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## ARTICLE TYPE

# Finite-gain $\mathcal{L}_1$ Event-triggered Interval Observers design for Continuous-time Linear Systems

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## Summary

This work introduces the design of finite-gain  $\mathcal{L}_1$  interval observers based on an event-triggered mechanism for linear continuous-time systems in the presence of unknown-but-bounded uncertainties with *a priori* known bounds (state disturbances and measurement noises). In addition, the times when the measurements are required in the estimation procedure are driven by an event-triggered mechanism. A dynamic event-triggered condition, which depends on the width of the feasible domain of the system's uncertainties and the width of the estimated state enclosures, is established. The proposed event-triggered strategy guarantees the existence of a positive lower bound on the inter-event times, which avoids the Zeno phenomenon. Moreover, the  $\mathcal{L}_1$ -stability of the estimation error is ensured while using as less as possible sensors data. Finally, simulation results are given to illustrate the effectiveness of the proposed method.

## KEYWORDS:

Interval observers,  $\mathcal{L}_1$ -stability, Event-triggered measurements, Hybrid systems

## 1 | INTRODUCTION

Immense effort has been made to enhance the performance of networked control systems (NCS) while ensuring the minimal use of the (shared) network. Alternative approaches to the traditional periodic data-sampling method were proposed, such as, the aperiodic data-sampling scheme where sampling is allowed within a predetermined time interval<sup>1</sup>, and the event/self-triggered sampling strategy<sup>2,3,4</sup> where the sampling time instant depends on the behaviour of the system.

In this work, we propose the design of interval observers based on an event-triggering mechanism allowing to use measurement only when necessary. The event-based estimation can relax the regularity assumption on the availability of the measurements commonly used by the traditional state estimation methods. Thanks to this mechanism, the amount of transmitted data over the network can be considerably reduced. In this non-uniformly sampled data framework, the interval/set-membership state estimation problem has not been fully investigated in the literature. Some preliminary works have considered the self-triggered<sup>5</sup> and the event-based<sup>6</sup> sampling of measurements using the prediction-correction approach. However, in both cited works, the intrinsic structures properties of the systems only are exploited in order to design converging set-membership state estimators. The main advantage of the observation approach that we propose in this paper is that the correction stage is based on a pre-calculated observation gain that ensures the stability of the estimation error along with some performance specifications. The latter is achieved using Input-Output Stability (IOS) analysis, in particular the  $\mathcal{L}_p$  stability concept<sup>7</sup>.

The  $\mathcal{L}_p$ -gain concept is an interesting approach to analyze the stability along with the performance of dynamical systems. This concept has been already applied to evaluate the performance of interval observers<sup>8,9</sup>. For instance, interval observers

for LPV systems with  $\mathcal{L}_1/\mathcal{L}_2$  performance analysis were proposed in<sup>8</sup>. Interval observers providing tight state enclosures are designed using peak-to-peak approach in<sup>9</sup> for discrete and continuous systems. In<sup>10</sup>, based on the minimization of the norm-1 of the steady state of the observation error, tight interval observers are designed that minimize the impact of the perturbations on the accuracy of the state estimates.

In this work, we combine the  $\mathcal{L}_1$ -gain approach for positive systems<sup>11,12</sup> and the  $\mathcal{L}_p$ -gain approach for hybrid systems<sup>13</sup> to study the stability and the performance of the proposed event-triggered interval observers viewed as impulsive systems. The positivity of the estimation error is guaranteed by using the internal positivity of the system. This interval observer is designed under the assumption that the system perturbations and the measurement noise are unknown-but-bounded with a priori known bounds. In addition, the measurements are not continuously available and a triggering mechanism is designed to request measurement only when needed to enhance the accuracy of the estimates, that is, the measurements are requested whenever a performance criterion, involving the width of the feasible domain of the system's uncertainties and the width of the estimated state enclosures, is violated. In the case of disturbed continuous-time linear systems, the novelty of this work is the co-design of an event-triggered mechanism for measurement sampling, and the observer gains that ensure the convergence of the width of the estimated state enclosures.

The outline of this paper is as follows. Some preliminaries are given in Section 2. The structure of the interval observers for continuous-time linear systems and the design of the event-triggered mechanism with guaranteed finite-gain  $\mathcal{L}_1$  are proposed in Section 3. The co-design of the event-triggered mechanism and the interval observer gain for continuous-time linear systems is given in Section 4 which differs from Section 3 where we assume that the interval observer gains are given a priori. An illustrative numerical example shows the efficiency of the proposed approach in Section 5. Finally, we provide concluding remarks in Section 6.

## 2 | PRELIMINARIES

### 2.1 | Notations

The set  $\mathbb{R}$ ,  $\mathbb{R}_+$  and  $\mathbb{N}$  are the set of real scalars, positive real scalars and positive integers including zero, respectively. Any  $p \times m$  matrix whose elements are all ones or zeros are simply denoted by  $\mathbb{1}_{p,m}$  or  $\mathbf{0}$ , respectively.  $I_p$  denotes the identity matrix in  $\mathbb{R}^{p \times p}$ . Throughout this paper the inequalities must be understood component-wise, for matrices as well as for vectors, i.e.  $A = (a_{i,j}) \in \mathbb{R}^{p \times m}$  and  $B = (b_{i,j}) \in \mathbb{R}^{p \times m}$  such that  $A \geq B$  if and only if,  $a_{i,j} \geq b_{i,j}$  for all  $i \in \{1, \dots, p\}, j \in \{1, \dots, m\}$ .  $M = \max\{A, B\}$  is the matrix where each entry is  $m_{i,j} = \max\{a_{i,j}, b_{i,j}\}$ . Let us define  $A^+ = \max\{A, \mathbf{0}\}$ ,  $A^- = A^+ - A$ ; thus, the element-wise absolute value will be denoted as  $|A| = A^+ + A^-$ . A matrix  $M \in \mathbb{R}^{n \times n}$  is said to be Metzler if all its off-diagonal entries are nonnegative. The  $i$ -th element of the vector  $v \in \mathbb{R}^n$  is denoted  $v_i$ .

Since the proposed interval-based estimation method relies on the hybrid systems framework, basic definitions about this framework are briefly introduced in the following subsection.

### 2.2 | The hybrid system framework

Here we present the main definition of the hybrid system formalism of<sup>14</sup> which allows the use of the well-defined notion of solutions and the tools provided within. Basic concepts and analysis of these classes of dynamical systems are given below. Consider the hybrid system of the following form

$$\dot{x} \in \mathcal{F}(x) \quad \forall x \in C, \quad x^+ \in \mathcal{G}(x) \quad \forall x \in D, \quad (1)$$

where  $x \in \mathbb{R}^n$  is the state,  $C, D \in \mathbb{R}^n$  and  $\mathcal{F}, \mathcal{G}$  are two set-valued functions.  $\mathcal{F}$ ,  $C$ ,  $\mathcal{G}$  and  $D$  are the *flow map*, the *flow set*, the *jump map* and the *jump set*, respectively. In our case, we consider that the flow and jump maps are single-valued functions instead of set-valued ones (e.g.,  $\dot{x} = \mathcal{F}(x)$  instead of  $\dot{x} \in \mathcal{F}(x)$ ). This model shows that the state  $x$  of the hybrid system evolves according to the differential equation  $\dot{x} = \mathcal{F}(x)$  as long as  $x \in C$ , and it experiences an instantaneous change according to the difference equation  $x^+ = \mathcal{G}(x)$  when  $x \in D$ . The solutions to system (1) are defined on so-called hybrid time domains.

- *Hybrid time domains* – A subset  $\mathcal{E} \in \mathbb{R}_{\geq 0} \times \mathbb{N}$  is a compact hybrid time domain if  $\mathcal{E} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$  for some finite sequence of times  $0 = t_0 \leq t_1 \leq \dots \leq t_J$ . It is a hybrid time domain if for all  $(T, J) \in \mathcal{E}$ ,  $\mathcal{E} \cap ([0, T] \times \{0, 1, \dots, J\})$  is a compact hybrid domain.

- *Hybrid arc* – A function  $\phi : \mathcal{E} \rightarrow \mathbb{R}^n$  is a hybrid arc if  $\mathcal{E}$  is a hybrid time domain and if for each  $j \in \mathbb{N}$ , the function  $t \mapsto \phi(t, j)$  is locally absolutely continuous on the interval  $I^j = \{t : (t, j) \in \mathcal{E}\}$ .

In the sequel, the hybrid time domain  $\mathcal{E}$  of the hybrid arc  $\phi$  will be noted by  $\text{dom}\phi$ . A hybrid arc  $\phi$  is a solution to the hybrid system  $(C, F, D, G)$  if: (i)  $\phi(0, 0) \in C \cup D$ ; (ii) for all  $j \in \mathbb{N}$ ,  $\phi(t, j) \in C$  and  $\dot{\phi}(t, j) = F(\phi(t, j))$  for almost all  $t \in I^j$ ; (iii) for all  $(t, j) \in \text{dom}\phi$ , such that  $(t, j+1) \in \text{dom}\phi$ ,  $\phi(t, j) \in D$  and  $\phi(t, j+1) = G(\phi(t, j))$ . A solution  $\phi$  to system (1) is *maximal* if it cannot be extended, *complete* if its domain,  $\text{dom}\phi$ , is *unbounded*. Also, it is *Zeno* if it is complete and  $\sup_t \text{dom}\phi < \infty$ .

Next, we will provide some important mathematical tools which play an essential role in showing the non-negativity of the estimation error of the interval observers.

### 2.3 | Positive (Cooperative) systems

**Definition 1.**<sup>15</sup> A discrete-time linear system  $x(k+1) = A_d x(k)$ , with the state  $x \in \mathbb{R}^n$  and  $A_d \in \mathbb{R}^{n \times n}$ , is said to be cooperative if  $A_d$  is a nonnegative matrix. A continuous-time linear system  $\dot{x}(t) = Ax(t)$ , with  $A \in \mathbb{R}^{n \times n}$ , is said to be cooperative if  $A$  is a Metzler matrix.

