^{2K+4})$ instead of $O(\varepsilon^{K+3})$.

The most essential point of our methodology is the derivation of the non-classical ansatz (2.12) of step (3). Let us stress that in the available works of the literature concerned with high order homogenization of scalar conductivity equations and its variants [15, 52, 12], the criminal ansatz (analogous to (2.12)) is readily obtained from the classical one (analogous to (1.11)) because the tensors $N^k$ and $\mathcal{X}^k$ coincide in these contexts (check for instance [15, 9]). Our case is very different because the heterogeneity comes from the Dirichlet boundary condition on the holes $\omega_\varepsilon$.

## 3. Derivation of the infinite order homogenized equation and of the criminal ansatz

This section now presents the steps (1) to (3) of section 2.2 in detail. We start in section 3.1 by reviewing the definition of the family of cell tensors $(\mathcal{X}^k(y))_{k \in \mathbb{N}}$ which allows to identify the functions $u_i(x,y)$ involved in the "usual" two-scale ansatz (1.8) and to obtain an error estimate for its truncation in proposition 2. This part is not new; it is a review of classical results available in a number of works, see e.g. [40, 41, 19]. We then establish in section 3.2 several properties of the tensor $\mathcal{X}^k$ which are less found in the literature; the most important being that the averages of all odd order tensors vanish: $\mathcal{X}^{2p+1*} = 0$ for any $p \in \mathbb{N}$.

The next section 3.3 focuses on the definition and on the properties of the family of tensors $M^k$ and $N^k(y)$ which allow to we infer the infinite order homogenized equation (1.9) and the criminal ansatz (1.11).

Finally, we investigate in section 3.4 how symmetries of the obstacle $\eta T$ with respect to the axes of the unit cell $P$ reflect into a decrease of the number of independent components of the homogenized tensors $\mathcal{X}^{k*}$ and $M^k$.

### 3.1. The traditional ansatz: definition of the cell tensors $\mathcal{X}^k$

Classically [50, 40, 41], the first step of our analysis is to insert formally the two-scale expansion (1.11) into the Poisson system (1.1). Because it will help highlight the occurrence of Cauchy products, we also assume (for the purpose of the derivation only) that the right-hand side $f \in \mathcal{C}^\infty(D)$ depends on $\varepsilon$ and admits the following formal expansion:

$$f(x) = \sum_{i=0}^{+\infty} \varepsilon^i f_i(x), \qquad x \in D.$$

Evaluating the Laplace operator against (1.11), we obtain formally

$$-\Delta u_\varepsilon = \sum_{i=-2}^{+\infty} \varepsilon^{i+2}(-\Delta_{yy} u_{i+2} - \Delta_{xy} u_{i+1} - \Delta_{xx} u_i),$$

where we use the convention $u_{-2}(x,y) = u_{-1}(x,y) = 0$, and where $-\Delta_{yy}$, $-\Delta_{xy}$, $-\Delta_{xx}$ are the operators

$$-\Delta_{xx} := -\mathrm{div}_x(\nabla_x \cdot),\ -\Delta_{xy} := -\mathrm{div}_x(\nabla_y \cdot) - \mathrm{div}_y(\nabla_x \cdot),\ -\Delta_{yy} := -\mathrm{div}_y(\nabla_y \cdot).$$

Identifying all powers in $\varepsilon$ yields then the traditional cascade of equations (obtained e.g. in [41]):

$$\begin{cases} -\Delta_{yy} u_{i+2} = f_{i+2} + \Delta_{xy} u_{i+1} + \Delta_{xx} u_i & \text{for all } i \geq -2 \\ u_{-2}(x,y) = u_{-1}(x,y) = 0. \end{cases} \tag{3.1}$$

This system of equations is solved by introducing an appropriate family of cell tensors [40, 41].

**Definition 1.** We define a family of tensors $(\mathcal{X}^(y))_{k\in\mathbb{N}}$ of order $k$ by recurrence as follows:

$$\begin{cases} -\Delta_{yy} \mathcal{X}^0 = 1 \text{ in } Y \\ -\Delta_{yy} \mathcal{X}^1 = 2\partial_j \mathcal{X}^0 \otimes e_j \text{ in } Y \\ -\Delta_{yy} \mathcal{X}^{k+2} = 2\partial_j \mathcal{X}^{k+1} \otimes e_j + \mathcal{X}^k \otimes I \text{ in } Y, & \text{for all } k \geq 0 \\ \mathcal{X}^k = 0 \text{ on } \partial(\eta T) \\ \mathcal{X}^k \text{ is } P\text{–periodic.} \end{cases} \tag{3.2}$$

The tensors $\mathcal{X}^k$ are extended by 0 inside the hole $\eta T$ in the whole unit cell $P$, namely $\mathcal{X}^k(y) = 0$ for $y \in \eta T$.

**Remark 1.** In view of (2.2), the third line of (3.2) is a short-hand notation for

$$-\Delta \mathcal{X}^{k+2}_{i_1 \dots i_{k+2}} = 2\partial_{i_{k+2}} \mathcal{X}^{k+1}_{i_1 \dots i_{k+1}} + \mathcal{X}^k_{i_1 \dots i_k} \delta_{i_{k+1} i_{k+2}}, \qquad \text{for all } k \geq 0.$$

**Proposition 1.** *The solutions $(u_i(x,y))_{i\geq 0}$ to the cascade of equations (3.1) are given by*

$$\forall i \geq 0,\ u_i(x,y) = \sum_{k=0}^{i} \mathcal{X}^k(y) \cdot \nabla^k f_{i-k}(x), \qquad x \in D,\ y \in Y. \tag{3.3}$$

*Recognizing a Cauchy product, the ansatz (1.11) can be formally written as the following infinite power series product:*

$$u_\varepsilon(x) = \sum_{i=0}^{+\infty} \varepsilon^{i+2} \mathcal{X}^i(x/\varepsilon) \cdot \nabla^i f(x) = \varepsilon^2 \left( \sum_{i=0}^{+\infty} \varepsilon^i \mathcal{X}^i(x/\varepsilon) \cdot \nabla^i \right) \left( \sum_{i=0}^{+\infty} \varepsilon^i f_i(x) \right). \tag{3.4}$$

*Proof.* See [40, 41, 32]. □

We complete our review by stating a classical error estimate result which justifies in some sense the formal power series expansion (3.4).

**Proposition 2.** *Denote $u_{\varepsilon,K}$ the truncated ansatz of (3.4) at order $K \in \mathbb{N}$:*

$$u_{\varepsilon,K}(x) := \sum_{i=0}^{K} \varepsilon^{i+2}\mathcal{X}^i(x/\varepsilon)\cdot\nabla^i f(x), \qquad x \in D_\varepsilon. \tag{3.5}$$

*Then assuming $f \in \mathcal{C}^\infty(D)$ is $D$–periodic, the following error bound holds:*

$$||u_\varepsilon - u_{\varepsilon,K}||_{L^2(D_\varepsilon)} + \varepsilon||\nabla(u_\varepsilon - u_{\varepsilon,K})||_{L^2(D_\varepsilon)} \le C_K\varepsilon^{K+3}||f||_{H^{K+2}(D)} \tag{3.6}$$

*for a constant $C_K$ independent of $f$ and $\varepsilon$ (but depending on $K$).*

*Proof.* See [40, 41, 32]. □

3.2. **Properties of the tensors $\mathcal{X}^k$: odd order tensors $\mathcal{X}^{2p+1*}$ are zero.**

Following our conventions of section 2.1, the average of the functions $u_i$ and $\mathcal{X}^i$ with respect to the $y$ variable are respectively denoted:

$$u_i^*(x) := \int_Y u_i(x,y)\mathrm{d}y, \qquad x \in D, \tag{3.7}$$

$$\mathcal{X}^{i*} := \int_Y \mathcal{X}^i(y)\mathrm{d}y. \tag{3.8}$$

In the next proposition, we show that $\mathcal{X}^{2p+1*} = 0$ are of zero average for any $p \in \mathbb{N}$ and that $\mathcal{X}^{2p*}$ depends only on the lower order tensors $\mathcal{X}^p$ and $\mathcal{X}^{p-1}$. Similar formulas have been obtained for the wave equation in heterogeneous media, see e.g. Theorem 3.5 in [2], and also [1, 48].

**Proposition 3.** *For any $0 \le p \le k$, the following identity holds for the tensor $\mathcal{X}^{k*}$:*

$$\mathcal{X}^{k*} = \int_Y \mathcal{X}^k\mathrm{d}y = (-1)^p\int_Y (\mathcal{X}^{k-p}\otimes(-\Delta_{yy}\mathcal{X}^p) - \mathcal{X}^{k-p-1}\otimes\mathcal{X}^{p-1}\otimes I)\mathrm{d}y, \tag{3.9}$$

*with the convention that $\mathcal{X}^{-1} = 0$. In particular, for any $p \in \mathbb{N}$:*

- *$\mathcal{X}^{2p+1*} = 0$*
- *$\mathcal{X}^{2p*}$ depends only on the tensors $\mathcal{X}^p$ and $\mathcal{X}^{p-1}$:*

$$\mathcal{X}^{2p*} = (-1)^p\int_Y (\partial_j\mathcal{X}^p\otimes\partial_j\mathcal{X}^p - \mathcal{X}^{p-1}\otimes\mathcal{X}^{p-1}\otimes I)\mathrm{d}y. \tag{3.10}$$

*Proof.* The result is proved by induction. Formula (3.9) holds true for $p = 0$ by using the convention $\mathcal{X}^{-1} = 0$ and $-\Delta_{yy}\mathcal{X}^0 = 1$. Assuming now the result to be true for $0 \le p < k$, we perform the following integration by parts where we use the boundary conditions satisfied by the tensors $\mathcal{X}^k$ and the commutativity property (2.6) of the tensor product:

$$\begin{aligned}
\mathcal{X}^{k*} =&(-1)^p\int_Y(-\Delta_{yy}\mathcal{X}^{k-p}\otimes\mathcal{X}^p - \mathcal{X}^{k-p-1}\otimes\mathcal{X}^{p-1}\otimes I)\mathrm{d}y\\
=&(-1)^p\int_Y((2\partial_j\mathcal{X}^{k-p-1}\otimes e_j + \mathcal{X}^{k-p-2}\otimes I)\otimes\mathcal{X}^p - \mathcal{X}^{k-p-1}\otimes\mathcal{X}^{p-1}\otimes I)\mathrm{d}y\\
=&(-1)^p\int_Y(-2\partial_j\mathcal{X}^p\otimes e_j - \mathcal{X}^{p-1}\otimes I)\otimes\mathcal{X}^{k-p-1} + \mathcal{X}^{k-p-2}\otimes\mathcal{X}^p\otimes I)\mathrm{d}y\\
=&(-1)^{p+1}\int_Y((-\Delta_{yy}\mathcal{X}^{p+1})\otimes\mathcal{X}^{k-p-1} - \mathcal{X}^{k-p-2}\otimes\mathcal{X}^p\otimes I)\mathrm{d}y.
\end{aligned}$$

Hence the formula is proved at order $p+1$.
Now, the formula at order $p = k$ reads

$$\mathcal{X}^{k*} = (-1)^k\int_Y \mathcal{X}^0(-\Delta_{yy}\mathcal{X}^k)\mathrm{d}y = (-1)^k\int_Y \mathcal{X}^k(-\Delta_{yy}\mathcal{X}^0)\mathrm{d}y = (-1)^k\mathcal{X}^{k*},$$

which implies $\mathcal{X}^{k*} = 0$ if $k$ is odd. Formula (3.10) follows easily from (3.9) with $k = 2p$. □

For completeness, we provide a minor result which implies that there is no order $k$ (even odd) such that $\mathcal{X}^k(y)$ is identically equal to zero. However, let us remark that some components $\mathcal{X}^k_{i_1\dots i_k}(y)$ may vanish for some set of indices $i_1,\dots,i_k$, e.g. in case of invariances of the obstacle $\eta T$ along the cell axes.

**Proposition 4.** *The following identity holds:*

$$-\Delta_{yy}(\partial^k_{i_1\dots i_k}\mathcal{X}^k_{i_1\dots i_k}) = (-1)^k(k+1), \tag{3.11}$$

*where we recall the implicit summation convention over the repeated indices $i_1\dots i_k$.*

*Proof.* The results clearly holds true for $k=0$. For $k=1$, it holds

$$-\Delta_{yy}\partial_i\mathcal{X}^1_i = \partial_i(2\partial_i\mathcal{X}^0) = 2\Delta\mathcal{X}^0 = -2.$$

Assuming the result holds true till rank $k-1$, the formula still holds at rank $k\geq 2$ because

$$\begin{aligned}
-\Delta_{yy}\partial^k_{i_1\dots i_k}\mathcal{X}^k_{i_1\dots i_k} &= \partial^k_{i_1\dots i_k}(2\partial_{i_k}\mathcal{X}^{k-1}_{i_1\dots i_{k-1}} + \mathcal{X}^{k-2}_{i_1\dots i_{k-2}}\delta_{i_{k-1}i_k})\\
&= 2\Delta_{yy}(\partial^{k-1}_{i_1\dots i_{k-1}}\mathcal{X}^{k-1}_{i_1\dots i_{k-1}}) + \Delta_{yy}(\partial^{k-2}_{i_1\dots i_{k-2}}\mathcal{X}^{k-2}_{i_1\dots i_{k-2}})\\
&= -2(-1)^{k-1}k - (-1)^{k-2}(k-1)\\
&= (-1)^k(k+1).
\end{aligned}$$

□

### 3.3. Infinite order homogenized equation and criminal ansatz: tensors $M^k$ and $N^k$

This part outlines the steps (2) and (3) of the procedure outlined section 2.2. Let us recall that the first tensor $\mathcal{X}^{0*}$ is a strictly positive number, since (3.10) implies $\mathcal{X}^{0*} = \int_Y |\nabla\mathcal{X}^0|^2\mathrm{d}y > 0$.

