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Djahid Rabehi[a], Nacim Meslem[b], Nacim Ramdani[a,*]

[a] *Université d'Orléans, INSA Centre Val de Loire (CVL), PRISME, EA 4229, F45072 Orléans, France.*
[b] *Université de Grenoble Alpes, CNRS, GIPSA-lab, F-38000 Grenoble, France.*

Abstract

This paper presents simple design methods of interval observers for discrete-time linear switched systems in the presence of output noise and system perturbation. The interval observers are designed using the internal positivity of the system without need to a coordinate transformation. This approach is proposed for arbitrary switching signals and it can be generalized to dwell-time switching rules. The observation gain that guarantees both stability and positivity of the interval estimate errors is synthesized by solving a Linear Matrix Inequality (LMI) feasibility problem. The theoretical result is supported by numerical simulations.

Keywords: Interval observers, Discrete-time linear switched systems, LMI design.

1. Introduction

In many engineering areas, state estimation for dynamical systems in the presence of system uncertainties (modeling error, state disturbance and measurement noise) is a fundamental and challenging problem. Theoretically, from the available data of the systems' inputs and outputs, observers have the ability to reconstruct the unknown internal state of these systems. However, in practice, the accuracy of the estimated state depends mainly on the quality of the models used to design state observers and the precision of the available data. In the case of linear systems, under some statistical assumptions on the uncertain parts of the systems, numerous approaches have been proposed in the literature to solve efficiently this problem. For instance, the classical Kalman filter [1], [2] for discrete-time linear systems and Linear Quadratic Optimal Luenberger observer [3], [4] for continuous-time linear systems. However, usually in practice, systems uncertainties are poorly-known and no probability density functions could describe them accurately. To overcome this problem, the concept of interval observer is introduced in [5], [6], [7], [8], [9] for biological systems, where the objective is to estimate trajectory tubes that contain, in a guaranteed way, the actual state variables of the system. After these seminal works, the interval observers design problem has witnessed increasing interest. Many approaches have been

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* Corresponding author
Email addresses: djahid.rabehi@univ-orleans.fr (Djahid Rabehi), nacim.meslem@grenoble-inp.fr (Nacim Meslem), nacim.ramdani@univ-orleans.fr (Nacim Ramdani)

proposed to deal with different classes of continuous/discrete-time dynamical systems such that linear systems [10], [11], [12], [13], [14], LTV/LPV systems [15], [16], [17], [18], and nonlinear systems [19], [10], [20], [21].

Recently, the concept of interval observer has been extended to classes of Linear switched systems [22], [23], [24], [25], [26], [27], [28], where the proposed approaches are mainly relied on the properties of positive systems [29], [30]. The fundamental idea applied is to combine constant/time-varying similarity transformations with the observer gains design method to ensure both positivity and practical-stability of the switched dynamics of the estimation errors. For instance, in [22], [23], [24], [25] the case of continuous-time switched systems is considered while in [26], [27], [28] authors have treated the case of discrete-time switched systems. Moreover, some proposed methods solve the interval observers design problem by including others performance criteria. For instance, to obtain tight estimates of the exact state enclosure, an H_∞ synthesis method is used in [28] to computed the best observers gains that on one hand, satisfy the positivity and the stability of the estimation error, and on the other hand, minimize the distance between the upper and lower estimated state trajectory.

In this work, we propose new interval observer design methods for discrete-time linear switched systems, where the similarity transformation constraints are avoided. More precisely, we introduce new LMI-based conditions whose solutions allow to compute directly the upper and lower state trajectories of the actual state trajectory of the switched system without using any kind of similarity transformation. This feature represents the main difference of the proposed approach with respect to the aforementioned works and is the main contribution of this paper.

The paper is organized as follows. Some preliminaries are given in Section 2. The interval observer structure for a class of discrete-time linear switched systems is proposed in Section 3. The BMI-based and the LMI-based synthesis procedure of the interval observer gain matrices is detailed in Section 4. Numerical results are presented in Section 5 to show the effectiveness of the proposed approach and to compare its performance with that obtained by an optimal interval observer selected from the literature for its good performance.

2. Preliminaries

2.1. Notations

The set $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{N}$ are the set of real scalars, positive real scalars and positive integers including zero, respectively. The induced matrix norm for a matrix $A \in \mathbb{R}^{n\times n}$ will be denoted as $||\cdot||$. Any $p \times m$ matrix whose elements are all zeros are simply denoted by 0, respectively. I_p denotes the identity matrix in $\mathbb{R}^{p\times p}$.

Throughout this paper the inequalities must be understood component-wise, for matrices as well as for vectors, i.e. $A = (a_{i,j}) \in \mathbb{R}^{p\times m}$ and $B = (b_{i,j}) \in \mathbb{R}^{p\times m}$ such that $A \geq B$ if and only if, $a_{i,j} \geq b_{i,j}$ for all $i \in \{1,\ldots,p\}, j \in \{1,\ldots,m\}$. $M = \max\{A,B\}$ is the matrix where each entry is $m_{i,j} = \max\{a_{i,j}, b_{i,j}\}$. Let us define $A^+ = \max\{A,0\}$, $A^- = A^+ - A$; thus, the element-wise absolute value will be denoted as $|A| = A^+ + A^-$. A matrix $M \in \mathbb{R}^{n\times n}$ is said to be M-matrix if it has nonpositive off-diagonal and nonnegative diagonal entries. A matrix $P \in \mathbb{R}^{n\times n}$ is said to be negative definite if $\upsilon^T P \upsilon < 0$ for all non-zero real vectors $\upsilon \in \mathbb{R}^n$ and it will be denoted by $P \prec 0$. Similarly, $P \preceq 0$ means semi-negative definite matrix. $\mathscr{L}_\infty$ denotes the set of measurable and locally essentially bounded signal $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$, whose L_∞-norm is bounded.