The solutions of cooperative systems, initialized at  $x(0) \geq 0$ , stay nonnegative:  $x(k) \geq 0$  for all  $k \in \mathbb{N}$  (resp.  $x(t) \geq 0$  for all  $t \geq 0$ ).

*Remark 1.* A system  $\dot{x} = f(x)$  of dimension  $n$  is cooperative if its *Jacobian* matrix  $(\frac{\partial f}{\partial x})$  is Metzler.

**Definition 2.**<sup>16</sup> If there exist  $\underline{f}$  and  $\bar{f}$  such that the inclusion  $\underline{f}(\underline{x}(t), \bar{x}(t)) \leq f(x(t)) \leq \bar{f}(\underline{x}(t), \bar{x}(t))$  hold, then a framer for the system  $\dot{x}(t) = f(x(t))$  is the pair of coupled dynamical systems

$$\begin{cases} \dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), \bar{x}(t)) \\ \dot{\bar{x}}(t) = \bar{f}(\underline{x}(t), \bar{x}(t)) \end{cases} \quad (2)$$

such that, for  $\underline{x}(0) \leq x(0) \leq \bar{x}(0)$ , we have  $\underline{x}(t) \leq x(t) \leq \bar{x}(t)$  for all  $t \geq 0$ .

*Remark 2.* In the case of linear cooperative system, a framer can be designed using copies of the system (i.e.,  $\underline{f}(\underline{x}, \bar{x}) = A\underline{x}$  and  $\bar{f}(\underline{x}, \bar{x}) = A\bar{x}$  for the lower and upper bounding system, respectively). In case of non-cooperative systems, the two functions  $\underline{f}, \bar{f}$  of the system (2) could be calculated using change of coordinates<sup>17</sup> or more general using methods based on the Müller's existence theorem<sup>18</sup> as proposed in<sup>19</sup>.

It is important to point out the fact that a framer is conceived to give an upper and a lower bounds for the unknown state. Stability can be considered as an additional feature for a framer to guarantee the convergence of the bounds to the system state and thus obtain an interval observer.

In our application, we combine the notion of framer for continuous-time and discrete-time systems to obtain a framer for the so-called *impulsive (or hybrid)* system.

Consider the following hybrid system:

$$\begin{cases} \dot{x} = f(x, d), & \forall (x, d) \in C \\ x_{t_k^+} = g(x, d), & \forall (x, d) \in D \\ y = h(x, d), \end{cases} \quad (3)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^{n_u}$  and  $y \in \mathbb{R}^{n_y}$  are the state vector, the exogenous input and the output vector, respectively, and  $d \in \mathbb{R}^{n_d}$  represents the exogenous input assumed to be unknown-but-bounded.  $x_{t_k^+} = \lim_{t \rightarrow t_k^+} x(t)$  refers to the state variable after jumps. The following definition generalizes the definition of framer of discrete-time and continuous-time systems.

**Definition 3.** If there exist functions  $\underline{f}, \bar{f}, \underline{g}, \bar{g} \in \mathbb{R}^{2(n+n_d)} \rightarrow \mathbb{R}^n$  and  $\underline{d}, \bar{d} \in \mathbb{R}^{n_d}$  such that the inclusions

$$\underline{d}(t) \leq d(t) \leq \bar{d}(t), \quad \forall t \geq 0, \quad (4a)$$

$$\underline{f}(\underline{x}, \bar{x}, \underline{d}, \bar{d}, y) \leq f(x, d) \leq \bar{f}(\underline{x}, \bar{x}, \underline{d}, \bar{d}, y), \quad \forall (x, d) \in C, \quad (4b)$$

$$\underline{g}(\underline{x}, \bar{x}, \underline{d}, \bar{d}, y) \leq g(x, d) \leq \bar{g}(\underline{x}, \bar{x}, \underline{d}, \bar{d}, y), \quad \forall (x, d) \in D. \quad (4c)$$

are satisfied, then the solution of the system

$$\begin{cases} \dot{\underline{x}} = \underline{f}(\underline{x}, \bar{x}, \underline{d}, \bar{d}, y), & \forall (x, d) \in \mathcal{C} \\ \dot{\bar{x}} = \bar{f}(\underline{x}, \bar{x}, \underline{d}, \bar{d}), \\ \underline{x}_{t_k^+} = \underline{g}(\underline{x}, \bar{x}, \underline{d}, \bar{d}, y), & \forall (x, d) \in \mathcal{D} \\ \bar{x}_{t_k^+} = \bar{g}(\underline{x}, \bar{x}, \underline{d}, \bar{d}, y), \\ \underline{x}(0) \leq x(0) \leq \bar{x}(0) \end{cases} \quad (5)$$

is a *framer* for the system (3) that means  $\underline{x}(t) \leq x(t) \leq \bar{x}(t) \quad \forall t \geq 0$ .

**Definition 4.** The impulsive system (5) is an interval observer for the system (3) if it is a framer for system (3) and the dynamics of the estimation errors  $\underline{e} = x - \underline{x}$ ,  $\bar{e} = \bar{x} - x$  are stable.

The stability of impulsive system can be studied using several approaches, e.g., using the Lyapunov approach, or the dissipative systems approach<sup>20</sup>. In this work, we adopt the  $\mathcal{L}_p$  stability approach developed for hybrid systems.<sup>13</sup>

## 2.4 | $\mathcal{L}_p$ stability: Input-Output sense

An Input-Output model relates the output of the system directly to its input, with no knowledge of the internal structure that can be represented by the state space model. The  $\mathcal{L}_p$  stability is a concept of stability in the input-output sense (for details see<sup>7</sup>, Chapter 5).

To this end, we use  $\|z\|_p$  the  $\mathcal{L}_p$ -norm for the hybrid signal (arc)  $z$  as is defined in<sup>13</sup>.

**Definition 5** ( $\mathcal{L}_p$ -norm). For a hybrid signal  $z$ , with domain  $\text{dom}(z) \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ , and a scalar  $T \in \mathbb{R} \geq 0$ , the  $T$ -truncated  $\mathcal{L}_p$ -norm of  $z$  is given by

$$\|z_{[T]}\|_p := \left( \sum_{i=1}^{j(T)} |z(t_i, i-1)|^p + \sum_{i=0}^{j(T)} \int_{t_i}^{\sigma_i} |z(s, i)|^p ds \right)^{\frac{1}{p}} \quad (6)$$

where  $t_0 = 0$ ,  $j(T) = \max\{k : (t, k) \in \text{dom}(z), t + k \leq T\}$ , and,  $\forall i \in \{0, \dots, j\}$ ,  $\sigma_i = \min(t_{i+1}, T - i)$ . Based on (6), the  $\mathcal{L}_p$ -norm of  $z$  is defined as

$$\|z\|_p = \lim_{T \rightarrow T^*} \|z_{[T]}\|_p, \quad (7)$$

where  $T^* = \sup\{t + j : (t, j) \in \text{dom}(z)\}$ . Moreover, we have that  $z \in \mathcal{L}_p$  whenever the above limit exists and is finite.

The  $\mathcal{L}_p$ -norm for hybrid signals in Definition 5 is a generalized definition of the  $\mathcal{L}_p$ -norm for continuous-time signals<sup>7</sup> and the  $l_p$ -norm for the discrete-time signal<sup>21</sup>. This norm can be simplified for particular cases; if the signal  $z(t, j)$  is continuous-time signal then the first part of the right hand side of expression (6) equals to zero, if  $z(t, j)$  is discrete-time signal then the second part of the right hand side of (6) equals to zero.

**Remark 3.** For multidimensional signals  $z \in \mathbb{R}^{n_z}$ , we write  $z \in \mathcal{L}_p^{n_z}$ , where  $\mathcal{L}_1$ -space is defined as the set of absolute-value integrable signals, the  $\mathcal{L}_2$ -space is defined as the set of square integrable signals, and the  $\mathcal{L}_\infty$ -space is defined as the set of signals bounded in amplitude. Note that the  $\mathcal{L}_p^{n_z}$ -norm of vector signals,  $\|z\|_p$  as defined in (7) differs from the  $p$ -norm of vectors which is a norm at time  $t$  defined as  $|z(t)|_p = \left( \sum_{i=1}^{n_z} z_i(t) \right)^{\frac{1}{p}}$ . For instance the 1-norm of the vector  $z(t)$  is written as  $|z(t)|_1$  which will be often used in the sequel.

The general version that combines the  $\mathcal{L}_p$  stability<sup>7</sup> and the  $l_p$  stability<sup>21</sup>, respectively for continuous-time and discrete-time systems is given in the following.

**Definition 6.** Given  $p \in [1, +\infty)$ , system (3) is finite-gain  $\mathcal{L}_p$  stable from  $d$  to  $y$  with gain upper bounded by  $\gamma_p \geq 0$  if there exists a scalar  $\beta \geq 0$  such that any solution to (3) satisfies

$$\|y\|_p \leq \beta \|x(0, 0)\|_p + \gamma_p \|d\|_p \quad (8)$$

for all  $d \in \mathcal{L}_p^{n_d}$ .

The  $\mathcal{L}_p$  stability characterizes the input-to-output stability of systems. This characterization can be obtained using Lyapunov methods. The Lyapunov function in this context is called *storage function* which is defined next for hybrid systems.

**Definition 7.**<sup>13</sup> Given  $p \in [1, +\infty)$ , a positive semi-definite continuously differentiable function  $V : \mathbb{R}^{n_v} \rightarrow \mathbb{R}_+$  is a finite-gain  $\mathcal{L}_p$  storage function for the system (3) if there exist positive constants  $c_2, \gamma_{yf}$  and  $\gamma_{yg}$ , and nonnegative constants  $\gamma_{dg}, \gamma_{df}$ , such that

$$0 \leq V(x) \leq c_2|x|^p, \quad \forall(x, d) \in C \cup D, \quad (9a)$$

$$\langle \nabla V(x), f(x, d) \rangle \leq -\gamma_{yf}|h(x, d)|^p + \gamma_{df}|d|^p, \quad \forall(x, d) \in C, \quad (9b)$$

$$V(g(x, d)) - V(x) \leq -\gamma_{yg}|h(x, d)|^p + \gamma_{dg}|d|^p, \quad \forall(x, d) \in D. \quad (9c)$$

Based on Definition 7, the  $\mathcal{L}_p$  stability of the hybrid system (3) is set in the following proposition.