**Proposition 5.** *Let $(M^k)_{i\in\mathbb{N}}$ be the family of $k$-th order tensors defined by induction as follows:*

$$\begin{cases} M^0 = (\mathcal{X}^{0*})^{-1},\\ M^k = -(\mathcal{X}^{0*})^{-1}\displaystyle\sum_{p=0}^{k-1}\mathcal{X}^{k-p*}\otimes M^p. \end{cases} \tag{3.12}$$

*Then it holds, given the definitions (3.1) and (3.7) of $u^*_i$:*

$$\forall i\in\mathbb{N},\ f_i(x) = \sum_{k=0}^{i} M^k\cdot\nabla^k u^*_{i-k}(x). \tag{3.13}$$

*Recognizing a Cauchy product, (3.13) can be rewritten formally in terms of the following "infinite order" homogenized equation for the "infinite order" homogenized average $u^*_\varepsilon$ of (2.8):*

$$\sum_{i=0}^{+\infty}\varepsilon^{i-2}M^i\cdot\nabla^i u^*_\varepsilon = f. \tag{3.14}$$

*Proof.* We proceed by induction. The case $i=0$ results from the identity $u^*_0(x) = \mathcal{X}^{0*}f_0(x)$ which yields $f_0(x) = (\mathcal{X}^{0*})^{-1}u^*_0(x)$. Then, assuming (3.13) holds till rank $i-1$ with $i\geq 1$, we average (3.3) with respect to the $y$ variable to obtain

$$u^*_i = \sum_{p=0}^{i}\mathcal{X}^{p*}\cdot\nabla^p f_{i-p} = \mathcal{X}^{0*}f_i + \sum_{p=1}^{i}\mathcal{X}^{p*}\cdot\nabla^p f_{i-p}.$$

By using (3.13) at ranks $i-p$ with $1 \leq p \leq i$, we obtain the following expression for $f_i$:

$$
\begin{aligned}
f_i &= (\mathcal{X}^{0*})^{-1} \left( u_i^* - \sum_{p=1}^{i} \sum_{q=0}^{i-p} (\mathcal{X}^{p*} \otimes M^q) \cdot \nabla^{p+q} u_{i-p-q}^* \right) \\
&= (\mathcal{X}^{0*})^{-1} \left( u_i^* - \sum_{p=1}^{i} \sum_{k=p}^{i} (\mathcal{X}^{p*} \otimes M^{k-p}) \cdot \nabla^{k} u_{i-k}^* \right) \qquad \text{(change of index } k = p+q \text{ )} \\
&= (\mathcal{X}^{0*})^{-1} \left( u_i^* - \sum_{k=1}^{i} \sum_{p=1}^{k} (\mathcal{X}^{p*} \otimes M^{k-p}) \cdot \nabla^{k} u_{i-k}^* \right) \qquad \text{(inversion of summation)} \\
&= (\mathcal{X}^{0*})^{-1} \left( u_i^* - \sum_{k=1}^{i} \left( \sum_{p=0}^{k-1} \mathcal{X}^{k-p*} \otimes M^{p} \right) \cdot \nabla^{k} u_{i-k}^* \right) \qquad \text{(change of index } p \leftrightarrow k-p) \\
&= M^0 u_i^* + \sum_{k=1}^{i} M^k \cdot \nabla^k u_{i-k}^*,
\end{aligned}
$$

which yields the result at rank $i$. □

**Corollary 1.** *$M^k = 0$ for any odd value of $k$.*

*Proof.* If $k$ is odd, then $k-p$ and $p$ have distinct parities in (3.12). Therefore, the result follows by induction and by using $\mathcal{X}^{k-p*} = 0$ for even values of $p$ (proposition 3). □

It is possible to write a more explicit formula for the tensors $M^k$:

**Proposition 6.** *The tensors $M^k$ are explicitly given by $M^0 = (\mathcal{X}^{0*})^{-1}$ and:*

$$
\forall k \geq 1,\ M^k = \sum_{p=1}^{k} \frac{(-1)^p}{(\mathcal{X}^{0*})^{p+1}} \sum_{\substack{i_1+\cdots+i_p=k \\ 1 \leq i_1 \ldots i_p \leq k}} \mathcal{X}^{i_1*} \otimes \cdots \otimes \mathcal{X}^{i_p*}. \tag{3.15}
$$

*Proof.* For $k=1$, the result is true because

$$
M^1 = -(\mathcal{X}^{0*})^{-1} M^0 \mathcal{X}^{1*} = -(\mathcal{X}^{0*})^{-2} \mathcal{X}^{1*}
$$

which is exactly (3.15). Assuming (3.15) holds till rank $k \geq 1$, we compute

$$
\begin{aligned}
M^{k+1} &= -(\mathcal{X}^{0*})^{-1} \sum_{p=0}^{k} \mathcal{X}^{k+1-p*} \otimes M^p \\
&= -(\mathcal{X}^{0*})^{-1} M^0 \mathcal{X}^{k+1*} - (\mathcal{X}^{0*})^{-1} \sum_{p=1}^{k} \sum_{q=1}^{p} \frac{(-1)^q}{(\mathcal{X}^{0*})^{q+1}} \mathcal{X}^{k+1-p*} \otimes \sum_{\substack{i_1+\cdots+i_q=p \\ 1 \leq i_1 \ldots i_q \leq p}} \mathcal{X}^{i_1*} \otimes \cdots \otimes \mathcal{X}^{i_q*} \\
&= -(\mathcal{X}^{0*})^{-2} \mathcal{X}^{k+1*} - (\mathcal{X}^{0*})^{-1} \sum_{q=1}^{k} \frac{(-1)^q}{(\mathcal{X}^{0*})^{q+1}} \sum_{p=q}^{k} \sum_{\substack{i_1+\cdots+i_q=p \\ 1 \leq i_1 \ldots i_q \leq p}} \mathcal{X}^{k+1-p*} \otimes \mathcal{X}^{i_1*} \otimes \cdots \otimes \mathcal{X}^{i_q*} \\
&= -(\mathcal{X}^{0*})^{-2} \mathcal{X}^{k+1*} - (\mathcal{X}^{0*})^{-1} \sum_{q=1}^{k} \frac{(-1)^q}{(\mathcal{X}^{0*})^{q+1}} \sum_{\substack{i_1+\cdots+i_{q+1}=k+1 \\ 1 \leq i_1 \ldots i_{q+1} \leq k+1}} \mathcal{X}^{i_{q+1}*} \otimes \mathcal{X}^{i_1*} \otimes \cdots \otimes \mathcal{X}^{i_q*} \\
&= -(\mathcal{X}^{0*})^{-2} \mathcal{X}^{k+1*} + \sum_{q=2}^{k+1} \frac{(-1)^q}{(\mathcal{X}^{0*})^{q+1}} \sum_{\substack{i_1+\cdots+i_q=k+1 \\ 1 \leq i_1 \ldots i_q \leq k+1}} \mathcal{X}^{i_1*} \otimes \cdots \otimes \mathcal{X}^{i_q*},
\end{aligned}
$$

from where the result follows. □

**Remark 2.** This result essentially states that $\sum_{k=0}^{+\infty} \varepsilon^k M^k \cdot \nabla^k$ is the formal series expansion of

$$\left(\sum_{k=0}^{+\infty} \varepsilon^k \mathcal{X}^k \cdot \nabla^k\right)^{-1}.$$

Indeed, it is elementary to show the following identity for the inverse of a power series $\sum_{k=0}^{+\infty} a_k z^k$ with $(a_k) \in \mathbb{C}^{\mathbb{N}}$, $z \in \mathbb{C}$ and radius of convergence $R > 0$:

$$\left(\sum_{k=0}^{+\infty} a_k z^k\right)^{-1} = a_0^{-1} + \sum_{k=1}^{+\infty}\left(\sum_{p=1}^{k} \frac{(-1)^p}{a_0^{p+1}} \sum_{\substack{i_1+\dots+i_p=k\\ 1\le i_1\dots i_p\le k}} a_{i_1}a_{i_2}\dots a_{i_p}\right) z^k. \tag{3.16}$$

We now turn on the derivation of the "criminal" ansatz (2.12). Being guided by (2.11), this ansatz is obtained by writing the oscillatory part $u_i(x,y)$ in terms of the non oscillatory part $u_i^*(x)$:

**Proposition 7.** *Given the previous definitions (3.1) and (3.7) of respectively $u_i$ and $u_i^*$, the following identity holds:*

$$\forall i \ge 0,\ u_i(x,y) = \sum_{k=0}^{i}\left(\sum_{p=0}^{k} M^p \otimes \mathcal{X}^{k-p}(y)\right) \cdot \nabla^k u_{i-k}^*(x). \tag{3.17}$$

*Proof.* We substitute (3.13) into (3.3), which yields

$$\begin{aligned} u_i(x,y) &= \sum_{p=0}^{i}\sum_{q=0}^{i-p} (\mathcal{X}^p(y) \otimes M^q) \cdot \nabla^{p+q} u_{i-p-q}^*(x) \\ &= \sum_{p=0}^{i}\sum_{k=p}^{i} (\mathcal{X}^p(y) \otimes M^{p-k}) \cdot \nabla^k u_{i-k}^*(x) \quad \text{(change of indices } k = p+q) \\ &= \sum_{k=0}^{i}\sum_{p=0}^{k} (\mathcal{X}^p(y) \otimes M^{p-k}) \cdot \nabla^k u_{i-k}^*(x) \quad \text{(interversion of summation)} \end{aligned} \tag{3.18}$$

The result follows by performing a last change of indices $p \leftrightarrow k-p$. □

This result motivates the definition of the tensors $N^k$ of (2.12):

**Definition 2.** For any $k \ge 0$, we denote by $N^k(y)$ the $k$-th order tensor

$$N^k(y) := \sum_{p=0}^{k} M^p \otimes \mathcal{X}^{k-p}(y), \qquad y \in Y. \tag{3.19}$$

Recognizing a Cauchy product, the identity (3.17) rewrites as expected as the "criminal" ansatz (2.12) which expresses the oscillating solution $u_\varepsilon$ in terms of its formal homogenized averaged $u_\varepsilon^*$ (defined in (2.8)):

$$u_\varepsilon(x) = \sum_{k=0}^{+\infty} \varepsilon^k N^k(x/\varepsilon) \cdot \nabla^k u_\varepsilon^*(x), \quad x \in D_\varepsilon. \tag{3.20}$$

The last proposition of this section gathers several important properties for the tensors $N^k$ that are dual to those of the tensors $\mathcal{X}^k$ stated in sections 3.1 and 3.2.

**Proposition 8.** *The tensor $N^k(y)$ satisfies:*

1. *$\int_Y N^0(y)\mathrm{d}y = 1$ and $\int_Y N^k(y)\mathrm{d}y = 0$ for any $k \ge 1$.*
2. *For any $k \ge 0$ and assuming the convention $N^{-1} = N^{-2} = 0$:*

$$-\Delta_{yy} N^{k+2} = 2\partial_j N^{k+1} \otimes e_j + N^k \otimes I + M^{k+2}. \tag{3.21}$$

3. *For any $k \ge 0$,*

$$-\Delta_{yy}(\partial^k_{i_1\dots i_k} N^k_{i_1\dots i_k}) = (-1)^k (k+1) M^0. \tag{3.22}$$

4. *For any $k \geq 1$ and $1 \leq p \leq k-1$,*

$$M^k = (-1)^{p+1} \int_Y (N^{k-p} \otimes (-\Delta_{yy} N^p) - N^{k-p-1} \otimes N^{p-1} \otimes I) \mathrm{d}y, \tag{3.23}$$

*In particular, $M^{2p}$ depends only on the tensors $N^p$ and $N^{p-1}$, which depend themselves only on the first $p+1$ tensors $\mathcal{X}^0 \dots \mathcal{X}^p$.*

*Proof.* 1. For $k=0$, it holds $\int_Y N^0(y)\mathrm{d}y = M^0 \mathcal{X}^{0*} = 1$. Furthermore, the definition (3.12) of the tensor $M^k$ can be rewritten as

$$\forall k \geq 1, \int_Y N^k(y)\mathrm{d}y = \sum_{p=0}^{k} \mathcal{X}^{k-p*} \otimes M^p = 0.$$

2. The cases $k=0$ and $k=1$ are easily verified. The case $k \geq 2$ is obtained by writing

$$\begin{aligned}
-\Delta_{yy} N^k &= \sum_{p=0}^{k} M^{k-p} \otimes (-\Delta_{yy} \mathcal{X}^p) \\
&= \sum_{p=2}^{k} M^{k-p} \otimes (2\partial_j \mathcal{X}^{p-1} \otimes e_j + \mathcal{X}^{p-2} \otimes I) + M^{k-1} \otimes (2\partial_j \mathcal{X}^0 \otimes e_j) + M^k \\
&= 2\partial_j \left( \sum_{p=1}^{k} M^{k-p} \otimes \mathcal{X}^{p-1} \right) \otimes e_j + \left( \sum_{p=0}^{k-2} M^p \otimes \mathcal{X}^{k-p-2} \right) \otimes I + M^k,
\end{aligned}$$

from where the result follows.

