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A class of short-term models for the oil industry addressing speculative storage

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This is a work in progress.

The aim is to propose a plausible mechanism for the short term dynamics of the oil market based on the interaction of economic agents. This is a theoretical research which by no means aim at describing all the aspects of the oil market. In particular, we use the tools and terminology of game theory, but we do not claim that this game actually exists in the real world.

In parallel, we are currently studying and calibrating a long term model for the oil industry, which addresses the interactions of a monopolists with a competitive fringe of small producers. It is the object of another paper that will be available soon.

The present preliminary version does not contain all the economic arguments and all the connections with our long term model. It mostly addresses the description of the model, the equations and numerical simulations focused on the oil industry short term dynamics.

A more complete version will be available soon.

1 Introduction

To be completed

2 The models and the systems of partial differential equations

We consider a monopolist producing a natural resource and facing both a competitive fringe of small producers and the business of speculative storage. Even though our motivation is to understand some aspects of the oil market at short or middle term (of the order of a year) and we may sometimes use a terminology linked to the oil industry (for example, oil for the resource, OPEC for the monopolist), the models proposed below may be applied in many other situations.

There are four types of agents: the consumers, a large producer or monopolist (OPEC), minor producers forming a competitive fringe, and the arbitragists who buy, store and then sell a part of the resource. The arbitragists will most often determine the price.

For simplicity, we are not going to address the decision making process of the competitive fringe of small producers; we rather assume that the dynamics of their global production rate is given.

Hence, the class of models described below involves two state variables:

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1. the level of speculative storage, that will be named k
2. the global production rate of the competitive fringe, that will be named z .

While k , i.e. the level of speculative storage, takes its value in a given interval, say $[k_{\min}, k_{\max}]$, the second state variable, i.e. z , may be either discrete or continuous, depending on the considered variant of the model.

The physical constraints on the storage capacity will play a key role. Indeed, it will be shown that in some situations and when the storage level is either minimal or maximal, the monopolist directly controls the price of the resource.

Mathematically, all the variants of the model lead to systems of partial differential equations coupling a Hamilton-Jacobi-Bellman equation for the monopolist and an equation of the type “master equation” for the price of the resource, see [4, 2, 1]. Note that in the present case, the master equation does not model a crowd of players as in mean field games, but rather an equilibrium reached by a crowd of arbitragists. To the best of our knowledge, the boundary conditions arising from the state constraints are completely new.

Our model has several variants:

1. In the first variant, the global production rate z_t of the competitive fringe takes its values in the interval $[z_{\min}, z_{\max}]$
2. In the second variant, z_t can take a finite number of values z_j , $j = 0, \dots, J - 1$, for a positive integer J (we will only discuss the cases when $J = 2$ and $J = 1$).

2.1 A model with two continuous state variables

2.1.1 The dynamics of k_t and z_t

The global production rate z_t of the competitive fringe is assumed to follow the dynamics

$$dz_t = b(k_t, z_t, p_t)dt,$$

where p_t stands for the unitary price of the resource and $b : [k_{\min}, k_{\max}] \times \mathbb{R}_+ \times \mathbb{R}_+$ is a given smooth function.

In the case of the oil industry, the choice of the function b comes from a long term model which is independent from the present one and currently investigated and calibrated by the same authors, and on which a paper will be available soon.

We know that the strategy of the monopolist consists of returning to a target market share which is of the order of 42%. Since we have chosen as a state variable z_t , the market share of the competitive fringe, the strategy of the monopolist consists of having z_t return to 58%. The monopolist controls the return of z_t to its target value by playing on the prices. More precisely, our long term model (with no storage) takes b as follows:

$$b(z, p) = -az + \lambda p - \mu, \tag{1}$$

with $\lambda = 0.4$, $\mu = 10$ et $a = 10\%$.

In the present model, since the aim is to study the interaction of the monopolist with the storage business, for which the time scale is of the order of a few semesters, we may neglect the variations of z in the definition of b because they are small on this time scale. Hence, we replace the term az by az_0 , where z_0 is the target value of the market share of the competitive fringe ($z_0 = 58\%$).

Besides, we are going to introduce a correction that will play the role of a proxy for the time delays

between the investment decisions of the producers belonging to the competitive fringe and the actual creation of new capacities of production. Indeed, (1) comes from the equation $b(z, p) = -az + c\sqrt{I}$, where I stands for the instantaneous investments that are conditioned by constraints on the credit; the latter are themselves connected to the price p . Hence I is a function of p . However, as already mentioned, there must be a delay between investments and the creation of capacity. It is therefore a strong simplification to assume that the investments have an *instantaneous* effect. This simplification permits to keep the theoretical complexity at a reasonable level and proves acceptable in a very long term model. The situation is much different when one deals with short terms: it seems necessary to model inertia effects, memory effects and anticipations of delays between investments and the creation of capacity.

Nevertheless, since we wish to keep the model as simple as possible, we limit ourselves to a proxy when addressing the above mentioned delay effects. Note that an accurate model is possible, but it would increase the difficulty of the numerical simulations significantly, in particular because it would increase the dimension of the problem.

The correction consists of modifying b by adding a function $f(k, z)$ which stands for a little increase (respectively decrease) in the production capacities when the storage facilities are close to empty (respectively full). Indeed, close to empty storage facilities must follow a period when the price is high, thus the investments of the producers in the competitive fringe are at a high level; the latter result in an increase of production capacity, i.e. an increase of z , even if the instantaneous price has decreased. The mechanism has to be reversed when the storage facilities are close to full. This is why we add to b the function $f(k, z)$ that is of the order of 2% for $k \sim k_{\min}$ and -2% for $k \sim k_{\max}$.

We shall suppose that there exist z_{\min}, z_{\max} , $0 \leq z_{\min} < z_{\max}$, such that for all $t \geq 0$,

- the probability that $b(k_t, z_t, p_t) \leq 0$ conditionally on $z_t = z_{\max}$ is one.
- the probability that $b(k_t, z_t, p_t) \geq 0$ conditionally on $z_t = z_{\min}$ is one.

Therefore, we may suppose that z_t takes its values in the compact interval $[z_{\min}, z_{\max}]$.

The demand of the consumers is a decreasing function of the price of the resource; after a suitable choice of units, the simplest demand function is

$$D(p) = 1 - \epsilon p,$$

where the parameter ϵ stands for the elasticity of demand. Note that it would be more appropriate to set $D(p) = \max(0, 1 - \epsilon p)$, but in the regime that will be considered, the price p will never exceed $1/\epsilon$.

The control variable of the monopoly is its production rate q_t . Matching demand and supply yields $dk_t = (q_t + z_t - D(p_t))dt$. However, we rather consider a slightly more general dynamics of k_t , possibly including some noise in the storage capacities:

$$dk_t = (q_t + z_t - D(p_t))dt + \sigma(k_t)dW_t, \tag{2}$$

where (W_t) is a Brownian motion. We suppose that the volatility $k \mapsto \sigma(k)$ is a smooth nonnegative function that vanishes at $k = k_{\min}$ and $k = k_{\max}$ and that the quantities $\frac{\sigma(k)}{k - k_{\min}}$ and $\frac{\sigma(k)}{k_{\max} - k}$ are bounded. This assumption will play an important role in the discussion of the boundary conditions.

2.1.2 The equilibrium

We look for a stationary equilibrium. Given the unitary price of the resource, the monopolist solves an optimal control problem. Let $(k, z) \mapsto U(k, z)$ be the associated value function. The price, described by a function $p(k, z)$, is fixed by ruling out opportunities of arbitrage. We will see that the

functions U and p satisfy a system of two coupled partial differential equations.

The optimal control problem solved by the monopolist knowing the trajectory of p_t is:

$$U(k, z) = \sup_{q_t} \mathbb{E} \left(\int_0^\infty e^{-rt} \left((p_t - c)q_t - \alpha \frac{|q_t - q_0|^2}{2} \right) dt \middle| (k_0, z_0) = (k, z) \right),$$

where r is a positive discount factor, c is the cost related to the production of a unit of resource, and $\alpha \frac{|q_t - q_0|^2}{2}$ is a penalty term used to avoid imposing bounds on q_t . The dynamic programming principle yields that the value function is a solution of the following Hamilton-Jacobi-Bellman equation:

$$-rU + \sup_{q \geq 0} \left(-\alpha \frac{|q - q_0|^2}{2} + (p - c)q + (q + z - D(p)) \partial_k U \right) + b(k, z, p) \partial_z U + \frac{\sigma^2(k)}{2} \partial_{kk} U = 0. \quad (3)$$

Introducing the Hamiltonian

$$H(z, p, \xi) = \sup_{q \geq 0} \left(-\alpha \frac{|q - q_0|^2}{2} + (p - c)q + \xi (q + z - D(p)) \right)$$

in which the maximum is reached by $q^* = \max(0, q_0 + \frac{1}{\alpha}(p - c + \xi))$, (3) can be written:

$$-rU + H(z, p, \partial_k U) + b(k, z, p) \partial_z U + \frac{\sigma^2(k)}{2} \partial_{kk} U = 0.$$