**Proposition 1.**<sup>13</sup> Consider system (3), and suppose that there exists a function  $V$  that satisfies (9). Then the system is finite gain  $\mathcal{L}_p$  stable, and the gain of the operator  $d \rightarrow y$  is upper bounded by  $\gamma_p = \sqrt[p]{\gamma_d/\gamma_y}$ , where  $\gamma_d = \max\{\gamma_{df}, \gamma_{dg}\}$ ,  $\gamma_y = \min\{\gamma_{yf}, \gamma_{yg}\}$ .

In Definition 6 and 7, we have presented the existing results for the  $\mathcal{L}_p$  stability of hybrid systems<sup>13</sup>. In this work, we use the finite-gain  $\mathcal{L}_1$  stability for hybrid systems, which is a special case of  $\mathcal{L}_p$  stability when  $p = 1$ .

In the following section, we will exploit these results to analyze and design event-triggered interval observers with  $\mathcal{L}_1$ -gain performance for continuous-time linear systems.

### 3 | EVENT-TRIGGERED INTERVAL OBSERVER FOR LINEAR SYSTEMS

Consider the linear time invariant system of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ed(t), & t \in [t_k, t_{k+1}], k \in \mathbb{N} \\ y(t_k) = Cx(t_k) + Fd(t_k), & k \in \mathbb{N} \end{cases} \quad (10)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^{n_u}$  and  $y \in \mathbb{R}^{n_y}$  is the state variables, the input, the output of the system, respectively, and  $d \in \mathbb{R}^{n_d}$  represents the exogenous input assumed to be unknown-but-bounded with *a priori* know bounds. This assumption on boundedness of perturbation is a standard assumption in interval estimation and is introduced as follows

**Assumption 1.** Let two vectors  $\bar{d}(t), \underline{d}(t) \in \mathbb{R}^{n_d}$  be given such that

$$\underline{d}(t) \leq d(t) \leq \bar{d}(t) \quad (11)$$

is verified  $\forall t \geq 0$ .

*Remark 4.* The exogenous input  $d(t)$  in the system (10) is a generalization of the measurements noises and the system disturbances with adapted input matrices  $E$  and  $F$ .

In this section, the goal is to estimate an upper  $\bar{x}$ , and a lower bound  $\underline{x}$  for the actual state of the system (10). More precisely, the aim is to compute a guaranteed enclosure of the set of admissible values for the actual state vector of the disturbed system. The measurements are supposed neither continuous nor periodic but are taken according to desired performance specifications on the estimation that will be introduced later. The advantage of this technique is two-fold: first, in networked systems, it may reduce the communication rate between the computers and the sensors, second, it can provide an estimate of the system state with only few sensor data.

The proposed interval observer includes two dynamic behaviors; the first part concerns the estimation without feedback information from the system, i.e., without measurement, and the second part improves the accuracy of the estimated state enclosure when a measurement is available.

The first part of the interval observer is proposed as follows:

$$\begin{cases} \dot{\underline{x}}(t) = A^M \underline{x}(t) - A^N \bar{x}(t) + Bu(t) + E^+ \underline{d}(t) - E^- \bar{d}(t) \\ \dot{\bar{x}}(t) = A^M \bar{x}(t) - A^N \underline{x}(t) + Bu(t) + E^+ \bar{d}(t) - E^- \underline{d}(t) \end{cases} \quad \forall t \in [t_k, t_{k+1}], \forall k \in \mathbb{N} \quad (12)$$

where  $A^M = d_A + (A - d_A)^+$  and  $A^N = A^M - A$  with  $d_A$  is a diagonal matrix contains only the diagonal elements of  $A$ , with the initial values

$$x(0) \in [\underline{x}(0), \bar{x}(0)] \quad (13)$$

The correction part at the time  $t = t_k$  is introduced by the following discrete-time system:

$$\begin{cases} \underline{x}(t_k^+) = (I_n + LC)^+ \underline{x}(t_k) - (I_n + LC)^- \bar{x}(t_k) \\ \quad + (LF)^+ \underline{d}(t_k) - (LF)^- \bar{d}(t_k) - Ly(t_k) \\ \bar{x}(t_k^+) = (I_n + LC)^+ \bar{x}(t_k) - (I_n + LC)^- \underline{x}(t_k) \\ \quad + (LF)^+ \bar{d}(t_k) - (LF)^- \underline{d}(t_k) - Ly(t_k) \end{cases} \quad \forall k \in \mathbb{N} \quad (14)$$

Using the output model in (10), the system state can be re-written as

$$x(t_k^+) = x(t_k) + L[Cx(t_k) + Fd(t_k) - y(t_k)] \quad k \in \mathbb{N} \quad (15)$$

The estimation error dynamics, of both bounds  $\underline{e}(t) = x(t) - \underline{x}(t)$  and  $\bar{e}(t) = \bar{x}(t) - x(t)$  can be described as follows:

From (10) and (12), one has

$$\begin{bmatrix} \dot{\underline{e}}(t) \\ \dot{\bar{e}}(t) \end{bmatrix} = \mathcal{M}(A) \begin{bmatrix} \underline{e}(t) \\ \bar{e}(t) \end{bmatrix} + \tilde{E}\psi(t) \quad (16)$$

where

$$\mathcal{M}(A) = \begin{bmatrix} A^M & A^N \\ A^N & A^M \end{bmatrix}, \quad \tilde{E} = \begin{bmatrix} E^+ & E^- \\ E^- & E^+ \end{bmatrix}, \quad \psi(t) = \begin{bmatrix} d(t) - \underline{d}(t) \\ \bar{d}(t) - d(t) \end{bmatrix}$$

And from (14) and (15) one obtains

$$\begin{bmatrix} \underline{e}(t_k^+) \\ \bar{e}(t_k^+) \end{bmatrix} = \Gamma(L) \begin{bmatrix} \underline{e}(t_k) \\ \bar{e}(t_k) \end{bmatrix} + \tilde{F}(L)\psi(t_k) \quad (17)$$

where

$$\Gamma(L) = \begin{bmatrix} (I_n + LC)^+ & (I_n + LC)^- \\ (I_n + LC)^- & (I_n + LC)^+ \end{bmatrix}, \quad \tilde{F}(L) = \begin{bmatrix} (LF)^+ & (LF)^- \\ (LF)^- & (LF)^+ \end{bmatrix}.$$

*Remark 5.* The choice of matrices  $A^M$  and  $A^N$  in (12) is based on the Müller's existence theorem. In general, one can pick any Metzler matrix  $A^M$  and nonnegative matrix  $A^N$  that satisfy the constraint  $A = A^M - A^N$ . This choice guarantees the Metzler property of the matrix  $\mathcal{M}(A)$  in (16).

*Remark 6.* The matrices  $\tilde{E}$ ,  $\tilde{F}(L)$  and  $\Gamma(L)$  are structurally nonnegative. The matrix  $\mathcal{M}(A)$  is Metzler. Thus, if Assumption 1 is satisfied then the dynamics (16) and (17) are nonnegative.

### 3.1 | Event-triggered interval Observer formulation

In this subsection, we study the interval observer under the event-triggered mechanism (ETM) that we propose next.

Let us now consider an augmented error vector for the interval estimation as  $\xi = (\underline{e}, \bar{e})$ . The hybrid system for the system (16)-(17) is given by

$$\begin{cases} \dot{\xi}(t) = \mathcal{M}(A)\xi(t) + \tilde{E}\psi(t) & \forall \xi \in \mathcal{C} \\ \xi(t_k^+) = \Gamma(L)\xi(t_k) + \tilde{F}(L)\psi(t_k) & \forall \xi \in \mathcal{D} \end{cases} \quad (18)$$

Before providing the sets  $\mathcal{C}$  and  $\mathcal{D}$  that represent the triggering mechanism, we define the width of the estimated state enclosure and the width of the feasible domain of uncertainties, respectively, as follows:

$$\begin{aligned} \omega(t) &= \bar{x}(t) - \underline{x}(t) = \bar{e}(t) + \underline{e}(t) \\ \delta(t) &= \bar{d}(t) - \underline{d}(t) \end{aligned} \quad (19)$$

Let us now define the flow and jump sets for the system (18) as

$$\begin{aligned} \mathcal{C} &= \{(\omega, \delta) \in \mathbb{R}^n \times \mathbb{R}^{n_d} : |\omega(t)|_1 \leq \beta |\delta(t)|_1\} \\ \mathcal{D} &= \{(\omega, \delta) \in \mathbb{R}^n \times \mathbb{R}^{n_d} : |\omega(t)|_1 \geq \beta |\delta(t)|_1\} \end{aligned} \quad (20)$$

where  $\beta$  is a positive scalar to be tuned.

**Remark 7.** Note that the variable  $\delta(t)$  is known a priori and the estimate width  $\omega(t)$  can be estimated from the external information at  $t_k$ , i.e., the measurements  $y(t_k)$ . Consequently, we can consider that the proposed triggering mechanism (20) as an *implicit self-triggering mechanism*. It is implicit because the triggering time  $t_{k+1}$  is not given explicitly.