3. The proof of (3.22) is identical to that of proposition 4.
4. We start by proving the result for $p=1$ with $k>1$: using the point 1., we write

$$\begin{aligned}
M^k &= \int_Y N^0 \otimes M^k \mathrm{d}y = \int_Y N^0 \otimes (-\Delta N^k - 2\partial_j N^{k-1} \otimes e_j - N^{k-2} \otimes I)\mathrm{d}y \\
&= \int_Y M^0 \otimes N^k \mathrm{d}y + \int_Y (2\partial_j N^0 \otimes e_j \otimes N^{k-1} - N^0 \otimes N^{k-2} \otimes I)\mathrm{d}y \\
&= \int_Y ((-\Delta_{yy} N^1) \otimes N^{k-1} - N^0 \otimes N^{k-2} \otimes I)\mathrm{d}y.
\end{aligned}$$

Assuming now the result holds until rank $p$ with $1 \leq p \leq k-2$, we prove it at rank $p+1$ thanks to analogous computations:

$$\begin{aligned}
M^k &= (-1)^{p+1} \int_Y (M^{k-p} + 2\partial_j N^{k-p-1} \otimes e_j + N^{k-p-2} \otimes I) \otimes N^p - N^{k-p-1} \otimes N^{p-1} \otimes I)\mathrm{d}y \\
&= (-1)^{p+1} \int_Y ((-2\partial_j N^p \otimes e_j - N^{p-1} \otimes I) \otimes N^{k-p-1} + N^{k-p-2} \otimes N^p \otimes I)\mathrm{d}y \\
&= (-1)^{p+1} \int_Y ((\Delta_{yy} N^{p+1} + M^{p+1}) \otimes N^{k-p-1} + N^{k-p-2} \otimes N^p \otimes I)\mathrm{d}y.
\end{aligned}$$

□

### 3.4. Simplifications for the tensors $\mathcal{X}^{k*}$ and $M^k$ in case of symmetries

In this last part, we analyze how the homogenized tensors $\mathcal{X}^{k*}$ and $M^k$ reduce to small number of effective coefficients when the obstacle $\eta T$ is symmetric with respect to axes of the unit cell $P$. Such results are classical in the theory of homogenization; our methodology follows e.g. section 6 in [13].

In all what follows, we denote by $S := (S_{ij})_{1 \leq i,j \leq d}$ an arbitrary orthogonal symmetry (satisfying $S = S^T$ and $SS = I$). We shall specialize $S$ in corollary 3 below to either of the following cell symmetries:

- for $1 \leq l \leq d$, $S^l$ denotes the symmetry with respect to the hyperplane orthogonal to $\boldsymbol{e}_l$:

$$S^l := I - 2\boldsymbol{e}_l \boldsymbol{e}_l^T; \tag{3.24}$$

- for $1 \leq m \neq l \leq d$, $S^{lm}$ denotes the symmetry with respect to the diagonal hyperplane that is orthogonal to $\boldsymbol{e}_l - \boldsymbol{e}_m$:

$$S^{lm} := I - \boldsymbol{e}_l \boldsymbol{e}_l^T - \boldsymbol{e}_m \boldsymbol{e}_m^T + \boldsymbol{e}_l \boldsymbol{e}_m^T + \boldsymbol{e}_m \boldsymbol{e}_l^T. \tag{3.25}$$

Recall the Laplace operator is invariant under such orthogonal symmetries $S$: for any smooth scalar field $\mathcal{X}$,

$$-\Delta(\mathcal{X} \circ S) = -(\Delta \mathcal{X}) \circ S. \tag{3.26}$$

**Proposition 9.** *If the cell $Y = P \backslash \eta T$ is invariant with respect to a symmetry $S$,* i.e. *$S(Y) = Y$, then the following identity holds for the components of the solutions $\mathcal{X}^k$ to the cell problem (3.2):*

$$\mathcal{X}^k_{i_1 \dots i_k} \circ S = S_{i_1 j_1} \dots S_{i_k j_k} \mathcal{X}^k_{j_1 \dots j_k}, \tag{3.27}$$

*where we recall the implicit summation convention over the repeated indices $j_1 \dots j_k$.*

*Proof.* The result is proved by induction on $k$. For $k = 0$, it holds

$$-\Delta_{yy}(\mathcal{X}^0 \circ S) = 1 \circ S = 1$$

and the symmetry of $Y$ implies that $\mathcal{X}^0 \circ S$ also satisfies the boundary conditions of (3.2). This implies $\mathcal{X}^0 \circ S = \mathcal{X}^0$. For $k = 1$, we write

$$-\Delta_{yy}(\mathcal{X}^1_{i_1} \circ S) = 2(\partial_{i_1} \mathcal{X}^0) \circ S = 2\partial_{j_1}(\mathcal{X}^0 \circ S) S_{i_1 j_1} = 2\partial_{j_1} \mathcal{X}^0 S_{i_1 j_1},$$

which implies similarly $\mathcal{X}^1_{i_1} \circ S = S_{i_1 j_1} \mathcal{X}^1_{j_1}$. Finally, if the result holds till rank $k+1$ with $k \geq 0$, then

$$\begin{aligned}
-\Delta_{yy}(\mathcal{X}^{k+2}_{i_1 \dots i_{k+2}} \circ S) &= 2(\partial_{i_{k+2}} \mathcal{X}^{k+1}_{i_1 \dots i_{k+1}}) \circ S + \delta_{i_{k+1} i_{k+2}} \mathcal{X}^k_{i_1 \dots i_k} \circ S \\
&= 2 S_{i_{k+2} j_{k+2}} \partial_{j_{k+2}} (\mathcal{X}^{k+1}_{i_1 \dots i_{k+1}} \circ S) + S_{i_{k+1} j_{k+1}} S_{i_{k+2} j_{k+2}} \delta_{j_{k+1} j_{k+2}} \mathcal{X}^k_{i_1 \dots i_k} \circ S \\
&= -S_{i_1 j_1} \dots S_{i_k j_{k+2}} \Delta_{yy} \mathcal{X}^{k+2}_{j_1 \dots j_{k+2}},
\end{aligned}$$

whence the result at rank $k+2$. $\square$

**Corollary 2.** *If the cell $Y = P \backslash \eta T$ is invariant with respect to a symmetry $S$, then the components of the tensors $\mathcal{X}^{k*}$ and $M^k$ of respectively (3.8) and (3.12) satisfy:*

$$\mathcal{X}^{k*}_{i_1 \dots i_k} = S_{i_1 j_1} \dots S_{i_k j_k} \mathcal{X}^{k*}_{j_1 \dots j_k} \tag{3.28}$$

$$M^k_{i_1 \dots i_k} = S_{i_1 j_1} \dots S_{i_k j_k} M^k_{j_1 \dots j_k} \tag{3.29}$$

*where we recall the implicit summation over the repeated indices $j_1 \dots j_k$.*

*Proof.* Equality (3.28) results from the previous proposition and from the following change of variables:

$$\mathcal{X}^{k*}_{i_1 \dots i_k} = \int_Y \mathcal{X}^k_{i_1 \dots i_k} \mathrm{d}y = \int_Y \mathcal{X}^k_{i_1 \dots i_k} \circ S \mathrm{d}y.$$

Equality (3.29) can be obtained by using (3.28) in the formula (3.15). $\square$

**Corollary 3.** *(1) If the cell $Y$ is symmetric with respect to all cell axes $\boldsymbol{e}_l$,* i.e. *$S^l(Y) = Y$ for any $1 \leq l \leq d$, then*

$$\mathcal{X}^{k*}_{i_1 \dots i_k} = 0 \text{ and } M^k_{i_1 \dots i_k} = 0$$

*whenever there exists a number $r$ occurring with an odd multiplicity in the indices $i_1 \dots i_k$, i.e. whenever*

$$\exists r \in \{1, \dots, d\}, \operatorname{Card}\{j \in \{1, \dots, k\} \,|\, i_j = r\} \text{ is odd }.$$

*(2) If the cell $Y$ is symmetric with respect to all diagonal axes orthogonal to $(\boldsymbol{e}_l - \boldsymbol{e}_m)$, i.e. $S^{l,m}(Y) = Y$ for any $1 \leq l < m \leq d$, then for any permutation $\sigma \in \mathfrak{S}_d$,*

$$\mathcal{X}^{k*}_{\sigma(i_1) \dots \sigma(i_k)} = \mathcal{X}^{k*}_{i_1 \dots i_k}.$$

$$M^k_{\sigma(i_1) \dots \sigma(i_k)} = M^k_{i_1 \dots i_k}.$$

*Proof.* (1) The symmetry $S^l$ is a diagonal matrix satisfying $S^l\boldsymbol{e}_l = -\boldsymbol{e}_l$ and $S^l\boldsymbol{e}_q = \boldsymbol{e}_q$ for $q \neq l$. Hence, replacing $S$ by $S^l$ in (3.28), it holds

$$\mathcal{X}^{k*}_{i_1\dots i_k} = (-1)^{\delta_{i_1 l}+\dots+\delta_{i_k l}}\mathcal{X}^{k*}_{i_1\dots i_k},$$

which implies the result.

(2) Applying (3.28) to the symmetry $S^{l,m}$ yields the result for $\sigma = \tau$ where $\tau$ is the transposition exchanging $l$ and $m$. Since this holds for any transposition $\tau \in \mathfrak{S}_d$, this implies the statement for any permutation $\sigma \in \mathfrak{S}_d$.

□

Let us illustrate how the previous corollary reads for the tensors $M^2$ and $M^4$:

- if $Y$ is symmetric with respect to the cell axes $(\boldsymbol{e}_l)_{1\leq l\leq d}$, then only the coefficients $M^2_{ii}$, $M^4_{iijj}$, $M^4_{iiii}$ with $1 \leq i,j \leq d$ and $i \neq j$ are non zeros (in particular $M^2$ is diagonal).
- if in addition $Y$ is symmetric with respect to the hyperplane orthogonal to $\boldsymbol{e}_l - \boldsymbol{e}_m$, then these coefficients do not depend on the values of the distinct indices $i$ and $j$. As a result, $M^2$ is a multiple of the identity and $M^4$ reduces to two effective coefficients: there exists constants $\alpha$, $\beta$, $\nu \in \mathbb{R}$ such that

$$M^2 \cdot \nabla^2 = \nu\Delta \text{ and } M^4 \cdot \nabla^4 = \alpha\sum_{i=1}^{d}\partial^4_{iiii} + \beta\sum_{1\leq i\neq j\leq d}\partial^4_{iijj}.$$

## 4. Homogenized equations of order $2K+2$: tensors $\mathbb{D}^{2k}_K$

This section details the steps (4) and (5) of section 2.2 concerned with the process of truncating the infinite order homogenized equation (2.10) so as to obtain well-posed effective models of finite order. Recall that this process is needed because (1.10) is in general ill-posed, since the tensors $M^k$ do not have any particular sign (in view of (3.15)).

The first section 4.1 introduces the main technical results which allow to derive error estimates. More particularly we show in section 4.1 that for any integer $K' \in \mathbb{N}$, any family of non-oscillating functions $(v^*_\varepsilon)_{\varepsilon>0}$ yields an error estimate of order $O(\varepsilon^{K'+3})$ provided (i) $v^*_\varepsilon$ is of order $O(\varepsilon^2)$ and (ii) $v^*_\varepsilon$ solves the infinite order homogenized equation (1.9) up to a remainder of order $O(\varepsilon^{K'+1})$:

$$\sum_{k=0}^{K'}\varepsilon^{k-2}M^k\cdot\nabla^k v^*_\varepsilon = f + O(\varepsilon^{K'+1}). \tag{4.1}$$

In particular, this result reminds us that higher order models are generally not unique, they differ by the choice of extra differential operators of order greater than $K'$ which turn (4.1) into a well-posed model.

Leaving momentarily these considerations aside, we propose in section 4.2, a "variational" method inspired from [52, 23] which allows to construct a well-posed effective model (1.6) of order $2K+2$ for any $K \in \mathbb{N}$. The procedure relies on an energy minimization principle based on the criminal ansatz (1.11); the coefficients $\mathbb{D}^{2k}_K$ are inferred from an effective energy $J^*_K$ (definition 4) and are *a priori* distinct from the tensors $M^{2k}$. These properties enable us to establish that the obtained model is *elliptic* (in particular, well-posed), hence amenable to numerical computations.

Finally, we obtain in section 4.3 that surprisingly, it holds $\mathbb{D}^{2k}_K = M^{2k}$ for any $0 \leq 2k \leq K$, which implies that (4.1) is satisfied by the solution $v^*_\varepsilon \equiv v^*_{\varepsilon,K}$ with $K' = 2K+1$ (recall $M^{2k+1} = 0$ for any $k \in \mathbb{N}$ from corollary 1). The error estimate (1.14) of order $O(\varepsilon^{2K+4})$ follows in corollary 5.

### 4.1. Sufficient conditions under which an effective solution yields higher order approximations

The main result of this part is proposition 10 where we provide sufficient conditions under which a sequence of sequence of macroscopic functions $v^*_\varepsilon \in \mathcal{C}^\infty(D)$ (depending on $\varepsilon$, $K$ and $f$) yields a high order approximation of $u_\varepsilon$. The proof is based on the properties of the tensors $N^k$ stated in proposition 8 and the next three results.