2.2. Definitions

Definition 1. [31] Any matrix $M \in \mathbb{R}^{n\times n}$ of the form

$$M = \alpha I_n - N,$$
$$N \geq 0,\ \alpha > 0,$$

for which $\alpha > \rho(N)$, the spectral radius of N, is called an *M-matrix*.

Theorem 1. *[31] Let $M \in \mathbb{R}^{n\times n}$. Then each of the following conditions is equivalent to the statement: "M is non-singular M-matrix."*

(i) *$M + D$ is inverse-positive for each nonnegative diagonal matrix D.*
(ii) *$M + sI_n$ is inverse-positive for each scalar $s \geq 0$.*

In statement (ii) of Theorem 1, it is shown that any non-singular M-matrix has a nonnegative inverse matrix.

Definition 2. A symmetric nonsingular M-matrix is called a *Stieltjes* matrix.

The set of Stieltjes matrices of dimension $n \times n$ will be denoted by $\mathscr{S}^{n\times n}$

Definition 3 (Schur complement, [32])**.** Suppose Q and R are square and symmetric. If the matrix R is invertible, then the condition

$$\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \preceq 0$$

is equivalent to

$$R \preceq 0, \quad Q - SR^{-1}S^\top \preceq 0.$$

Definition 4 (Positive dynamics, [29])**.** A discrete-time linear system $x(k+1) = Ax(k) + \phi(k)$, with the state $x \in \mathbb{R}^n$ and the state matrix $A \in \mathbb{R}^{n\times n}$, is said to be positive systems if A is a nonnegative matrix and $\phi(k)$ is a nonnegative vector.

The solutions of positive systems, initiated at $x(k_0) \geq 0$, stay nonnegative: $x(k) \geq 0$ for all $k \geq k_0$.

3. Interval observer for discrete-time switched linear systems

Consider discrete-time switched linear systems of the form

$$\begin{cases} x(k+1) = A_\sigma x(k) + B_\sigma u(k) + w(k), \\ y(k) = C_\sigma x(k) + v(k), \end{cases} \qquad k \in \mathbb{N}, \sigma \in \mathscr{I} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^n$, and $y \in \mathbb{R}^p$ are respectively the state variables, the input, and the output of the system. $\sigma : \mathbb{R}_{\geq 0} \to \mathscr{I}$ is a right-continuous, piece-wise constant switching signal where $\mathscr{I}$ is a finite index set for subsystems. $w \in \mathbb{R}^n$ and $v \in \mathbb{R}^p$ represent the perturbation and the output uncertainty, respectively.

The goal here is to estimate an upper bound and a lower bound, that enclose the set of admissible values of the state variables of system (1) while ensuring the convergence of the estimation errors. To do so, let first introduce an assumption on the boundedness of the measurement uncertainty and perturbation.

Assumption 1. *Let $\overline{v}(k) \in \mathscr{L}_\infty^p$ be a given vector such that*

$$\forall k \in \mathbb{N} \quad |v(k)| \leq \overline{v}(k). \tag{2}$$

Let two vectors $\overline{w}(k), \underline{w}(k) \in \mathscr{L}_\infty^n$ be given such that

$$\forall k \in \mathbb{N} \quad \underline{w}(k) \leq w(k) \leq \overline{w}(k). \tag{3}$$

Assumption 2. *The initial state vector $x(k_0)$ satisfies*

$$\underline{x}(k_0) \ \leq x(k_0) \leq \overline{x}(k_0),$$

where $\underline{x}(k_0)$, $\overline{x}(k_0) \in \mathbb{R}^n$ are given constant vectors.

The interval observer that we propose in the present work has the following form:

$$\begin{cases} \underline{x}(k+1) = \underline{A}_\sigma^+ \underline{x}(k) - \underline{A}_\sigma^- \overline{x}(k) + B_\sigma u(k) \\ \qquad\qquad + \underline{w}(k) - |\underline{L}_\sigma| \overline{v}(k) - \underline{L}_\sigma y(k), \\ \overline{x}(k+1) = \overline{A}_\sigma^+ \overline{x}(k) - \overline{A}_\sigma^- \underline{x}(k) + B_\sigma u(k) \qquad k \in \mathbb{N}, \sigma \in \mathscr{I} \\ \qquad\qquad + \overline{w}(k) + |\overline{L}_\sigma| \overline{v}(k) - \overline{L}_\sigma y(k), \\ \quad x(k_0) \in [\underline{x}(k_0), \overline{x}(k_0)]. \end{cases} \tag{4}$$

where

$$\begin{aligned} \underline{A}_\sigma &= (A_\sigma + \underline{L}_\sigma C_\sigma) \\ \overline{A}_\sigma &= (A_\sigma + \overline{L}_\sigma C_\sigma) \end{aligned} \tag{5}$$

with $\underline{L}_\sigma$, $\overline{L}_\sigma \in \mathbb{R}^{n \times p}$ are the observer gains to be designed corresponding to the lower and the upper estimate, respectively.