In order to analyze the stability of the estimation error, we use the  $\mathcal{L}_1$  stability (a particular case of the  $\mathcal{L}_p$  stability introduced in Definition 6 with  $p = 1$ ). To this aim, we consider the  $\mathcal{L}_1$ -gain of the operator  $\delta \rightarrow \omega$ , in other words the  $\mathcal{L}_1$ -gain from the width of the known interval of the exogenous input  $d(t)$  to the width of the estimated interval of  $x(t)$ . It is worth noting that the  $\mathcal{L}_1$ -gain of the operator  $\delta \rightarrow \omega$  is the same  $\mathcal{L}_1$ -gain of the operator  $\psi \rightarrow \xi$ . This property is true if and only if the variables  $\delta$ ,  $\omega$ ,  $\psi$  and  $\xi$  are nonnegative, which is structurally satisfied. From (19) one can simply get

$$\omega(t) = [I_n \ I_n] \xi(t); \quad \delta(t) = [I_{n_d} \ I_{n_d}] \psi(t). \quad (21)$$

Finding an expression of an upper bound  $\gamma_{\mathcal{L}_1}$  of the  $\mathcal{L}_1$ -gain of the operator  $\delta \rightarrow \omega$  allows to synthesize this upper bound in a way to minimize the effect of uncertainties width  $\delta$  onto the estimate width  $\omega$ . We can thus obtain an interval estimate as tight as possible.

To simplify the computation procedure in the sequel, we use the  $\mathcal{L}_1$ -gain of the operator  $\psi \rightarrow \xi$  instead of using the  $\mathcal{L}_1$ -gain of the operator  $\delta \rightarrow \omega$ . An equivalence of these two operators is based on the following equalities  $|\delta(t)|_1 = |\psi(t)|_1$  and  $|\omega(t)|_1 = |\xi(t)|_1$ , which is detailed in Appendix A.1.

The triggering condition in (20), defined as  $|\omega(t)|_1 \geq \beta |\delta(t)|_1$ , is equivalent to  $\begin{bmatrix} \xi(t) \\ \psi(t) \end{bmatrix}^\top \begin{bmatrix} \mathbb{1}_{2n} \\ -\beta \mathbb{1}_{2n_d} \end{bmatrix} \geq 0$ . Thus, the flow and jump sets (20) can be written as follows

$$\begin{aligned} C &= \left\{ (\xi, \psi) : \begin{bmatrix} \xi(t) \\ \psi(t) \end{bmatrix}^\top \begin{bmatrix} \mathbb{1}_{2n} \\ -\beta \mathbb{1}_{2n_d} \end{bmatrix} \leq 0 \right\} \\ D &= \left\{ (\xi, \psi) : \begin{bmatrix} \xi(t) \\ \psi(t) \end{bmatrix}^\top \begin{bmatrix} \mathbb{1}_{2n} \\ -\beta \mathbb{1}_{2n_d} \end{bmatrix} \geq 0 \right\} \end{aligned} \quad (22)$$

Let us now state the next contribution of this paper. The following theorem provides a design methodology of the ETM (22).

**Theorem 1.** Let Assumption 1 hold. For a given matrix  $L \in \mathbb{R}^{n \times n_y}$ , if there exist a nonnegative vector  $\lambda \in \mathbb{R}_{\geq 0}^{2n}$ , and nonnegative scalars  $\zeta_c, \zeta_D, \gamma_{\delta f}, \gamma_{\delta g}, \gamma_{\omega f}, \gamma_{\omega g}$  and  $\beta$ , satisfying the following inequalities

$$\begin{bmatrix} \mathcal{M}^\top(A)\lambda + (\gamma_{\omega f} - \zeta_c)\mathbb{1}_{2n} \\ \tilde{E}^\top \lambda - (\gamma_{\delta f} - \zeta_c \beta)\mathbb{1}_{2n_d} \end{bmatrix} \leq 0, \quad (23a)$$

$$\begin{bmatrix} \Gamma^\top(L)\lambda - \lambda + (\gamma_{\omega g} + \zeta_D)\mathbb{1}_{2n} \\ \tilde{F}^\top(L)\lambda - (\gamma_{\delta g} + \zeta_D \beta)\mathbb{1}_{2n_d} \end{bmatrix} \leq 0, \quad (23b)$$

then, the system (12)-(14) with the event-triggering mechanism (20) is a finite  $\mathcal{L}_1$ -gain interval observer for the system (10). Furthermore, the  $\mathcal{L}_1$ -gain from  $\delta$  to  $\omega$  is upper bounded by  $\gamma_{\mathcal{L}_1} = \gamma_{\delta} / \gamma_{\omega}$  where  $\gamma_{\delta} = \max\{\gamma_{\delta f}, \gamma_{\delta g}\}$  and  $\gamma_{\omega} = \min\{\gamma_{\omega f}, \gamma_{\omega g}\}$ .

*Proof.* This proof is split into two main part; the first part shows the observation error non-negativity, and the second one is about the  $\mathcal{L}_1$  stability and the performance of the interval observation error.

### Observation error non-negativity

This is based on Definition 3.

For given initial conditions  $\underline{x}(0)$  and  $\bar{x}(0)$  that satisfy  $\underline{x}(0) \leq x(0) \leq \bar{x}(0)$ , the initial values of the estimation error  $\underline{e}(0) = x(0) - \underline{x}(0)$  and  $\bar{e}(0) = \bar{x}(0) - x(0)$  are non-negative. By construction the matrix  $\mathcal{M}(A)$  and  $\tilde{E}$  are Metzler and non-negative matrices, respectively. In addition, based on Assumption 1, the vector  $\psi(t)$  is non-negative. Thus, the continuous dynamics (16) of the impulsive observer is non-negative between two successive measurement time instants (i.e.  $\forall t \in [t_k^+, t_{k+1}]$ ). To ensure the non-negativity of the estimation error for all  $t \in \mathbb{R}_+$  we add a condition that guarantees that the inclusion  $\underline{x}(t_k^+) \leq x(t_k) \leq \bar{x}(t_k^+)$  is satisfied provided that  $\underline{x}(t_k) \leq x(t_k) \leq \bar{x}(t_k)$ , that is, at measurement time instants the observer updates the values of the interval bounds by means of a correction jump represented by the discrete dynamics (17), which should stay non-negative after the jump. By construction, the matrices  $\tilde{F}(L)$  and  $\Gamma(L)$  are non-negative and based on Assumption 1, the vector  $\tilde{F}(L)\psi(t_k)$  is non-negative. This allows to preserve the ordering relation for the estimation error after experiencing the reset. Consequently, the errors  $\underline{e}(t)$ ,  $\bar{e}(t)$  of the system (18) are non-negative  $\forall t \geq 0$  provided that  $\underline{e}(0) \geq 0$ ,  $\bar{e}(0) \geq 0$ .



### $\mathcal{L}_1$ stability of the observation error

This is based on the non-negativity nature of the variables used, which allows us to use some positive system properties. In the sequel, we will thus use linear co-positive Lyapunov functions of the form  $V(\xi) = \xi^\top \lambda$  where  $\lambda \in \mathbb{R}_{\geq 0}^{2n}$  which can reduce the complexity of the design problem. First of all, we pick  $p = 1$  as a special case of the results obtained in<sup>13</sup> given in Definition 7. Based on the first part of the proof, we have  $\xi(t, j) \geq 0$ ,  $\forall(t, j) \in \text{dom}\xi$ , thus, the function  $V(\xi)$  is non-negative. By choosing  $c_2 = \max\{\lambda_i\}$ , the inclusion (9a) is satisfied.

Now, we first analyze the behavior of the interval observer between two successive measurements (the continuous dynamics of (18)). For simplicity of presentation, we drop the time index for all variables i.e.,  $|\omega(t)|_1 = |\omega|_1$ . The variation of the proposed Lyapunov function is given by

$$\langle \nabla V(\xi), \mathcal{M}(A)\xi + \tilde{E}\psi \rangle = \xi^\top \mathcal{M}^\top(A)\lambda + \psi^\top \tilde{E}^\top \lambda \quad (24)$$

Using the fact that  $|\omega|_1 = |\xi|_1 = \xi^\top \mathbb{1}_{2n}$  and  $|\delta|_1 = |\psi|_1 = \psi^\top \mathbb{1}_{2n_d}$  and by designing an upper bound of the  $\mathcal{L}_1$ -gain of the operator  $\delta \rightarrow \omega$  as defined by (9b), one can write

$$\langle \nabla V(\xi), \mathcal{M}(A)\xi + \tilde{E}\psi \rangle \leq -\gamma_{\omega f} |\xi|_1 + \gamma_{\delta f} |\psi|_1$$

Now using (24), we obtain

$$\xi^\top \mathcal{M}^\top(A)\lambda + \psi^\top \tilde{E}^\top \lambda \leq -\gamma_{\omega f} \xi^\top \mathbb{1}_{2n} + \gamma_{\delta f} \psi^\top \mathbb{1}_{2n_d} \quad (25)$$

and (25) can be represented under a vector form as follows

$$\begin{bmatrix} \xi \\ \psi \end{bmatrix}^\top \begin{bmatrix} \mathcal{M}^\top(A)\lambda + \gamma_{\omega f} \mathbb{1}_{2n} \\ \tilde{E}^\top \lambda - \gamma_{\delta f} \mathbb{1}_{2n_d} \end{bmatrix} \leq 0 \quad (26)$$

The inequality (26) should be satisfied when the observer is flowing (i.e.,  $\forall(\xi, \psi) \in C$  in (22)). By a similar reasoning to the S-procedure<sup>1</sup> used in<sup>13</sup>, Corollary 1, this is equivalent to

$$\begin{bmatrix} \xi \\ \psi \end{bmatrix}^\top \begin{bmatrix} \mathcal{M}^\top(A)\lambda + \gamma_{\omega f} \mathbb{1}_{2n} \\ \tilde{E}^\top \lambda - \gamma_{\delta f} \mathbb{1}_{2n_d} \end{bmatrix} - \zeta_c \begin{bmatrix} \xi \\ \psi \end{bmatrix}^\top \begin{bmatrix} \mathbb{1}_{2n} \\ -\beta \mathbb{1}_{2n_d} \end{bmatrix} \leq 0 \quad (27)$$

Based on the fact that  $\begin{bmatrix} \xi \\ \psi \end{bmatrix}^\top \begin{bmatrix} \xi \\ \psi \end{bmatrix} \geq 0$  then the inequality (27) holds if the inequality (23a) is satisfied.