**Lemma 1** (see e.g. Lions (1981) [41])**.** *There exists a constant $C$ independent of $\varepsilon$ such that for any $\phi \in H^1(D_\varepsilon)$ satisfying $\phi = 0$ on the boundary $\partial\omega_\varepsilon$ of the holes, the following Poincaré inequality holds:*

$$||\phi||_{L^2(D_\varepsilon)} \leq C\varepsilon||\nabla\phi||_{L^2(D_\varepsilon,\mathbb{R}^d)}.$$

**Corollary 4.** *For any $h \in L^2(D_\varepsilon)$, let $r_\varepsilon \in H^1(D_\varepsilon)$ be the unique solution to the Poisson problem*

$$\begin{cases} -\Delta r_\varepsilon = h \text{ in } D_\varepsilon \\ \quad r_\varepsilon = 0 \text{ on } \partial\omega_\varepsilon \\ \quad r_\varepsilon \text{ is } D\text{–periodic.} \end{cases} \tag{4.2}$$

*There exists a constant $C$ independent of $\varepsilon$ and $h$ such that*

$$||r_\varepsilon||_{L^2(D_\varepsilon)} + \varepsilon||\nabla r_\varepsilon||_{L^2(D_\varepsilon,\mathbb{R}^d)} \leq C||h||_{L^2(D_\varepsilon)}\varepsilon^2. \tag{4.3}$$

*Proof.* This result is classical; it is a consequence of lemma 1 and of the energy estimate

$$\int_{D_\varepsilon} |\nabla r_\varepsilon|^2 \mathrm{d}x = \int_{D_\varepsilon} h r_\varepsilon \mathrm{d}x \leq ||h||_{L^2(D_\varepsilon)}||r_\varepsilon||_{L^2(D_\varepsilon)} \leq C\varepsilon||h||_{L^2(D_\varepsilon)}||\nabla r_\varepsilon||_{L^2(D_\varepsilon,\mathbb{R}^d)}$$

which implies $||\nabla r_\varepsilon||_{L^2(D_\varepsilon,\mathbb{R}^d)} \leq C||h||_{L^2(D_\varepsilon)}\varepsilon$, then (4.3). □

**Lemma 2.** *Assume that the boundary of obstacle $\eta T$ is smooth. Then, for any $k \in \mathbb{N}$, the tensor $\mathcal{X}^k$ is well-defined and is smooth, namely it holds $\mathcal{X}^k \in \mathcal{C}^\infty(\overline{Y})$. In particular, $\mathcal{X}^k \in L^\infty(Y) \cap H^1(Y)$.*

*Proof.* Since the constant function 1 is smooth, standard regularity theory for the Laplace operator $-\Delta_{yy}$ (see [35, 22, 31]) implies $\mathcal{X}^0 \in \mathcal{C}^\infty(\overline{Y})$. The result follows by induction by repeating this argument to $\mathcal{X}^1$ and $\mathcal{X}^{k+2}$ for any $k \geq 0$. □

**Proposition 10.** *Let $v_\varepsilon^* \in \mathcal{C}^\infty(D)$ be a $D$–periodic function depending on $\varepsilon$ (and possibly on $K'$ and $f$) satisfying the following two hypotheses:*

*1. for any $m \in \mathbb{N}$, there exists a constant $C_{K',m}(f)$ depending only on $m$, $K'$ and $f \in \mathcal{C}^\infty(D)$ such that*

$$||v_\varepsilon^*||_{H^m(D)} \leq C_{K',m}(f)\varepsilon^2 \tag{4.4}$$

*2. $v_\varepsilon^*$ solves the infinite order homogenized equation (1.9) up to a remainder of order $O(\varepsilon^{K'+1})$:*

$$\left|\left|\sum_{k=0}^{K'} \varepsilon^{k-2} M^k \cdot \nabla^k v_\varepsilon^* - f\right|\right|_{L^2(D)} \leq C_{K'}(f)\varepsilon^{K'+1}. \tag{4.5}$$

*Then the reconstructed function $W_{\varepsilon,K'}(v_\varepsilon^*)$ of (1.12) approximates the solution $u_\varepsilon$ of the perforated Poisson problem (1.1) at order $O(\varepsilon^{K'+3})$, viz. there exists a constant $C_{K'}(f)$ independent of $\varepsilon$ such that*

$$||u_\varepsilon - W_{\varepsilon,K'}(v_\varepsilon^*)||_{L^2(D_\varepsilon)} + \varepsilon||\nabla(u_\varepsilon - W_{\varepsilon,K'}(v_\varepsilon^*))||_{L^2(D_\varepsilon,\mathbb{R}^d)} \leq C_{K'}(f)\varepsilon^{K'+3}.$$

*Proof.* Let us compute

$$\begin{aligned} -\Delta W_{\varepsilon,K'}(v_\varepsilon^*) = &\sum_{k=0}^{K'} \varepsilon^{k-2}(-\Delta N^k - 2\partial_l N^{k-1} \otimes e_l - N^{k-2} \otimes I)(\cdot/\varepsilon) \cdot \nabla^k v_\varepsilon^* \\ &- \varepsilon^{K'-1}(2\partial_l N^{K'} \otimes e_l + N^{K'-1} \otimes I)(\cdot/\varepsilon) \cdot \nabla^{K'+1} v_\varepsilon^* - \varepsilon^{K'} N^{K'}(\cdot/\varepsilon) \otimes I \cdot \nabla^{K'+2} v_\varepsilon^* \\ = &\sum_{k=0}^{K'} \varepsilon^{k-2} M^k \cdot \nabla^k v_\varepsilon^* \\ &- \varepsilon^{K'-1}(2\partial_l N^{K'} \otimes e_l + N^{K'-1} \otimes I)(\cdot/\varepsilon) \cdot \nabla^{K'+1} v_\varepsilon^* - \varepsilon^{K'} N^{K'}(\cdot/\varepsilon) \otimes I \cdot \nabla^{K'+2} v_\varepsilon^*, \end{aligned}$$

where we have used (3.21) to obtain the second equality. Since the functions $(N^k)_{k\in\mathbb{N}}$ are smooth (lemma 2 and (3.19)), assumption (4.4) implies that the last two terms are lower than $\varepsilon^{K'+1}$:

$$\left|\left|\varepsilon^{K'-1}(2\partial_l N^{K'} \otimes e_l + N^{K'-1} \otimes I)(\cdot/\varepsilon) \cdot \nabla^{K'+1} v_\varepsilon^* + \varepsilon^{K'} N^{K'}(\cdot/\varepsilon) \otimes I \cdot \nabla^{K'+2} v_\varepsilon^*\right|\right|_{L^2(D_\varepsilon)} \leq C_{K'}(f)\varepsilon^{K'+1}.$$

Using now assumption (4.5) and applying corollary 4 to $r_\varepsilon := u_\varepsilon - W_{\varepsilon,K'}(v_\varepsilon^*)$ yields the result. □

### 4.2. **Construction of a well-posed higher order effective model by mean of a variational principle**

Leaving momentarily aside the result of proposition 10, we now detail the construction of our effective model (1.6) of finite order which is inspired from the works [15, 52, 23]. The construction of the coefficients $\mathbb{D}_K^{2k}$ from an effective energy is exposed in section 4.2.1, and the well-posedness of the effective model is established in section 4.2.2.

#### 4.2.1. *The method of Smyshlyaev and Cherednichenko* [52]

According to the ideas outlined in step (4) of section 2.2, we consider truncations $W_{\varepsilon,K}(v)$ of the "criminal" ansatz (2.12) of the form:

$$W_{\varepsilon,K}(v)(x) := \sum_{k=0}^{K} \varepsilon^k N^k(x/\varepsilon) \cdot \nabla^k v(x), \quad x \in D, \tag{4.6}$$

where we seek a function $v \in H^{K+1}(D)$ which does not depend on the fast variable $x/\varepsilon$ and which approximates the formal homogenized average $u_\varepsilon^*$ of (2.8). The tensors $N^k(y)$ is extended by 0 in $P \backslash Y$ for (4.6) to make sense in $D \backslash D_\varepsilon$.

For any $u \in H^1(D)$ and $f \in L^2(D)$, we denote by $J(u,f)$ the energy

$$J(u,f) := \int_D \left( \frac{1}{2} |\nabla u|^2 - fu \right) \mathrm{d}x.$$

Following the lines of the point (4) of section 2.2, we consider the problem (2.15) of finding a minimizer of $v \mapsto J(W_{\varepsilon,K}(v), f)$. The key step of the strategy is to eliminate the fast variable $x/\varepsilon$ in $J(W_{\varepsilon,K}(v), f)$ so as to obtain an effective energy $J_K^*(v,f,\varepsilon) \simeq J(W_{\varepsilon,K}(v), f)$ which does not involve oscillating functions. The main technical tool which allows us to perform this operation is the following classical lemma of two-scale convergence (see e.g. Appendix C. of [52] or [6]).

**Lemma 3.** *Let $\phi$ be a $P = [0,1]^d$–periodic function and $f \in \mathcal{C}^\infty(D)$ be a smooth $D$–periodic function. Then for any $k \in \mathbb{N}$ arbitrarily large, the following inequality holds:*

$$\left| \int_D f(x)\phi(x/\varepsilon)\mathrm{d}x - \int_D \int_P f(x)\phi(y)\mathrm{d}y\mathrm{d}x \right| \leq \frac{L^{d/2}}{|2\pi|^p} \left|\left| \phi - \int_P \phi \mathrm{d}y \right|\right|_{L^2(P)} ||f||_{H^p(D)} \varepsilon^p. \tag{4.7}$$

*Proof.* See Lemma 7.3 in [32] for a proof of this exact statement. □

Before providing the definition of $J_K^*$ based on the application of lemma 3, we introduce several additional tensors that arise in the averaging process.

**Definition 3** (Tensors $\mathbb{B}_K^{l,m}$)**.** For any $K \in \mathbb{N}$, $1 \leq j \leq d$ and $0 \leq k \leq K+1$, let $\widetilde{N}_j^k(y)$ (with implicit dependence with respect to $K$) be the $k$-th order tensor defined by

$$\widetilde{N}_j^k(y) = \begin{cases} \partial_j N^0(y) \text{ if } k = 0 \\ \partial_j N^k(y) + N^{k-1}(y) \otimes e_j \text{ if } 1 \leq k \leq K \\ N^K(y) \otimes e_j \text{ if } k = K+1. \end{cases} \tag{4.8}$$

We define a family of constant bilinear tensors $\mathbb{B}_K^{l,m}$ of order $l+m$ by the formula

$$\mathbb{B}_K^{l,m} := \int_Y \widetilde{N}_j^l(y) \otimes \widetilde{N}_j^m(y) \mathrm{d}y, \text{ for any } 0 \leq l,m \leq K+1, \tag{4.9}$$

where the Einstein summation convention is still assumed over the repeated subscript index $1 \leq j \leq d$.

**Definition 4** (Approximate energy $J_K^*$)**.** For any $f \in L^2(D)$ and periodic function $v \in H^{K+1}(D)$, we define

$$J_K^*(v,f,\varepsilon) := \int_D \left( \frac{1}{2} \sum_{l,m=0}^{K+1} \varepsilon^{l+m-2} \mathbb{B}_K^{l,m} \nabla^l v \nabla^m v - fv \right) \mathrm{d}x. \tag{4.10}$$

where we recall (2.5) for the definition of $\mathbb{B}_K^{l,m} \nabla^l v \nabla^m v$.

The definition of the energy $J_K^*(v,f,\varepsilon)$ is motivated by the following asymptotic—provided by lemma 3—which holds with any $p \geq 0$ arbitrarily large

$$J(W_{\varepsilon,K}(v),f) = J_K^*(v,f,\varepsilon) + o(\varepsilon^p).$$

More precisely, we have the following result:

**Proposition 11.** *Assume $f \in \mathcal{C}^\infty(D)$ is $D$–periodic. Let $v \in \mathcal{C}^\infty(D)$ be a smooth $D$–periodic function and $W_{\varepsilon,K}(v) \in \mathcal{C}^\infty(D_\varepsilon)$ be the truncated ansatz of the form of (4.6). The following estimate holds true with $p \geq 0$ arbitrarily large:*

$$|J(W_{\varepsilon,K}(v),f) - J_K^*(v,f,\varepsilon)| \leq C_{K,p}(||v||^2_{H^{p+2}(D)} + ||f||^2_{H^p(D)})\varepsilon^p.$$

*for a constant $C_{K,p}$ depending only on $p$ and $K$ (and $\eta$).*

*Proof.* For any $1 \leq j \leq d$, the partial derivative $\partial_{x_j} W_{\varepsilon,K}(v)$ reads

$$\begin{aligned}\partial_{x_j} W_{\varepsilon,K}(v) &= \sum_{k=0}^{K} \left(\varepsilon^{k-1}\partial_{y_j} N^k(\cdot/\varepsilon)\cdot\nabla^k v + \varepsilon^k N^k(\cdot/\varepsilon)\otimes e_j\cdot\nabla^{k+1}v\right)\\ &= \sum_{i=0}^{K+1} \varepsilon^{k-1}\widetilde{N}_j^k(\cdot/\varepsilon)\cdot\nabla^k v,\end{aligned}$$

by the definition (4.8) of the tensors $\widetilde{N}_j^k$. The computation of the energy $J(W_{\varepsilon,K}(v),f)$ yields then

$$\begin{aligned}&J(W_{\varepsilon,K}(v),f)\\ &\quad= \int_D \left(\frac{1}{2}\sum_{l,m=0}^{K+1}\varepsilon^{l+m-2}(\widetilde{N}_j^l(\cdot/\varepsilon)\cdot\nabla^l v)(\widetilde{N}_j^m(\cdot/\varepsilon)\cdot\nabla^m v) - \sum_{l=0}^{K}\varepsilon^l(N^l(\cdot/\varepsilon)\cdot\nabla^l v)f\right)\mathrm{d}x.\end{aligned} \tag{4.11}$$