Since, in the regime that will be considered, $q_0 + \frac{1}{\alpha}(p - c + \partial_k U)$ will always be nonnegative, we omit for simplicity the constraint $q \geq 0$ in the definition of the Hamiltonian: hereafter, we set

$$\begin{aligned} H(z, p, \xi) &= \sup_{q \in \mathbb{R}} \left(-\alpha \frac{|q - q_0|^2}{2} + (p - c)q + \xi (q + z - D(p)) \right) \\ &= \frac{1}{2\alpha} (p - c + \xi)^2 + \xi (z - D(p)) + q_0 (p - c - \xi), \end{aligned} \quad (4)$$

and the optimal production rate at $k_t = k$ and $z_t = z$ is given by the feedback law

$$\begin{aligned} q^*(k, z) &= D_\xi H(z, p(k, z), \partial_k U(z, p)) - z + D(p(k, z)) \\ &= q_0 + \frac{1}{\alpha} (p(k, z) - c + \partial_k U(k, z)). \end{aligned} \quad (5)$$

Let us now turn to the price of a unit of resource : ruling out opportunities of arbitrage implies that the price process obeys the following relation,

$$p_t = \mathbb{E} \left(e^{-r\delta t} p_{t+\delta t} - \int_t^{t+\delta t} g(k_s) ds \middle| (k_t, z_t) \right),$$

where $g(k)$ is the cost of storing a unit of resource per unit of time when the level of storage is k . Recalling that $p_t = p(k_t, z_t)$, Ito formula yields:

$$-rp + D_\xi H(z, p, \partial_k U) \partial_k p + b(k, z, p) \partial_z p + \frac{\sigma^2(k)}{2} \partial_{kk} p - g(k) = 0.$$

To summarize, the system of PDEs satisfied by U, p is

$$0 = -rU + H(z, p, \partial_k U) + b(k, z, p) \partial_z U + \frac{\sigma^2(k)}{2} \partial_{kk} U, \quad (6)$$

$$0 = -rp + D_\xi H(z, p, \partial_k U) \partial_k p + b(k, z, p) \partial_z p + \frac{\sigma^2(k)}{2} \partial_{kk} p - g(k), \quad (7)$$

for $k_{\min} < k < k_{\max}$ and $z_{\min} < z < z_{\max}$ and with H given by (4). Note that equation (7) is nonlinear with respect to p . It is reminiscent of the master equations discussed in ? .

Note also that it seems possible to refine the present model by considering that the arbitragists running the speculative storage business are rational agents playing a mean field game. This would lead to a more involved model of a mean field game with a major agent, see [3]. Yet, the resulting system of partial differential equations would have the same structure as (6-7).

2.2 A variant in which the production of the fringe is a two-state Poisson process

We consider a situation in which the production rate z_t can take only two values $0 \leq z_0 < z_1$ and is described by a stochastic Poisson process with intensities that may depend on k_t and p_t :

$$\begin{cases} \mathbb{P}(z_{t+\Delta t} = z_0 | z_t = z_0) &= 1 - \lambda_0(k_t, p_t)\Delta t + o(\Delta t), \\ \mathbb{P}(z_{t+\Delta t} = z_1 | z_t = z_0) &= \lambda_0(k_t, p_t)\Delta t + o(\Delta t), \\ \mathbb{P}(z_{t+\Delta t} = z_1 | z_t = z_1) &= 1 - \lambda_1(k_t, p_t)\Delta t + o(\Delta t), \\ \mathbb{P}(z_{t+\Delta t} = z_0 | z_t = z_1) &= \lambda_1(k_t, p_t)\Delta t + o(\Delta t). \end{cases} \quad (8)$$

All the other features of the model are the same as in paragraph 2.1, in particular, the dynamics of k_t is still given by (2). The optimal value of the monopolist and the price are described by $U(k, z_j) = U_j(k)$ and $p(k, z_j) = p_j(k)$, where for $j = 0, 1$, the real values functions U_j, p_j are defined on $[k_{\min}, k_{\max}]$ and satisfy a system of four coupled differential equations.

Introducing the Hamiltonians

$$H_j(p, \xi) = \frac{1}{2\alpha}(p - c + \xi)^2 + \xi(z_j - D(p)) + q_0(p - c - \xi),$$

(we still omit the constraint that the production rate is nonnegative), and repeating the arguments contained in paragraph 2.1.2, we get the following system of differential equations:

$$0 = -rU_j + H_j(p_j, U'_j) + \lambda_j(k, p_j)(U_\ell - U_j) + \frac{\sigma^2(k)}{2}U''_j, \quad (9)$$

$$0 = -rp_j + D_\xi H_j(p_j, U'_j)p'_j + \lambda_j(k, p_j)(p_\ell - p_j) - g(k) + \frac{\sigma^2(k)}{2}p''_j, \quad (10)$$

for $j = 0, 1$, $\ell = 1 - j$, and $k \in (k_{\min}, k_{\max})$. The optimal drift of k_t in (2) is then given by $D_\xi H_j(p_j(k_t), U'_j(k_t)) = \frac{1}{\alpha}(U'_j(k_t) - c + p_j(k_t)) + z_j - D(p_j(k_t)) + q_0$ if $z_t = z_j$.

2.3 An even simpler model

It is possible to simplify further the model by assuming that the production rate of the competitive fringe is a constant z . Introducing the Hamiltonian

$$H(p, \xi) = \frac{1}{2\alpha}(p - c + \xi)^2 + \xi(z - D(p)) + q_0(p - c - \xi),$$

and repeating the arguments contained in paragraph 2.1.2, we get the following system of two differential equations:

$$0 = -rU + H(p, U') + \frac{\sigma^2(k)}{2}U'', \quad (11)$$

$$0 = -rp + D_\xi H(p, U')p' - g(k) + \frac{\sigma^2(k)}{2}p'', \quad (12)$$

for $k \in (k_{\min}, k_{\max})$.

3 Boundary conditions

The systems of partial differential equations have to be supplemented with boundary conditions. We are going to discuss the latter in full details within the framework described in paragraph 2.1, and more briefly for the variants proposed in 2.2 and 2.3.

3.1 The boundary conditions associated with the model discussed in paragraph 2.1

3.1.1 Boundary conditions at $z = z_{\min}, z_{\max}$

No boundary conditions are needed at $z = z_{\min}$ and $z = z_{\max}$, from the assumptions made on the dynamics of z_t in § 2.1.

3.1.2 Boundary conditions at $k = k_{\min}$ and $k = k_{\max}$

Preliminary : monotone envelopes of $\xi \mapsto H(z, p, \xi)$. For describing the boundary conditions linked to the state constraints $k_{\min} \leq k_t \leq k_{\max}$, it is useful to introduce the nonincreasing and nondecreasing envelopes of the function $\xi \mapsto H(z, p, \xi)$: we set

$$H_{\downarrow}(z, p, \xi) = \max_{q \leq D(p) - z} \left(-\frac{\alpha}{2}(q - q_0)^2 + (p - c)q + \xi(q + z - D(p)) \right), \quad (13)$$

and

$$H_{\uparrow}(z, p, \xi) = \max_{q \geq D(p) - z} \left(-\frac{\alpha}{2}(q - q_0)^2 + (p - c)q + \xi(q + z - D(p)) \right). \quad (14)$$

The Hamiltonian $H_{\downarrow}(z, p, \xi)$ (resp. $H_{\uparrow}(z, p, \xi)$) corresponds to the controls q such that the drift of k_t in (2) is nonpositive (resp. nonnegative). It may also be convenient to set

$$H_{\min}(z, p) = \min_{\xi} H(z, p, \xi) = -\frac{\alpha}{2}(D(p) - z - q_0)^2 + (p - c)(D(p) - z) \quad (15)$$

which corresponds to the control $q = D(p) - z$ for which the drift of k_t in (2) vanishes. Note that $p \mapsto H_{\min}(z, p)$ is strongly concave with respect p . It is easy to check that

$$H(z, p, \xi) = H_{\downarrow}(z, p, \xi) + H_{\uparrow}(z, p, \xi) - H_{\min}(z, p).$$

The optimal values of q in the definition of $H_{\downarrow}(z, p, \xi)$ and $H_{\uparrow}(z, p, \xi)$ are

$$q_{\downarrow}^*(z, p, \xi) = \min \left(D(p) - z, q_0 + \frac{p - c + \xi}{\alpha} \right), \quad (16)$$

$$q_{\uparrow}^*(z, p, \xi) = \max \left(D(p) - z, q_0 + \frac{p - c + \xi}{\alpha} \right). \quad (17)$$

Hence,

$$H_{\downarrow}(z, p, \xi) = \frac{1}{2} \left(\left(\sqrt{\alpha}(z - D(p) + q_0) + \frac{1}{\sqrt{\alpha}}(p - c + \xi) \right)_- \right)^2 + H_{\min}(z, p), \quad (18)$$

$$H_{\uparrow}(z, p, \xi) = \frac{1}{2} \left(\left(\sqrt{\alpha}(z - D(p) + q_0) + \frac{1}{\sqrt{\alpha}}(p - c + \xi) \right)_+ \right)^2 + H_{\min}(z, p). \quad (19)$$

Assumption 3.1. Hereafter, we assume that for all $k \in [k_{\min}, k_{\max}]$, $z \in [z_{\min}, z_{\max}]$, $\xi \in \mathbb{R}$, the function $p \mapsto H_{\min}(z, p) + \xi b(k, z, p)$ is strongly concave.

Boundary conditions at $k = k_{\min}$. In view of the assumptions made on σ , it is not restrictive to focus on the deterministic case: we take $\sigma = 0$ for simplicity.