Using the output model in (1), the state equation of this system can be re-written for all $k \in \mathbb{N}$ and $\sigma \in \mathscr{I}$ as follows:

$$\begin{aligned} x(k+1) =& A_\sigma x(k) + B_\sigma u(k) + w(k) + L_\sigma^\bullet [C_\sigma x(k) + v(k) - y(k)] \\ =& (A_\sigma + L_\sigma^\bullet C_\sigma) x(k) + B_\sigma u(k) + w(k) + L_\sigma^\bullet [v(k) - y(k)] \\ =& [(A_\sigma + L_\sigma^\bullet C_\sigma)^+ - (A_\sigma + L_\sigma^\bullet C_\sigma)^-] x(k) + B_\sigma u(k) + w(k) + L_\sigma^\bullet [v(k) - y(k)] \end{aligned} \tag{6}$$

where $L_\sigma^\bullet \in \{\underline{L}_\sigma, \overline{L}_\sigma\}$.

The dynamics of both estimation error bounds $\underline{e}(k) = x(k) - \underline{x}(k)$ and $\overline{e}(k) = \overline{x}(k) - x(k)$ can be described by the following dynamical system:

$$\begin{cases} \underline{e}(k+1) = \underline{A}_\sigma^+ \underline{e}(k) + \underline{A}_\sigma^- \overline{e}(k) + (w(k) - \underline{w}(k)) \\ \qquad\qquad + (\underline{L}_\sigma v(k) + |\underline{L}_\sigma| \overline{v}(k)) \\ \overline{e}(k+1) = \overline{A}_\sigma^+ \overline{e}(k) + \overline{A}_\sigma^- \underline{e}(k) + (\overline{w}(k) - w(k)) \\ \qquad\qquad + (|\overline{L}_\sigma| \overline{v}(k) - \overline{L}_\sigma v(k)) \end{cases} k \in \mathbb{N}, \sigma \in \mathscr{I} \tag{7}$$

Due to the coupling between the upper and the lower estimate errors, we define the augmented error of the interval estimation as $\xi = [\underline{e}^\top, \overline{e}^\top]^\top$, then the estimation errors dynamics (7) can be written as a discrete-time system of the following form

$$\xi(k+1) = \Lambda_\sigma \xi(k) + \Psi_\sigma(k), \quad \forall \sigma \in \mathscr{I} \tag{8}$$

where

$$\Lambda_\sigma = \begin{bmatrix} \underline{A}_\sigma^+ & \underline{A}_\sigma^- \\ \overline{A}_\sigma^- & \overline{A}_\sigma^+ \end{bmatrix}; \quad \Psi_\sigma(k) = \begin{bmatrix} w(k) - \underline{w}(k) \\ \overline{w}(k) - w(k) \end{bmatrix} + \begin{bmatrix} \underline{L}_\sigma v(k) - |\underline{L}_\sigma| \underline{v} \\ |\overline{L}_\sigma| \overline{v} - \overline{L}_\sigma v(k) \end{bmatrix}. \tag{9}$$

Theorem 2. *Let Assumptions 1 and 2 hold. For given gains matrices* $\underline{L}_\sigma$, $\overline{L}_\sigma \in \mathbb{R}^{n\times p}$ *and a positive scalar* δ*, if there exist a symmetric positive semi-definite matrix* $P \in \mathbb{R}^{2n\times 2n}$ *and a positive scalar* β *such that*

$$\begin{bmatrix} -P + \beta I_{2n} & \Lambda_\sigma^\top P \\ \star & -\frac{P}{(1+\delta)} \end{bmatrix} \preceq 0 \quad \forall \sigma \in \mathscr{I} \tag{10}$$

are satisfied, then the system (4) *is an interval observer for the discrete-time switched linear system* (1)*.*

Proof. Positivity of the estimation errors:

Based on inequalities (2) and (3) of Assumption 1, we have that the vectors $\Psi_\sigma(k)$ are nonnegative for all $k \in \mathbb{N}$ and $\sigma \in \mathscr{I}$. The matrices Λ_σ are nonnegative by construction. From the augmented error equation (8)-(9), the positivity property of matrices $\Psi_\sigma(k)$ and Λ_σ implies the nonnegativity of the augmented error $\xi(k)$ provided that $\xi(k_0)$ is nonnegative, which in turn implies the nonnegativity of the upper and the lower estimation errors $\overline{e}(k)$ and $\underline{e}(k)$, respectively. Consequently, the interval observer (4) with the system (1) preserve the order relation $\underline{x}(k) \leq x(k) \leq \overline{x}(k)$ for all $k \geq k_0$.