The result follows by estimating both term after applying lemma 3:

$$\begin{aligned}\forall 0 \leq l,m \leq K+1, &\left|\int_D \varepsilon^{l+m-2}\left((\widetilde{N}_j^l(\cdot/\varepsilon)\cdot\nabla^l v)(\widetilde{N}_j^m(\cdot/\varepsilon)\cdot\nabla^m v) - \mathbb{B}_K^{l,m}\nabla^l v\nabla^m v\right)\mathrm{d}x\right|\\ &\leq C_p\varepsilon^p||\nabla^l v\otimes\nabla^m v||_{H^{p-(l+m-2)}(D,\mathbb{R}^{d^{l+m}})}\\ &\leq C_p'\varepsilon^p\left(||\nabla^l v||^2_{H^{p-(l+m-2)}(D,\mathbb{R}^{d^l})} + ||\nabla^m v||^2_{H^{p-(l+m-2)}(D,\mathbb{R}^{d^m})}\right)\\ &\leq C_p''\varepsilon^p||v||^2_{H^{p+2}(D)},\end{aligned} \tag{4.12}$$

$$\begin{aligned}\forall 0 \leq l \leq K, \left|\int_D \varepsilon^l\left(N^l(\cdot/\varepsilon)\cdot\nabla^l vf - fv\right)\mathrm{d}x\right| &\leq C_p\varepsilon^p||f\nabla^l v||_{H^{p-l}(D,\mathbb{R}^{d^l})}\\ &\leq C_p'\varepsilon^p(||f||^2_{H^{p-l}(D)} + ||v||^2_{H^p(D)}) \leq C_p'\varepsilon^p(||f||^2_{H^p(D)} + ||v||^2_{H^p(D)}),\end{aligned} \tag{4.13}$$

where we used that $N^l$ is of average 1 if $l = 0$ and 0 otherwise (point 1. of proposition 8). □

The approximate energy (4.10) is used (instead of $J(W_{\varepsilon,K}(v),f)$ in (2.15)) in order to construct our higher order homogenized model.

**Definition 5.** For any $K \in \mathbb{N}$, we call homogenized equation of order $2K+2$ the Euler–Lagrange equation associated with the minimization problem

$$\begin{aligned}\min_{v\in H^{K+1}(D)} &\quad J_K^*(v,f,\varepsilon)\\ s.t. &\quad v \text{ is } D\text{–periodic.}\end{aligned} \tag{4.14}$$

This equation reads explicitly in terms of a higher order homogenized solution $v_K^* \in H^{K+1}(D)$ as

$$\begin{cases}\displaystyle\sum_{k=0}^{K+1}\varepsilon^{2k-2}\mathbb{D}_K^{2k}\cdot\nabla^{2k}v^*_{\varepsilon,K} = f\\ v^*_{\varepsilon,K} \text{ is } D\text{–periodic,}\end{cases} \tag{4.15}$$

where the constant tensors $\mathbb{D}_K^{2k}$ are defined for any $0 \leq 2k \leq 2K+2$ by:

$$\mathbb{D}_K^{2k} := \sum_{l=0}^{2k} (-1)^l \mathbb{B}_K^{l,2k-l}, \tag{4.16}$$

assuming the convention $\mathbb{B}_K^{l,m} = 0$ for any $l > K+1$ or $m > K+1$.

*Proof.* Let us detail slightly the derivation of (4.15). The Euler–Lagrange equation of (4.14) reads, after an integration by parts:

$$\sum_{l,m=0}^{K+1} \varepsilon^{l+m-2} \frac{1}{2}((-1)^m \mathbb{B}_K^{l,m} + (-1)^l \mathbb{B}_K^{m,l}) \nabla^{l+m} v_{\varepsilon,K}^* = f. \tag{4.17}$$

Since $\mathbb{B}_K^{l,m} = \mathbb{B}_K^{m,l}$ and $(-1)^l + (-1)^m$ vanishes when $l$ and $m$ are not of the same parity, only terms such that $l+m$ is even are possibly not zero in the above equation. Hence, (4.17) rewrites as (4.15) with

$$\mathbb{D}_K^{2k} = \sum_{l+m=2k} \frac{1}{2}((-1)^l + (-1)^m)\mathbb{B}_K^{l,m} = \sum_{l=0}^{2k} \frac{1}{2}((-1)^l + (-1)^{2k-l})\mathbb{B}_K^{l,2k-l}, \quad \text{for any } 0 \leq 2k \leq 2K+2,$$

which eventually yields the desired expression (4.16). □

**Remark 3.** As announced in the introduction, (4.15) turns out to be a simple correction of (1.10), see proposition 13 below.

4.2.2. *Well-posedness of the homogenized model of order $2K+2$*

We now establish the well-posedness of the high order homogenized model (4.15). More precisely, we prove its ellipticity, which implies the existence and the uniqueness of the effective solution $v_{\varepsilon,K}^*$. Before stating the result, let us stress the following observation which is obvious, but a somewhat important consequence of the definition (4.9).

**Lemma 4.** *The dominant tensor $\mathbb{B}_K^{K+1,K+1}$ is symmetric and non-negative.*

**Proposition 12.** *Assume further that the dominant tensor $\mathbb{B}_K^{K+1,K+1}$ is non-degenerate, viz. there exists a constant $\nu > 0$ such that*

$$\forall \xi = \xi_{i_1 \ldots i_{K+1}} \in \mathbb{R}^{d^{K+1}},\ \mathbb{B}_K^{K+1,K+1} \cdot \xi\xi \geq \nu |\xi|^2. \tag{4.18}$$

*Then (4.15) is elliptic and there exists a unique solution $v_K^* \in H^{K+1}(D)$ to the homogenized equation (4.15) of order $2K+2$.*

*Proof.* Let us consider the space $V_K := \{v \in H^{K+1}(D) \,|\, v \text{ is } D\text{–periodic }\}$ and introduce $a : V_K \times V_K \to \mathbb{R}$ and $b : V_K \to \mathbb{R}$ the respective bilinear and linear forms defined for any $v \in V_K$ by

$$a(v,v) = \int_D \sum_{l,m=0}^{K+1} \varepsilon^{l+m-2} \mathbb{B}_K^{l,m} \nabla^l v \nabla^m v \mathrm{d}x, \tag{4.19}$$

$$b(v) = \int_D f v \mathrm{d}x. \tag{4.20}$$

The homogenized equation (4.15) reduces to the following variational problem:

$$\text{find } v_{\varepsilon,K}^* \in V_K \text{ such that } \forall v \in V_K,\ a(v_{\varepsilon,K}^*, v) = b(v). \tag{4.21}$$

From there, one could directly rely on the theory of Fredholm operators [42] to conclude to the existence of a solution $v_{\varepsilon,K}^*$. However we are going to show that $a$ is coercive (meaning (4.15) is elliptic), which will allow us to apply Lax–Milgram theorem [30].
Under the non-degeneracy assumption (4.18), it is readily obtained that there exists a constant $C_\varepsilon$ (depending on $\varepsilon$) such that

$$\forall v \in V_K(D),\ a(v,v) \geq (\varepsilon^{2K}\nu)||\nabla^{K+1} v||^2_{L^2(D,\mathbb{R}^d)} + \varepsilon^{-2} M^0 ||v||^2_{L^2(D)} - C_\varepsilon ||v||_{H^{K+1}(D)} ||v||_{H^K(D)}.$$

Remembering $M^0 > 0$ and applying the following Young's inequality

$$\forall x, y \in \mathbb{R},\ |xy| \le \frac{x^2}{2\gamma} + \frac{\gamma y^2}{2}$$

for a sufficiently small $\gamma > 0$, we obtain the existence of two constants $\alpha_{\varepsilon,K} > 0$ and $\beta_{\varepsilon,K} > 0$ (that depend on $\varepsilon$ and $K$) such that

$$\forall v \in V_K(D),\ a(v,v) \ge \alpha_{\varepsilon,K}||v||^2_{H^{K+1}(D)} - \beta_{\varepsilon,K}||v||^2_{H^K(D)}. \tag{4.22}$$

Furthermore, (4.9) together with the proof of proposition 11 allow to rewrite $a(v,v)$ as

$$a(v,v) = \int_D \int_Y \left| (\varepsilon^{-1}\nabla_y + \nabla_x) \left( \sum_{k=0}^{K} \varepsilon^k N^k(y) \cdot \nabla^k v(x) \right) \right|^2 \mathrm{d}x. \tag{4.23}$$

Then, $\int_D \int_Y u(x,y)^2 \mathrm{d}y\mathrm{d}x \ge \int_D \left| \int_Y u(x,y)\mathrm{d}y \right|^2 \mathrm{d}x$ and point 1. of proposition 8 imply the the following inequality:

$$\forall v \in V_K,\ a(v,v) \ge ||\nabla v||^2_{L^2(D,\mathbb{R}^d)}. \tag{4.24}$$

We now prove that (4.22) and (4.24) together imply the coercivity of $a$ on the space $V_K$, that is we claim there exists a constant $c_{\varepsilon,K} > 0$ such that

$$\forall v \in V_K,\ a(v,v) \ge c_{\varepsilon,K}||v||^2_{H^{K+1}(D)}. \tag{4.25}$$

Assume the contrary is true, then one can find a sequence $(v_n)$ of functions satisfying $||v_n||_{H^{K+1}(D)} = 1$ and such that $a(v_n, v_n) \to 0$. Up to extracting a relevant subsequence, we may assume that $v_n \rightharpoonup v$ weakly in $H^{K+1}(D)$ and $v_n \to v$ strongly in $H^K(D)$. Then the polarization identity together with (4.22) and the positivity of $a$ allow to show that $(v_n)$ is a Cauchy sequence in $V_K$:

$$\begin{aligned}
\forall p, q \in \mathbb{N},\ \alpha_{\varepsilon,K}||v_p - v_q||^2_{H^{K+1}(D)} &\le a(v_p - v_q, v_p - v_q) + \beta_{\varepsilon,K}||v_p - v_q||^2_{H^K(D)} \\
&= 2a(v_p, v_p) + 2a(v_q, v_q) - a(v_p + v_q, v_p + v_q) + \beta_{\varepsilon,K}||v_p - v_q||^2_{H^K(D)} \\
&\le 2a(v_p, v_p) + 2a(v_q, v_q) + \beta_{\varepsilon,K}||v_p - v_q||^2_{H^K(D)} \xrightarrow{p,q\to\infty} 0.
\end{aligned}$$

Therefore $v_n \to v$ strongly in $V_K$. Using the continuity of $a$, we infer then $a(v,v) = \lim_{n\to+\infty} a(v_n, v_n) = 0$. The property (4.24) yields then that $v$ is a constant. Therefore, $0 = a(v,v) = \varepsilon^{-2}M^0||v||^2_{L^2(D)}$, which implies $v = 0$. This is in contradiction with the fact that $||v_n||_{H^{K+1}(D)} = 1$ for any $n \ge 0$ and the strong convergence of $(v_n)$ in $H^{K+1}(D)$.

Finally, the coercivity (4.25) and the continuity of $a$ and $b$ over $V_K$ ensure that all the assumptions of the Lax–Milgram theorem are fulfilled, which yields existence and uniqueness to the problem (4.21). □

**Remark 4.** It is always possible to add to $\mathbb{D}_K^{2K+2}$ a small perturbation making (4.18) satisfied while keeping an "admissible" higher order homogenized equation. Indeed, since the other $2K+1$ coefficients are kept unaffected, the error estimate provided by proposition 10 and corollary 5 below remain valid. Let us remark, however, that this non-degeneracy condition is automatically fulfilled for any shape of obstacle $\eta T$ when $K = 0$ because it is easily shown that $\mathbb{D}_0^2 = -(M^0)^2 \int_Y |\mathcal{X}^0(y)|^2 \mathrm{d}y > 0$ (see (4.29)). In the general case $K \ge 1$, it could fail to be satisfied for particular obstacle shapes (e.g. in case of invariance of $\eta T$ along some of the cell axes).

### 4.3. Asymptotic surprise: $\mathbb{D}_K^{2k} = M^{2k}$; error estimates for the homogenized model of order $2K+2$

We terminate this section by verifying the assumptions of proposition 10 which ensure the validity of the error estimate (1.14) claimed in the introduction. The next proposition establishes the point 1.