Similarly, the stability condition for the discrete dynamics (9c), using the proposed Lyapunov function, is given as

$$[\Gamma(L)\xi + \tilde{F}(L)\psi]^\top \lambda - \xi^\top \lambda \leq -\gamma_{\omega g} \xi^\top \mathbb{1}_{2n} + \gamma_{\delta g} \psi^\top \mathbb{1}_{2n_d}, \quad \forall(\xi, \psi) \in \mathcal{D} \quad (28)$$

which is equivalent to

$$\begin{bmatrix} \xi \\ \psi \end{bmatrix}^\top \begin{bmatrix} \Gamma^\top(L)\lambda - \lambda + \gamma_{\omega g} \mathbb{1}_{2n} \\ \tilde{F}^\top(L)\lambda - \gamma_{\delta g} \mathbb{1}_{2n_d} \end{bmatrix} - \zeta_D \begin{bmatrix} \xi \\ \psi \end{bmatrix}^\top \begin{bmatrix} -\mathbb{1}_{2n} \\ \beta \mathbb{1}_{2n_d} \end{bmatrix} \leq 0 \quad (29)$$

Thus we have that (23b) implies (9c). And this concludes the proof.  $\square$

In the Theorem 1, the stability and the non-negativity of the interval observer error is proved. Another important property of the ETM is the existence of minimum inter-event time (MIET), but has not been analyzed yet. In fact, the proposed ETM cannot guarantee the existence of MIET. The argument for this remark is the following: if the width of the estimates is too large compared to the width of perturbation, i.e.,  $|\omega(t_k, k-1)|_1 \gg \beta |\delta(t_k, k-1)|_1$ , and if the jump correction cannot lead to the flow set where  $(\omega(t_k, k), \delta(t_k, k)) \notin C$ , then the ETM will exhibit a Zeno behaviour.

Next, we will tackle this problem by adding an auxiliary dynamical variable to the proposed ETM.

## 3.2 | Minimum inter-event time

The flow and jump dynamics of the hybrid system (18) are linear. Using the fact that linear systems are globally Lipschitz<sup>7</sup> we can bound the flow of (18) as follows<sup>2</sup>

$$|\xi(t)|_1 \leq \mathcal{K}_\xi (|\xi(t)|_1 + |\psi(t)|_1) \quad (30)$$

<sup>1</sup>The simple definition of the S-procedure is ; the inequality  $F_0(x) \geq 0 \quad \forall x$  such that  $F_1(x) \geq 0$  holds if  $\exists \zeta \geq 0$  such that  $\forall x, \quad F_0(x) - \zeta F_1(x) \geq 0$ . for details see<sup>22</sup>, Section 2.6.3

<sup>2</sup>The Lipschitz property is correct with any norm.<sup>7</sup>

where  $\mathcal{K}_\xi = \max\{|\mathcal{M}(A)|_\infty, |\tilde{E}|_\infty\}$ , and by the definition of the 1-norm<sup>3</sup> of  $\xi$  and  $\psi$  we can write

$$|\dot{\omega}(t)|_1 \leq \mathcal{K}_\xi(|\omega(t)|_1 + |\delta(t)|_1) \quad (31)$$

The new dynamic event-triggering mechanism that we propose is as follows

$$\begin{aligned} C_\eta &= \left\{ (\omega, \delta, \eta) \in \mathbb{R}^n \times \mathbb{R}^{n_d} : |\omega(t)|_1 \leq \beta|\delta(t)|_1 + \frac{\eta(t)}{\theta} \right\} \\ D_\eta &= \left\{ (\omega, \delta, \eta) \in \mathbb{R}^n \times \mathbb{R}^{n_d} : |\omega(t)|_1 \geq \beta|\delta(t)|_1 + \frac{\eta(t)}{\theta} \right\} \end{aligned} \quad (32)$$

where  $\eta$  is the state of an auxiliary scalar dynamical system given as follows

$$\begin{aligned} \dot{\eta}(t) &= -\alpha\eta(t) + \beta|\delta(t)|_1 - |\omega(t)|_1 \\ \eta(0) &\geq |\omega(0)|_1 - \beta|\delta(0)|_1 \end{aligned} \quad (33)$$

where the initial condition of the auxiliary system is chosen in a way to initialize the observer in the flow set  $C_\eta$ .

**Remark 8.** The solution of the auxiliary dynamics (33) with the event-triggering condition (32) is nonnegative. Its nonnegativity is proved similarly to<sup>4</sup>, Lemma 1. The proof is as follows: When the hybrid system is flowing (i.e.  $(\omega, \delta) \in C_\eta$ ) the auxiliary variable dynamics (33) satisfies the inequality  $\dot{\eta} \geq -\alpha\eta - \frac{1}{\theta}\eta$ . And using the comparison theorem, the solution of the obtained dynamics is lower bounded by the solution of  $\dot{\eta} = -\alpha\eta - \frac{1}{\theta}\eta$  which has a nonnegative dynamics provided that  $(\omega, \delta) \in C_\eta$  and the initial condition  $\eta(0)$  satisfies (33). The initial condition of  $\eta(t)$  is the main difference between the estimation problem in this study and the control one in<sup>4</sup> (e.g., in general the initialization of estimate which width is  $\omega(0)$  cannot always satisfy  $(\omega, \delta) \in C_\eta$  if  $\eta(0) = 0$ ).

In the hybrid framework in general we have the choice to flow or jump when the variables are in  $C \cap D$ , but in our study we force the ETM to jump when  $(\omega, \delta) \in C_\eta \cap D_\eta$ . In the following, we will show how the use of this event-triggering mechanism can ensure stability of the estimation error and, under mild conditions, guarantees the existence of MIET.

**Assumption 2.** Let  $\mathcal{K}_\delta < \infty$  be a bounded positive scalar. The width of perturbation  $\delta(t)$  defined in (19) satisfies

$$|\dot{\delta}(t)|_1 \leq \mathcal{K}_\delta |\delta(t)|_1. \quad (34)$$

**Corollary 1.** Let Assumption 1 and 2 hold. For a given matrix  $L \in \mathbb{R}^{n \times n_y}$  and a given positive scalar  $\theta$ , if there exist a nonnegative vector  $\lambda \in \mathbb{R}_{\geq 0}^{2n}$ , and nonnegative scalars  $\zeta_c, \zeta_D, \gamma_{\delta f}, \gamma_{\delta g}, \gamma_{\omega f}, \gamma_{\omega g}, \alpha$  and  $\beta$ , satisfying the following inequalities

$$\begin{bmatrix} \mathcal{M}^\top(A)\lambda + (-1 + \gamma_{\omega f} - \zeta_c)\mathbb{1}_{2n} \\ \tilde{E}^\top \lambda + (\beta - \gamma_{\delta f} + \zeta_c \beta)\mathbb{1}_{2n_d} \\ -\alpha + \zeta_c \frac{1}{\theta} \end{bmatrix} \leq 0, \quad (35a)$$

$$\begin{bmatrix} \Gamma^\top(L)\lambda - \lambda + (\gamma_{\omega g} + \zeta_D)\mathbb{1}_{2n} \\ \tilde{F}^\top(L)\lambda - (\gamma_{\delta g} + \zeta_D \beta)\mathbb{1}_{2n_d} \end{bmatrix} \leq 0, \quad (35b)$$

$$\gamma_{\delta g} - \beta\gamma_{\omega g} \leq 0, \quad (35c)$$

then, the hybrid system (18), (32)–(33) is finite  $\mathcal{L}_1$ -gain stable. Thus, the system (12)–(14) with the triggering mechanism (32)–(33) is a finite  $\mathcal{L}_1$ -gain interval observer for the system (10), where the ETM guarantees the existence of positive inter-measurement times. Furthermore, the  $\mathcal{L}_1$ -gain from  $\delta$  to  $\omega$  is upper bounded by  $\gamma_{\mathcal{L}_1} = \gamma_\delta / \gamma_\omega$  where  $\gamma_\delta = \max\{\gamma_{\delta f}, \gamma_{\delta g}\}$ ,  $\gamma_\omega = \min\{\gamma_{\omega f}, \gamma_{\omega g}\}$ .

*Proof.* The nonnegativity property of  $\eta(t)$  is provided in Remark 8. The estimate error  $\xi(t)$  is nonnegative as shown in Theorem 1. Based on the nonnegativity property of  $\eta$  and  $\xi$ , we can study the stability using a new Lyapunov function of the form  $W(\xi, \eta) = V(\xi) + \sigma\eta = \xi^\top \lambda + \sigma\eta$ , with  $\eta \in \mathbb{R}_+$ , where  $\sigma$  is a positive constant. Without loss of generality, we pick  $\sigma = 1$ .

The function  $V(\xi)$  is no longer a Lyapunov function for the event-triggered mechanism (32) because the decrease of the auxiliary variable  $\eta$  allows the function  $V(\xi)$  to increase while  $W(\xi, \eta)$  decreases.

<sup>3</sup>This norm is for a vector at time  $t$ . It defers from the signal norm  $\mathcal{L}_1$ .