Stability of the estimation errors:

For studying the stability, we propose the following quadratic Lyapunov function for the augmented estimation error

$$V(\xi) = \xi^\top P \xi \tag{11}$$

The variation of $V(\xi)$ is given as follows

$$\Delta V(\xi) = \xi^\top \Lambda_\sigma^\top P \Lambda_\sigma \xi + 2\xi^\top \Lambda_\sigma^\top P \Psi_\sigma - \xi^\top P \xi$$

For any positive scalar δ, by using the Young's inequality $2a^\top xb \leq \delta a^\top xa + \frac{1}{\delta} b^\top xb$, one gets

$$\Delta V(\xi) \leq \xi^\top \left[(1+\delta)\Lambda_\sigma^\top P \Lambda_\sigma - P\right]\xi + (1 + \frac{1}{\delta})\Psi_\sigma^\top P \Psi_\sigma$$

By the Schur complement, the inequality (10) is equivalent to

$$\begin{gathered} (1+\delta)\Lambda_\sigma^\top P \Lambda_\sigma - P \preceq -\beta I_{2n} \\ P \succeq 0 \end{gathered} \tag{12}$$

Consequently, the variation of the Lyapunov function is upper bounded as follows:

$$\Delta V(\xi) \leq -\beta \xi^\top \xi + (1 + \frac{1}{\delta})\Psi_\sigma^\top P \Psi_\sigma \tag{13}$$

So, by similar argument of [33, Definition 4.4], the estimation errors are Input-to-State Stable ISS relatively to the perturbation w and the noise v. Moreover, the estimation errors are exponentially stable in the case where the system (1) is free from state disturbances and measurement noises. □

So far, a verification method has been given. In the following section we propose a method to design the interval observer gain matrices.

4. Design methodology

In this section we will present the design of the interval observer gain matrices based on the internal positivity of the estimation error.

4.1. BMI-based design

Proposition 1. *Let $A_d \in \mathbb{R}^{n\times n}$ be a nonnegative matrix. If there exist two nonnegative (positive) matrices $A_a, E \in \mathbb{R}^{n\times n}$ such that $A_d = A_a + E$, then the solution $\chi_d(k)$ of the system $\chi_d(k+1) = A_d\chi_d(k)$ and the solution $\chi(k)$ of the system $\chi(k+1) = A_a\chi(k)$ satisfy $\chi_d(k) \geq \chi(k)$ ($\chi_d(k) > \chi(k)$) $\forall k \in \mathbb{N}$ provided that $\chi_d(0) \geq \chi(0) \geq 0$.*

Proof. Starting from the solution sequence $\chi_d(k)$, we have

$$\begin{aligned} \chi_d(k+1) &= A_d\chi_d(k) && (14)\\ &= (A_a+E)\chi_d(k) && (15)\\ &= A_a\chi_d(k) + E\chi_d(k), && (16) \end{aligned}$$

Based on the nonnegativity (positivity) of the matrices E and A_a and $\chi_d(0) \geq \chi(0) \geq 0$, we can say that the following system

$$\chi_d(k+1) = A_d\chi_d(k)$$

has a nonnegative dynamics and the term $\Delta\chi_d(k)$ is nonnegative. Consequently, from equation (16) we have

$$\begin{aligned} \chi_d(k+1) &\geq A_a\chi_d(k) && (17)\\ &\geq A_a\chi(k) = \chi(k+1). && (18) \end{aligned}$$

This concludes the proof. □

Proposition 2. *Let $M \in \mathbb{R}^{n\times n}$, for any two nonnegative matrices $M_P, M_N \in \mathbb{R}^{n\times n}_{\geq 0}$ satisfy $M = M_P - M_N$, there exits a nonnegative matrix $\Delta \in \mathbb{R}^{n\times n}_{\geq 0}$ such that $M_P = (M^+ + \Delta)$ and $M_N = (M^- + \Delta)$.*

Proof. For any element M_{ij} of the matrix M, if $M_{ij} \leq 0$ then $M^-_{ij} = |M_{ij}|$ and $M^+_{ij} = 0$, and we have $M_{ij} = M_{Pij} - M_{Nij} = (+\Delta) - (M^-_{ij} + \Delta)$, consequently, the condition $M_{Pij} \geq 0$ implies $\Delta \geq 0$. The same property for $M_{ij} \geq 0$. □

Theorem 3. *Let Assumptions 1 and 2 hold. For a given $\delta > 0$, if there exist symmetric positive definite matrix $P \in \mathbb{R}^{2n\times 2n}$, nonnegative matrices $H_{i,\sigma} \in \mathbb{R}^{n\times n}_{\geq 0}$, $i = \{1,\dots,4\}$, two matrices $\underline{L}_\sigma$, $\overline{L}_\sigma$ and a positive scalar β such that*

$$\begin{bmatrix} -P + \beta I_{2n} & H_\sigma^\top P \\ \star & -\frac{P}{(1+\delta)} \end{bmatrix} \preceq 0, \tag{19a}$$

$$H_{1,\sigma} - H_{2,\sigma} = A_\sigma + \underline{L}_\sigma C_\sigma, \tag{19b}$$

$$H_{4,\sigma} - H_{3,\sigma} = A_\sigma + \overline{L}_\sigma C_\sigma, \tag{19c}$$

with $H_\sigma = \begin{bmatrix} H_{1,\sigma} & H_{2,\sigma} \\ H_{3,\sigma} & H_{4,\sigma} \end{bmatrix}$, are satisfied $\forall \sigma \in \mathscr{I}$, then the system (4) is an interval observer for the system (1).