**Lemma 5.** *Assume the non-degeneracy condition (4.18). The solution $v^*_{\varepsilon,K}$ of (4.15) belongs to $\mathcal{C}^\infty(D)$ and for any $m \in \mathbb{N}$, there exists a constant $C_{m,K}$ that does not depend on $\varepsilon$ such that*

$$||v^*_{\varepsilon,K}||_{H^m(D)} \le C_{m,K}||f||_{H^m(D)}\varepsilon^2. \tag{4.26}$$

*Proof.* We solve (4.15) explicitly with Fourier expansions in the periodic domain $D = [0, L]^d$. Let $\widehat{f}(\xi)$ be the Fourier coefficients of $f$ and $c(\xi, \varepsilon)$ the symbol of the differential operator of (4.15), namely:

$$c(\xi, \varepsilon) = \sum_{k=0}^{K+1} (-1)^k |2\pi/L|^{2k} \varepsilon^{2k-2} \mathbb{D}_K^{2k} \cdot \xi^{2k},$$

where $\xi^0 = 1$ by convention and where we have used the short-hand notation $\mathbb{D}_K^{2k} \cdot \xi^{2k} := \mathbb{D}^{2k}_{i_1 \ldots i_{2k}} \xi_{i_1 \ldots i_{2k}}$. For $\xi \in \mathbb{Z}^d$, the Fourier coefficients $\widehat{v}^*_{\varepsilon,K}$ of $v^*_K$ read:

$$\widehat{v}^*_{\varepsilon,K}(\xi) = \frac{\widehat{f}(\xi)}{c(\xi, \varepsilon)}. \tag{4.27}$$

Applying Parseval's identity to the bilinear form (4.23), we obtain

$$\begin{aligned} \forall \xi \in \mathbb{Z}^d,\ \forall \varepsilon > 0,\ c(\xi, \varepsilon) &= \int_Y \left| (\varepsilon^{-1} \nabla_y + (2i\pi/L)\xi) \left( \sum_{k=0}^{K} (2i\pi/L)^k \varepsilon^k N^k(y) \cdot \xi^k \right) \right|^2 \mathrm{d}y \\ &= \varepsilon^{-2} \Phi(\varepsilon \xi). \end{aligned}$$

where $\Phi$ is the function defined by:

$$\begin{aligned} \Phi :\ & \mathbb{R}^d \longrightarrow \mathbb{R} \\ & \lambda \longmapsto \int_Y \left| (\nabla_y + (2i\pi/L)\lambda) \left( \sum_{k=0}^{K} (2i\pi/L)^k N^k(y) \cdot \lambda^k \right) \right|^2 \mathrm{d}y. \end{aligned}$$

In order to obtain (4.26), it is enough to prove the existence of some positive constant $C_K > 0$ such that $\Phi(\lambda) \geq C_K$ for any $\lambda \in \mathbb{R}^d$. The function $\Phi$ is clearly continuous, nonnegative and satisfies $\Phi(\lambda) \to +\infty$ when $|\lambda| \to +\infty$ because of (4.18). Therefore it admits a minimum point $\lambda_0 \in (\mathbb{R}_+)^d$. Let us assume that $\Phi(\lambda_0) = 0$. Using the Cauchy-Schwartz inequality, we obtain

$$0 = \Phi(\lambda_0) \geq \left| \int_Y (\nabla_y + (2i\pi/L)\lambda_0) \left( \sum_{k=0}^{K} (2i\pi/L)^k N^k(y) \cdot \lambda_0^k \right) \mathrm{d}y \right|^2 = 4\pi^2/L^2 |\lambda_0|^2.$$

Therefore it must hold $\lambda_0 = 0$, however this is not possible because

$$\Phi(0) = \int_Y \left| \nabla_y N^0(y) \right|^2 \mathrm{d}y = M^0 > 0.$$

Consequently, $\forall \lambda \in \mathbb{R}^d$, $\Phi(\lambda) \geq \Phi(\lambda_0) > 0$ which concludes the proof. □

The point 2 of proposition 10 is the object of the next result. More precisely, we prove that surprisingly, the coefficients $\mathbb{D}_K^{2k}$ and $M^{2k}$ coincide for $0 \leq 2k \leq 2K$. A similar fact was also observed for the (scalar) antiplane elasticity model considered in [52], but only for the first half $0 \leq 2k \leq K$ of the coefficients (and without providing a proof).

**Proposition 13.** *All the coefficients of the homogenized equation (1.6) of order $2K + 2$ (defined in (4.15)) coincide with those of the formal infinite order homogenized equation (1.9), except the leading order one:*

$$\mathbb{D}_K^{2k} = M^{2k} \textit{ for any } 0 \leq 2k \leq 2K, \tag{4.28}$$

$$\mathbb{D}_K^{2K+2} = (-1)^{K+1} \int_Y N^K \otimes N^K \otimes I \mathrm{d}y. \tag{4.29}$$

*Proof.* First of all, (4.29) is only a rewriting of (4.16). We show the following, slightly more general, result:

$$\forall 0 \leq k \leq 2K,\ M^k = \sum_{l=0}^{k} (-1)^l \mathbb{B}_K^{l,k-l}, \tag{4.30}$$

which is enough for our purpose because of (4.16). We distinguish two cases.

1. *Case* $0 \le k \le K$. For $0 \le k, l \le K$, the coefficient $\mathbb{B}_K^{l,k-l}$ is given by (from (4.8))

$$\mathbb{B}_K^{l,k-l} = \int_Y (\partial_j N^l + N^{l-1} \otimes e_j) \otimes (\partial_j N^{k-l} + N^{k-l-1} \otimes e_j) \mathrm{d}y, \tag{4.31}$$

where we use the convention $N^{-1} = N^{-2} = 0$. After an integration by parts, we rewrite $\mathbb{B}_K^{l,k-l}$ as follows:

$$\begin{aligned} \mathbb{B}_K^{l,k-l} &= \int_Y (-\Delta N^l - 2\partial_j N^{l-1} \otimes e_j - N^{l-2} \otimes I) \otimes N^{k-l} \mathrm{d}y \\ &+ \int_Y (\partial_j N^l \otimes N^{k-l-1} \otimes e_j + N^{l-1} \otimes N^{k-l-1} \otimes I + \partial_j N^{l-1} \otimes N^{k-l} \otimes e_j + N^{l-2} \otimes N^{k-l} \otimes I) \mathrm{d}y \\ &= \int_Y (M^l \otimes N^{k-l}) \mathrm{d}y + B^{k,l} + B^{k,l-1} \end{aligned} \tag{4.32}$$

where $B^{k,l}$ is the $k$-th order tensor defined by

$$B^{k,l} := \int_Y (\partial_j N^l \otimes N^{k-l-1} \otimes e_j + N^{l-1} \otimes N^{k-l-1} \otimes I) \mathrm{d}y.$$

Using now the point 1. of proposition 8 and recognizing a telescopic series, we obtain

$$\begin{aligned} \sum_{l=0}^{k} (-1)^l \mathbb{B}_K^{l,k-l} &= (-1)^k M^k + \sum_{l=0}^{k} ((-1)^l B^{k,l} - (-1)^{l-1} B^{k,l-1}) \\ &= (-1)^k M^k + (-1)^k B^{k,k} - (-1)^{-1} B^{k,-1}. \end{aligned}$$

This implies (4.30) by using the facts that $M^k = 0$ when $k$ is odd and $B^{k,k} = B^{k,-1} = 0$ owing to our convention $N^{-1} = 0$.

*2. Case* $K+1 \le k \le 2K$. The equality (4.31) is valid for any $K+1 \le k \le 2K$ and $0 \le l \le k$ if we assume by convention (*in this part only*) that $N^m = 0$ whenever $m > K$, because of the definition (4.9). Then (4.32) remains true provided $0 \le l \le K$. Therefore, recognizing the same telescopic series, we are able to compute

$$\sum_{l=0}^{K} (-1)^l \mathbb{B}_K^{l,k-l} = (-1)^K B^{k,K} = (-1)^K \int_Y (\partial_j N^K \otimes N^{k-K-1} \otimes e_j + N^{K-1} \otimes N^{k-K-1} \otimes I) \mathrm{d}y.$$

Remembering that $\mathbb{B}_K^{l,m} = 0$ whenever $l > K+1$ or $m > K+1$, we eventually obtain

$$\begin{aligned} \sum_{l=0}^{k} (-1)^l \mathbb{B}_K^{l,k-l} &= \sum_{l=0}^{K} (-1)^l \mathbb{B}_K^{l,k-l} + (-1)^{K+1} \mathbb{B}_K^{K+1,k-K-1} \\ &= (-1)^K \int_Y (\partial_j N^K \otimes N^{k-K-1} \otimes e_j + N^{K-1} \otimes N^{k-K-1} \otimes I) \mathrm{d}y \\ &+ (-1)^{K+1} \int_Y (N^K \otimes e_j) \otimes (\partial_j N^{k-K-1} + N^{k-K-2} \otimes e_j) \mathrm{d}y \\ &= (-1)^K \int_Y ((2\partial_j N^K \otimes e_j + N^{K-1} \otimes I) \otimes N^{k-K-1} - N^K \otimes N^{k-K-2} \otimes I) \mathrm{d}y \\ &= (-1)^K \int_Y ((-\Delta_{yy} N^{K+1} - M^{K+1}) \otimes N^{k-K-1} - N^K \otimes N^{k-K-2} \otimes I) \mathrm{d}y, \end{aligned} \tag{4.33}$$

where we have used (3.21) in the last equality. We now consider two cases:

- if $k = K+1$, then the above expression reads

  $$\begin{aligned} \sum_{l=0}^{K+1} (-1)^l \mathbb{B}_K^{l,K+1-l} &= (-1)^K \int_Y ((-\Delta_{yy} N^{K+1} - M^{K+1}) \otimes N^0) \mathrm{d}y \\ &= (-1)^{K+1} M^{K+1} + (-1)^K \int_Y N^{K+1} \otimes M^0 \mathrm{d}y = (-1)^{K+1} M^{K+1} = M^{K+1}, \end{aligned}$$

  where the last equality is a consequence of corollary 1.
- if $K+2 \le k \le 2K$, then (4.33) coincides with $M^k$ by using (3.23) (with $p = K+1$).

$\square$

**Remark 5.** As is highlighted by the proof proposition 13, the "surprising" fact that $\mathbb{D}_K^{2k} = M^{2k}$ even for $K+1 \leq 2k \leq 2K$ is partly due to the fact that for any $p \in \mathbb{N}$, the tensor $M^{2p}$ can be computed from the first $p$ homogenized tensors $(\mathcal{X}^i(y))_{0\leq i\leq p}$ only (point 4. of proposition 8).

Since we have verified that all the requirements of proposition 10 are satisfied with $v_\varepsilon^* \equiv v_{\varepsilon,K}^*$ and $L \equiv 2K+1$ (remember $M^{2K+1} = 0$ from corollary 1), we are in position to state our main result.

**Corollary 5.** *The error estimate (1.14) holds for the reconstructed solution $W_{\varepsilon,2K+1}(v_{\varepsilon,K}^*)$ where $v_{\varepsilon,K}^*$ is the solution of (4.15).*

## 5. Retrieving the classical regimes: low volume fraction limits when the size of the obstacles tends to zero

The goal of this section is to show that our higher order homogenized models (1.6) have the potential of being valid in any regime of size of holes. For this purpose, we obtain asymptotics for the tensors $\mathcal{X}^{k*}$ and $M^k$ in the low volume fraction limit when the scaling $\eta$ of the obstacle vanishes to zero: $\eta \to 0$. Our main results are stated in corollaries 6 and 7; they imply that both the infinite order homogenized equation (3.14) as well as the effective model (1.6) of order $2K+2$ converge coefficient-wise to either of the three classical regimes of the literature (namely, to the original Laplace equation (1.1), or to the analogue of the Brinkman or Darcy equation (1.3) and (1.4)) if $K = 0$, and to either of (1.3) or (1.4) for $K \geq 1$ if $\eta$ remains greater or comparable to the critical size $\eta_{\text{crit}} \sim \eta^{2/(d-2)}$.

In this whole subsection, it is assumed, for simplicity, that the space dimension is greater than 3:

$$d \geq 3. \tag{5.1}$$

We do not consider the case $d = 2$ which requires a specific treatment, although very similar results could be stated (see e.g. [4]).

In all what follows, the hole $\eta T$ is assumed to be strictly included in the unit cell for any $\eta \leq 1$ (it does not touch the boundary): $\eta T \subset\subset P$. Functions of the rescaled cell $\eta^{-1}P$ are indicated by a tilde $\widetilde{\ }$ notation. For a given function $\widetilde{v} \in L^2(\eta^{-1}P)$, we denote by $\langle\widetilde{v}\rangle$ the average $\langle\widetilde{v}\rangle := \eta^d \int_{\eta^{-1}P} \widetilde{v}(y)\mathrm{d}y$. With a small abuse of notation, when $\widetilde{v} \in L^2(\eta^{-1}P\backslash T)$, we still denote by $\langle\widetilde{v}\rangle$ this quantity where we implicitly extend $\widetilde{v}$ by 0 within $T$.