Similarly to the proof of Theorem 1, the stability condition on the continuous dynamics is given as follows

$$\langle \nabla W(\xi), \begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} \rangle = \xi^\top \mathcal{M}^\top(A) \lambda + \psi^\top \tilde{E}^\top \lambda + (-\alpha \eta + \beta \psi^\top \mathbb{1}_{2n_d} - \xi^\top \mathbb{1}_{2n}) \leq -\gamma_{\omega f} \xi^\top \mathbb{1}_{2n} + \gamma_{\delta f} \psi^\top \mathbb{1}_{2n_d} \quad \forall (\xi, \psi) \in \mathcal{C}_\eta \quad (36)$$

which is equivalent to

$$\begin{bmatrix} \xi \\ \psi \\ \eta \end{bmatrix}^\top \begin{bmatrix} \mathcal{M}^\top(A) \lambda - \mathbb{1}_{2n} + \gamma_{\omega f} \mathbb{1}_{2n} \\ \tilde{E}^\top \lambda + \beta \mathbb{1}_{2n_d} - \gamma_{\delta f} \mathbb{1}_{2n_d} \\ -\alpha \end{bmatrix} - \zeta_c \begin{bmatrix} \xi \\ \psi \\ \eta \end{bmatrix}^\top \begin{bmatrix} \mathbb{1}_{2n} \\ -\beta \mathbb{1}_{2n_d} \\ -\frac{1}{\theta} \end{bmatrix} \leq 0 \quad (37)$$

Finally, if the inequality (35a) is satisfied then the inequality (37) holds.

It is worth noting that jumps do not impact the auxiliary variable  $\eta$  (i.e.,  $\eta(t_k, k) = \eta(t_k, k-1)$ ). Similarly, the stability condition on the discrete dynamics can thus be written as

$$W(\xi^+, \eta^+) - W(\xi, \eta) = V(\xi^+) - V(\xi) \leq -\gamma_{\omega g} |\omega|_1 + \gamma_{\delta g} |\delta|_1, \quad \forall (\omega, \delta) \in \mathcal{D}_\eta \quad (38)$$

which is satisfied if the inequality (35b) holds.

In order to investigate the existence of MIET, we study the variation of the following ratio:

$$\kappa(t, j) = \frac{|\omega(t, j)|_1}{\beta |\delta(t, j)|_1 + \frac{\eta(t, j)}{\theta}} \quad (39)$$

It is the ratio between the 1-norm of the estimation error width and the threshold  $\beta |\delta(t, j)|_1 + \frac{\eta(t, j)}{\theta}$ . Based on the ETM (32), this ratio is larger than or equal to 1 when  $(\omega, \delta, \eta) \in \mathcal{D}_\eta$ , and it is lower than 1 when  $(\omega, \delta, \eta) \in \mathcal{C}_\eta$ . To simplify the analysis we consider that at times of jump, this ratio satisfies  $\kappa(t_k, k-1) = 1$  which fits the triggering condition in (32). After the jump, it will be reset to  $\kappa(t_k, k) \in [0, 1)$  based on the condition that the width of estimate is contracted using measurement.

This condition is proved in two steps as following: In the first step, we show that the width of the estimation error is contracting at jump times. Then, in the second step, by analyzing the dynamics of the ratio  $\kappa(t, j)$  we will show how this contraction can guarantee the existence of a lower bound of the inter-event times.

**Step 1:** Contraction of the estimate width after jump.

As mentioned above the stability condition at jump instants satisfies (38). Replacing the event-triggered condition  $|\omega|_1 \geq \beta |\delta|_1 + \frac{\eta}{\theta}$  in the right hand side of the inequality (38), one gets

$$\xi^{+\top} \lambda - \xi^\top \lambda \leq -(\gamma_{\omega g} - \frac{\gamma_{\delta g}}{\beta}) |\omega|_1 - \frac{\gamma_{\delta g}}{\beta \theta} \eta \quad (40)$$

Thus, satisfying the inequality (35c) implies that  $\xi^{+\top} \lambda - \xi^\top \lambda < 0$ . Based on the non-negativity of both the vector  $\lambda$  and the estimation error  $\xi$ , one can deduce that  $\xi$  is decreasing at measurement times.

**Step 2:** The existence of a positive time interval that let the estimate width to increase before entering the jump set  $\mathcal{D}_\eta$ .

Now, we return to study the dynamics of the ratio  $\kappa(t, j)$  in between two successive measurement times.

$$\begin{aligned} \frac{d}{dt} \kappa(t) &= \frac{d}{dt} \frac{|\omega(t)|_1}{\beta |\delta(t)|_1 + \frac{\eta(t)}{\theta}} = \frac{d}{dt} \frac{|\xi(t)|_1}{\beta |\delta(t)|_1 + \frac{\eta(t)}{\theta}} \\ &= \frac{\dot{\xi}(t)^\top \mathbb{1}_{2n} (\beta |\delta(t)|_1 + \frac{\eta(t)}{\theta}) - |\xi(t)|_1 (\beta \dot{\delta}(t)^\top \mathbb{1}_{n_d} + \frac{\dot{\eta}(t)}{\theta})}{(\beta |\delta(t)|_1 + \frac{\eta(t)}{\theta})^2} \end{aligned}$$

Using the property (30), the dynamics (33), and Assumption 2, one gets

$$\begin{aligned}
\frac{d}{dt}\kappa(t) &\leq \frac{\mathcal{K}_\xi(|\omega(t)|_1 + |\delta(t)|_1)}{(\beta|\delta(t)|_1 + \frac{\eta(t)}{\theta})} + \frac{|\omega(t)|_1\beta\mathcal{K}_\delta|\delta(t)|_1}{(\beta|\delta(t)|_1 + \frac{\eta(t)}{\theta})^2} - \frac{|\omega(t)|_1(-\alpha\eta(t) + \beta|\delta(t)|_1 - |\omega(t)|_1)}{\theta(\beta|\delta(t)|_1 + \frac{\eta(t)}{\theta})^2} \\
&\leq \mathcal{K}_\xi \frac{|\omega(t)|_1}{(\beta|\delta(t)|_1 + \frac{\eta(t)}{\theta})} + \frac{\mathcal{K}_\xi}{\beta} + \frac{|\omega(t)|_1\beta\mathcal{K}_\delta|\delta(t)|_1}{(\beta|\delta(t)|_1 + \frac{\eta(t)}{\theta})^2} + \frac{|\omega(t)|_1\alpha\eta(t)}{\theta(\beta|\delta(t)|_1 + \frac{\eta(t)}{\theta})^2} - \frac{|\omega(t)|_1(\beta|\delta(t)|_1)}{\theta(\beta|\delta(t)|_1 + \frac{\eta(t)}{\theta})^2} + \frac{|\omega(t)|_1^2}{\theta(\beta|\delta(t)|_1 + \frac{\eta(t)}{\theta})^2} \\
&\leq \frac{\mathcal{K}_\xi}{\beta} + \left[ \mathcal{K}_\xi + \frac{\alpha\eta(t)}{\theta(\beta|\delta(t)|_1 + \frac{\eta(t)}{\theta})} + \frac{\beta\mathcal{K}_\delta|\delta(t)|_1}{(\beta|\delta(t)|_1 + \frac{\eta(t)}{\theta})} - \frac{(\beta|\delta(t)|_1)}{\theta(\beta|\delta(t)|_1 + \frac{\eta(t)}{\theta})} \right] \frac{|\omega(t)|_1}{(\beta|\delta(t)|_1 + \frac{\eta(t)}{\theta})} + \frac{|\omega(t)|_1^2}{\theta(\beta|\delta(t)|_1 + \frac{\eta(t)}{\theta})^2} \\
&\leq \frac{\mathcal{K}_\xi}{\beta} + \left[ \mathcal{K}_\xi + \alpha + \mathcal{K}_\delta - \frac{1}{\theta} \right] \frac{|\omega(t)|_1}{(\beta|\delta(t)|_1 + \frac{\eta(t)}{\theta})} + \frac{|\omega(t)|_1^2}{\theta(\beta|\delta(t)|_1 + \frac{\eta(t)}{\theta})^2} \\
&\leq \frac{\mathcal{K}_\xi}{\beta} + [\mathcal{K}_\xi + \alpha + \mathcal{K}_\delta]\kappa(t) + \frac{1}{\theta}\kappa^2(t)
\end{aligned}$$

Based on the fact that  $0 \leq \kappa(t) < 1$ , one can simplify the above inequality as following

$$\frac{d}{dt}\kappa(t) \leq \frac{\mathcal{K}_\xi}{\beta} + [\mathcal{K}_\xi + \alpha + \mathcal{K}_\delta + \frac{1}{\theta}]\kappa(t) \quad (41)$$

Thus, an upper bound trajectory for the ratio  $\kappa(t)$ ,  $\forall t \in [t_k, t_{k+1}]$  is given by

$$\dot{\bar{\kappa}} = a + b\bar{\kappa} \quad (42)$$

where  $a = \frac{\mathcal{K}_\xi}{\beta}$  and  $b = \mathcal{K}_\xi + \alpha + \mathcal{K}_\delta + \frac{1}{\theta}$  with  $\bar{\kappa}(t_k, k) = \kappa(t_k) < 1$  and  $\bar{\kappa}(t_{k+1}, k) = \kappa(t_{k+1}, k) = 1$ . By using the fact that  $\kappa(t, j)$  is a monotone increasing solution function as shown by lemma 1 of Appendix A.2, one can deduce that the ratio  $\kappa(t, j)$  solution to (42) guarantees the existence of  $\tau_{min}$  such that  $0 < \tau_{min} \leq t_{k+1} - t_k$  for all  $k \in \mathbb{N}$  when initial and final conditions are given as  $\bar{\kappa}(t_k, k) = \kappa(t_k) \in [0, 1)$ ,  $\bar{\kappa}(t_{k+1}, k) = 1$ . Consequently, as the solution  $\bar{\kappa}(t, j)$  is an over-approximation of the ratio  $\kappa(t, j)$ , the time  $\tau_{min}$  is the MIET for the ETM (32).  $\square$

The previous result shows the event-triggered mechanism design for a pre-calculated observer gain. In the following section we will show how to co-design the ETM and the observer gain.

## 4 | CO-DESIGN OF EVENT-TRIGGERED MECHANISMS AND INTERVAL OBSERVER GAINS

In this section, we will co-design the ETM and the observation gain using an over-approximate of the reset matrix  $\Gamma(L)$  of the estimation error dynamics (18). This reset matrix is not easy to synthesized due to the non-smooth operator in (17) that use the gain  $L$ . To tackle this problem we propose to use a nonnegative realization of the matrix  $[I_n + LC]$  to over-approximate the reset matrix.