Proof. From equation (6), one can write

$$\begin{aligned} x(k+1) =& \big(A_\sigma^{\bullet+} - A_\sigma^{\bullet-}\big)x(k) \\ &+ B_\sigma u(k) + w(k) + L_\sigma^\bullet[v(k) - y(k)] \\ =& \big([A_\sigma^{\bullet+} + \Delta_\sigma^\bullet] - [A_\sigma^{\bullet-} + \Delta_\sigma^\bullet]\big)x(k) \\ &+ B_\sigma u(k) + w(k) + L_\sigma^\bullet[v(k) - y(k)] \end{aligned} \tag{20}$$

where $\Delta^\bullet \in \{\underline{\Delta},\, \overline{\Delta}\}$. Recalling $\underline{A}_\sigma$ and $\overline{A}_\sigma$ are defined in (5), picking

$$H_{1,\sigma} = \underline{A}_\sigma^+ + \underline{\Delta}, \tag{21a}$$
$$H_{2,\sigma} = \underline{A}_\sigma^- + \underline{\Delta}, \tag{21b}$$
$$H_{3,\sigma} = \overline{A}_\sigma^- + \overline{\Delta}, \tag{21c}$$
$$H_{4,\sigma} = \overline{A}_\sigma^+ + \overline{\Delta}, \tag{21d}$$

constraints (19b)-(19c) are satisfied, and using Proposition 2, the nonnegativity of H_σ implies the nonnegativity of the matrix $\Delta_\sigma = \begin{bmatrix} \underline{\Delta}_\sigma & \overline{\underline{\Delta}}_\sigma \\ \underline{\Delta}_\sigma & \overline{\Delta}_\sigma \end{bmatrix}$. It is worth noting that the matrix H_σ is an internal positive realization of the matrix Λ_σ. Thus, using equation (20) the following system

$$\begin{cases} \underline{x}(k+1) =& H_{1,\sigma}\underline{x}(k) - H_{2,\sigma}\overline{x}(k) + B_\sigma u(k) \\ & + \underline{w}(k) - |\underline{L}_\sigma|\overline{v}(k) - \underline{L}_\sigma y(k) \\ \overline{x}(k+1) =& H_{4,\sigma}\overline{x}(k) - H_{3,\sigma}\underline{x}(k) + B_\sigma u(k) \\ & + \overline{w}(k) + |\overline{L}_\sigma|\overline{v}(k) - \overline{L}_\sigma y(k) \\ & \underline{x}(k_0) \le x(k_0) \le \overline{x}(k_0) \end{cases} \tag{22}$$

is a interval observer for the systems (1) provided that the constraints (19) are satisfied, for which the estimation error is given by

$$\begin{aligned} \xi(k+1) &= H_\sigma \xi(k) + \Psi_\sigma(k) \\ &= \big(\Lambda_\sigma + \Delta_\sigma\big)\xi(k) + \Psi_\sigma(k) \end{aligned} \tag{23}$$

By argument of Proposition 1, the stability of the dynamics of (23) implies the stability of estimation error (8). Consequently, the system (4), with the gains $\underline{L}_\sigma$, $\overline{L}_\sigma$ designed using (19), is an interval observer for the system (1), and this concludes the proof. □

4.2. LMI-based design

Corollary 4. *Let Assumptions 1 and 2 hold. For a given $\delta > 0$, if there exist two Stieltjes matrices $P_1, P_2 \in \mathscr{S}^{n\times n}$, nonnegative matrices $\Omega_{i,\sigma}$, $i = \{1,\dots,4\}$, two matrices $\underline{U}_\sigma$, $\overline{U}_\sigma$ and a positive scalar β such that*

$$\begin{bmatrix} -P + \beta I_{2n} & \Omega_\sigma^\top \\ \star & -\frac{P}{(1+\delta)} \end{bmatrix} \preceq 0, \tag{24a}$$
$$\Omega_{1,\sigma} - \Omega_{2,\sigma} = P_1 A_\sigma + \underline{U}_\sigma C_\sigma, \tag{24b}$$
$$\Omega_{4,\sigma} - \Omega_{3,\sigma} = P_2 A_\sigma + \overline{U}_\sigma C_\sigma, \tag{24c}$$

where $\Omega_\sigma = \begin{bmatrix} \Omega_{1,\sigma} & \Omega_{2,\sigma} \\ \Omega_{3,\sigma} & \Omega_{4,\sigma} \end{bmatrix}, P = \begin{bmatrix} P_1 & \mathbf{0} \\ \mathbf{0} & P_2 \end{bmatrix}$, *are satisfied* $\forall \sigma \in \mathscr{I}$, *then the system* (4) *with the gains matrices* $\underline{L}_\sigma = P_1^{-1}\underline{U}_\sigma$, $\overline{L}_\sigma = P_2^{-1}\overline{U}_\sigma$ *is an interval observer for the system* (1).