Let us recall that for any $v \in H^1(P\backslash(\eta T))$, if $\widetilde{v}$ is the rescaled function defined by $\widetilde{v}(y) := v(\eta y)$ in the rescaled cell $\eta^{-1}P\backslash T$, then the $L^2$ norms of $v$ and $\widetilde{v}$ and their gradients are related by the following identities:

$$||v||_{L^2(P\backslash(\eta T))} = \eta^{d/2}||\widetilde{v}||_{L^2(\eta^{-1}P\backslash T)} \text{ and } ||\nabla v||_{L^2(P\backslash(\eta T),\mathbb{R}^d)} = \eta^{d/2-1}||\nabla\widetilde{v}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)}.$$

We follow the methodology of [4] (and also [37]), which relies extensively on the use of the next two lemmas:

**Lemma 6.** *Assume $d \geq 3$. There exists a constant $C > 0$ independent of $\eta > 0$ such that for any $\widetilde{v} \in H^1(\eta^{-1}P\backslash T)$ which vanishes on the hole $\partial T$ and which is $\eta^{-1}P$ periodic, the following inequalities hold:*

$$||\widetilde{v}||_{L^2(\eta^{-1}P\backslash T)} \leq C\eta^{-d/2}||\nabla\widetilde{v}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)}, \tag{5.2}$$

$$|\langle\widetilde{v}\rangle| \leq C||\nabla\widetilde{v}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)}, \tag{5.3}$$

$$||\widetilde{v} - \langle\widetilde{v}\rangle||_{L^2(\eta^{-1}P\backslash T)} \leq C\eta^{-1}||\nabla\widetilde{v}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)}, \tag{5.4}$$

$$||\widetilde{v} - \langle\widetilde{v}\rangle||_{L^{2d/(d-2)}(\eta^{-1}P\backslash T)} \leq C||\nabla\widetilde{v}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)}. \tag{5.5}$$

*Proof.* See [4, 38]. $\square$

**Lemma 7.** *Consider $\widetilde{h} \in L^2(\eta^{-1}P\backslash T)$ and let $\widetilde{v} \in H^1(\eta^{-1}P\backslash T)$ be the unique solution to the Poisson problem:*

$$\begin{cases} -\Delta\widetilde{v} = \widetilde{h} \text{ in } \eta^{-1}P\backslash T \\ \widetilde{v} = 0 \text{ on } \partial T \\ \widetilde{v} \text{ is } \eta^{-1}P\backslash T\text{–periodic.} \end{cases} \tag{5.6}$$

*There exists a constant $C>0$ independent of $\eta$ and $\widetilde{h}$ such that*

$$||\nabla \widetilde{v}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)} \leq C(\eta^{-1}||\widetilde{h}-\langle\widetilde{h}\rangle||_{L^2(\eta^{-1}P\backslash T)}+\eta^{-d}|\langle\widetilde{h}\rangle|). \tag{5.7}$$

*Proof.* An integration by parts and the use of lemma 6 yields

$$\begin{aligned}||\nabla \widetilde{v}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)} &= \int_{\eta^{-1}P\backslash T}\widetilde{h}\widetilde{v}\mathrm{d}y = \int_{\eta^{-1}P\backslash T}(\widetilde{h}-\langle\widetilde{h}\rangle)(\widetilde{v}-\langle\widetilde{v}\rangle)\mathrm{d}y + \int_{\eta^{-1}P\backslash T}\langle\widetilde{h}\rangle\langle\widetilde{v}\rangle\mathrm{d}y\\ &\leq C(||\widetilde{h}-\langle\widetilde{h}\rangle||_{L^2(\eta^{-1}P\backslash T)}||\widetilde{v}-\langle\widetilde{v}\rangle||_{L^2(\eta^{-1}P\backslash T)}+\eta^{-d}|\langle\widetilde{h}\rangle|\,|\langle\widetilde{v}\rangle|)\\ &\leq C(\eta^{-1}||\widetilde{h}-\langle\widetilde{h}\rangle||_{L^2(\eta^{-1}P\backslash T)}+\eta^{-d}|\langle\widetilde{h}\rangle|)||\nabla\widetilde{v}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)}.\end{aligned} \tag{5.8}$$

□

We also need to consider the so-called Deny-Lions space $\mathcal{D}^{1,2}(\mathbb{R}^d\backslash T)$ whose definition is recalled below (the reader is referred to [4, 3, 5] and also [44], p.59. for more details).

**Definition 6** (Deny-Lions space)**.** The Deny-Lions space $\mathcal{D}^{1,2}(\mathbb{R}^d\backslash T)$ is the completion of the space of smooth functions by the $L^2$ norm of their gradients:

$$\mathcal{D}^{1,2}(\mathbb{R}^d\backslash T) := \overline{\mathcal{D}(\mathbb{R}^d\backslash T)}^{||\nabla\cdot||_{L^2(\mathbb{R}^d\backslash T,\mathbb{R}^d)}}.$$

When $d\geq 3$, it is admits the following characterization:

$$\mathcal{D}^{1,2}(\mathbb{R}^d\backslash T) = \{\phi \text{ measurable } \mid ||\phi||_{L^{2d/(d-2)}(\mathbb{R}^d\backslash T)} < +\infty \text{ and } ||\nabla\phi||_{L^2(\mathbb{R}^d\backslash T,\mathbb{R}^d)} < +\infty\}.$$

We introduce $\Psi$ the unique solution to the exterior problem

$$\begin{cases} -\Delta\Psi = 0 \text{ in } \mathbb{R}^d - T\\ \Psi = 0 \text{ on } \partial T\\ \Psi \to 1 \text{ at } \infty,\end{cases} \tag{5.9}$$

and we denote by $F$ the normal flux

$$F := \int_{\mathbb{R}^d\backslash T}|\nabla\Psi|^2\mathrm{d}x = -\int_{\partial T}\nabla\Psi\cdot\boldsymbol{n}\mathrm{d}s, \tag{5.10}$$

where $\boldsymbol{n}$ is the normal pointing *inward* $T$. The condition $\Psi\to 1$ at $\infty$ is to be understood in the sense that $\Psi - 1\in\mathcal{D}^{1,2}(\mathbb{R}^d\backslash T)$.

The following result provides asymptotics for the tensors $\mathcal{X}^k$ and their averages $\mathcal{X}^{k*}$. It extends Theorem 3.1 of [4] (see also [37]) where the special case $k=0$ was obtained for the Stokes system.

**Proposition 14.** *Assume $d\geq 3$. For any $k\geq 0$, denote by $\widetilde{\mathcal{X}}^{2k}$ and $\widetilde{\mathcal{X}}^{2k+1}$ the rescaled tensors in $\eta^{-1}P\backslash T$ defined by:*

$$\forall x\in\eta^{-1}P\backslash T,\ \widetilde{\mathcal{X}}^{2k}(x) := \eta^{(d-2)(k+1)}\mathcal{X}^{2k}(\eta x) \text{ and } \widetilde{\mathcal{X}}^{2k+1}(x) := \eta^{(d-2)(k+1)}\mathcal{X}^{2k+1}(\eta x).$$

*Then:*

*(1) there exists a constant $C>0$ independent of $\eta>0$ such that:*

$$\forall\eta>0,\ ||\nabla\widetilde{\mathcal{X}}^{2k}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)}\leq C \text{ and } ||\nabla\widetilde{\mathcal{X}}^{2k+1}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)}\leq C; \tag{5.11}$$

*(2) the following convergences hold as $\eta\to 0$:*

$$\widetilde{\mathcal{X}}^{2k} \rightharpoonup \frac{\Psi}{F^{k+1}}J^{2k},\ \text{weakly in } H^1_{loc}(\mathbb{R}^d\backslash T) \tag{5.12}$$

$$\widetilde{\mathcal{X}}^{2k+1} \rightharpoonup 0 \text{ weakly in } H^1_{loc}(\mathbb{R}^d\backslash T) \tag{5.13}$$

$$\mathcal{X}^{2k*} \sim \frac{1}{\eta^{(d-2)(k+1)}F^{k+1}}J^{2k}, \tag{5.14}$$

*where we recall $J^{2k} := \overbrace{I\otimes I\otimes\cdots\otimes I}^{k \text{ times}}$ (definition (2.4)).*

**Remark 6.** Let us recall that we already know $\mathcal{X}^{2k+1*}=0$ for any $k\in\mathbb{N}$ (proposition 3).

*Proof.* The result is proved by induction.

(1) *Case $2k$ with $k=0$.* The tensor $\widetilde{\mathcal{X}}^0$ satisfies

$$-\Delta\widetilde{\mathcal{X}}^0=\eta^d \text{ in } \eta^{-1}P\backslash T \tag{5.15}$$

as well as the other boundary conditions of (5.6), hence lemma 7 yields

$$||\nabla\widetilde{\mathcal{X}}^0||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)}\le C\eta^{-d}\eta^d|\langle 1\rangle|\le C.$$

From (5.3), the average $\langle\widetilde{\mathcal{X}}^0\rangle$ is bounded. Therefore, there exists a constant $c^0\in\mathbb{R}$ and a function $\widehat{\Psi}^0$ such that, up to extracting a subsequence,

$$\langle\widetilde{\mathcal{X}}^0\rangle\to c^0 \text{ and } \widetilde{\mathcal{X}}^0\rightharpoonup\widehat{\Psi}^0 \text{ weakly in } H^1_{loc}(\mathbb{R}^d\backslash T) \text{ when } \eta\to 0.$$

Furthermore, the lower semi-continuity of the $H^1_{loc}(\mathbb{R}^d\backslash T)$ norm and (5.5) imply that $\widehat{\Psi}^0-c^0$ belongs to $\mathcal{D}^{1,2}(\mathbb{R}^d\backslash T)$ (see [4] for a detailed justification). Multiplying (5.15) by a compactly supported test function $\Phi\in\mathcal{C}^\infty_c(\mathbb{R}^d\backslash\overline{T})$ and integrating by part yields

$$\int_{\eta^{-1}P\backslash T}\nabla\widetilde{\mathcal{X}}^0\cdot\nabla\Phi\mathrm{d}y=\eta^d\int_{\eta^{-1}P\backslash T}\Phi\mathrm{d}y. \tag{5.16}$$

Passing to the limit when $\eta\to 0$ entails that $\widehat{\Psi}^0$ is solution to the exterior problem

$$\begin{cases}-\Delta\widehat{\Psi}^0=0 \text{ in } \mathbb{R}^d\backslash T\\ \widehat{\Psi}^0=0 \text{ on } \partial T\\ \widehat{\Psi}^0\to c^0 \text{ at } \infty.\end{cases} \tag{5.17}$$

By linearity, the unique solution to this problem is given by $\widehat{\Psi}=c^0\Psi$ where $\Psi$ is the solution to (5.9). Finally, the constant $c^0$ can be identified by integrating (5.15) against the constant test function $\Phi=1$, which implies

$$c^0F=-\int_{\partial T}\nabla(c^0\Psi^0)\cdot\boldsymbol{n}\mathrm{d}s=\lim_{\eta\to 0}-\int_{\partial T}\nabla\widetilde{\mathcal{X}}^0\cdot\boldsymbol{n}\mathrm{d}s=1.$$

Therefore, $c^0=1/F$ from where (5.12) follows for $k=0$. Since the obtained limit is unique, the convergence holds for the whole sequence. Then (5.14) follows from a simple change of variable.

(2) *Case $2k+1$ with $k=0$.* A simple computation yields

$$-\Delta\widetilde{\mathcal{X}}^1=2\eta\partial_j\widetilde{\mathcal{X}}^0\otimes e_j. \tag{5.18}$$

Applying lemma 7 and remarking that $\langle 2\eta\partial_j\widetilde{\mathcal{X}}^0\rangle=0$, we obtain

$$||\nabla\widetilde{\mathcal{X}}^1||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)}\le C\eta^{-1}\eta||\nabla\widetilde{\mathcal{X}}^0||\le C'.$$

Integrating (5.18) against a compactly supported test function $\Phi\in\mathcal{C}^\infty_c(\mathbb{R}^d\backslash T)$ and passing to the limit as $\eta\to 0$, we obtain with similar arguments the existence of a constant tensor $c^1$ (of order 1) such that, up to the extraction of a subsequence,

$$\widetilde{\mathcal{X}}^1\rightharpoonup c^1\Psi \text{ weakly in } H^1_{loc}(\mathbb{R}^d\backslash T) \text{ and } \langle\widetilde{\mathcal{X}}^1\rangle\to c^1.$$

From proposition 3, it holds that $\langle\widetilde{\mathcal{X}}^1\rangle=0$ which implies $c^1=0$, hence (5.13) for $k=0$.

(3) *General case.* We now complete the proof by induction on $k$. Assuming the result holds till rank $k\ge 0$, we compute

$$-\Delta\widetilde{\mathcal{X}}^{2k+2}=2\eta^{d-1}\partial_j\widetilde{\mathcal{X}}^{2k+1}\otimes e_j+\eta^d\widetilde{\mathcal{X}}^{2k}\otimes I,$$
$$-\Delta\widetilde{\mathcal{X}}^{2k+3}=2\eta\partial_j\widetilde{\mathcal{X}}^{2k+2}\otimes e_j+\eta^d\widetilde{\mathcal{X}}^{2k+1}\otimes I.$$

Applying lemmas 6 and 7 and (5.11) at rank $k$, we obtain

$$\begin{aligned}&||\nabla\widetilde{\mathcal{X}}^{2k+2}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)}\\ &\quad\le C(\eta^{d-2}||\nabla\widetilde{\mathcal{X}}^{2k+1}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)}+\eta^{d-1}||\widetilde{\mathcal{X}}^{2k}-\langle\widetilde{\mathcal{X}}^{2k}\rangle||_{L^2(\eta^{-1}P\backslash T)}+||\langle\widetilde{\mathcal{X}}^{2k}\rangle||_{L^2(\eta^{-1}P\backslash T)})\\ &\quad\le C(\eta^{d-2}||\nabla\widetilde{\mathcal{X}}^{2k+1}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)}+\eta^{d-2}||\nabla\widetilde{\mathcal{X}}^{2k}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)}+||\nabla\widetilde{\mathcal{X}}^{2k}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)})\le C',\end{aligned} \tag{5.19}$$

$$
\begin{aligned}
&||\nabla\widetilde{\mathcal{X}}^{2k+3}||^2_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)}\\
&\quad\leq C(||\nabla\widetilde{\mathcal{X}}^{2k+2}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)}+\eta^{d-1}||\widetilde{\mathcal{X}}^{2k+1}-\langle\widetilde{\mathcal{X}}^{2k+1}\rangle||_{L^2(\eta^{-1}P\backslash T)}+|\langle\widetilde{\mathcal{X}}^{2k+1}\rangle|)\\
&\quad\leq C(||\nabla\widetilde{\mathcal{X}}^{2k+2}||_{L^2(\eta^{-1}P\backslash T,\mathbb{R}^d)}+\eta^{d-2}||\nabla\widetilde{\mathcal{X}}^{2k+1}||)\leq C'.
\end{aligned}
$$

This implies (5.11) at rank $k+1$. Using similar arguments as previously, we infer the existence of constant tensors $c^{2k+2}$ and $c^{2k+3}$ such that

$$
\begin{aligned}
\widetilde{\mathcal{X}}^{2k+2}&\rightharpoonup c^{2k+2}\Psi \text{ weakly in } H^1_{loc}(\mathbb{R}^d\backslash T) \text{ and } \langle\widetilde{\mathcal{X}}^{2k+2}\rangle\to c^{2k+2}\\
\widetilde{\mathcal{X}}^{2k+3}&\rightharpoonup c^{2k+3}\Psi \text{ weakly in } H^1_{loc}(\mathbb{R}^d\backslash T) \text{ and } \langle\widetilde{\mathcal{X}}^{2k+3}\rangle\to c^{2k+3}.
\end{aligned}
$$

Since we know from proposition 3 that $\langle\widetilde{\mathcal{X}}^{2k+3}\rangle=0$, we obtain $c^{2k+3}=0$ which yields (5.13) at rank $k+1$. Finally, we integrate (5.19) by parts against the test function $\Phi=1$ in order to identify $c^{2k+2}$:

$$
c^{2k+2}F=-\int_{\partial T}\nabla(c^{2k+2}\Psi)\cdot\boldsymbol{n}\mathrm{d}s=\lim_{\eta\to0}-\int_{\partial T}\nabla\widetilde{\mathcal{X}}^{2k+2}\cdot\boldsymbol{n}\mathrm{d}s=\lim_{\eta\to0}\langle\widetilde{\mathcal{X}}^{2k}\rangle\otimes I=c^{2k}\otimes I.
$$

This implies $c^{2k+2}=J^{2k+2}/F^{k+2}$ and $c^{2k+3}=0$, which concludes the proof.