The state constraint $k_t \geq k_{\min}$ implies that $q^*(k_{\min}, z) + z - D(p(k_{\min}, z)) \geq 0$. Two situations may occur:

1. If $\frac{\partial_k U(k, z) - c + p(k, z)}{\alpha} + z - D(p) + q_0 > 0$ for k near k_{\min} , then the optimal strategy results in increasing the level of storage. This means that in (7), the drift $D_{\xi} H(z, p, \partial_k U)$ is positive for k near k_{\min} , and no boundary condition is needed for p .

2. On the contrary, if $\frac{\partial_k U(k, z) - c + p(k, z)}{\alpha} + z - D(p) + q_0 \leq 0$ for k near k_{\min} , then the optimal drift of k_t in (2) must vanish at $k = k_{\min}$, i.e. $q + z - D(p(k_{\min}, z)) = 0$. This relation and the strict monotonicity of D imply that p can be considered as the control variable at $k = k_{\min}$. In other words, the monopoly directly controls the price in this situation.

On the other hand, ruling out opportunities of arbitrage and taking into account the suboptimality due to the state constraints, we see that the price process obeys the following inequality: if $k_t = k_{\min}$, then

$$p_t \geq \mathbb{E} \left(e^{-r\delta t} p_{t+\delta t} - \int_t^{t+\delta t} g(k_s) ds \middle| (k_t, z_t) \right).$$

Since the optimal drift of k_t is 0, we obtain that

$$rp(k_{\min}, z) - b(k_{\min}, z, p(k_{\min}, z)) \partial_z p(k_{\min}, z) + g(k_{\min}) \geq 0. \quad (20)$$

Another way to understand (20) is as follows: we expect that, in the present case, p is nonincreasing with respect to k for k near $k = k_{\min}$. Indeed, if p was increasing with respect to k for k near $k = k_{\min}$, then the arbitragists would increase the level of storage, i.e. dk_t would be positive, in contradiction with the assumption. Then, plugging this information in (7) implies (20).

Turning back to the monopolist, we deduce from the considerations above that, among the strategies consisting of keeping k_t fixed at k_{\min} for $z_t = z$, the optimal one is

$$q^* = D(p^*) - z, \quad (21)$$

$$p^* = \operatorname{argmax}_{\pi: r\pi \geq b(k_{\min}, z, \pi) \partial_z p - g(k_{\min})} F(\pi, \partial_z U), \quad (22)$$

where

$$F(\pi, \partial_z U) = H_{\min}(z, \pi) + b(k_{\min}, z, \pi) \partial_z U, \quad (23)$$

and $H_{\min}(z, \pi)$ is defined in (15). Note that π^* is unique from Assumption 3.1 and depends on $z, \partial_z U, \partial_z p$. In this situation, the nonlinear boundary condition

$$p = p^*(z, \partial_z U, \partial_z p) \quad (24)$$

must be imposed at (k_{\min}, z) .

Summary. Setting $p(k_{\min, +}, z) = \lim_{k \rightarrow k_{\min}^+} p(k, z)$, another way of formulating the boundary conditions at $k = k_{\min}$ is:

- The nonlinear condition (24), i.e.

$$p = p^*(z, \partial_z U, \partial_z p), \quad (25)$$

understood in a weak sense, (i.e. it holds only if the optimal drift $\partial_k U(k, z) - c + p(k, z)$ is ≤ 0 near $k = k_{\min}$), and where $p^*(z, \partial_z U, \partial_z p)$ achieves the maximum in (28) below

- the equation for U can be written

$$-rU + \max(A, B) = 0, \quad (26)$$

with

$$A = H_{\uparrow}(z, p(k_{\min, +}, z), \partial_k U) + b(k_{\min}, z, p(k_{\min, +}, z)) \partial_z U, \quad (27)$$

$$B = \max_{\pi: r\pi \geq b(k_{\min}, z, \pi) \partial_z p - g(k_{\min})} F(\pi, \partial_z U), \quad (28)$$

and F is given by (23).

Note that, to the best of our knowledge, this set of boundary conditions, associated to the system (6-7) and to the state constraint $k \geq k_{\min}$, has never been proposed and a fortiori analyzed.

Boundary conditions at $k = k_{\max}$. Arguing as above and setting $p(k_{\max,-}, z) = \lim_{k \rightarrow k_{\max}^-} p(k, z)$, the boundary conditions at $k = k_{\max}$ can be written as follows:

- A nonlinear condition for p of the form

$$p = p^{**}(z, \partial_z U, \partial_z p), \quad (29)$$

understood in a weak sense (i.e. it holds only if the optimal drift $\partial_k U(k, z) - c + p(k, z)$ is ≥ 0 near $k = k_{\max}$), where $p^{**}(z, \partial_z U, \partial_z p)$ achieves the maximum in (32) below (it is unique from Assumption 3.1).

- An equation for U :

$$-rU + \max(C, D) = 0, \quad (30)$$

with

$$C = H_{\downarrow}(z, p(k_{\max,+}, z), \partial_k U) + b(k_{\max}, z, p) \partial_z U, \quad (31)$$

$$D = \max_{\pi: r\pi \leq b(k_{\max}, z, \pi) \partial_z p - g(k_{\max})} G(\pi, \partial_z U), \quad (32)$$

and

$$G(\pi, \partial_z U) = H_{\min}(z, \pi) + b(k_{\max}, z, \pi) \partial_z U. \quad (33)$$

3.2 The boundary conditions associated with the model discussed in paragraph 2.2

The boundary conditions associated with the system (9-10) are obtained in the same manner as in the previous case. To avoid repetitions, we focus on the boundary $k = k_{\min}$, because the needed modifications with respect to paragraph 3.1.2 are similar for $k = k_{\max}$ and $k = k_{\min}$. The interested reader will easily find the boundary conditions at $k = k_{\max}$ from paragraph 3.1.2 and what follows. As above, we set

$$H_{j,\min}(p) = \min_{\xi} H_j(p, \xi) = -\frac{\alpha}{2} (D(p) - z_j - q_0)^2 + (p - c)(D(p) - z_j), \quad (34)$$

$$\begin{aligned} H_{j,\downarrow}(p, \xi) &= \max_{q \leq D(p) - z_j} \left(-\frac{\alpha}{2} (q - q_0)^2 + (p - c)q + \xi(q + z_j - D(p)) \right) \\ &= \frac{1}{2} \left(\left(\sqrt{\alpha}(z_j - D(p) + q_0) + \frac{1}{\sqrt{\alpha}}(p - c + \xi) \right)_- \right)^2 + H_{j,\min}(p), \end{aligned} \quad (35)$$

$$\begin{aligned} H_{j,\uparrow}(p, \xi) &= \max_{q \geq 1 - z_j - \epsilon p} \left(-\frac{\alpha}{2} (q - q_0)^2 + (p - c)q + \xi(q + z_j - D(p)) \right) \\ &= \frac{1}{2} \left(\left(\sqrt{\alpha}(z_j - D(p) + q_0) + \frac{1}{\sqrt{\alpha}}(p - c + \xi) \right)_+ \right)^2 + H_{j,\min}(p). \end{aligned} \quad (36)$$

The optimal values of q in the definition of $H_{j,\downarrow}(p, \xi)$ and $H_{j,\uparrow}(p, \xi)$ are

$$q_{j,\downarrow}^*(p, \xi) = \min \left(D(p) - z_j, q_0 + \frac{p - c + \xi}{\alpha} \right), \quad (37)$$

$$q_{j,\uparrow}^*(p, \xi) = \max \left(D(p) - z_j, q_0 + \frac{p - c + \xi}{\alpha} \right). \quad (38)$$

Boundary conditions at $k = k_{\min}$. Setting $p_{\ell,+} = \lim_{k \rightarrow k_{\min}^+} p_{\ell}(k)$, $\ell = 0, 1$, the boundary conditions at $k = k_{\min}$ are as follows: for $i = 0, 1$ and $j = 1 - i$,

- a condition of the form

$$p_j = p_j^*(U_j, U_\ell, p_{\ell,+}), \quad \text{with } \ell = 1 - j, \quad (39)$$

understood in a weak sense, (i.e. it holds only if $U_j'(k) - c + p_j(k) \leq 0$ for k near k_{\min}), where $p_j^*(U_j, U_\ell, p_{\ell,+})$ achieves the maximum in (42) below (it is supposed to be unique).