Proof. To relax the BMI design problem (19) to an LMI one, we consider the Lyapunov function (11) where $P = \begin{bmatrix} P_1 & \mathbf{0} \\ \mathbf{0} & P_2 \end{bmatrix}$ such that P_1 and P_2 are Stieltjes matrices. Based on the property of the inverse of Stieltjes matrix given in Theorem 1, we have that P_1^{-1} and P_2^{-1} are nonnegative matrices.
By writing $\Omega_\sigma = PH_\sigma$, we have that the set of BMI (19a) are equivalent to the set of LMIs (24a). Thus, one can write

$$H_\sigma = \begin{bmatrix} H_{1,\sigma} & H_{2,\sigma} \\ H_{3,\sigma} & H_{4,\sigma} \end{bmatrix} = \begin{bmatrix} P_1^{-1}\Omega_{1,\sigma} & P_1^{-1}\Omega_{2,\sigma} \\ P_2^{-1}\Omega_{3,\sigma} & P_2^{-1}\Omega_{4,\sigma} \end{bmatrix}$$

So, the nonnegativity of Ω_σ and the Stieltjes property of the matrix P imply the nonnegativity of the matrix H_σ.
Pre-multiplying both sides of equality (24b) and (24c) by P_1^{-1} and P_2^{-1}, respectively, we get

$$\begin{aligned} P_1^{-1}\Omega_{1,\sigma} - P_1^{-1}\Omega_{2,\sigma} &= A_\sigma + \underline{L}_\sigma C_\sigma, \\ P_2^{-1}\Omega_{4,\sigma} - P_2^{-1}\Omega_{3,\sigma} &= A_\sigma + \overline{L}_\sigma C_\sigma, \end{aligned} \tag{25}$$

Consequently, equations (25) are equivalent to constraints (19b)-(19c), and this concludes the proof. □

5. Illustrative examples

In order to illustrate the efficiency of the proposed design methods of interval observers, we consider the following examples.

5.1. Example 1

Consider the discrete-time linear switched system given in [28], of the form (1) without output noise ($v(k) = 0\ \forall k \in \mathbb{N}$) where

$$A_1 = \begin{bmatrix} -2.5 & 1 \\ -2 & -1.1 \end{bmatrix}, A_2 = \begin{bmatrix} 0.3 & -2 \\ 0 & 0.6 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -2.5 & 1 \\ -2 & -1.1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, B_2 = \begin{bmatrix} 6 \\ 1 \end{bmatrix}, B_3 = \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.2 & 0.8 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, C_3 = \begin{bmatrix} 0.1 & 1 \end{bmatrix}$$

In this example, we compare our design methods of Theorem 3 and Corollary 4 to the design methodology proposed in [28].
The interval observer gains designed in Theorem 3 are found by solving the BMI-based problem (19) using the YALMIP toolbox [34] based on Penlab 1.04 solver. The feasible solution is obtained as follows

$$\underline{L}_1 = \overline{L}_1 = \begin{bmatrix} 0.6104 \\ -0.2353 \end{bmatrix}, \underline{L}_2 = \overline{L}_2 = \begin{bmatrix} -0.7357 \\ 0.1239 \end{bmatrix},$$
$$\underline{L}_3 = \overline{L}_3 = \begin{bmatrix} 1.6435 \\ -0.0379 \end{bmatrix}.$$

The interval observer gains designed in Corollary 4 are found by solving the LMI-based problem (24) using the YALMIP toolbox [34] based on SeDuMi 1.3 solver. The feasible solution is obtained as follows

$$\underline{L}_1 = \overline{L}_1 = \begin{bmatrix} 0.0645 \\ -0.1975 \end{bmatrix}, \underline{L}_2 = \overline{L}_2 = \begin{bmatrix} -0.2967 \\ -0.0012 \end{bmatrix},$$
$$\underline{L}_3 = \overline{L}_3 = \begin{bmatrix} 0.6974 \\ -0.1512 \end{bmatrix}.$$

The design approach in [28] minimizes the upper bound of the H_∞ norm of the operator $\rho \to e$, where $e = \overline{x} - \underline{x}$ and $\rho = \overline{w} - \underline{w}$ are the width of the estimated interval and the width of perturbation uncertainty, respectively. Their solution is given as follows

$$L_1 = \begin{bmatrix} 0.026 \\ 0.0914 \end{bmatrix}, L_2 = \begin{bmatrix} 0.5416 \\ 0.0758 \end{bmatrix}, L_3 = \begin{bmatrix} 0.857 \\ 0.1028 \end{bmatrix},$$

and the similarity transformation matrices T_σ that make $T_\sigma(A_\sigma - L_\sigma C_\sigma)T_\sigma^{-1}$ nonnegative $\forall \sigma \in \mathscr{I}$ are given as

$$T_1 = \begin{bmatrix} 0.0901 & 0.6930 \\ 0.0901 & 0.3070 \end{bmatrix}, T_2 = \begin{bmatrix} 0.2372 & 0.8173 \\ 0.2372 & 1.8173 \end{bmatrix},$$
$$T_3 = \begin{bmatrix} 0.0191 & 0.9913 \\ 0.0191 & 0.0087 \end{bmatrix}.$$

In order to compare our results to the result of Dinh et al., in [28], we use the same known conditions of their simulation: the initial conditions $x_0 = [1,2]^\top$, $\overline{x}_0 = [1.5,\ 2.5]^\top$, $\underline{x}_0 = [0.5, 1.5]^\top$, the system perturbation and its bounds

$$\underline{w}(k) = -\begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \leq w(k) = \begin{bmatrix} 0.1\sin(0.5k) \\ 0.1\cos(0.5k) \end{bmatrix} \leq \overline{w}(k) = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$$

and the switching signal is given in Figure 1. The system input is chosen here as $u = 0.5\sin(0.1k)$.