**Proposition 2.** Consider the following nonnegative discrete-time system

$$\chi_D(k+1) = A_D\chi_D(k) \quad (43)$$

where  $\chi_D \in \mathbb{R}^n$  is the state variables, with  $A_D \in \mathbb{R}_{\geq 0}^{n \times n}$ . Let assume that the matrix  $A_D$  is Schur stable<sup>4</sup>. If there exist two nonnegative matrices  $A_d, E_d \in \mathbb{R}_{\geq 0}^{n \times n}$  such that  $A_D = A_d + E_d$ , then  $\chi_D(k)$  solution to (43) and  $\chi_d(k)$  solution to the system  $\chi_d(k+1) = A_d\chi_d(k)$  satisfy  $\chi_D(k) \geq \chi_d(k) \forall k \in \mathbb{N}$  provided that  $\chi_D(0) \geq \chi_d(0) \geq 0$ .

*Proof.* Starting from the solution sequence  $\chi_D(k)$ , we have

$$\chi_D(k+1) = (A_d + E_d)\chi_D(k) = A_d\chi_D(k) + E_d\chi_D(k), \quad (44)$$

Based on the nonnegativity of the matrices  $E_d$  and  $A_d$  and the initial conditions  $\chi_d(0) \geq \chi(0) \geq 0$ , one have that the system  $\chi_D(k+1) = A_D\chi_D(k)$  has a nonnegative dynamics and the term  $E_d\chi_D(k)$  is nonnegative. Thus, from (44) one gets

$$\chi_D(k+1) \geq A_d\chi_D(k) \geq A_d\chi_d(k) = \chi_d(k+1)$$

<sup>4</sup>All its eigenvalues are contained in the open unit disk in the complex plane.

and this concludes the proof.  $\square$

**Proposition 3.** Let  $H \in \mathbb{R}^{n \times n}$ , for any two nonnegative matrices  $H_p, H_n \in \mathbb{R}_{\geq 0}^{n \times n}$  satisfy  $H = H_p - H_n$ , there exists a nonnegative matrix  $\Delta \in \mathbb{R}_{\geq 0}^{n \times n}$  such that  $H_p = (H^+ + \Delta)$  and  $H_n = (H^- + \Delta)$ .

*Proof.* For any element  $H_{ij}$  of the matrix  $H$ , if  $H_{ij} \leq 0$  then  $H_{ij}^- = |H_{ij}|$  and  $H_{ij}^+ = 0$ , and we have  $H_{ij} = H_{pij} - H_{nij} = (+\Delta) - (H_{ij}^- + \Delta)$ , consequently, the condition  $H_{pij} \geq 0$  implies  $\Delta \geq 0$ . The same property for  $H_{ij} \geq 0$ .  $\square$

In the following Theorem, we will provide a co-design methodology of the observer gain and some parameters of the event-triggered mechanism. Comparing to Corollary 1, in the latter one the interval observer gains are supposed to be given a priori, but here we co-design them along with the ETM.

**Theorem 2.** Let Assumption 1 hold, if there exist a matrix  $L \in \mathbb{R}^{n \times n_y}$ , and nonnegative matrices  $G_p, G_n \in \mathbb{R}_{\geq 0}^{n \times n}$  and  $R_p, R_n \in \mathbb{R}_{\geq 0}^{n \times n_d}$ , a nonnegative vector  $\lambda \in \mathbb{R}_{\geq 0}^{2n}$ , and nonnegative scalars  $\zeta_c, \zeta_D, \gamma_{\delta f}, \gamma_{\delta g}, \gamma_{\omega f}, \gamma_{\omega g}$  and  $\beta$ , satisfying inequalities (35a) and (35c) and the following inequality

$$\begin{bmatrix} \Gamma^\top(G_p, G_n)\lambda - \lambda + (\gamma_{\omega g} + \zeta_D)\mathbb{1}_{2n} \\ \tilde{F}^\top(R_p, R_n)\lambda - (\gamma_{\delta g} + \zeta_D\beta)\mathbb{1}_{2n_d} \end{bmatrix} \leq 0, \quad (45a)$$

$$G_p - G_n = I_n + LC, \quad (45b)$$

$$R_p - R_n = LF, \quad (45c)$$

where

$$\Gamma(G_p, G_n) = \begin{bmatrix} G_p & G_n \\ G_n & G_p \end{bmatrix}, \quad \tilde{F}(R_p, R_n) = \begin{bmatrix} R_p & R_n \\ R_n & R_p \end{bmatrix}$$

Then, the system (12)-(14) with the event-triggered mechanism (32) is a finite  $\mathcal{L}_1$ -gain interval observer for the system (10), where the ETM (32), (33) guarantees the existence of positive inter-measurement times. Furthermore, the  $\mathcal{L}_1$ -gain from  $\delta$  to  $\omega$  is upper bounded by  $\gamma_{\mathcal{L}_1}$  given in Theorem 1.

*Proof.* The difference between Corollary 1 and Theorem 2 is the design of the observer gain  $L$ . Thus we need to prove that the constraints (45) imply (35b). Pick  $G = [I_n + LC]$  and  $R = LF$ , based on equations 45b and (45c), an upper bound of the estimation error of the correction part (14) can be written as follows:

$$\begin{cases} \underline{x}(t_k^+) = G_p \underline{x}(t_k) - G_n \bar{x}(t_k) + R_p \underline{d}(t_k) - R_n \bar{d}(t_k) - Ly(t_k) \\ \bar{x}(t_k^+) = G_p \bar{x}(t_k) - G_n \underline{x}(t_k) + R_p \bar{d}(t_k) - R_n \underline{d}(t_k) - Ly(t_k) \end{cases} \quad \forall k \in \mathbb{N} \quad (46)$$

Satisfying inequality (45a), the correction part (46) with the obtained parameters  $G_p, G_n, R_p$  and  $R_n$  can ensure the stability and the positivity of the interval observer error as given in Corollary 1.

Using the result of Proposition 3, the existence of nonnegative matrices  $G_p, G_n, R_p$  and  $R_n$  satisfying (45b) implies the existence of nonnegative matrices  $\Delta_G$  and  $\Delta_R$  such that  $G_p = (I_n + LC)^+ + \Delta_G$ ,  $G_n = (I_n + LC)^- + \Delta_G$ ,  $R_n = (LF)^- + \Delta_R$  and  $R_p = (LF)^+ + \Delta_R$ . Consequently, the estimation error at jump (17) can be seen as

$$\begin{aligned} \Gamma(G_p, G_n) &= \begin{bmatrix} (I_n + LC)^+ & (I_n + LC)^- \\ (I_n + LC)^- & (I_n + LC)^+ \end{bmatrix} + \begin{bmatrix} \Delta_G & \Delta_G \\ \Delta_G & \Delta_G \end{bmatrix} = \Gamma(L) + \begin{bmatrix} \Delta_G & \Delta_G \\ \Delta_G & \Delta_G \end{bmatrix} \\ \tilde{F}(R_p, R_n) &= \tilde{F}(L) + \begin{bmatrix} \Delta_R & \Delta_R \\ \Delta_R & \Delta_R \end{bmatrix} \end{aligned}$$

Now, based on the property of uncertain discrete time system given in Proposition 2, the estimation error at jump of the correction part (14) is upper bounded by the one of the correction part (46). Finally, the inequality (45) implies the inequality (35b) which allows to implement the correction part given by (14). This concludes the proof.  $\square$

**Remark 9.** It is worth noting that the matrices  $G_p, G_n, R_p$  and  $R_n$  are only intermediate variables allowing the synthesis of the interval observer gains  $L$ .

## 5 | ILLUSTRATIVE EXAMPLE

In order to illustrate the performance of the proposed observer, we consider the following example.

## 5.1 | Double spring-mass-damper system

Consider a mechanical system consisting of two masses  $m_1$  and  $m_2$  that are sliding over an horizontal surface. Suppose that the masses are attached to one another, and to two immovable walls, by means of three horizontal springs of stiffness constants " $k_1$ ,  $k_2$  and  $k_3$ " and dampers of damping ratio " $c_1$ ,  $c_2$  and  $c_3$ ".

Let  $x^\top = [x_{m_1} \dot{x}_{m_1} x_{m_2} \dot{x}_{m_2}]$  be the state variables containing the position and the velocity of each mass and  $u^\top = [f_{m_1} f_{m_2}]$  be the force applied to the object, respectively. We introduce an LTI model (10) for the double spring-mass-damper system as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k_1+k_2)}{m_1} & -\frac{(c_1+c_2)}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_2} & \frac{c_2}{m_2} & -\frac{(k_2+k_3)}{m_2} & -\frac{(c_2+c_3)}{m_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}, \quad E = \begin{bmatrix} 0.1 & -0.2 \\ -0.7 & 0.6 \\ 0.2 & -0.2 \\ -0.5 & 0.6 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0.6 & -0.8 \\ -0.4 & 0.5 \end{bmatrix}$$

with  $d(t) = [d_1(t) d_2(t)]^\top$  is the disturbance which is assumed unknown-but-bounded  $-\bar{d} \leq d(t) \leq \bar{d}$  with  $\bar{d} = [0.5 \ 0.5]^\top$ .

The practical parameters of the system are given as  $m_1 = 0.6\text{kg}$ ,  $m_2 = 1\text{kg}$ ,  $k_1 = k_2 = k_3 = 1\text{N/m}$  and  $c_1 = 2\text{Ns/m}$ ,  $c_2 = 1.4\text{Ns/m}$ ,  $c_3 = 1.2\text{Ns/m}$ .

The dynamics of the interval estimation error in between two consecutive measurements defined in (16) has the following matrices

$$A^M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{(c_1+c_2)}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_2} & \frac{c_2}{m_2} & 0 & -\frac{(c_2+c_3)}{m_2} \end{bmatrix}, \quad A^N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{(k_1+k_2)}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(k_2+k_3)}{m_2} & 0 \end{bmatrix}.$$

Note that although the plant model is stable, the state estimation given by the predictor in between measurements is unstable.