- the equation for U_j can be written

$$-rU_j + \max(A, B) = 0, \quad (40)$$

with

$$A = H_{j,\uparrow}(p_{j,+}, U_j') + \lambda_j(k_{\min}, p_{j,+})(U_\ell - U_j), \quad (41)$$

$$B = \max_{p: (r + \lambda_j(k_{\min}, p))p - \lambda_j(k_{\min}, p)p_{\ell,+} + g(k_{\min}) \geq 0} F_j(p, U_j, U_\ell), \quad (42)$$

with $\ell = 1 - j$, and

$$F_j(p, U_j, U_\ell) = H_{j,\min}(p) + \lambda_j(k_{\min}, p)(U_\ell - U_j). \quad (43)$$

3.3 The boundary conditions associated with the model discussed in paragraph 2.3

Here also, we focus on $k = k_{\min}$ to avoid repetitions. Let us set

$$H_{\min}(p) = \min_{\xi} H(p, \xi) = -\frac{\alpha}{2} (D(p) - z - q_\circ)^2 + (p - c)(D(p) - z), \quad (44)$$

$$\begin{aligned} H_{\downarrow}(p, \xi) &= \max_{q \leq D(p) - z} \left(-\frac{\alpha}{2} (q - q_\circ)^2 + (p - c)q + \xi(q + z - D(p)) \right) \\ &= \frac{1}{2} \left(\left(\sqrt{\alpha}(z - D(p) + q_\circ) + \frac{1}{\sqrt{\alpha}}(p - c + \xi) \right) \right)_-^2 + H_{\min}(p), \end{aligned} \quad (45)$$

$$\begin{aligned} H_{\uparrow}(p, \xi) &= \max_{q \geq 1 - z - \epsilon p} \left(-\frac{\alpha}{2} (q - q_\circ)^2 + (p - c)q + \xi(q + z - D(p)) \right) \\ &= \frac{1}{2} \left(\left(\sqrt{\alpha}(z - D(p) + q_\circ) + \frac{1}{\sqrt{\alpha}}(p - c + \xi) \right) \right)_+^2 + H_{\min}(p). \end{aligned} \quad (46)$$

Boundary conditions at $k = k_{\min}$. Setting $p_+ = \lim_{k \rightarrow k_{\min}^+} p(k)$, the boundary conditions at $k = k_{\min}$ are as follows:

- a condition of the form

$$p = p^*, \quad (47)$$

understood in a weak sense, (i.e. it holds only if $U'(k) - c + p(k) \leq 0$ for k near k_{\min}), where p^* achieves the maximum in (50) below (p^* is unique).

- the equation for U can be written

$$-rU + \max(A, B) = 0, \quad (48)$$

with

$$A = H_{\uparrow}(p_+, U'), \quad (49)$$

$$B = \max_{p: rp + g(k_{\min}) \geq 0} H_{\min}(p). \quad (50)$$

4 Mathematical analysis of the boundary conditions in the one dimensional model

In what follows, we explain why the boundary conditions discussed in paragraph 3.3 permit to determine in a unique way the behavior of a solution of (11)-(12) near the boundary. Although the argument proposed below is rather formal, we think that it gives useful information on the solutions. More precisely, we are going to see that the boundary conditions induce a unique expansion of the function p and of the derivative of the value function $V = \partial_k U$ near the boundary. The system of PDEs satisfied by p and V is as follows:

$$0 = -rV + \left(\sigma(k)\sigma'(k) + D_\xi H(p, V) \right) V' + D_p H(p, U') p' + \frac{\sigma^2(k)}{2} V'', \quad (51)$$

$$0 = -rp + D_\xi H(p, V) p' - g(k) + \frac{\sigma^2(k)}{2} p'', \quad (52)$$

for $k \in (k_{\min}, k_{\max})$. We focus on the boundary conditions at $k = k_{\min}$.

First, in the case in which the drift $D_\xi H(p, V)$ is positive near k_{\min} , p and V are expected to be smooth at the boundary.

Hence, we focus on the case in which the drift $D_\xi H(p, V)$ points toward the boundary (i.e. $D_\xi H(p, V) \leq 0$). We make the following ansatz:

$$V(k) = V(k_{\min}) + \gamma(k - k_{\min})^n + o((k - k_{\min})^n), \quad (53)$$

$$p(k) = p(k_{\min}) - \beta(k - k_{\min})^m + o((k - k_{\min})^m), \quad (54)$$

with $n, m \leq 1$. For shortening the notation, let us define the pair $(V_0, p_0) := (V(k_{\min}), p(k_{\min}))$.

4.1 A singularity is expected

Let us explain why a singular behavior should be expected near the boundary $k = k_{\min}$. Indeed, assume that this is not the case and that $m = n = 1$; in this situation, from the assumption made on the sign of the drift near the boundary and the constraint $k_t \geq k_{\min}$, we deduce that

$$0 = D_\xi H(p_0, V_0) = \left(\frac{1}{\alpha} + \epsilon \right) p_0 + \frac{1}{\alpha} V_0 - \frac{c}{\alpha} + z - 1 - q_0. \quad (55)$$

Then, plugging the ansatz for V and p into (51)-(52) and focusing on the zeroth order terms, we obtain that

$$\begin{cases} rV_0 = \beta \left(\frac{1}{\alpha} (p_0 - c + V_0) + \epsilon V_0 + q_0 \right), \\ rp_0 = -g(k_{\min}). \end{cases} \quad (56)$$

The equations in (56) and (55) form a linear system which is overdetermined except for a single value of β . Thus, the values of V_0 , p_0 and β are determined. Passing to the first order terms in the expansion of the system, we obtain two second order polynomial equations in γ and β , while β is already known. It is then easy to observe that for a generic choice of the parameters, this system of second order equations is not consistent with the already obtained values of V_0 , p_0 and β .

Remark 4.1. *Recall that in the case in which the drift is positive near k_{\min} , no singularity is expected.*

4.2 Characterization of the singularity

Proposition 4.2. *If V and p satisfy (53) and (54), then $n = m = 1/2$ and the pair (V_0, p_0) is completely determined by the values of z, ϵ, α, q_0 . Moreover, if $(\alpha\epsilon)^2 + \alpha\epsilon > 1$, then there is at most one pair (γ, β) such that (53) and (54) hold.*

Remark 4.3. *The latter condition on $\alpha\epsilon$ will be fulfilled in the numerical simulations in § 6 below.*

Remark 4.4. *The value of p_0 is obviously p^* which has been defined in paragraph 3.3.*

Proof. Plugging the ansatz into (51)-(52), and using both the boundedness of g and the fact that σ vanishes near k_{\min} , we deduce that

$$(1 + \alpha\epsilon)V_0 + p_0 = c - \alpha q_0 \quad (57)$$

by identifying the higher order terms in the expansion. From the state constraint and the sign assumption on the drift, the following also holds:

$$(1 + \alpha\epsilon)p_0 + V_0 = c + \alpha(1 - z) - \alpha q_0. \quad (58)$$

Since $\alpha\epsilon \notin \{-2; 0\}$, we deduce that

$$\begin{cases} V_0 = \frac{z - 1 + \epsilon(c - \alpha q_0)}{\epsilon(2 + \alpha\epsilon)}, \\ p_0 = \frac{\epsilon(c - \alpha q_0) + (1 + \alpha\epsilon)(1 - z)}{\epsilon(2 + \alpha\epsilon)}. \end{cases} \quad (59)$$

Identifying the higher order terms in the expansion, we see that if $n \neq m$, then $\beta = \gamma = 0$. Therefore, $n = m$. Now, if $2n - 1 \notin \{0; 1\}$, identifying the terms of order $2n - 1$ leads to

$$\begin{cases} -(1 + \alpha\epsilon)\beta + \gamma = 0, \\ (1 + \alpha\epsilon)\gamma - \beta = 0. \end{cases} \quad (60)$$

The latter system yields that $\gamma = \beta = 0$. Thus $n \in \{1/2; 1\}$. The only possible value of n is $1/2$ since the case $n = 1$ has already been ruled out.

Considering the zeroth order terms, we conclude that

$$\begin{cases} 0 = -rV_0 + \frac{1}{2\alpha}(\gamma - \beta)^2 - \epsilon\gamma\beta, \\ 0 = -rp_0 - \frac{\beta}{2\alpha}(\gamma - \beta) + \frac{\epsilon}{2}\beta^2 - g(k_{\min}). \end{cases} \quad (61)$$

Let us introduce the parameter

$$\lambda = \frac{rV_0}{rp_0 + g(k_{\min})}, \quad (62)$$

which is well defined since $g \geq 0, p_0 > 0$. Observe that $0 \geq \lambda \geq -1$. We deduce that

$$\frac{1}{2\alpha}(\gamma - \beta)^2 - \epsilon\gamma\beta = -\lambda\frac{\beta}{2\alpha}(\gamma - \beta) + \lambda\frac{\epsilon}{2}\beta^2, \quad (63)$$

then that

$$\gamma^2 + (1 - \lambda(1 + \alpha\epsilon))\beta^2 - (1 + 2\alpha\epsilon - \lambda)\gamma\beta = 0. \quad (64)$$

Defining the numbers x_{\pm} by

$$x_{\pm} = \frac{1 + 2\alpha\epsilon - \lambda \pm \sqrt{(1 + 2\alpha\epsilon - \lambda)^2 - 4(1 - \lambda(1 + \alpha\epsilon))}}{2}, \quad (65)$$

we finally obtain that

$$(\gamma - x_+\beta)(\gamma - x_-\beta) = 0. \quad (66)$$

Rewriting the second equation in (61), we obtain

$$2\alpha g(k_{\min}) + 2\alpha rp_0 = \beta((\alpha\epsilon + 1)\beta - \gamma). \quad (67)$$

Thus, $\gamma = x_+\beta$ is impossible if $x_+ > 1 + \alpha\epsilon$. An easy calculation leads to the fact that this last condition is satisfied (i.e. x_+ is large enough) if

$$(\alpha\epsilon)^2 + \alpha\epsilon - 1 > 0. \quad (68)$$

□

5 Approximation by a finite difference method

Let us focus on the case when

$$\begin{aligned} b(k, z, p) \frac{\partial p}{\partial z} &= (\phi(k) + \kappa(\lambda p - \mu)) \frac{\partial p}{\partial z} \\ &= \frac{\partial}{\partial z} \left(\phi(k)p + \frac{\kappa}{2\lambda} (\lambda p - \mu)^2 \right). \end{aligned} \quad (69)$$