The simulation results are presented in Figure 2 and Figure 3. The upper and the lower estimate bounds of the state variables are presented in Figure 2 for the LMI-based design of Corollary 4 and the BMI-based design of Theorem 3 along with the estimates obtained by the approach in [28].

In order to compare those approaches, we have also plotted the $2-$norm of the width of the interval estimate, defined as $|e|_2 = |\overline{x} - \underline{x}|_2$, for each approach in Figure 3. From Figure 3, one can see that our approaches provide tighter interval estimate than the one in [28]. Moreover, the LMI-based design in Corollary 4 provides the tightest interval estimates.

So far, we have compared our approach to the one in [28], in which the measurement noise is not considered. With the same simulation conditions of Example 1, we show in Figure 4 the

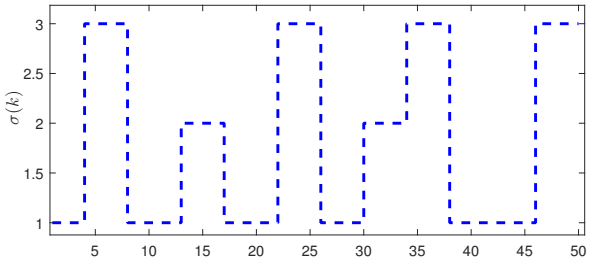

Figure 1: Switching signal.

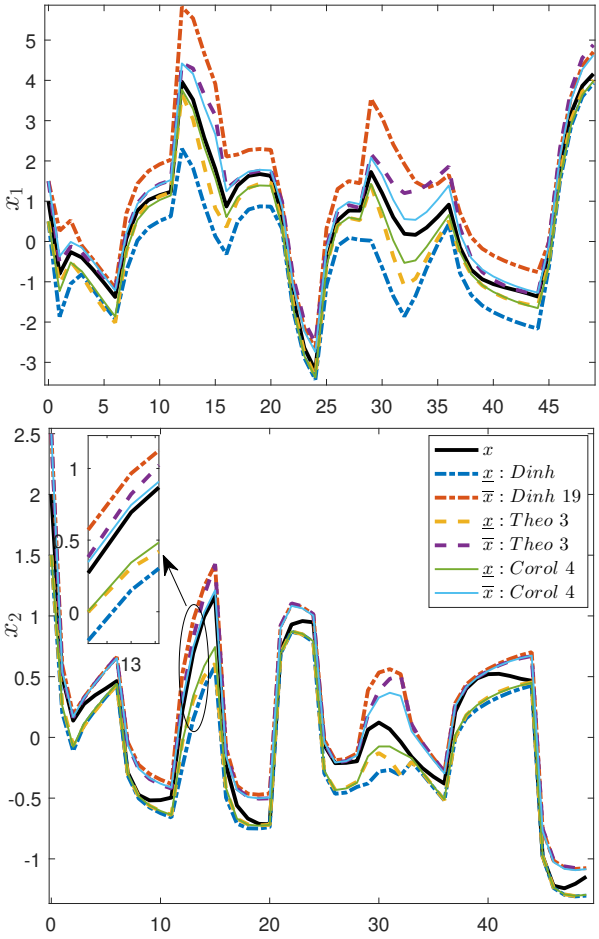

Figure 2: Interval estimates from different approaches: Corollary 4 (solid), Theorem 3 (dashed) and method of Dinh et al'19 [28] (dash-dotted).

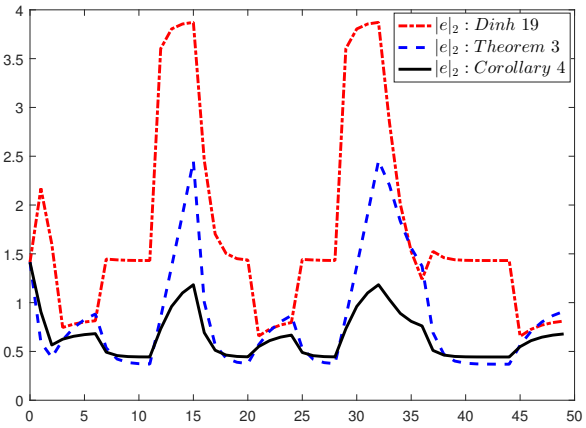

Figure 3: Widths of the interval estimates from different approaches: Corollary 4 (solid), Theorem 3 (dashed) and method of Dinh et al'19 [28] (dash-dotted).