The synthesis problem of the observer gain in Theorem 2 is solved using the YALMIP toolbox<sup>23</sup> based on the FMINCON solver. For  $\theta = 2$ , the solution of the co-design of the event-triggered mechanism and the observer gain is as follows: the obtained coefficients of the ETM (33) are  $\alpha = 1.3081$  and  $\beta = 3.9244$ , and the computed observation gain matrix is

$$L = \begin{bmatrix} -0.4535 & -0.0558 \\ 0 & 0 \\ -0.0528 & -0.5166 \\ 0 & 0 \end{bmatrix}$$

The upper bound of the  $\mathcal{L}_1$ -gain from  $\delta$  to  $\omega$  is found as  $\gamma_{\mathcal{L}_1} = 190$ .

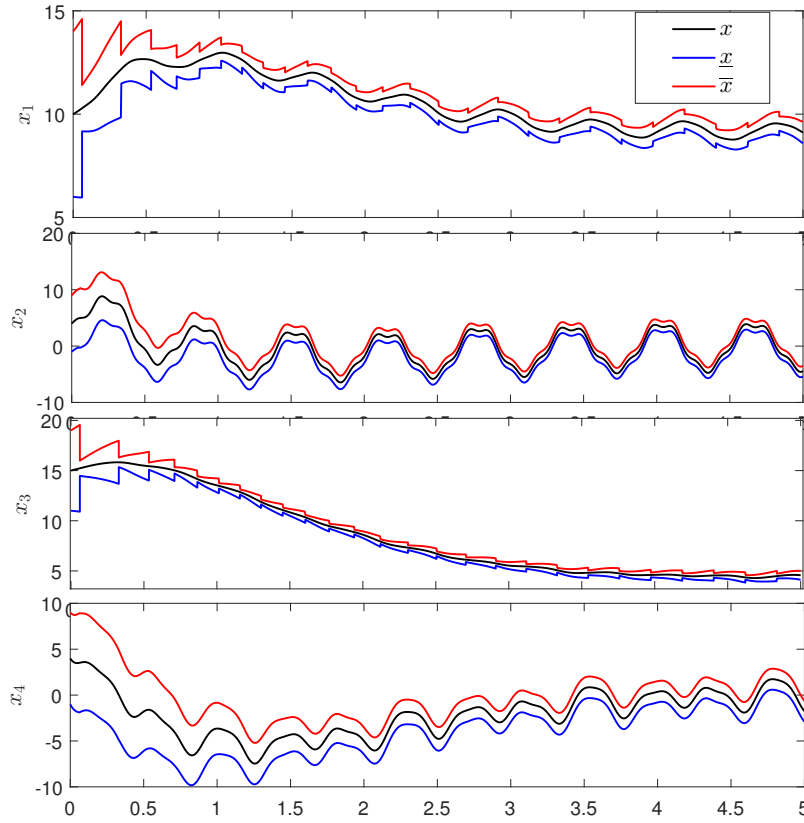
For generating the pseudo-actual data, the system inputs are taken as  $f_{m_1}(t) = 14[1 + 2\sin(10t) + \cos(40t)]$ ,  $f_{m_2}(t) = 10[2\sin(15t) + \sin(30t)]$ , and the initial values of the system state taken as  $x(0) = [10 \ 4 \ 15 \ 4]^\top$ .

The observer uses the following lower and upper bounds:  $\underline{x}(0) = [6 \ -1 \ 11 \ -1]^\top$  and  $\bar{x}(0) = [14 \ 9 \ 19 \ 9]^\top$ , respectively. The simulation results are given in Figure 1 and 2. In Figure 1, it is noticeable that the correction part of the interval observer contracts the estimate bounds, even though the open-loop dynamics of the interval estimator error is unstable. This behaviour corresponds to the condition given in Corollary 1.

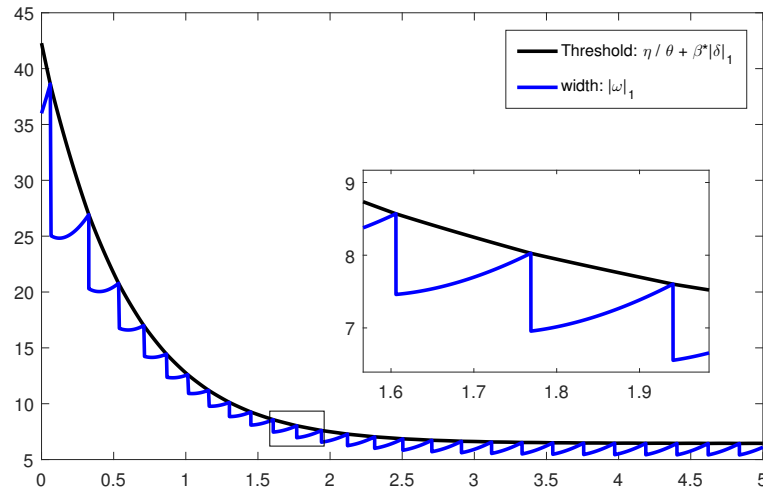
In Figure 2, we can see that the observer triggers the measurements whenever the norm of the width violates the dynamic threshold as described by (32). We can see also that, after jump, the width is strictly inferior than its threshold which fits the condition of Corollary 1. Moreover, this condition allows the existence of positive inter-measurement times. From the simulation, the MIET obtained for this example is  $\tau_{min} = 0.0609 \leq t_{k+1} - t_k$  while the maximum inter-measurement time is  $t_{k+1} - t_k \leq \tau_{max} = 0.2668$ .

## 6 | CONCLUSIONS

In this paper, event-triggered interval observers for linear continuous-time systems are proposed. The proposed event-triggered mechanism is based on the positivity property of the interval observation errors. Moreover, the existence of positive inter-measurement times is guaranteed. The proposed observers ensure also a finite  $\mathcal{L}_1$ -gain between the width of perturbation



**FIGURE 1** Simulation results for the double spring-mass-damper system: the upper and the lower estimate bounds for the masses position ( $x_1, x_3$ ), and masses velocity ( $x_2, x_4$ ).



**FIGURE 2** The evolution of the triggering mechanism : (blue) the width of the state estimate, (black) the threshold for the width.

bounds and the width of the estimated state intervals . Future work can focus on the design of stabilizing control law based on the proposed interval observers, and the extension of these approaches to other classes of uncertain systems such that linear parameter variant (LPV) and non linear systems.

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## APPENDIX

### A

#### A.1 Relation between errors $\psi(t)$ , $\xi(t)$ and widths $\delta(t)$ , $\omega(t)$ norms

Suppose that Assumption 1 is satisfied, then we have  $\psi(t) = \begin{bmatrix} d(t) - \underline{d}(t) \\ \bar{d}(t) - d(t) \end{bmatrix} \geq 0$ . By definition  $|\delta(t)|_1 = \sum_{i=1}^{n_d} |\delta_i(t)| = \mathbb{1}_{n_d}^\top \delta(t)$  and using (21) we obtain

$$\begin{aligned} |\delta(t)|_1 &= \left| [I_{n_d} \ I_{n_d}] \psi(t) \right|_1 \\ &= \left| [I_{n_d} \ I_{n_d}] \begin{bmatrix} d(t) - \underline{d}(t) \\ \bar{d}(t) - d(t) \end{bmatrix} \right|_1 \\ &= \left| (d(t) - \underline{d}(t)) + (\bar{d}(t) - d(t)) \right|_1 \\ &= \left| \bar{d}(t) - \underline{d}(t) \right|_1 \end{aligned}$$

and

$$\begin{aligned} |\psi(t)|_1 &= \left| \begin{bmatrix} d(t) - \underline{d}(t) \\ \bar{d}(t) - d(t) \end{bmatrix} \right|_1 \\ &= \left| d(t) - \underline{d}(t) \right|_1 + \left| \bar{d}(t) - d(t) \right|_1 \\ &= \mathbb{1}_{n_d}^\top [d(t) - \underline{d}(t)] + \mathbb{1}_{n_d}^\top [\bar{d}(t) - d(t)] \\ &= \mathbb{1}_{n_d}^\top [(d(t) - \underline{d}(t)) + (\bar{d}(t) - d(t))] = \left| \bar{d}(t) - \underline{d}(t) \right|_1 \end{aligned}$$

Thus, it is explicit that  $|\delta(t)|_1 = |\psi(t)|_1$ . By the same steps, based on the inclusions  $\underline{x} \leq x \leq \bar{x}$ , we deduce that  $|\omega(t)|_1 = |\xi(t)|_1$ .

#### A.2 Integral of rational functions

**Lemma 1.** Given the differential equation  $\dot{\bar{\kappa}} = a + b\bar{\kappa}$  with  $\bar{\kappa}(t_0) = \bar{\kappa}_0$ ,  $\forall t_0 \geq 0$ . If  $a$  and  $b$  are positive, then the differential equation has a monotone increasing solution  $\bar{\kappa}(t, \bar{\kappa}_0)$ ,  $\forall t \geq t_0$ .

*Proof.* The system can be rewritten as

$$\frac{d\bar{\kappa}}{dt} = a + b\bar{\kappa}$$

By integration from  $\bar{\kappa}_0$  to  $\bar{\kappa}(t, \bar{\kappa}_0)$ , one gets

$$t - t_0 = \left[ \ln(|a + b\bar{\kappa}|) \right]_{\bar{\kappa}(t_0)}^{\bar{\kappa}(t)}$$

The function  $f_1(\bar{\kappa}) = \ln(|a + b\bar{\kappa}|)$  is an increasing function, thus, for any  $t_1, t_2 \in \mathbb{R}_+$  the relation  $\bar{\kappa}(t_1) < \bar{\kappa}(t_2)$  implies  $t_1 < t_2$ .  $\square$