We are going to use the latter conservative form in the numerical scheme for (7). Note that

$$\phi(k)p + \frac{\kappa}{2\lambda} (\lambda p - \mu)^2 = \frac{\kappa\lambda}{2} \left(p - \frac{\mu}{\lambda} + \frac{\phi(k)}{\kappa\lambda} \right)^2 - \frac{\phi^2(k)}{2\kappa\lambda} + \frac{\mu}{\lambda} \phi(k)$$

It is useful to introduce the following numerical flux function:

$$\Psi(k, p_\ell, p_r) = \begin{cases} \max_{p_\ell \leq p \leq p_r} \left(\phi(k)p + \frac{\kappa}{2\lambda} (\lambda p - \mu)^2 \right), & \text{if } p_\ell \leq p_r, \\ \min_{p_r \leq p \leq p_\ell} \left(\phi(k)p + \frac{\kappa}{2\lambda} (\lambda p - \mu)^2 \right), & \text{if } p_\ell \geq p_r, \end{cases} \quad (70)$$

and straightforward calculus leads to

$$\Psi(k, p_\ell, p_r) = -\frac{\phi^2(k)}{2\kappa\lambda} + \frac{\mu}{\lambda} \phi(k) + \frac{\kappa\lambda}{2} \max \left(\left(p_r - \frac{\mu}{\lambda} + \frac{\phi(k)}{\kappa\lambda} \right)_+^2, \left(p_\ell - \frac{\mu}{\lambda} + \frac{\phi(k)}{\kappa\lambda} \right)_-^2 \right). \quad (71)$$

Consider a uniform grid on the rectangle $[k_{\min}, k_{\max}] \times [z_{\min}, z_{\max}]$: we set $k_i = k_{\min} + i\Delta k$, $i = 0, \dots, N$, with $\Delta k = \frac{k_{\max} - k_{\min}}{N}$ and $z_j = z_{\min} + j\Delta z$, $j = 0, \dots, M$, with $\Delta z = \frac{z_{\max} - z_{\min}}{M}$. The discrete approximation of $U(k_i, z_j)$ and $p(k_i, z_j)$ are respectively named $U_{i,j}$ and $p_{i,j}$.

5.1 The discrete version of the system (6-7)

We use the following notation for the three nodes centered finite difference approximation of the second order derivative with respect to k :

$$(D_k^2 U)_{i,j} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\Delta k^2}.$$

Consider also the first order one sided finite difference approximations of $\partial_k U$ and $\partial_z U$, namely

$$\begin{aligned} (D_{k,\ell} U)_{i,j} &= \frac{U_{i,j} - U_{i-1,j}}{\Delta k}, & (D_{k,r} U)_{i,j} &= \frac{U_{i+1,j} - U_{i,j}}{\Delta k}, \\ (D_{z,\ell} U)_{i,j} &= \frac{U_{i,j} - U_{i,j-1}}{\Delta z}, & (D_{z,r} U)_{i,j} &= \frac{U_{i,j+1} - U_{i,j}}{\Delta z}. \end{aligned}$$

The advection term with respect to z in (6) will be discretized with a first order upwind scheme. The discrete version of the Hamiltonian H involves the function $\mathcal{H} : \mathbb{R}^4 \rightarrow \mathbb{R}$,

$$\mathcal{H}(z, p, \xi_\ell, \xi_r) = H_\downarrow(z, p, \xi_\ell) + H_\uparrow(z, p, \xi_r) - H_{\min}(z, p),$$

where H_\downarrow , H_\uparrow and H_{\min} are respectively defined in (18), (19) and (15). Note that \mathcal{H} is nonincreasing with respect to ξ_ℓ and nondecreasing with respect to ξ_r .

The discrete version of (6) (monotone and first order scheme) is as follows:

$$0 = \begin{cases} -rU_{i,j} + \frac{\sigma^2(k_i)}{2} (D_k^2 U)_{i,j} \\ + \mathcal{H}(z_j, p_{i,j}, (D_{k,\ell} U)_{i,j}, (D_{k,r} U)_{i,j}) \\ + \max(0, b(k_i, z_j, p_{i,j})) (D_{z,r} U)_{i,j} + \min(0, b(k_i, z_j, p_{i,j})) (D_{z,\ell} U)_{i,j} \end{cases} \quad (72)$$

for $i = 1, \dots, N - 1$ and $j = 0, \dots, M$. Note that the scheme is actually well defined at $j = 0$ (with a slight abuse of notation), because, since $b(k_i, z_{\min}, p_{i,j}) \geq 0$, it does not involve $U_{i,-1}$. A similar remark can be made in the case when $j = M$.

We choose the following discrete version of (7):

$$\begin{aligned} g(k_i) &= -rp_{i,j} + \frac{\sigma^2(k_i)}{2}(D_k^2 p)_{i,j} \\ &\quad + \frac{\partial H_\downarrow}{\partial \xi}(z_j, p_{i,j}, (D_{k,\ell} U)_{i,j})(D_{k,\ell} p)_{i,j} + \frac{\partial H_\uparrow}{\partial \xi}(z_j, p_{i,j}, (D_{k,r} U)_{i,j})(D_{k,r} p)_{i,j} \\ &\quad + \frac{1}{\Delta z}(\Psi(k_i, p_{i,j}, p_{i,j+1}) - \Psi(k_i, p_{i,j-1}, p_{i,j})), \end{aligned} \quad (73)$$

for $i = 1, \dots, N - 1$ and $j = 0, \dots, M$.

5.2 The discrete scheme at $i = 0$

In order to write the discrete version of the boundary conditions at $k = k_{\min}$, we set

$$A_j = \begin{cases} H_\uparrow(z_j, p_{0,j}, (D_{k,r} U)_{0,j}) \\ + \max(0, b(k_{\min}, z_j, p_{0,j}))(D_{z,r} U)_{0,j} + \min(0, b(k_{\min}, z_j, p_{0,j}))(D_{z,\ell} U)_{0,j}, \end{cases} \quad (74)$$

$$B_j = \max_{p: K_j(p_{0,j-1}, p, p_{0,j+1}) \leq g(k_{\min})} F_j(p, U), \quad (75)$$

where

$$K_j(p_{0,j-1}, p, p_{0,j+1}) = -rp + \frac{1}{\Delta z}(\Psi(k_{\min}, p, p_{0,j+1}) - \Psi(k_{\min}, p_{0,j-1}, p)), \quad (76)$$

and

$$F_j(p, U) = H_{\min}(z_j, p) + \max(0, b(k_{\min}, z_j, p))(D_{z,r} U)_{0,j} + \min(0, b(k_{\min}, z_j, p))(D_{z,\ell} U)_{0,j}. \quad (77)$$

The numerical scheme corresponding to the boundary condition at $i = 0$ consists of two equations for each $0 \leq j \leq M$:

1. the first equation is

$$-rU_{0,j} + \max(A_j, B_j) = 0, \quad (78)$$

2. the second equation is either (79) or (80) below:

- (a) if the maximum in (78) is achieved by A_j , then

$$g(k_{\min}) = \begin{cases} -rp_{0,j} + \frac{\partial H_\uparrow}{\partial \xi}(z_j, p_{0,j}, (D_{k,r} U)_{0,j})(D_{k,r} p)_{0,j} \\ + \frac{1}{\Delta z}(\Psi(k_{\min}, p_{0,j}, p_{0,j+1}) - \Psi(k_{\min}, p_{0,j-1}, p_{0,j})) \end{cases} \quad (79)$$

- (b) otherwise (the maximum in (78) is achieved by B_j),

$$p_{0,j} = p_j^*(U, P), \quad (80)$$

where $p_j^*(U, P)$ achieves the maximum in (75).

Remark 5.1. *There is a unique solution to*

$$rp = \frac{1}{\Delta z}(\Psi(k_{\min}, p, p_{0,j+1}) - \Psi(k_{\min}, p_{0,j-1}, p)) - g(k_{\min}). \quad (81)$$

Indeed, (81) can be written $\chi(q) = -g(k_{\min})$, with

$$\begin{aligned} \chi(q) = & r \left(\frac{\mu}{\lambda} - \frac{\phi(k_{\min})}{\kappa\lambda} \right) + rq \\ & - \frac{\kappa\lambda}{2\Delta z} \max \left((q_{0,j+1})_+^2, (q)_-^2 \right) + \frac{\kappa\lambda}{2\Delta z} \max \left((q)_+^2, (q_{0,j-1})_-^2 \right). \end{aligned} \quad (82)$$

The function χ is increasing and $\lim_{q \rightarrow \pm\infty} \chi(q) = \pm\infty$. Note also that

$$\chi'(q) = r + \frac{\kappa\lambda}{\Delta z} q \left(-1_{q \leq -(q_{0,j+1})_+} + 1_{q \geq (q_{0,j-1})_-} \right).$$

Setting

$$Q = - \left(\frac{\mu}{\lambda} - \frac{\phi(k_{\min})}{\kappa\lambda} \right) + \frac{\kappa\lambda}{2r\Delta z} (q_{0,j+1})_+^2 - \frac{\kappa\lambda}{2r\Delta z} (q_{0,j-1})_-^2 - \frac{g(k_{\min})}{r},$$

we see that

$$q = \begin{cases} \frac{r - \sqrt{r^2 + \frac{2\kappa\lambda}{\Delta z} \left(r \left(\frac{\mu}{\lambda} - \frac{\phi(k_{\min})}{\kappa\lambda} \right) + \frac{\kappa\lambda}{2\Delta z} (q_{0,j-1})_-^2 + g(k_{\min}) \right)}}{\frac{\kappa\lambda}{\Delta z}} & \text{if } Q < -(q_{0,j+1})_+, \\ Q, & \text{if } Q \in [-(q_{0,j+1})_+, (q_{0,j-1})_-], \\ \frac{-r + \sqrt{r^2 - \frac{2\kappa\lambda}{\Delta z} \left(r \left(\frac{\mu}{\lambda} - \frac{\phi(k_{\min})}{\kappa\lambda} \right) - \frac{\kappa\lambda}{2\Delta z} (q_{0,j+1})_+^2 + g(k_{\min}) \right)}}{\frac{\kappa\lambda}{\Delta z}} & \text{if } Q > (q_{0,j-1})_-. \end{cases} \quad (83)$$

Then p satisfies the constraint in (75) if and only if $p \geq q + \frac{\mu}{\lambda} - \frac{\phi(k_{\min})}{\kappa\lambda}$, and B_j is computed by maximizing a concave and quadratic function on the set $p \geq q + \frac{\mu}{\lambda} - \frac{\phi(k_{\min})}{\kappa\lambda}$.