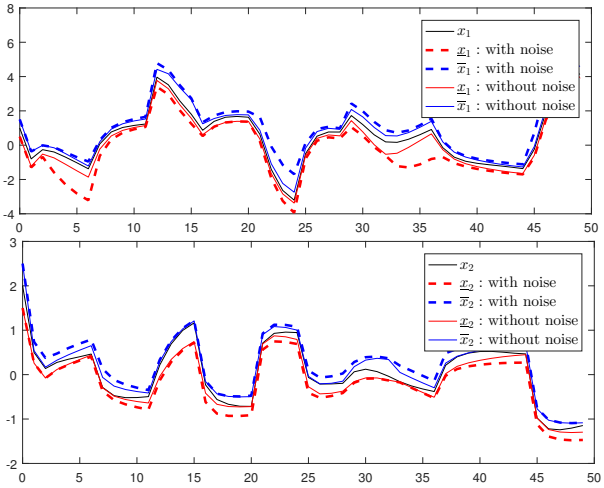

Figure 4: Interval estimates of the approach in Corollary 4 in the presence of measurement noise (dashed) and without noise (solid).

interval estimate of the approach based on Corollary 4 in the presence of the measurement noise. This noise is chosen as $v(k) = 0.5\cos(0.2k)$. The known upper bound of the noise is given as $\overline{v}(k) = 0.5$. Note that the system perturbation differs from zero for all cases of the simulation (with/ without noise).

5.2. Example 2

This example is borrowed from [27] where authors designed an interval observer for discrete-time linear switched systems without output noise using time-varying similarity transformations.

The considered system is as follows

$$A_1 = \begin{bmatrix} 0.34 & 0.05 \\ 0.40 & 0.27 \end{bmatrix}, A_2 = \begin{bmatrix} 0.15 & 0.24 \\ 0.08 & 0.32 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix}, B_2 = \begin{bmatrix} 0.6 \\ 0.3 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}, C_2 = \begin{bmatrix} 0.4 & 0.3 \end{bmatrix}$$

The observer gain matrices are found by solving the LMI-based design (24) in Corollary 4 as follows

$$\underline{L}_1 = \overline{L}_1 = \begin{bmatrix} -1.2813 \\ 2.1448 \end{bmatrix}, \underline{L}_2 = \overline{L}_2 = \begin{bmatrix} -0.5346 \\ 0.2297 \end{bmatrix}$$

we use the same known conditions of the simulation of Guo et al., in [27]: the initial conditions are $x_0 = [1,4]^\top$, $\overline{x}_0 = [5,\ 10]^\top$, $\underline{x}_0 = [-5,0]^\top$, the system input is $u = 10\sin(5k)$, the system perturbation and its bounds are given as follows

$$\underline{w}(k) = -\begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix} \leq w(k) = \begin{bmatrix} 0.8\cos(k) \\ 0.6\cos(k) \end{bmatrix} \leq \overline{w}(k) = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix},$$

and the switching signal is given in Figure 5. The simulation result of the interval estimates for

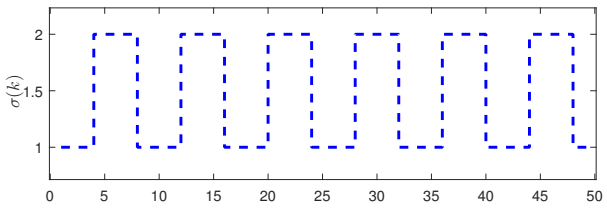

Figure 5: Switching signal.

the chosen systems is shown in Figure 6. Without reproducing the simulation of the approach in [27] but only by comparing their graphical results to the curves of our simulation in Figure 6, it is possible to say that our approach provides better interval estimates. Further comparisons with the aforementioned work, that should be pointed out, are as follows:

- Our interval observer has a simple structure and is designed in the original basis, while the one in [27] uses a time-varying similarity transformation.
- In this example, our approach could design the observer gain matrices for any arbitrary switching among systems (due to the use of a common Lyapunov function), while in [27], the switching among systems is restricted by an average dwell-time (due to the use of multiple Lyapunov functions).

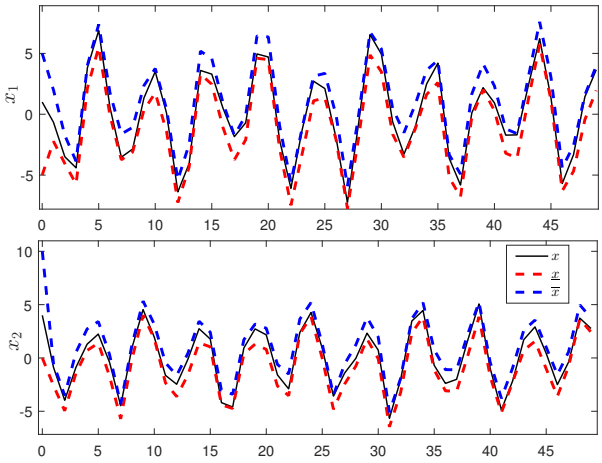

Figure 6: Interval estimates for Example 2.

6. CONCLUSIONS

This paper has presented a new simple methodology to design interval observers for a class of discrete-time linear switched systems. The interval observer is designed in the original coordinates using the internal positivity representation of the system without introducing any similarity transformations to satisfy the positivity property. The proposed structure of the interval observer here is simpler than the structures in [28] and [27]. Moreover, we take into account the presence of output noise which is not considered in the above-cited works. A numerical example shows the efficiency of the proposed interval observer comparing to the recent result in [28].

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