□

**Remark 7.** The convergence (5.13) seems to indicate that we may have not found the optimal scaling for the odd order tensors $\mathcal{X}^{2k+1}$ since we are able to identify only a zero weak limit in (5.13). A more elaborate analysis could be carried on by different techniques, see e.g. the use of layer potentials in [37].

We are now able to identify the asymptotic behavior of the constant tensors $M^k$. Recall already know that $M^{2k+1}=0$ from corollary 1.

**Corollary 6.** *Assume $d\geq3$. The following convergences hold for the tensors $M^k$ as $\eta\to0$:*

$$
M^0\sim\eta^{d-2}F, \tag{5.20}
$$

$$
M^2\to-I, \tag{5.21}
$$

$$
\forall k>1,\ M^{2k}=o\left(\frac{1}{\eta^{(d-2)(k-1)}}\right). \tag{5.22}
$$

*Proof.* We replace the asymptotics of proposition 14 in the explicit formula (3.15) for the tensor $M^k$. (5.20) is a consequence of the definition $M^0=(\mathcal{X}^{0*})^{-1}$. (5.21) is obtained by writing

$$
M^2=-((\mathcal{X}^{0*})^{-1})^2\mathcal{X}^{2*}\sim-\frac{\eta^{2(d-2)}F^2}{\eta^{2(d-2)}F^2}I=-\mathrm{I}.
$$

Let us now prove (5.22). By eliminating terms of odd orders in (3.15), we may write for any $k\geq1$,

$$
\begin{aligned}
M^{2k}&=\sum_{p=1}^{2k}\frac{(-1)^p}{(\mathcal{X}^{0*})^{p+1}}\sum_{\substack{i_1+\dots+i_p=k\\1\leq i_1\dots i_p\leq k}}\mathcal{X}^{2i_1*}\otimes\dots\otimes\mathcal{X}^{2i_p*}\\
&=\sum_{p=1}^{2k}(-1)^p\eta^{(p+1)(d-2)}F^{p+1}\sum_{\substack{i_1+\dots+i_p=k\\1\leq i_1\dots i_p\leq k}}\frac{J^{2i_1}}{\eta^{(d-2)(i_1+1)}F^{i_1+1}}\otimes\dots\otimes\frac{J^{2i_p}}{\eta^{(d-2)(i_p+1)}F^{i_p+1}}+o\left(\frac{1}{\eta^{(k-1)(d-2)}}\right)\\
&=\frac{J^{2k}}{\eta^{(k-1)(d-2)}F^{k-1}}\left(\sum_{p=1}^{2k}(-1)^p\sum_{\substack{i_1+\dots+i_p=k\\1\leq i_1\dots i_p\leq k}}1\right)+o\left(\frac{1}{\eta^{(k-1)(d-2)}}\right).
\end{aligned}
$$

Then (5.22) results from the last summation being zero:

$$
\forall k>1,\sum_{p=1}^{k}(-1)^p\sum_{\substack{i_1+\dots+i_p=k\\1\leq i_1\dots i_p\leq k}}1=0. \tag{5.23}
$$

There are several ways to obtain the latter formula. A rather direct argument in the spirit of the proof of proposition 6 is to apply the identity (3.16) to the power series $1/(1-z) = \sum_{k\in\mathbb{N}} z^k$ which yields

$$1 - z = 1 + \sum_{k=1}^{+\infty} \left( \sum_{p=1}^{k} (-1)^p \sum_{\substack{i_1+\cdots+i_p=k \\ 1\le i_1 \dots i_p \le k}} 1 \right) z^k,$$

from where (5.23) follows by identifying the powers in $z^k$. $\square$

**Remark 8.** We retrieve formally in corollary 6 the different classical asymptotic regimes (1.3) to (1.5) for the perforated problem (1.1) (at least for $K = 0$), because the coefficients of (1.9) read, as $\eta \to 0$:

$$\begin{aligned} \varepsilon^{-2} M^0 &\sim \frac{\eta^{d-2}}{\varepsilon^2} F, \\ \varepsilon^0 M^2 &\to -I, \\ \varepsilon^{2k-2} M^{2k} &= o\left( \left( \frac{\varepsilon^2}{\eta^{d-2}} \right)^{k-1} \right) \text{ for } k \ge 1. \end{aligned}$$

These asymptotics bring into play the ratio $\varepsilon^2/\eta^{d-2}$ and so the critical scaling $\eta \sim \varepsilon^{2/(d-2)}$ corresponding to the "Brinkman" regime (1.3) (which implies $\varepsilon^{-2}M^0 \to F$ and $\varepsilon^{2k+2}M^{2k} = o(1)$). The Darcy regimes (1.4) and (1.5) correspond to the situation where $\eta^{d-2}/\varepsilon^2 \to +\infty$; in that case the zeroth order term $\varepsilon^{-2}M^0$ is dominant. The Stokes regime (1.2) is found for $K = 0$ and $\eta = o(\varepsilon^{2/(d-2)})$ (which implies $\varepsilon^{-2}M^0 \to 0$). Note that $\varepsilon^0 M^2 \to -I$ whatever the scaling $\eta \to 0$. Unfortunately, the inverse of the critical ratio $\varepsilon^2/\eta^{d-2}$ could blow up as $\eta$ vanishes at the rate $\eta = o(\varepsilon^{2/(d-2)})$: a more elaborate analysis shall be performed in future works to determine whether $\varepsilon^{2k-2}M^{2k}$ with $k > 1$ still converges to zero in this regime.

We conclude this paper with the statement that the regimes (1.4) and (1.5) are also captured in the low volume fraction limit $\eta \to 0$ by the homogenized equation (1.6) of finite order $2K + 2$. Because of proposition 13, it is enough to establish that $\mathbb{D}_K^{2K+2}$ satisfies the same asymptotics than $M^{2K+2}$ (hence all the coefficients $\mathbb{D}_K^{2k}$ for $0 \le 2k \le 2K + 2$ because of proposition 13). The proof of this result requires the following asymptotic bounds for the tensors $N^k$:

**Proposition 15.** *Assume $d \ge 3$. For any $k \ge 0$, denote $\widetilde{N}^{2k}$ and $\widetilde{N}^{2k+1}$ the rescaled tensors in $\eta^{-1}P\backslash T$ defined by:*

$$\forall y \in \eta^{-1}P\backslash T,\ \widetilde{N}^{2k}(y) := \eta^{(d-2)k} N^{2k}(\eta y) \text{ and } \widetilde{N}^{2k+1}(y) := \eta^{(d-2)k} N^{2k+1}(\eta y).$$

*Then there exists a constant $C$ independent of $\eta > 0$ such that*

$$\forall \eta > 0,\ ||\nabla \widetilde{N}^{2k}||_{L^2(\eta^{-1}P\backslash T)} \le C \text{ and } ||\nabla \widetilde{N}^{2k+1}||_{L^2(\eta^{-1}P\backslash T)} \le C. \tag{5.24}$$

*Moreover, the following convergences hold as $\eta \to 0$:*

$$\widetilde{N}^0 \rightharpoonup \Psi \text{ weakly in } H^1_{loc}(\mathbb{R}^d\backslash T) \tag{5.25}$$

$$\forall k \ge 1, \widetilde{N}^k \rightharpoonup 0 \text{ weakly in } H^1_{loc}(\mathbb{R}^d\backslash T). \tag{5.26}$$

*Proof.* Using the definition (3.19) of the tensors $N^k$ and eliminating odd order terms, the tensors $\widetilde{N}^{2k}$ and $\widetilde{N}^{2k+1}$ can be rewritten as:

$$\widetilde{N}^0 = \frac{M^0}{\eta^{d-2}} \widetilde{\mathcal{X}}^0,$$

$$\forall k \ge 1,\ \widetilde{N}^{2k} = \frac{M^0}{\eta^{d-2}} \widetilde{\mathcal{X}}^{2k} + M^2 \otimes \widetilde{\mathcal{X}}^{2k-2} + \sum_{p=2}^{k} \eta^{(d-2)(p-1)} M^{2p} \otimes \widetilde{\mathcal{X}}^{2(k-p)},$$

$$\forall k \ge 0,\ \widetilde{N}^{2k+1} = \frac{M^0}{\eta^{d-2}} \widetilde{\mathcal{X}}^{2k+1} + M^2 \otimes \widetilde{\mathcal{X}}^{2k-1} + \sum_{p=2}^{k} \eta^{(d-2)(p-1)} M^{2p} \otimes \widetilde{\mathcal{X}}^{2(k-p)+1}.$$

By using the results of proposition 14 and corollary 6, we obtain

$$\widetilde{N}^0 = F\widetilde{\mathcal{X}}^0 + o_{\mathcal{D}^{1,2}(\mathbb{R}^d\setminus T)}(1)$$
$$\forall k \geq 1,\ \widetilde{N}^{2k} = F\widetilde{\mathcal{X}}^{2k} - \widetilde{\mathcal{X}}^{2k-2} \otimes I + o_{\mathcal{D}^{1,2}(\mathbb{R}^d\setminus T)}(1)$$
$$\forall k \geq 1,\ \widetilde{N}^{2k+1} = F\widetilde{\mathcal{X}}^{2k+1} - \widetilde{\mathcal{X}}^{2k-1} \otimes I + o_{\mathcal{D}^{1,2}(\mathbb{R}^d\setminus T)}(1),$$

where $o_{\mathcal{D}^{1,2}(\mathbb{R}^d\setminus T)} \to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^d\setminus T)$. This implies (5.24) to (5.26) by using once again proposition 14. □

**Corollary 7.** *It holds, as $\eta \to 0$:*

$$\mathbb{D}_0^2 \to -I \tag{5.27}$$
$$\mathbb{D}_K^{2K+2} = o\left(\frac{1}{\eta^{(d-2)K}}\right) = o\left(\frac{1}{\eta^{(d-2)((K+1)-1)}}\right) \text{ for any } K \geq 1. \tag{5.28}$$

*Proof.* According to (4.29), $\mathbb{D}_0^2 = -\int_Y N^0 \otimes N^0 \otimes I\mathrm{d}y$. Then, proposition 14 and corollary 6 imply

$$-\int_Y N^0 \otimes N^0 \otimes I\mathrm{d}y = -(M^0)^2 \int_Y |\mathcal{X}^0|^2 \mathrm{d}yI = -(M^0)^2 \eta^{-2(d-2)} \eta^d \int_Y |\widetilde{\mathcal{X}}^0|^2 \mathrm{d}yI \sim -F^2 \langle \widetilde{\mathcal{X}}^0 \rangle^2 I \to -I.$$

Finally, for $K \geq 1$ we estimate $\mathbb{D}_K^{2K+2}$ as follows:

$$\begin{aligned}\mathbb{D}_K^{2K+2} &= (-1)^K \eta^d \eta^{-(d-2)2\lfloor K/2 \rfloor} \int_{\eta^{-1}P\setminus T} (\widetilde{N}^K - \langle \widetilde{N}^K \rangle) \otimes (\widetilde{N}^K - \langle \widetilde{N}^K \rangle)\mathrm{d}y \\ &= O(\eta^{-(d-2)(2\lfloor K/2 \rfloor - 1)}) = o(\eta^{-(d-2)K}).\end{aligned}$$

□

**Acknowledgements.** This work was supported by the Association Nationale de la Recherche et de la Technologie (ANRT) [grant number CIFRE 2017/0024] and by the project ANR-18-CE40-0013 SHAPO financed by the French Agence Nationale de la Recherche (ANR). The author is grateful towards G. Allaire, C. Dapogny and S. Fliss for insightful discussions and their helpful comments and revisions.

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