5.3 The discrete scheme at $i = N$

For brevity, we do not write the numerical scheme corresponding to the boundary condition at $k = k_{\max}$, because the equations (two equations for each value of j , $0 \leq j \leq M$,) may be obtained in exactly the same way as in the previous paragraph.

5.4 Solving the system of nonlinear equations: a long time approximation

The system of equations including (72, (73)) for $0 < i < N$ and $0 \leq j \leq M$, and the discrete versions of the boundary conditions at $k = k_{\min}$ and $k = k_{\max}$ described above, can be written in an abstract form as follows:

$$\mathcal{F}(\mathcal{U}, \mathcal{P}) = 0, \quad (84)$$

where \mathcal{F} is a nonlinear map from $\mathbb{R}^{2(N+1)(M+1)}$ to $\mathbb{R}^{2(N+1)(M+1)}$ such that the Jacobian matrix of $\mathcal{F}(\mathcal{U}, \mathcal{P})$ has negative diagonal entries.

We aim at solving the discrete system (84) by a long time approximation involving an explicit scheme. The reason for choosing an explicit scheme lies in the complexity of the boundary conditions. Finding an implicit or semi-implicit scheme consistent with the nonlinear boundary conditions seems challenging.

We fix a time step $\Delta t > 0$.

Setting $\mathcal{U}^\ell = (U_{i,j}^\ell)_{0 \leq i \leq N, 0 \leq j \leq M}$ and $\mathcal{P}^\ell = (p_{i,j}^\ell)_{0 \leq i \leq N, 0 \leq j \leq M}$, we compute the sequence $(\mathcal{U}^\ell, \mathcal{P}^\ell)$ by the induction:

$$(\mathcal{U}^{\ell+1}, \mathcal{P}^{\ell+1}) = (\mathcal{U}^\ell, \mathcal{P}^\ell) - \Delta t \mathcal{F}(\mathcal{U}^\ell, \mathcal{P}^\ell), \quad (85)$$

and expect that the sequence converges as $\ell \rightarrow +\infty$. In the latter case, the limit is a solution of (84).

6 Numerical simulations

The numerical simulations reported below aim at describing some aspects of the short term dynamics of the oil market. We believe that they make it possible to explain the sharp falls in the prices that have been observed in 2015 and 2020.

6.1 Case one

We take

$$b(k, z, p) = a \left(\frac{k_{\max} - k}{k_{\max} - k_{\min}} \right)^2 - 2a \left(\frac{k - k_{\min}}{k_{\max} - k_{\min}} \right)^2 + \kappa(\lambda p - \mu),$$

$$g = 10 \left(\frac{k - k_{\min}}{k_{\max} - k_{\min}} \right)^3,$$

with

$$\begin{aligned} r &= 0.1, \\ \epsilon &= 4 \cdot 10^{-4}, \\ a &= 0.02, \\ \kappa &= 6.66 \cdot 10^{-4}, \quad \lambda = 0.4, \quad \mu = 10, \\ q_o &= 0.42, \quad \alpha = 10^4, \\ c &= 10. \end{aligned}$$

We set $k_{\min} = 0$, $k_{\max} = 0.07$, $z_{\min} = 0.35$ and $z_{\max} = 0.75$.

The mesh parameters are $N = M = 200$, and the time step is $\Delta t = 0.00001$.

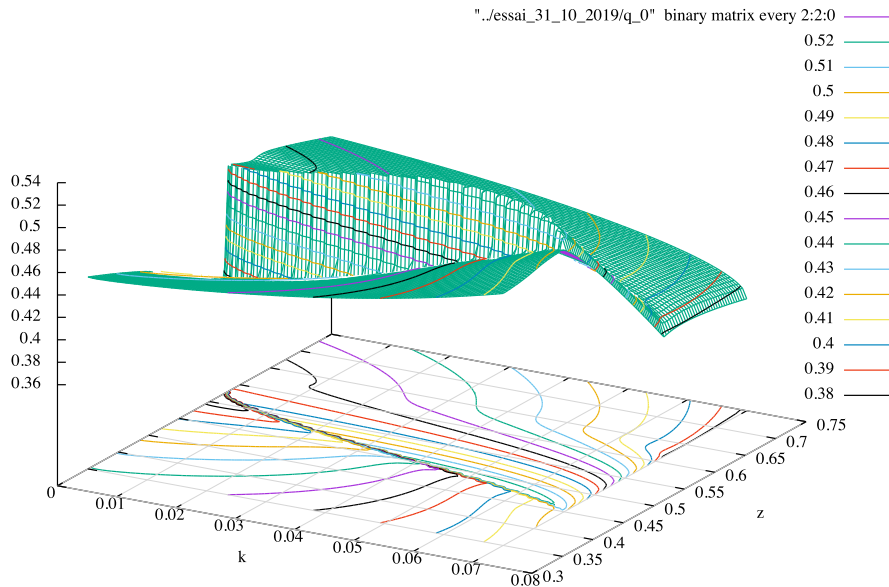


Figure 1: Case one: the optimal production level of the monopolist: we see that the optimal level of production displays a shock whose amplitude is maximal for at the $k = k_{\min}$ and vanishes at $k = k_{\max}$.

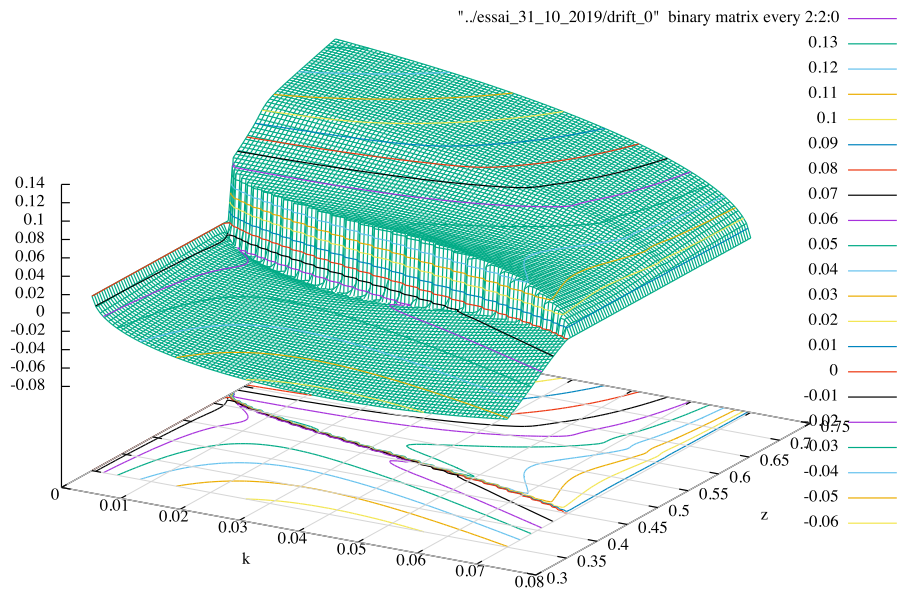


Figure 2: Case one: the k component of the optimal drift, which gives the dynamics of the storage level. Note that it behaves like $\sqrt{k - k_{\min}}$ when it is nonpositive near $k = k_{\min}$, and that it behaves like $\sqrt{k_{\max} - k}$ when it is nonnegative near $k = k_{\max}$.

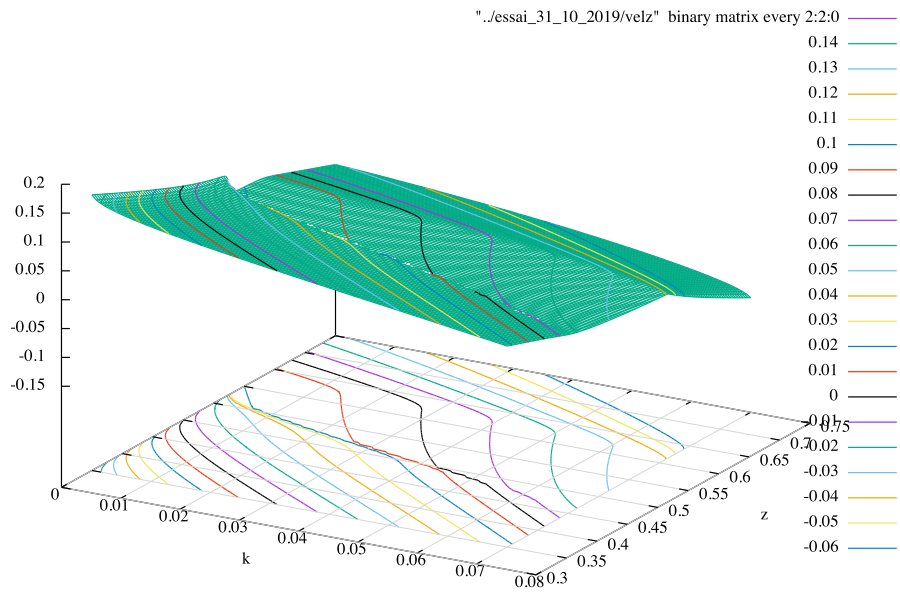


Figure 3: Case one: the z component of the optimal drift, which gives the dynamics of the production level of the fringe

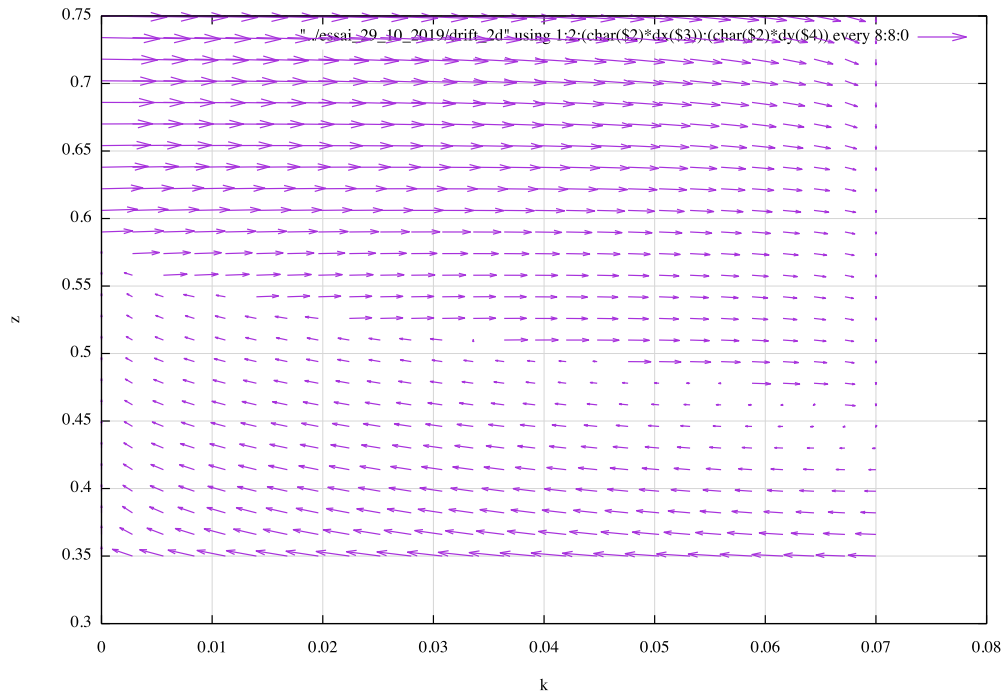


Figure 4: Case one: the optimal drift

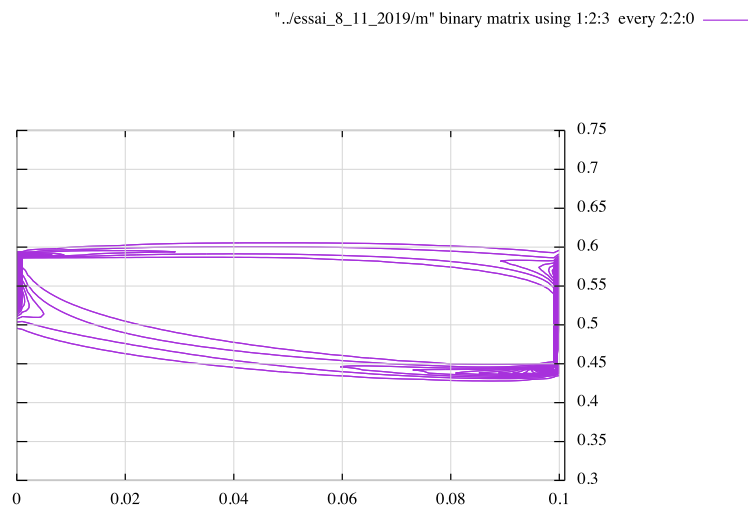


Figure 5: Case one: the contours of the invariant measure: it is concentrated around the stable cycle. Within the cycle, the density of the measure is much higher in the region close to the lines $k = k_{\min}$ and k_{\max} , because the drift is small there.

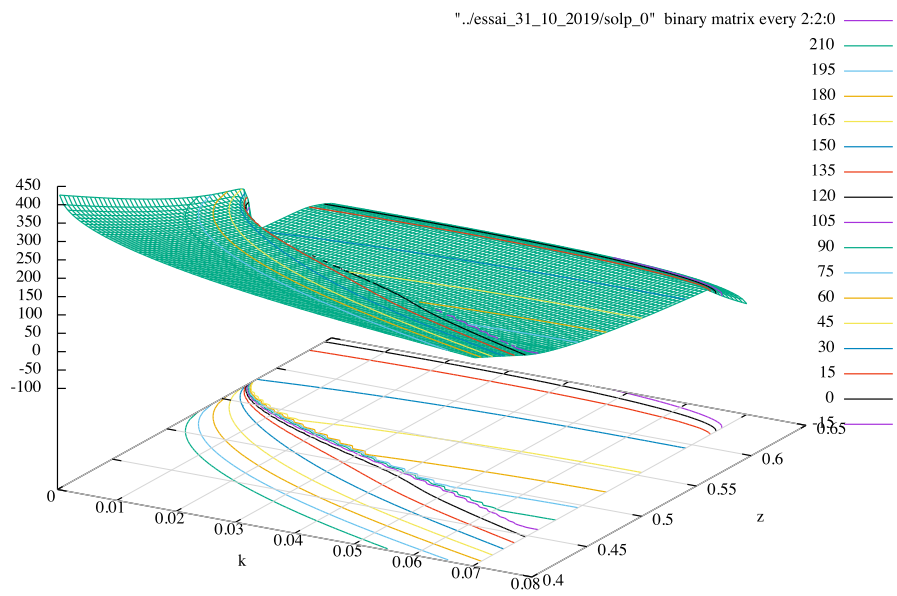


Figure 6: Case one: the price; there is also a shock in the price. Note that the price takes negative values for large values of z , but that such large values are irrelevant in the oil industry (the level of production of the competitive fringe is close to 0.58 and does not vary more than 5%). The negative values of p_t are due to the fact that we chose not to put any constraints on the production level q .

Comments on case 1 To be completed

6.2 Case two

We take

$$b(k, z, p) = a \left(\frac{k_{\max} - k}{k_{\max} - k_{\min}} \right)^2 - a \left(\frac{k - k_{\min}}{k_{\max} - k_{\min}} \right)^2 + \kappa(\lambda p - \mu),$$

$$g = 10 \left(\frac{k - k_{\min}}{k_{\max} - k_{\min}} \right)^3,$$

with

$$\begin{aligned} r &= 0.1, \\ \epsilon &= 4 \cdot 10^{-4}, \\ a &= 0.01, \\ \kappa &= 6.66 \cdot 10^{-4}, \quad \lambda = 0.4, \quad \mu = 10, \\ q_o &= 0.42, \quad \alpha = 10^4, \\ c &= 10. \end{aligned}$$

Compared to case one, we also add viscosity in the z -variable: $\nu_z = 10^{-4}$.

We set $k_{\min} = 0$, $k_{\max} = 0.07$, $z_{\min} = 0.35$ and $z_{\max} = 0.75$.

The mesh parameters are $N = M = 200$, and the time step is $\Delta t = 0.00001$.

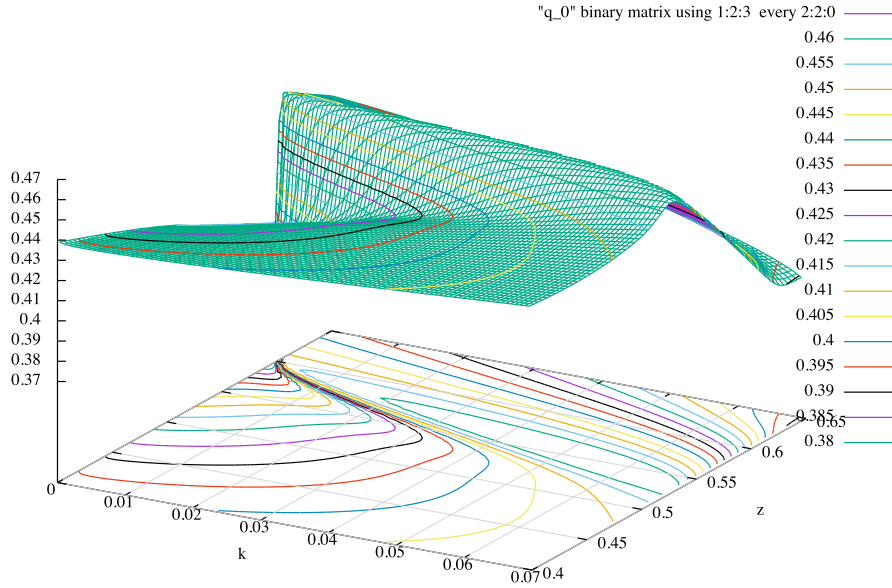


Figure 7: Case two: the optimal production level of the monopolist.

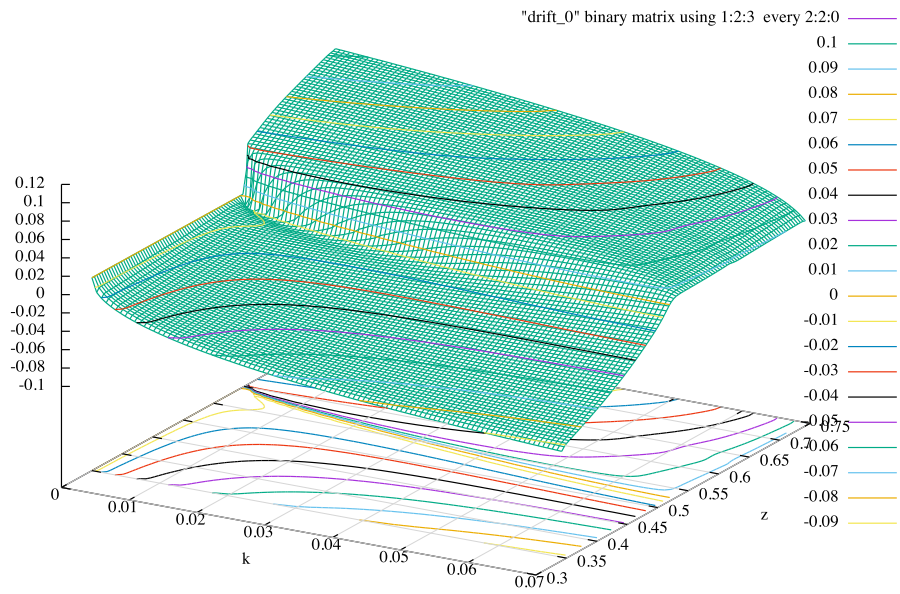


Figure 8: Case two: the k component of the optimal drift.

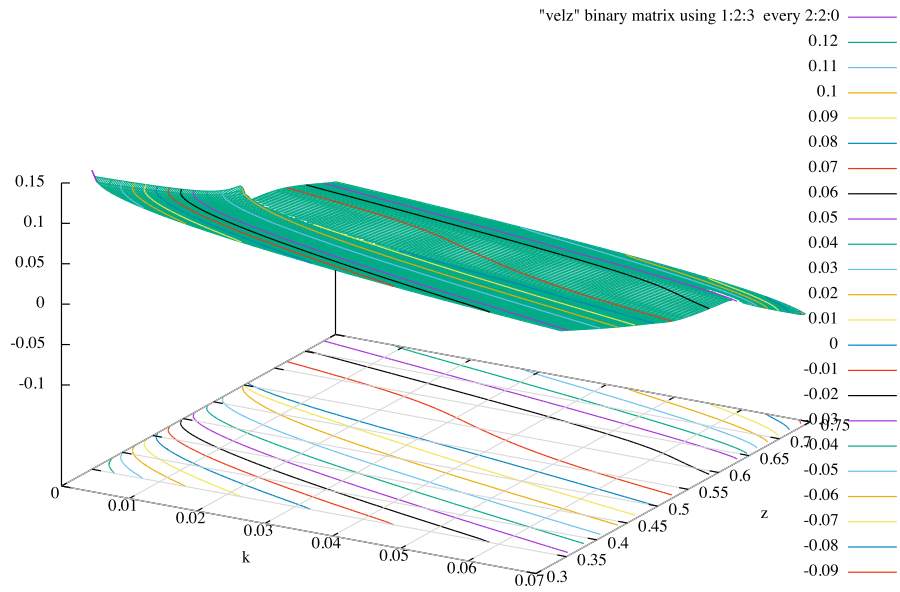


Figure 9: Case two: the z component of the optimal drift.

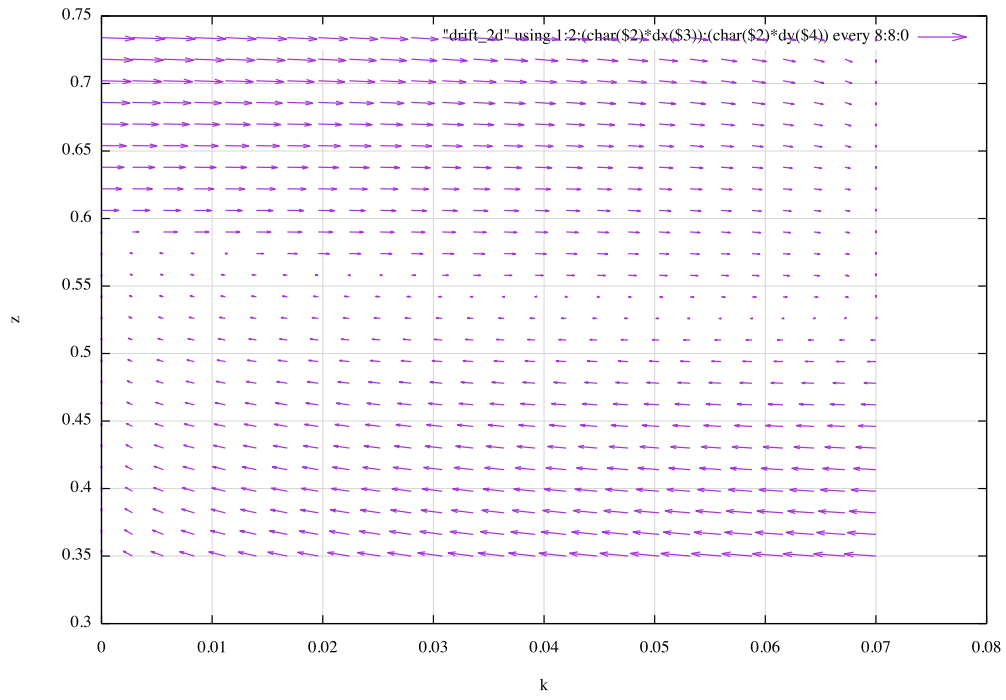


Figure 10: Case two: the optimal drift.

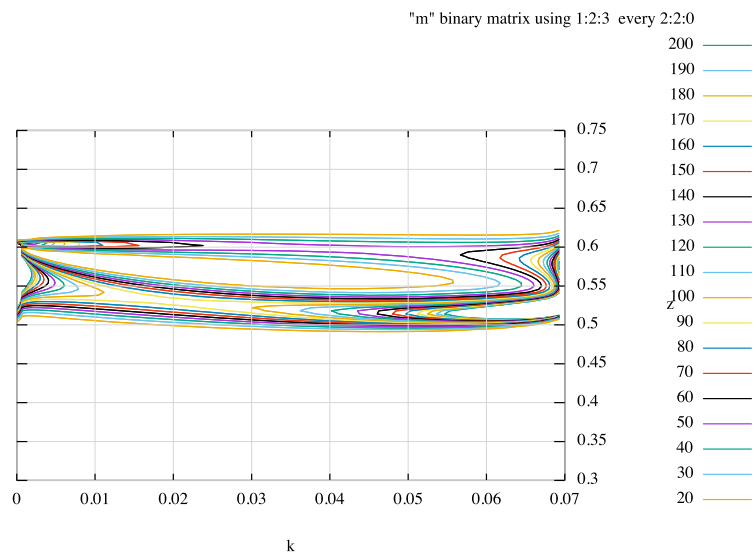


Figure 11: Case two: the contours of the invariant measure.

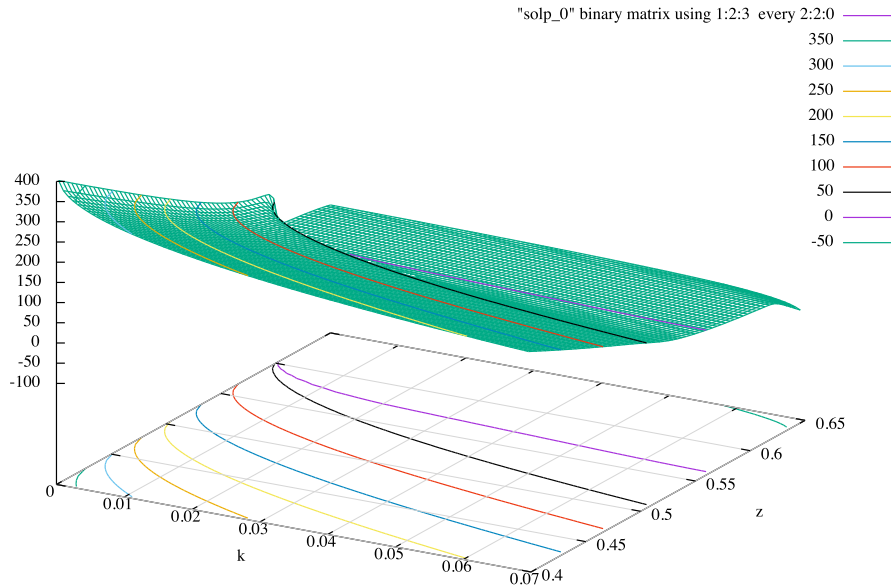


Figure 12: Case two: the price.

Comments on case 2 To be completed

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